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A GENERAL MECHANISM OF INSTABILITY IN HAMILTONIAN SYSTEMS: SKIPPING ALONG A NORMALLY HYPERBOLIC INVARIANT MANIFOLD

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Abstract. We describe a recent method to show instability in Hamiltonian systems. The main hypothesis of the method is that some explicit transversality conditions – which can be verified in concrete systems by finite calculations – are satisfied.

In particular, for several types of perturbations of integrable Hamiltonian systems, the hypothesis can be verified by just checking that some Melnikov-type integrals have non-degenerate zeros. This holds for Baire generic sets of perturbations in the C^r-topology, for $r \in [3, \infty) \cup \{\omega\}$. Our method does not require that the unperturbed Hamiltonian system is convex, or that the perturbation is polynomial, which are non-generic properties.

Provided that the transversality conditions are verified, one concludes the existence of orbits which change the action coordinate by a quantity independent of the size of the perturbation. In fact, one can obtain orbits that follow any path in action space, up to an error decreasing with the size of the perturbation.

1. **Introduction and informal description.** The goal of this expository paper is to describe some recent developments in the geometric program for Arnol′d diffusion. These results are presented in Sections 3.1 and 3.2. We also announce some new results in Section 3.3. For the moment, we just indicate that the problem of Arnol′d diffusion refers to situations in which small perturbations of a Hamiltonian system accumulate over time to produce chaotic motions at a scale much larger than the size of the perturbation.

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In the geometric program, the first step is to identify geometric structures whose presence implies the existence of very rich sets of motions. The second step is to develop tools which allow to verify the existence of such structures in concrete problems of interest (e.g. in astrodynamics) or in generic systems.

The fact that some simple structures – whose existence can be established by a finite calculation –, lead to a rich set of orbits has a long history in dynamics. A famous example is the result [51, 15, 54] that a transverse homoclinic intersection implies the existence of infinitely many periodic orbits, as well as of other dynamical features such as horseshoes etc. Of course, by now there are many other results and a variety of tools to establish the existence of features in concrete systems (fixed point equations with interval arithmetic, combinatorial topology, etc) or in generic systems.

For the problem of Arnold diffusion, the first geometric structures proposed were chains of whiskered tori [1], with transverse intersection between the stable manifolds and the unstable manifolds of nearby tori. By now, whiskered tori have been supplemented by several other geometric objects that can be more effective for diffusion.

In this paper, we focus on the use of normally hyperbolic invariant manifolds (NHIM's). See Section 2.1 for the definition.

There are several mechanisms of Arnold diffusion based on NHIM's. Since NHIM's exist even in non-Hamiltonian systems, these mechanisms apply to more general systems (e.g., magnetic fields [44] or dissipative systems [37]). The mechanisms based on NHIM's can accommodate also general time-dependent perturbations (e.g., quasiperiodic [19], or time-recurrent [30]). In many cases, the use of NHIM's leads to optimal estimates on the diffusion time, – i.e., the time it takes an orbit to change its action coordinate by $O(1)$ with respect to the size of the perturbation –, and can show that the set of diffusing orbits has large Hausdorff dimension. See [25, 26, 39, 40, 55, 56, 57, 50, 10].

Mechanisms based on NHIM's can be verified in concrete systems via non-perturbative methods – e.g., numerical computations [11, 23, 10, 17, 35] –, and can be applied in astrodynamics and space mission design.

This paper does not aim to be a review paper and its only goal is to present the ideas in a recent geometric mechanism [34].

The basic idea of the mechanism described here is the following. The NHIM's have stable and unstable manifolds. When these stable and unstable manifolds intersect transversally – one of the main hypothesis in the mechanism – they generate families of (homoclinic) orbits that converge asymptotically to the NHIM both in the future and in the past. If one has many homoclinic orbits, then one can produce pseudo-orbits of the system by alternatively combining long segments of homoclinic excursions and long orbits in the NHIM. In the mathematical treatment of Arnol′d diffusion, a good deal of work is devoted to produce true orbits out of the pseudo-orbits. In applications to space mission design, pseudo-orbits that require small maneuvers are acceptable. For example, one can have a spacecraft follow one segment of a homoclinic orbit at zero cost, and at the end of that segment use a low-cost maneuver to jump to another segment of a homoclinic orbit. Such

a low-cost maneuver would replace the orbit in the NHIM that is required in the theoretical treatment.[1].

The first result – see Theorem 3.1 – is that, when the intersections are transverse (plus some other transversality properties), one can shadow these pseudo-orbits (i.e., show that there are true orbits that follow them) provided that each of the orbit segments is long enough.

One convenient tool to quantify the homoclinic excursions and putting them in the same footing as the orbits in the manifold is the scattering map (discussed in more detail in Section 2.2). The main idea is that if we consider a long segment of a homoclinic orbit to the NHIM, in the future it resembles the orbit of a point in the NHIM and in the past it resembles another orbit. The orbit mimicked in the future may be different from the orbit mimicked in the past. The scattering map gives the asymptotic orbit in the future as a function of the asymptotic orbit in the past. If we denote the scattering map by σ, a long segment of the homoclinic orbit will be in a neighborhood of $f^{-N}(x)$ at time $-N$ and will be in a neighborhood of $(f^N \circ \sigma)(x)$ at time N. The content of Theorem 3.1 is that we can concatenate such excursions.

The power of the language of scattering maps comes from the fact that, as a consequence of Theorem 3.1, we can think of them as generators of a second dynamics, not just as book-keeping devices. The dynamics generated by the scattering map, however, allows to obtain consequences for the original dynamics. In Corollary 1, we obtain that if the system has recurrence, one can shadow orbits of the scattering map itself. Moreover, we present a dichotomy: either the dynamics in the NHIM is unbounded, or the system has recurrence and one can shadow the scattering map itself.

The second result – see Theorem 3.2 – is to show that, under some mild extra conditions, for nearly integrable Hamiltonian systems the orbits constructed above have large drift in the action coordinates. The precise statements and the proofs are contained in [34].

One remarkable aspect of the new developments is that there is almost no information required on the dynamics on the NHIM (just recurrence is more than enough). All the hypotheses used in the new mechanism regard properties of the family of homoclinic orbits.

It is important to note that the hypothesis on the existence of intersections of the stable and unstable manifolds of the NHIM, and the hypothesis on the family of homoclinic orbits (see Section 2.2) are transversality conditions, hence they are very robust and very generic. The use of symplectic properties is rather minimal, mainly that, in the symplectic case, the Melnikov potential has always critical points. If these points are non-degenerate, they give the transverse intersection between the manifolds.

We also note that the existence of NHIM's and homoclinic intersections are rather 'soft' methods which work with low regularity and even in infinite dimensional systems. The present method does not rely on sophisticated tools such as KAM theory, Aubry-Mather theory, etc.

[1]Of course, to translate these ideas into commercial technology one needs to develop a theoretical approach to optimize the maneuvers with respect to cost, time of flight, reliability, etc., which are deep mathematical problems now being actively explored.

The new results Theorems 3.3 and 3.4 can take advantage of having several scattering maps. If multiple scattering maps are available, we can shadow the orbits of the iterated function system generated by the scattering maps.

Note that, even if the dynamics of one map can have difficulties moving long distances (KAM and Nekhoroshev theorems), the dynamics of several maps do not have such restrictions. It is very unusual to have objects that are invariant for two maps.

As a consequence, we show that if the scattering vector fields (3.8) – the scattering map for infinitesimal values of the perturbation parameter – satisfy some non-degeneracy conditions that are generic, then there exist orbits that follow any prescribed path in the action space. In particular, they visit any neighborhood of a certain size – that goes to zero with the perturbation parameter – contained in a domain independent of the perturbation.

In the rest of this exposition we will give some more details on the nature of the hypotheses; they will typically be formulas that have to give a non-zero result. Such hypotheses can be verified with finite computations to establish the conclusions in concrete systems. The nature of the conditions is that they can be considered as transversality conditions in a space of mappings, so one can obtain rather strong genericity results.

For brevity, we can only hope to give the main ideas only of the skipping method in [34]. We cannot even hope to make justice to the geometric program (see [22] for an attempt to survey up to 2006, and [21] for extended list of references).

We omit important developments in the geometric program (e.g., the resonant cylinders, the anti-integrable limit, the separatrix map, normally hyperbolic laminations, etc.).

Of course, the geometric program is not the only program for the study of Arnol′d diffusion in the pure mathematics literature. A variational program was formulated in [45, 46, 47].

A hybrid program in which geometric methods are used to produce approximate orbits and then variational methods to produce shadowing orbits was started in [2, 3] using local variational methods. Global variational methods for shadowing have been used in [12, 16, 4, 42]. It seems that variational methods require a Hamiltonian structure, positive definiteness, and are restricted to time-periodic perturbations.

To simplify the exposition, we will present the general result Theorem 3.1 only for the case of maps. This allows us to lower the dimension[2] of the phase space. The corresponding results for flows can be deduced by the usual procedure of taking surfaces of section, reduction to lower dimensional invariant manifolds, or proved directly using the same ideas as for maps.

The rest of the results Theorems 3.2, 3.3 and 3.4 are presented for Hamiltonian flows because in this case more explicit formulas and conditions can be given.

2. **Preliminary notions.**

In this section, we collect some more or less standard definitions and results so that we can make the exposition largely self-contained and set the notation.

2.1. **Standard definitions and results on normal hyperbolicity.**

Let $f : M \to M$ be a C^r-diffeomorphism of a C^r-differentiable manifold M. Following [27, 28, 41, 49] we say that a $\Lambda \subset M$ with $f(\Lambda) = \Lambda$ is a normally hyperbolic

[2]It is well known that Arnol′d diffusion can only happen when the dimension of phase space is 6 for flows or 4 for maps.

invariant manifold (NHIM) if there exists a splitting of the tangent bundle of TM into Df-invariant sub-bundles

$$TM = E^u \oplus E^s \oplus T\Lambda,$$

and there exist a constant $C > 0$ and rates

$$0 < \lambda_+ < \eta_- \leq 1 \leq \eta_+ \leq \mu_-, \tag{2.1}$$

such that for all $x \in \Lambda$ we have

$$\begin{aligned} v \in E^s_x &\Leftrightarrow \|Df^k_x(v)\| \leq C\lambda^k_+\|v\| \text{ for all } k \geq 0, \\ v \in E^u_x &\Leftrightarrow \|Df^k_x(v)\| \leq C\mu^{-k}_-\|v\| \text{ for all } k \leq 0, \\ v \in T_x\Lambda &\Leftrightarrow \|Df^k_x(v)\| \leq C\eta^k_+\|v\|, \quad \|Df^{-k}_x(v)\| \leq C\eta^{-k}_-\|v\|, \text{ for all } k \geq 0. \end{aligned} \tag{2.2}$$

Even if many standard references assume that Λ is compact, this is not needed if one assumes that the definition of C^r spaces includes uniform continuity and uniform boundedness in the derivatives. [6, 5]. This allows to develop the theory even in infinite dimensional problems. We note that the proof of Theorem 3.1 presented in [34] based on nested balls works even in the infinite dimensional context.

If $Df(x), Df^{-1}(x)$ are uniformly bounded, we have that there are opposite inequalities, namely there exist $\lambda_- \leq \lambda_+$ and $\mu_+ \geq \mu_-$ such that

$$\begin{aligned} v \in E^s_x &\implies \|Df^k_x(v)\| \geq C\lambda^k_-\|v\| \text{ for all } k \geq 0, \\ v \in E^u_x &\implies \|Df^k_x(v)\| \geq C\mu^{-k}_+\|v\| \text{ for all } k \leq 0. \end{aligned} \tag{2.3}$$

The normal hyperbolicity of Λ implies that there exist stable and unstable invariant manifolds, $W^s(\Lambda)$, $W^u(\Lambda)$ formed by points whose orbits are asymptotic to orbits of points in Λ in the future or in the past.

In the case when f is symplectic, it is natural to consider hyperbolic manifolds with the property that

$$\eta_- = 1/\eta_+, \quad \lambda_+ = 1/\mu_-, \text{ and } \lambda_- = 1/\mu_+. \tag{2.4}$$

As shown in [20], normally hyperbolic invariant manifolds for symplectic maps with the restricted exponents as in (2.4) enjoy many geometric properties (e.g., the map restricted to the manifold is symplectic).

Assume that there exists an integer $\ell > 0$ such that

$$\ell \leq \min(r, \log \lambda_-^{-1}/\log \eta_+^{-1}, \log \eta_-/\log \mu_+).$$

Then Λ is C^ℓ-differentiable, and its stable and unstable manifolds $W^s(\Lambda)$, $W^u(\Lambda)$ are C^ℓ-differentiable manifolds. See [53]. It is also well known that one cannot expect, in general, more differentiability of the manifolds even if the map f is analytic.

From now on, we assume that r and the hyperbolic structure are so that $\ell \geq 2$.

The manifolds $W^s(\Lambda)$, $W^u(\Lambda)$ are foliated by stable and unstable manifolds of points $W^s(z)$, $W^u(z')$ respectively, with $z, z' \in \Lambda$, which are C^r-differentiable manifolds. The foliations are $C^{\ell-1}$-differentiable. Again this regularity is known to be optimal in general.

For each $x \in W^s(\Lambda)$ there exists a unique $x^+ \in \Lambda$ such that $x \in W^s(x^+)$, and for each $x \in W^u(\Lambda)$ there exists a unique $x^- \in \Lambda$ such that $x \in W^u(x^-)$. We define the wave maps:

$$\begin{aligned} \Omega^+ &: W^s(\Lambda) \to \Lambda \text{ by } \Omega^+(x) = x^+, \\ \Omega^- &: W^u(\Lambda) \to \Lambda \text{ by } \Omega^-(x) = x^-. \end{aligned}$$

The maps Ω^+ and Ω^- are $C^{\ell-1}$-smooth.

The interpretation of the maps $\Omega^\pm$ is that, given a point in the stable (unstable) manifold, its orbit gets close very fast in the future (or in the past) to the orbit of a unique point in Λ. The maps $\Omega^\pm$ applied to a point x give the point $x^\pm$ in the manifold whose orbit is asymptotic to the orbit of x (in the future or in the past). Note that in the intersections of the stable and unstable manifolds, we can define both $\Omega^\pm$.

2.1.1. *Dependence on parameters in the theory of normally hyperbolic invariant manifolds.* For our purposes, it is important to note that these objects depend smoothly on parameters. Given a family f_ε, if f_0 admits a NHIM Λ_0, for each $|\varepsilon|$ small enough, we can find a NHIM Λ_ε for f_ε, which form a smooth family[3].

Similarly, we obtain smooth dependence of the stable and unstable manifolds and of the projections $\Omega^\pm$.

One way to obtain Λ_ε is to find C^ℓ-smooth maps $k_\varepsilon : \Lambda_0 \to M$, with $k_0 = \mathrm{Id}_{\Lambda_0}$,

$$k_\varepsilon(\Lambda_0) = \Lambda_\varepsilon, \tag{2.5}$$

so that k_ε is a diffeomorphism from Λ_0 to Λ_ε. Indeed, there are many such k_ε's.

In [20] it is shown that there is a very natural way to select the parametrization k_ε; we require that it gives a graph of the hyperbolic variables over the central ones (which give the dynamics in the NHIM):

$$\frac{d}{d\varepsilon} k_\varepsilon(x) \in E^s_x \oplus E^u_x, \tag{2.6}$$

where the splitting $E^s_x \oplus E^u_x$ corresponds to the invariant manifold of f_ε.

Notice that the mapping k_ε giving the parameterization is not uniquely defined, since any reparametrization of Λ_0 will also give (2.5). The normalization (2.6) eliminates this possibility, so that the $\frac{d}{d\varepsilon}$ is determined uniquely. This makes it plausible that it is a good normalization.

In [20] it is shown that k_ε satisfying (2.5) and (2.6) exists, is unique and, more surprisingly, is symplectic if f is symplectic. That is, $k_\varepsilon^* \Omega_{\Lambda_\varepsilon} = \Omega_{\Lambda_0}$, where $\Omega_{\Lambda_\varepsilon}$, Ω_{Λ_0} denote the pull backs of the symplectic form on M to Λ_ε, Λ_0, respectively. Hence, if g_ε is symplectic on Λ_ε (i.e $g_\varepsilon^* \Omega_{\Lambda_\varepsilon} = \Omega_{\Lambda_\varepsilon}$), then

$$\left(k_\varepsilon^{-1} \circ g_\varepsilon \circ k_\varepsilon\right)^* \Omega_{\Lambda_0} = \Omega_{\Lambda_0}. \tag{2.7}$$

We note that the differentiability of these objects, which is established by the general theory, has an important consequence. When we consider perturbation theory, we start from the knowledge that the derivatives exist and we only need to identify the formulas for what the derivatives are. Since we know that they exist, this can be done by easy methods such as matching power expansions and solving the resulting equations.

2.2. **The scattering map.** The scattering map is an effective way to quantify homoclinic excursions, and enjoys remarkable geometric properties. See [20]. The scattering map is defined on the NHIM, hence it makes it very comfortable to discuss at the same time the homoclinic excursions and the orbits in the NHIM.

[3]The standard trick is to consider the extended map $F_0(x,\varepsilon) = (f_0, \varepsilon), F(x,\varepsilon) = (f_\varepsilon, \varepsilon)$. If F_0, F are C^r close, the NHIM for F is a smooth family of manifolds. The precise meaning of a smooth family of manifolds is that they are jointly smooth in the parameter and in the coordinates of the manifolds. This is straightforward but long to state and we refer to [20].

To define the scattering map we assume that there exists a transverse homoclinic manifold $\Gamma \subseteq M$, which is $C^{\ell-1}$-differentiable. This means that $\Gamma \subseteq W^u(\Lambda) \cap W^s(\Lambda)$ and, for each $x \in \Gamma$, we have the following transversality condition:

(A) The intersection of W^s_Λ and W^u_Λ is transverse along Γ:

$$\begin{aligned} T_x M &= T_x W^u(\Lambda) + T_x W^s(\Lambda), \\ T_x \Gamma &= T_x W^u(\Lambda) \cap T_x W^s(\Lambda). \end{aligned} \tag{2.8}$$

Moreover, to define the scattering map we require two extra conditions:

(B) The manifold $\Gamma \subset W^u_\Lambda \cap W^s_\Lambda$ is transverse to the foliations of the stable/unstable manifolds at each point:

$$\begin{aligned} T_x W^s(\Lambda) &= T_x W^s(x^+) \oplus T_x \Gamma, \\ T_x W^u(\Lambda) &= T_x W^u(x^-) \oplus T_x \Gamma, \end{aligned} \tag{2.9}$$

The final assumption for defining the scattering map is that

(C) The wave map $\Omega^- : \Gamma \to \Omega^-(\Gamma) \subset \Lambda$ is a $C^{\ell-1}$-diffeomorphism.

Note that given (2.8) and (2.9), the invertibility condition happens in an open set.

When the above assumptions are fulfilled we refer to Γ as a *homoclinic channel*, and define the associated scattering map as

$$\sigma^\Gamma = \Omega^+ \circ \left(\Omega^-|_\Gamma\right)^{-1} \tag{2.10}$$

If $\sigma^\Gamma(x^-) = x^+$, then there exits a unique $x \in \Gamma$ such that $W^u(x^-) \cap W^s(x^+) \cap \Gamma = \{x\}$. Note that the backwards orbit $f^{-n}(x)$ of x in M is asymptotic to the backwards orbit $f^{-n}(x^-)$ in Λ as $n \to \infty$, and the forward orbit $f^m(x)$ of x in M is asymptotic to the forward orbit $f^m(x^+)$ in Λ as $m \to +\infty$.

Informally, for the orbits in Γ, which are assymptotic in the future and in the past to orbits in Λ, σ^Γ gives the future asymptotics as a function of the past asymptotics.

From (2.10) one can show that the scattering map is $C^{\ell-1}$-differentiable.

Remark 1. For those readers with a background in mathematical physics, we note that if we consider the dynamics on the NHIM as the free dynamics and the homoclinic excursions as an interaction, this is quite analogous to the S matrix in quantum mechanics and the $\Omega^\pm$ are analogues of the Moeller wave operators. Theorem 2.1 below is analogous to the unitarity of the S matrix and the perturbation calculations in Section 2.2.1 are analogue to the Fermi Golden rule.

Remark 2. In general, the domain of the scattering map is not the whole manifold Λ. Even if we have the transversality conditions for all Γ, there could be global obstructions to considering the scattering map defined everywhere (monodromy). Continuing it along a closed path we obtain a different map. See [18].

Of course, from the point of view of establishing Arnol′d diffusion, non-trivial monodromy is a very favorable situation. It implies that there are several scattering maps that one can use to build more pseudo-orbits. Indeed, as we will see, the natural situation is that one has an infinite collection of scattering maps. In such case, it can be natural to define a scattering relation that consists of all pairs (x^-, x^+) for which $\sigma^\Gamma(x^-) = x^+$ for some scattering map σ^Γ. See [36].

A remarkable result in [20] is:

Theorem 2.1. *If f is a symplectic map, Λ is a symplectic NHIM, and Γ homoclinic channel satisfying (2.8), (2.9), then the corresponding scattering map σ^Γ is also symplectic.*

2.2.1. *Perturbative expansions of scattering map.* If we consider families of maps f_ε, the usual smooth dependence on parameters of transversal intersections shows that the intersections Γ_ε depend smoothly on parameters. The definition of the scattering map (2.10) shows also that the scattering map associated to Γ_ε depends smoothly on ε.

Via the parametrization k_ε of Λ_ε, defined as in (2.5) and (2.6), we can express the scattering map $\sigma_\varepsilon^{\Gamma_\varepsilon}$ in terms of the coordinates on Λ_0 as

$$s_\varepsilon = s_\varepsilon^{\Gamma_\varepsilon} \equiv k_\varepsilon^{-1} \circ \sigma_\varepsilon^{\Gamma_\varepsilon} \circ k_\varepsilon \tag{2.11}$$

From Theorem 2.1 and from (2.7) we obtain that s_ε is a smooth family of symplectic maps. Hence, we can describe s_ε in terms of a family of Hamiltonian functions using the standard deformation theory. More precisely, we have

$$\begin{aligned} &\frac{d}{d\varepsilon} s_\varepsilon = \mathcal{S}_\varepsilon \circ s_\varepsilon \\ &i(\mathcal{S}_\varepsilon)\Omega|_{\Lambda_0} = dS_\varepsilon \end{aligned} \tag{2.12}$$

where $\mathcal{S}_\varepsilon$ is a vector field, S_ε is a function, and $i(\cdot)$ denotes the interior product on differential forms; in our case, $i(\mathcal{S}_\varepsilon)$ maps the 2-form $\Omega|_{\Lambda_0}$ to the 1-form dS_ε.

In conclusion, because of the differentiability of the scattering map s_ε , its derivative respect to the parameter ε can be described by a Hamiltonian vector field $\mathcal{S}_\varepsilon$, which can be obtained from a Hamiltonian function S_ε.

Observe that, if we call $\psi(\varepsilon; \varepsilon_0, x)$ the general solution of the vector field $\mathcal{S}(\varepsilon, x) = \mathcal{S}_\varepsilon(x)$, with ε viewed as the 'time variable', ε_0 as the 'initial time', and x the 'initial point', then we have that $s_\varepsilon(x) = \psi(\varepsilon; 0, s_0(x))$. In particular, when $s_0 = \mathrm{Id}$ the scattering map is given by the time ε-map of the time-dependent Hamiltonian vector field $\mathcal{S}$.

The perturbation theory for the scattering map amounts to finding the function S_ε or at least formal expansions. This feature is of great advantage, since functions transform very easily under changes of coordinates, and allow for natural geometric computations. For example we can easily compute the effect of the perturbation on fast variables symplectically conjugate to slow variables.

At least in the first approximation it is easy to guess (up to a multiple) what the first order in ε of the Hamiltonian function S_ε should be taking into account: (i) It should depend linearly on the perturbing Hamiltonian, (ii) It has to depend only on the values of the perturbing Hamiltonian evaluated on the homoclinic orbit, (iii) It has to be independent of the choice of the origin of time.

In the case that the map f_ε is the Poincaré map of a Hamiltonian system H_ε, one has to integrate the Hamiltonian over the homoclinic orbit and subtract the integrals over the asymptotic orbits. In fact, [20] provides a perturbative formula for the scattering map:

$$s_\varepsilon = s_0 + \varepsilon J \nabla S \circ s_0 + O(\varepsilon^2) \tag{2.13}$$

where J is the almost complex structure compatible with the standard symplectic form, and S is the real valued C^ℓ-function on Λ_0 given by:

$$S(x) = \lim_{T\to+\infty} \int_{-T}^{0} \left[\frac{dH_\varepsilon}{d\varepsilon}_{|\varepsilon=0} \circ \phi_t(z) - \frac{dH_\varepsilon}{d\varepsilon}_{|\varepsilon=0} \circ \phi_t(s_0^{-1}(x)) \right] dt \qquad (2.14)$$
$$+ \lim_{T\to+\infty} \int_{0}^{T} \left[\frac{dH_\varepsilon}{d\varepsilon}_{|\varepsilon=0} \circ \phi_t(z) - \frac{dH_\varepsilon}{d\varepsilon}_{|\varepsilon=0} \circ \phi_t(x) \right] dt.$$

where $z \in W^u((s_0)^{-1}(x)) \cap W^s(x) \cap \Gamma_0$ and ϕ_t is the flow corresponding to the unperturbed Hamiltonian H_0. See also [31].

The normal hyperbolicity of Λ_0 ensures that $\phi_t(z) - \phi_t(s_0^{-1}(x))$ and $\phi_t(z) - \phi_t(x)$ converge to zero exponentially fast as $t \to \mp\infty$ respectively. This makes the integral in (2.14) absolutely convergent with its derivatives.

Similar calculations were done in the past using Melnikov theory, but this typically needed to assume properties of the asymptotic orbits. In contrast, the scattering map allows to compute the effect of the perturbation on all homoclinic orbits to Λ_ε, irrespective of their asymptotic behavior. Note also that the integrals giving the first order approximation of the perturbed scattering map are improper integrals that converge uniformly (exponentially fast) as well as its derivatives up to an order (in the case of perturbations of integrable mappings all derivatives converge).

2.3. **Arnol′d diffusion.** There are many definitions of Arnol′d diffusion[4]. The basic idea is that perturbations in Hamiltonian systems accumulate and lead to effects much larger than their strength. These effects are supposed to happen for 'typical' perturbations[5].

The most interesting case is that of nearly integrable systems, since for them there are two important results that say that – under appropriate hypotheses – perturbations do not accumulate:

(A) The KAM theorem, which says that perturbations of size ε can only cause changes in the actions of order $\varepsilon^{1/2}$ in infinite time, except for a set of orbits of measure[6] smaller than order $\varepsilon^{1/2}$.
(B) The Nekhoroshev theorem, which says that there are positive numbers a, b depending only on the dimension of the system, such that changes in the actions of order ε^b can only happen after times of order $\exp(C\varepsilon^{-a})$.

In this note, besides perturbative cases, we will also consider non-perturbative situations, and we give verifiable conditions on the system that imply large effects in the phase space.

In both cases, we provide general results that can be applied to concrete systems.

We start with integrable systems[7]. These systems preserve the action variables, denoted by I.

[4]Two of the authors remember the conference: Hamiltonian Systems with three or more degrees of freedom (Sagaró, Spain, 1995), where, having an after dinner event with V. Arnol′d, M. Herman, G. Gallavotti, J. Moser, Y. Sinai where they were posed the question of coming up with a standard, precise definition. Specially V. Arnol′d was adamant about not doing it.

[5]We note that there are many different possibilities for *typical*. One can think of typical as being exhibited by many systems on their own or being exhibited by many families of one or more parameters. The precise meaning of 'many' can be taken to be generic in the sense of Baire in different topologies, or some more or less ad-hoc definitions.

[6]Using that a good part of the domain to which the KAM theory does not apply is covered by secondary tori, one can reduce the measure of the orbits for which the action can change [58].

[7]There are many definitions of an integrable system. See Section 2.3.1.

We will say that a family of Hamiltonians H_ε, with H_0 integrable, exhibits weak Arnol′d diffusion when there is a number $a > 0$ such that for all $0 < \varepsilon \ll 1$ there are trajectories such that the actions change by more than a.

We will say that a family H_ε exhibits strong Arnol′d diffusion when we can find an open set U in action space and a continuous function $\delta(\varepsilon)$ with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$ such that given any C^1-path $\gamma(s)$ in U for every $0 < \varepsilon \ll 1$, we can find a trajectory $x_\varepsilon(t)$ of H_ε in such a way that $|I(x_\varepsilon(\Phi(s_j))) - \gamma(s_j)| \leq \delta(\varepsilon)$ for some monotone function $\Phi(s)$ and for some sequence of values s_j such that $|\gamma(s_{j+1}) - \gamma(s_j)| \leq \delta(\varepsilon)$. Above, $I(\cdot)$ denotes the action coordinate of a point.

The strong version of Arnol′d diffusion is very similar to the notion of transitivity. It implies that all neighborhoods of a certain size $\delta(\varepsilon)$ are getting visited. We allow some error in the times and we allow that there are intermediate segments of the orbit that we do not investigate.

We assert that for every H_ε in the family, there is a neighborhood U in action space that is visited by some orbits. There are more ambitious results that one could hope. One first specifies the open set U and wants to obtain conditions on the family H_ε such that for small enough values of the parameter there exist orbits that visit all the neighborhoods of size $\delta(\varepsilon)$ in U. One can prescribe the set U uniformly for all the families. We refer to this version as the 'uniform' version of Arnol′d diffusion.

We will give explicit sufficient conditions that guarantee that the family satisfies the conclusions of the existence of Arnol′d diffusion.

We will also discuss the abundance (in different meanings) of families satisfying the hypotheses sufficient to obtain diffusion.

With the method presented below, the much stronger, in principle, uniform version of diffusion, follows with minor modification of the proof.

2.3.1. *Remarks on the distinction between 'a-priori stable' and 'a-priori unstable' systems.* Sometimes, in the literature, one finds distinctions among 'a-priori stable' and 'a-priori unstable' systems. These definitions were introduced in [9] for one-parameter analytic systems. A-priori stable corresponds to the case when the unperturbed system is integrable in the sense that it can be expressed in action-angle variables, and then all the actions are preserved, and a-priori unstable corresponds to the case when the unperturbed system has a subsystem with some hyperbolic fixed point – e.g., a pendulum factor, and the rest of the actions are preserved.

When considering systems involving two (or more) parameters $H_0 + \mu_1 H_1 + \mu_2 H_2$ the distinction is more muddled. Moving one of the parameters (under some mild non-degeneracy conditions) one can get normally hyperbolic invariant manifolds with weak hyberbolicity. Moving the second parameter allows obtain a NHIM with stable and unstable manifolds. If these manifolds intersect, moving the second parameter allows to generate transversal intersections. Hence, the study of a-priori stable systems with two parameters can be done by averaging and using the techniques of a-priori unstable systems.

In the analytic case, one has to face the problem that the second parameter has to be exponentially small with respect to the first, but in the finitely differentiable case it is enough to take the second parameter to be just a power of the first one [29].

Finally, we note that systems with general perturbations $H_0 + H_1$ can be considered as systems involving infinitely many parameters since the H_1 range in an infinite dimensional space.

3. **Statement of results.** We will present two results. The first one, Theorem 3.1 will be a non-perturbative shadowing lemma showing that if we have a pseudo-orbit obtained by alternatively applying the scattering map[8] and the dynamics restricted to the NHIM, then we can obtain nearby orbits. If the system has recurrence, one immediately obtains that one can shadow pseudo-orbits of the scattering map itself (see Corollary 1).

The second result, Theorem 3.2, shows that one can verify the hypothesis of Theorem 3.1 in a perturbative setting. This can be accomplished by verifying two conditions: (i) We first verify a condition that ensures that the stable and unstable manifolds intersect transversally, and that they define a scattering map in a domain of size one. (ii) Second, we verify a sufficient condition so that the scattering map associated to the homoclinic intersection is not trivial.

3.1. **The main shadowing result.**

Theorem 3.1 ([34]). *Assume that $f : M \to M$ is a C^r-map, $r \geq r_0$, $\Lambda \subseteq M$ is a normally hyperbolic invariant manifold, $\Gamma \subseteq M$ is a homoclinic channel, and $\sigma^\Gamma : \Omega^-(\Gamma) \to \Omega^+(\Gamma)$ is the scattering map associated to Γ. Assume that Λ and Γ are compact.*

Then, for every $\delta > 0$ there exists $n^* \in \mathbb{N}$ depending on δ, and a family of functions $m_i^* : \mathbb{N}^{2i+1} \to \mathbb{N}$, $i \geq 0$, depending on δ, such that, for every pseudo-orbit $\{y_i\}_{i\geq 0}$ in Λ of the form

$$y_{i+1} = f^{m_i} \circ \sigma^\Gamma \circ f^{n_i}(y_i), \tag{3.1}$$

for all $i \geq 0$, with $n_i \geq n^*$ and $m_i \geq m_i^*(n_0, \ldots, n_{i-1}, n_i, m_0, \ldots, m_{i-1})$, there exists an orbit $\{z_i\}_{i\geq 0}$ of f in M such that, for all $i \geq 0$,

$$z_{i+1} = f^{m_i+n_i}(z_i),$$

and

$$d(z_i, y_i) < \delta.$$

In Theorem 3.1 and below we assume that r_0 is sufficiently large.

We call attention that the main assumption of Theorem 3.1 is that the time we spend among the jumps is long enough and depends on the previous history.

We emphasize that Theorem 3.1 is not tied up to having a Hamiltonian structure. If one can verify the hypotheses in a non-Hamiltonian system, then one obtains the conclusions. Of course, the abundance of the hypotheses and the methods to check them are different in the Hamiltonian systems from the non-Hamiltonian ones. An interesting class of systems is that of non-Hamiltonian perturbations of Hamiltonian systems [37, 32].

In the case that the system is Hamiltonian, the following corollary is very natural.

Corollary 1 ([34]). *(a) If the map $f|_\Lambda$ satisfies Poincaré recurrence, then we can shadow every finite orbit of the scattering map $\{(\sigma^\Gamma)^n(x)\}_{n=1}^N$.*

(b) Moreover, we have the following alternative:

- *If the manifold Λ has finite volume, then every finite orbit of the scattering map $\{(\sigma^\Gamma)^n(x)\}_{n=1}^N$ can be shadowed.*
- *If the manifold Λ is unbounded, then either $\{(\sigma^\Gamma)^n(x)\}_{n=1}^N$ can be shadowed, or there is a positive measure set of unbounded orbits of the inner map in any neighborhood of $\{(\sigma^\Gamma)^n(x)\}_{n=1}^N$.*

[8] or in the case that there are several scattering maps, an arbitrarily chosen scattering map

In other words, if a finite orbit $(\sigma^\Gamma)^n(x)$ experiences Arnol′d diffusion, then either we get Arnol′d diffusion for the true orbits of the Hamiltonian system – by shadowing the orbits of σ^Γ through homoclinic excursions –, or we get Arnol′d diffusion just by drifting along the NHIM.

It is important to remark that Theorem 3.1 and Corollary 1 can be formulated just as well for a finite collection of different scattering maps (or even for an infinite collection of scattering maps), but in this case we will need to require $m \geq m_i^*$, $n \geq n_i^*$, where m_i^*, n_i^* depend not only on the previous n and m's but also on the scattering maps used up to the $(i+1)$-st jump. Hence, from the point of view of diffusing orbits, we can use iterations of several scattering maps in an arbitrary order (this is sometimes called 'polysystem' or 'iterated function system'). Note that polysystems have few barriers to diffusion [48, 7, 36]. The KAM tori for one map do not in general constitute barriers for another one.

3.2. **Perturbative results.** In view of Theorem 3.1, the only thing we have to do is to verify that, given a family of nearly integrable systems, there are some perturbative calculations that establish that all the elements of the family satisfy the hypotheses of Theorem 3.1 and that the orbits obtained are rich enough. Since the main hypothesis of Theorem 3.1 is a transversality hypothesis (i.e., the existence of a homoclinic channel Γ), we note that a first order perturbative calculation is enough to establish it. After that, the rich variety in the behavior of orbits is a consequence. There is no need to construct the orbits one by one.

For simplicity, we consider the nearly-integrable time-periodic Hamiltonian

$$H_\varepsilon(I,\phi,p,q,t) = h(I) + \sum_{i=1}^{n} \pm(\frac{1}{2}p_i^2 + V_i(q_i)) + \varepsilon H_1(I,\phi,p,q,t) \tag{3.2}$$

where $I \in \mathbb{R}^d, \phi \in \mathbb{T}^d$ are symplectically conjugate variables, $(p_i, q_i) \in \mathbb{R}^2$, $i = 1, \dots n$, are symplectically conjugate variables, and the phase $t \in \mathbb{T}^1$.

We assume

- The functions h, V_i, H_1 are C^r-differentiable, $r \geq r_0$.
- $V_i'(0) = 0$, $V_i''(0) < 0$, and $(V_i'(q) = 0 \,\&\, V_i(q) = V_i(0)) \implies q = 0$.

The second part of the assumption is just that each V_i has a non-degenerate maximum, which, without loss of generality we set at zero; furthermore, 0 is the only critical point at this energy level. This assumption implies that each of the penduli has two homoclinic orbits to the critical point 0. When we consider the system of n penduli we obtain 2^n geometrically different homoclinic orbits. Of course, we can also shift the origin of time independently in each of them.

We choose one homoclinic orbit for the system of penduli:

$$\begin{aligned}(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1})) = \big(&p_1^0(\tau_1 + t), \dots, p_n^0(\tau_n + t),\\ &q_1^0(\tau_1 + t), \dots, q_n^0(\tau_n + t)\big)\,,\end{aligned}$$

where $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$, and $\bar{1} = (1, \dots, 1) \in \mathbb{R}^n$, and $t \in \mathbb{R}$ is the time. One can think of the components of τ as a choice of the times at which the penduli perform jumps. It seems clear that, depending on the times at which the jumps take place, the external forcing will make the system gain energy or loose energy. Notice that the homoclinic orbits in the coupled system are parameterized by the state of the action-angle variables I, ϕ, the phase t of the system, and the times τ of jump of the penduli.

The observation is that, for $\varepsilon = 0$, the manifold Λ_0 given by $p = 0$, $q = 0$ is a normally hyperbolic invariant manifold foliated by invariant d-dimensional tori $I =$ const. Its stable and unstable manifolds are union of the separatrices corresponding to the tori, and they coincide. Consequently, for every $\varepsilon > 0$ sufficiently small, there is a normally hyperbolic invariant manifold Λ_ε for the time-one map of the flow of (3.2), which persists from the unperturbed case. (In fact, one first shows the persistence of the NHIM for the flow, and then reduce the problem to the case of maps by taking the time-one map.)

To study the splitting of the stable and unstable manifolds of Λ_ε, we define the Melnikov potential associated to this family of homoclinic intersections:

$$L(\tau, I, \phi, s) = -\int_{-\infty}^{\infty} \big[H_1(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \phi + \omega(I)t, s + t; 0) - H_1(0, 0, I, \phi + \omega(I)t, s + t; 0)\big]\, dt. \tag{3.3}$$

where $\omega(I) = \partial h/\partial I$.

The physical meaning of the function L in (3.3) is the following: fixing (I, ϕ, s) the gradient of L as a function of τ gives the first order expansion of the gain of energy due to a jump of the penduli. The proof of this is very simple using that the energies of the penduli are slow variables. See [21]. A streamlined proof for more general dependence on time is in [31].

Theorem 3.2. [34] *Assume the following conditions:*

(A.0) We have Poincaré recurrence in the manifold Λ_ε for all ε sufficiently small.

(A.1) There exists a set $U^- := \mathcal{I} \times \mathcal{J} \subset \mathbb{R}^d \times \mathbb{T}^{d+1}$, such that $\mathcal{I}$ is an open ball in $\mathbb{R}^d$, and for any values $(I, \phi, s) \in U^-$, the map

$$\tau \in \mathbb{R}^n \to L(\tau, I, \phi, s) \in \mathbb{R} \text{ has a non-degenerate critical point } \tau^*. \tag{3.4}$$

By the implicit function theorem, we can find a family of such critical points indexed by the other variables.

$$\tau^* = \tau^*(I, \phi, s).$$

Define

$$\mathcal{L}(I, \phi, s) = L(\tau^*(I, \phi, s), I, \phi, s), \quad \mathcal{L}^*(I, \theta) = \mathcal{L}(I, \theta, 0). \tag{3.5}$$

We regard $\mathcal{L}^*(I, \theta)$ as a function on the set

$$Dom(\mathcal{L}^*) = \{(I, \theta) \in \mathbb{R}^d \times \mathbb{T}^d \,|\, \exists s \in \mathbb{T}^1 \text{ s.t. } (I, \theta + \omega(I)s, s) \in U^-\}.$$

(A.2) The reduced Poincaré function $\mathcal{L}^*(I, \theta)$ associated to the critical point τ^* satisfies that $J\nabla\mathcal{L}^*(I, \theta)$ is transverse, relative to $\mathbb{R}^d \times \mathbb{T}^d$, to the level set $\{I = I_*\}$ at some point $(I_*, \theta_*) = (I_*, \phi_* - \omega(I_*)s)$, with $(I_*, \phi_*, s) \in U^-$. That is:

$$\frac{\partial \mathcal{L}^*}{\partial \theta}(I_*, \theta_*) \neq 0. \tag{3.6}$$

Then, there exists $\varepsilon_0 > 0$, and $\rho > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there exists a trajectory $x(t)$ of the Hamiltonian flow of Hamiltonian (3.2) and $T > 0$ such that

$$\|I(x(T)) - I(x(0))\| > \rho.$$

The meaning of the assumption (A.1) is that the perturbation generates a transverse intersection between the stable and unstable manifolds of Λ_ε for the time-one map of the flow of (3.2). The main content of the assumption is that the critical points τ^* are not degenerate. In [24, 26] one can find simple arguments that show that there are always infinitely many critical points for each value of (I, ϕ, s). These

points may be degenerate, but this is a situation that can be removed by an small perturbation [8].

Let us recall that each of these critical points τ^* gives rise to a transverse homoclinic intersection to Λ_ε and therefore to a scattering map. The meaning of the assumption (A.2) follows from the fact that $\mathcal{L}^*$ is the Hamiltonian generating the scattering map (see (2.12), (2.13), (2.14)). Moreover, in this case the unperturbed scattering map is the identity map in Λ_0 and therefore formula (2.12) becomes:

$$s_\varepsilon = \mathrm{Id} + \varepsilon J \nabla \mathcal{L}^* + O(\varepsilon^2). \tag{3.7}$$

This formula shows that s_ε is $\mathcal{O}(\varepsilon^2)$-close to the time-$\varepsilon$ map of the Hamiltonian vector field:

$$\dot{x} = J \nabla \mathcal{L}^*(x). \tag{3.8}$$

We will call (3.8) the scattering vector field. It can be considered as an infinitesimal version of the scattering map.

Condition (A.2) ensures that the flow of (3.8) has trajectories whose action I moves a quantity independent of the parameter ε. Nearby these trajectories one can find orbits of the scattering map with similar behavior by an argument similar to the study of the convergence of the Euler method in numerical analysis of ODE.

The computation of $\mathcal{L}^*$ requires a non-trivial calculation and takes advantage of beautiful cancelations afforded by the symplectic structure. As we discussed before, the fact that a similar formula should be true can be guessed by the fact that we know that the scattering map is given, in first order, by the symplectic gradient of a function, that the function has to be linear in the perturbation and can only depend on the connecting orbits in the unperturbed map and be invariant under translation. It is important to note that (A.1) and (A.2) are transversality conditions. They, therefore, hold generically in open sets of variables (I, ϕ, s) and parameters. One can be more ambitious and specify the set of (I, ϕ, s). In such case, of course, transversality conditions may fail for one intersection, but we have infinitely many intersections. See Section 3.4 for a more detailed discussion.

3.3. **Accessibility.** If a finite number of scattering maps are available we can take advantage that Theorem 3.1 to obtain the following theorem, whose proof will appear in [33]:

Theorem 3.3. *Assume the following conditions in the model (3.2).*

(A.0) We have Poincaré recurrence in the manifold Λ_ε for all ε sufficiently small.

(A.3) Consider the set $U^- = \mathcal{I} \times \mathcal{J}$ given in (A.1), and assume that for each point in U^- we can find K non-degenerate critical points of the Melnikov potential L and such that the corresponding reduced functions $\mathcal{L}_1^*, \ldots, \mathcal{L}_K^*$ satisfy

$$\mathit{Span}\,(J\nabla \mathcal{L}_1^*, \ldots, J\nabla \mathcal{L}_K^*) = \mathbb{T}^d \times \mathbb{R}^d$$

Then we have the strong form of Arnold diffusion in the ball $\mathcal{I}$ in the action space.

A more sophisticated argument, whose complete proof will appear in [33] and which requires that the Hamiltonian (3.2) is differentiable enough, yields:

Theorem 3.4. *Assume the following conditions in the model (3.2).*

(A.0) We have Poincaré recurrence in the manifold Λ_ε for all ε sufficiently small.

(A.3') Consider the set $U^- = \mathcal{I} \times \mathcal{J}$ given in (A.1), and assume that for each point in U^- we can find K non-degenerate critical points of the Melnikov potential L and such that the corresponding reduced functions $\mathcal{L}_1^*, \ldots, \mathcal{L}_K^*$ satisfy

$$Span\left(J\nabla\mathcal{L}_i^*, \left[J\nabla\mathcal{L}_i^*, J\nabla\mathcal{L}_j^*\right]\left[J\nabla\mathcal{L}_i^*, \left[J\nabla\mathcal{L}_j^*, J\nabla\mathcal{L}_k^*\right]\right]\cdots\right) = \mathbb{R}^{2d}. \tag{3.9}$$

Then we have the strong form of Arnold diffusion in the ball $\mathcal{I}$ in the action space.

The condition (3.9), where $[\cdot,\cdot]$ denotes the Lie bracket, is just the well known condition for accessibility[9]. Chow's theorem also called Rashevsky theorem (see, [14, 52, 38]) asserts that we can join any two points via segments of trajectories of the scattering vector fields. One very subtle point is that the orbits of the scattering maps can follow closely positive-time orbits associated to the corresponding scattering vector fields (3.8). However, Chow's theorem uses trajectories followed in positive time, i.e., of the form ψ_s for $s \geq 0$, but also trajectories followed in negative time, i.e., of the form ψ_{-s} for $s \geq 0$. Hence, a crucial step in [33] is to show that, using the recurrence condition again, we can build pseudo orbits combining the inner and the scattering maps to follow the flows for negative time. The main difference between Theorem 3.2 and Theorem 3.3 is that in the former we need at least $K = 2d$ scattering vector fields, while in the latter it sufficient to have $K' = 2$ scattering vector fields satisfying (3.9); see [38].

3.4. **Genericity properties of the hypotheses of Theorems 3.2, 3.3, 3.4.** It is not difficult to show that if we fix I then there have to be critical points of the Melnikov potential. We just observe that the Melnikov potential (3.3) is periodic in τ if $\omega(I)$ is rational and quasi-periodic if $\omega(I)$ is not rational. See, for example, [24, 20].

The genericity of the hypothesis of Theorem 3.2 in the C^r-topology, for $r \in [3,\infty)\cup\{\omega\}$, is established in [8]. They note that, since Melnikov potential (3.3) is linear in H_1, it suffices to add a small deformation (e.g. trigonometric function) and compute that it indeed yields a non-degenerate critical point (3.4).

To verify the second hypothesis, once the first one is ensured, it suffices to argue that in the case that it is not satisfied, one can add another (smaller) perturbation that achieves (3.6). If we choose to describe the perturbed system without using parameters, i.e., as $H_0 + H_1$, we can formulate the genericity of the condition on H_1 that yield diffusion in the following way. Let

$$\mathcal{S}^1 = \{\tilde{H}_1 \in C^r \,|\, \|\tilde{H}_1\|_{C^r} = 1\}.$$

Then there exists an open-dense subset $\mathcal{G}^1 \subset \mathcal{S}^1$ and a lower-semicontinuous function $\varepsilon_0 : \mathcal{S}^1 \to [0,+\infty)$ with $\varepsilon_0 > 0$ on $\mathcal{G}^1$ and $\varepsilon_0 = 0$ on $\mathcal{S}^1 \setminus \mathcal{G}^1$, such that, for each H_1 in the generalized ball

$$\mathcal{B} = \left\{ H_1 \in C^r \,|\, 0 < \|H_1\| < \varepsilon_0\left(\frac{H_1}{\|H_1\|_{C^r}}\right)\right\},$$

the system $H_0 + H_1$ has Arnol′d diffusion.

Above, we note that the radius of the generalized ball depends on the direction in $\mathcal{S}^1$, and it can be zero in a nowhere dense set of directions.

[9]The condition (3.9) is sometimes also called in the literature the Hörmander condition since Hörmander showed it is equivalent to hypoellipticity of the sum of squares of the vector fields.

4. **Applications to concrete systems.** The main results in this paper can be applied to concrete systems with explicit perturbations, and can be used to obtain detailed quantitative information on diffusing orbits.

A concrete application to celestial mechanics can be found in [11]. The unperturbed system is the planar circular restricted three-body problem, on the motion of an infinitesimal mass under the gravity of two large masses (referred to as primaries) that move on circular Keplerian orbits. This model, when written in rotating coordinates is described by an autonomous Hamiltonian H_0.

For an open set of values of the mass ratio of the primaries, the Lagrange equilibrium point L_1 (which, by convention, we assume is between the primaries) is of center-saddle type. For the purpose of [11], the mass ratio is fixed to that of the physical system Sun-Jupiter. By the Lyapunov Theorem [43] for each energy level sufficiently close to that of L_1, there exists a periodic orbit around L_1 that is hyperbolic within the corresponding energy manifold (obtained by fixing the value of H_0). Moreover, for a suitable range of energy levels, the stable and unstable manifolds of each periodic orbit can be shown to intersect transversally within the energy manifold. This family of periodic orbits defines the Lyapunov manifold Λ_0, which is a NHIM as in Section 2.1. It can be naturally parametrized by symplectic action-angle coordinates (I, θ), with the action I being a proxy for the energy. The stable and unstable manifolds of Λ_0, $W^s(\Lambda_0)$ and $W^u(\Lambda_0)$, respectively, intersect transversally. One can choose finitely many homoclinic channels Γ_0^i, $i = 1, \dots, k$, and consider the unperturbed scattering maps σ_0^i associated to each Γ_0^i. Each scattering map turns out to give a phase shift along each periodic orbit in Λ_0:

$$\sigma_0^i(I, \theta) = (I, \theta + \Delta^i(I))$$

The perturbed system that is considered is the planar elliptic restricted three-body problem, obtained by adding to the previous model the eccentricity of the orbits of the primaries. The resulting system is described by a time-periodic Hamiltonian $H_\varepsilon = H_\varepsilon(\cdot, t)$. Similarly to Section 3.2, there is a NHIM Λ_ε that persists from the unperturbed case. Since Λ_0 is a manifold with boundary, to show its persistence as an invariant manifold (as opposed to only locally invariant), [11] checks that the conditions of the KAM theorem are verified on the Lyapunov manifold, so Λ_ε can be bounded by invariant tori, hence it is itself invariant. Since Λ_ε is 2-dimensional, the KAM tori block diffusion for the dynamics restricted to Λ_ε.

There are also homoclinic channels Γ_ε^i that persist from the unperturbed case, for all $i = 1, \dots, k$.

The paper [11] computes numerically the Melnikov potentials S^i associated to Γ_0^i, using the formula (2.14), and checks numerically that for each point in the Lyapunov manifold there exists an S^i with $-\frac{\partial S^i}{\partial \theta} > 0$ at that point. By (2.13), this implies that the corresponding scattering map σ_ε^i increases the action I by $O(\varepsilon)$. Applying Corollary 1 (a) in the case of multiple scattering maps, one concludes that for the perturbed system, for every sufficiently small (but non-zero) values of the eccentricity, there are orbits of the infinitesimal mass that start close to one Lyapunov orbit and get close to another Lyapunov orbit. Hence these orbits start at some energy level and end at some other energy level. The gain of energy that can be accomplished is independent of the eccentricity. The conclusion is that there exists weak Arnol′d diffusion in the energy.

We note that the KAM Theorem is only invoked to guarantee the persistence of Λ_ε as an invariant manifold. Alternatively one can obtain the persistence of Λ_ε

only as a locally invariant manifold. Even in this case, one can still obtain diffusing orbits by using Corollary 1 (b): either the orbits of the inner dynamics (restricted to Λ_ε) stay within some finite volume domain in Λ_ε, in which case the first statement of Corollary 1 (b) yields diffusing orbits under the combination of the inner and outer dynamics, or the orbits of the inner dynamics leave any suitable finite volume domain in Λ_ε, in which case the second statement of Corollary 1 (b) yields diffusing orbits under the inner dynamics alone.

Even if the verification of the conditions in [11] is only numerical, the computations of the integrals and the verification of the conditions that are needed are easy for today's computational tools, and standard numerical analysis heuristic estimates predict that the computations are much more accurate than needed.

The paper [10] studies the same model as in [11], albeit for a different physical system: Neptune-Triton. It provides a computer assisted proof, via validated numerics, for the existence of diffusing orbits in energy. Most importantly, it provides a very detailed quantitative information on these diffusing orbits. First, it gives explicit estimates for the range of values of the eccentricity ε for which there is diffusion, of the amount of energy growth, and of the diffusion time. Second, it shows the existence of symbolic dynamics in the energy, which amounts to strong Arnol′d diffusion in the energy. Third, it estimates that the Hausdorff dimension of the set of initial points that undergo symbolic dynamics is greater than 4 in the 5-dimensional extended phase space. Fourth, it shows that there exists a collection of sets Ω_ε of positive Lebesgue measure, for which the distribution of energies $H_0(\phi_t(x)) - H_0(x)$, for $x \in \Omega_\varepsilon$, approaches (relative to some appropriate topology) a Brownian motion with drift as $\varepsilon \to 0$. Moreover, any Brownian motion with drift (that is, for any drift and variance parameter) can be obtained by adequately choosing the sets Ω_ε. This latter result addresses a conjecture by Chirikov [13], that the energy growth follows a diffusion process (in the stochastic sense). The fact that there are collections of initial conditions that yield different Brownian motions with drift is consistent with Chirikov's numerical experiments and heuristic arguments.

The general shadowing theorem can be used even for non-Hamiltonian systems. In the paper, [37], the mechanism is applied to show existence of orbits transferring energy in some models of piezoelectric energy harvesters. These systems consist of coupled piezolectric oscillators, and they can convert ambient vibrations into electrical energy. An interesting problem that we are considering is to show that the Arnol′d diffusion mechanism can overcome the mechanical dissipation inherent to the system.

The results in this section illustrate that the geometric method presented in this paper is quite powerful, since it can be applied to concrete systems (with physically relevant parameters), and can provide, in principle, as much quantitative information on the diffusing orbits as desired.

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