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STABILITY PROBLEMS IN NONAUTONOMOUS LINEAR DIFFERENTIAL EQUATIONS IN INFINITE DIMENSIONS.

Dedicated to Tomás Caraballo for his 60th birthday.

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Abstract. In this paper we study the robustness of the stability in nonautonomous linear ordinary differential equations under integrally small perturbations in infinite dimensional Banach spaces. Some applications are obtained to the case of rapidly oscillating perturbations, with arbitrarily small periods, showing that even in this case the stability is robust. These results extend to infinite dimensions some results given in Coppel [3]. Based in Rodrigues [11] and in Kloeden & Rodrigues [10] we introduce a class of functions that we call Generalized Almost Periodic Functions that extend the usual class of almost periodic functions and are suitable tomodel these oscillating perturbations. We also present an infinite dimensional example of the previous results.

As counterparts, we show first in another example that it is possible to stabilize an unstable system by using a perturbation with a large period and a small mean value, and finally we give an example where we stabilize an unstable linear ODE with a small perturbation in infinite dimensions by using some ideas developed in Rodrigues & Solà-Morales [21] after an example due to Kakutani (see [13]).

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1. Introduction.

In several papers of some of us we have been extending or analysing in infinite dimensions some results that were known for finite-dimensional problems. This was the case of Kloeden & Rodrigues [10], Rodrigues [11], Rodrigues

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& Ruas [16], Rodrigues & Solà-Morales [17, 18, 19, 20], Rodrigues, Caraballo & Gameiro [14] and Rodrigues, Teixeira & Gameiro [15].

Following this philosophy, in this paper we study the relation between the stability properties of a system $\dot{x} = A(t)x$ of ordinary differential equations in an infinite dimensional Banach space $\mathbb{X}$ and a perturbed system $\dot{y} = A(t)y + B(t)y$, where $B(t)$ is supposed to be small in some sense. We suppose first that $A(t)$ and $B(t)$ are bounded operators, continuous and uniformly bounded with respect to $t \in \mathbb{R}$, that the first system is asymptotically stable and that $B(t)$ is integrally small in an arbitrary interval of length bounded by $h > 0$. We establish conditions on the smallness of $B(t)$ in such a way that the perturbed system will also be asymptotically stable. This is established in Theorem 2.1. Then we extend to some cases where $A : \mathcal{D} \to \mathbb{X}$ is unbounded and generatesa C^0-semigroup $T(t),\ t \geq 0$. This is established in Theorem 5.1.

In Daleckii & Krein [4] page 178 and in Carvalho et al. [1] similar results are presented about robustness of stability but with a stronger assumption, given by $\frac{1}{\tau_0}\int_t^{t+\tau_0} \|B(\tau)\| d\tau \ \leq \delta$, for some $\tau_0 > 0$, for every $t \in \mathbb{R}$ for sufficiently small δ. One observes that the smallness condition is imposed with the norm inside the integral and in our case the norm appears outside the integral and this makes a significant difference, as it is shown in Theorem (2.1).

Then we introduce in Section 3 a class of functions that we call Generalised Almost Periodic Functions, that contains the usual almost periodic functions. In fact, part of it was introduced in Kloeden & Rodrigues [10], where the authors studied perturbations of an hyperbolic equilibrium. This class of Generalized Almost Periodic Functions ($\mathcal{GAP}$) is suitable to define the concept of mean value, as it will be shown, which will be used in this paper .

This new class of functions has some important advantages compared with the usual almost periodic functions, namely, if we perturb an almost periodic function of a variable t with a local perturbation in t, then the perturbed function will no longer be almost periodic. Therefore, the usual class it is not robust with respect to this kind of perturbations. It is also not robust with respect to some more general perturbations, like chaotic functions. We understand that the class $\mathcal{GAP}$ is one of the natural classes for our perturbation $B(t)$ to belong.

As an application of Theorem 2.1 we study a system of the form $\dot{y} = A(t)y + B(\omega t)y$ and prove that if $\omega > 0$ is sufficiently large the the stability is preserved. When $B(t)$ is periodic the result says that for sufficiently small periods and large oscillations the stability is preserved. The function $B(t)$ does not need to be small and so if we have a linear perturbation with large oscillations the stability is preserved. This is shown in Theorem 3.12. In the periodic case the perturbation will have a very small period. In Section 4 we present an example in the infinite-dimensional space ℓ_2 where we show that the stability is preserved. These results extend to infinite dimensions some results of Coppel [3].

Then in Theorem 5.1 we extend the above results to the case where we have an unbounded infinitesimal generator. Henry [8] proves similar results with different applications, but using a different method where he passes from the continuous case to a discrete case and then recover the results for the continuous problem. Our method follows more the method of Coppel [3] (finite dimension).

As a counterpart of the previous results on the robustness of the stability, the last two sections are devoted to show, by means of examples, that instability is not so difficult to break. In Section 7 we present a two dimensional example where we

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show that it is possible to stabilise an unstable system with a periodic perturbation with large period and small mean value.

Finally in Section 8 using some ideas developed in Rodrigues &Solà-Morales [21] and in an example of Kakutani [13], we give an example in infinite dimensions where we textcolorredstabilize an unstable linear system using a linear perturbation $B(t)$ that tends to zero as t tends to infinity.

These two last examples seem to be new in the literature, to our knowledge.

2. **Robustness of Stability.**

This section is devoted to state and prove the following Theorem. It extends to infinite dimensional Banach spaces a result of W. A. Coppel [3], Proposition 6, p.6. We think that the key point is the three-terms integration by parts that appears in the beginning of the proof. This integration by parts shows also how the condition of $B(t)$ being integrally small appears along the proof.

Theorem 2.1. *Let $\mathbb{X}$ be a Banach space and $A,\ B:\ \mathbb{R}\to\ L(\mathbb{X})$ be continuous functions such that $\|A(t)\|\leq M$ and $\|B(t)\|\leq M$ for every $t\in\mathbb{R}$.*

Consider the equations:

$$\dot{x} = A(t)x \tag{1}$$

$$\dot{y} = A(t)y + B(t)y \tag{2}$$

Let $T(t,s)=X(t)X^{-1}(s)$ the evolution operator of (1). Suppose that $\|T(t,s)\|\leq Ke^{\alpha(t-s)}$ for $t\geq s,\ t,s\in\ \mathbb{R}$, where $\alpha\in\mathbb{R}$ and $K\geq 1$.

Let $\delta,\ h$ be two positive numbers.

If $\|\int_{t_1}^{t_2}B(t)dt\|\leq\delta$ for $|t_2-t_1|\leq h$, and $t_1,t_2\in\mathbb{R}$, then the evolution operator $S(t,s)=Y(t)Y^{-1}(s)$ of (2) satisfies the inequality:

$$\|S(t,s)\|\leq(1+\delta)Ke^{\beta(t-s)}\ \text{for}\ t\geq s,\ t,s\in\mathbb{R},\ \text{where}\ \beta=\alpha+3MK\delta+\frac{\log((1+\delta)K)}{h}.$$

If α is negative, h is sufficiently large and δ sufficiently small in such a way that $\beta<0$ then it follows that system (2) is asymptotically stable.

Proof: By the variation of constants formula

$$S(t,s)=T(t,s)+\int_s^t T(t,u)B(u)S(u,s)du,\ t\geq s.$$

If we let $C(u)=\int_t^u B(\tau)d\tau$

$$\int_s^t T(t,u)B(u)S(u,s)du=\int_s^t T(t,u)\frac{d}{du}\int_t^u B(\tau)d\tau S(u,s)du=\int_s^t T(t,u)\frac{d}{du}C(u)S(u,s)du$$

Taking derivatives,

$$\frac{d}{du}[T(t,u)C(u)S(u,s)]=$$

$$-T(t,u)A(u)C(u)S(u,s)+T(t,u)B(u)S(u,s)+T(t,u)C(u)(A(u)+B(u))S(u,s)$$

Integrating the above equation gives the three-terms integration by parts we commented above. Then we obtain

$$\int_s^t\frac{d}{du}[T(t,u)C(u)S(u,s)]du=-\int_s^t T(t,u)A(u)C(u)S(u,s)du+$$

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$$\int_s^t T(t,u)B(u)S(u,s)du+\int_s^t T(t,u)C(u)A(u)S(u,s)du+\int_s^t T(t,u)C(u)B(u)S(u,s)du$$

And so,

$$-T(t,s)C(s) = -\int_s^t T(t,u)A(u)C(u)S(u,s)du+$$

$$\int_s^t T(t,u)B(u)S(u,s)du+\int_s^t T(t,u)C(u)A(u)S(u,s)du+\int_s^t T(t,u)C(u)B(u)S(u,s)du$$

Therefore,

$$\int_s^t T(t,u)B(u)S(u,s)du = -T(t,s)C(s) + \int_s^t T(t,u)A(u)C(u)S(u,s)du$$

$$-\int_s^t T(t,u)C(u)A(u)S(u,s)du - \int_s^t T(t,u)C(u)B(u)S(u,s)du.$$

Therefore,

$$S(t,s) = T(t,s) + \int_s^t T(t,u)B(u)S(u,s)du =$$

$$T(t,s)(I-C(s)) + \int_s^t T(t,u)A(u)C(u)S(u,s)du$$

$$-\int_s^t T(t,u)C(u)A(u)S(u,s)du - \int_s^t T(t,u)C(u)B(u)S(u,s)du.$$

We first suppose that $s \le t \le s+h$ and estimate $|S(t,s)|$. Let $s \le u \le s+h$. Suppose

$$|C(u)| \le |\int_t^u B(\tau)d\tau| \le \delta.$$

Therefore,

$$|S(t,s)| \le K(1+\delta)e^{\alpha(t-s)} + 3MK\delta\int_s^t e^{-\alpha(t-u)}|S(u,s)|du.$$

and so, using Gronwall's inequlity it follows that in an arbitrary interval of length h, say for $s \le t \le s+h$ we have

$$|S(t,s)| \le K(1+\delta)e^{\alpha(t-s)}e^{3MK\delta(t-s)} = K(1+\delta)e^{(\alpha+3MK\delta)(t-s)}$$

For $t \ge s$ there exists $n \in \mathbb{N}$, $n = n(t,s)$ such that $s+nh \le t \le s+(n+1)h$ and so

$$|S(t,s+nh)| \le K(1+\delta)e^{(\alpha+3MK\delta)(t-s-nh)}.$$

We are going to prove by induction that for $s+nh \le t \le s+(n+1)h$

$$|S(t,s)| \le [K(1+\delta)]^{n+1}e^{(\alpha+3KM\delta)(t-s)}$$

The case $n=0$ has already been proved.

But $S(s+nh,s) = S(s+nh,s+(n-1)h)\cdots S(s+h,s)$ and so

$$|S(s+nh,s)| \le [K(1+\delta)]^n e^{(\alpha+3KM\delta)nh}$$

Therefore for $s+nh \le t \le s+(n+1)h$

$$|S(t,s)| \le |S(t,s+nh)||S(s+nh,s)| \le$$

$$K(1+\delta)e^{(\alpha+3KM\delta)(t-s-nh)}[K(1+\delta)]^n e^{(\alpha+3KM\delta)nh} = [K(1+\delta)]^{n+1}e^{(\alpha+3KM\delta)(t-s)}$$

Therefore for $s+nh \le t \le s+(n+1)h$ we have

$$|S(t,s)| \le [K(1+\delta)]^{n+1} e^{((\alpha+3KM\delta)(t-s)}.$$

Let $\gamma \doteq \frac{\ln((1+\delta)K)}{h}$. Since $t \ge s+nh$, we have

$$[(1+\delta)K]^n = e^{\gamma nh} \le e^{\gamma(t-s)}.$$

Therefore,

$$|S(t,s)| \le K(1+\delta) e^{(\alpha+3KM\delta+\frac{\ln((1+\delta)K)}{h})(t-s)} \quad \square$$

3. The space of generalised almost periodic functions.

In this section we introduce the class that we call Generalised Almost Periodic Functions that extends the usual concept of almost periodicity. As we said in the Introduction Section, this new class is more robust with respect to perturbations and it is a natural class for our function $B(t)$ to belong, as it will appear in Theorem 3.12 and its corollary.

Let $(\mathbb{X}, |\cdot|)$ be a Banach space and recall the definition of an almost periodic function [5].

Definition 3.1. *A continuous function $f:\mathbb{R}\to\mathbb{X}$ is said to be almost periodic if for every sequence (α'_n) there exists a subsequence (α_n) such that the $\lim_{n\to\infty} f(t+\alpha_n)$ exists uniformly in $\mathbb{R}$.*

Now let $BUC(\mathbb{R}, L(\mathbb{X})$ denote the space of bounded and uniformly continuous functions $A:\mathbb{R}\to L(\mathbb{X})$, which is a Banach space with the supremum norm $\|A\| \doteq \sup_{t\in\mathbb{R}} |A(t)|$, and define

$$\mathcal{F} \doteq \{A \in BUC(\mathbb{R}, L(\mathbb{X})) \ : \ A \text{ is uniformly continuous with precompact range } \mathcal{R}(A)\}.$$

The class $\mathcal{F}$ is quite large and includes both periodic and almost periodic functions as well as other nonrecurrent functions.

Proposition 3.2. *Let $A(t) \in L(\mathbb{X})$ be almost periodic. Then $A \in \mathcal{F}$.*

Proof: The proof is trivial.

Theorem 3.3. *$\mathcal{F}$ is a closed subspace of $BUC(\mathbb{R}, L(\mathbb{X}))$ and hence a Banach space.*

Proof: This proof can be found in Kloeden-Rodrigues [10].

Lemma 3.4. *Let $\sup_{t\in\mathbb{R}} |A(t)| \le M$, If there exists $\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt$ for some $a \in \mathbb{R}$ then it is independent of a.*

Proof: Let $a \in \mathbb{R}$.

$$|\frac{1}{T}\int_a^{a+T} A(t)dt - \frac{1}{T}\int_0^T A(t)dt| = |\frac{1}{T}[\int_a^{a+T} A(t)dt - \int_0^T A(t)dt]|$$

$$|\frac{1}{T}[\int_a^0 A(t)dt + \int_T^{a+T} A(t)dt]| \le \frac{2M|a|}{T} \to 0, \text{ as } T\to\infty.$$

$\square$

Then we define:

Definition 3.5. *We say that $A \in \mathcal{F}$ is a generalized almost periodic function if there exists the limit $\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt$ in $L(\mathbb{X})$, that is, there exists $\mathbf{A} \in L(\mathbb{X})$ such that, given $\varepsilon > 0$ there exists $T_0 = T_0(\varepsilon) > 0$ such that $|\frac{1}{T}\int_a^{a+T} A(t)dt - \mathbf{A}| < \varepsilon$ for every $T \ge T_0$ uniformly with respect do $a \in \mathbb{R}$.*

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Definition 3.6. *We define the class of generalized almost periodic functions as*

$$\mathcal{GAP} = \{A \in \mathcal{F} : \ A \text{ is a generalized almost periodic function}\}$$

Lemma 3.7. *$\mathcal{GAP}$ is a closed subspace of $\mathcal{F}$.*

Proof: Let $A_n \in \mathcal{GAP}$, $A_n \to A$ in $\mathfrak{F}$. We must prove that $A \in \mathcal{GAP}$. Given $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that $\|A_{n_0} - A\| = \sup_{t\in\mathbb{R}} |A_{n_0}(t) - A(t)| < \varepsilon$.

Since there exists the $\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A_{n_0}(t)dt = \mathbf{A}_{n_0}$, there exists $T_0 = T_0(\varepsilon)$ such that

$$T_1,\ T_2 > T_0 \ \Rightarrow |\frac{1}{T_2}\int_a^{a+T_2} A_{n_0}(t)dt - \frac{1}{T_1}\int_a^{a+T_1} A_{n_0}(t)dt| < \varepsilon,\ \forall a \in \mathbb{R}$$

Then

$$T_1,\ T_2 > T_0 \ \Rightarrow |\frac{1}{T_2}\int_a^{a+T_2} A(t)dt - \frac{1}{T_1}\int_a^{a+T_1} A(t)dt| \leq$$

$$|\frac{1}{T_2}\int_a^{a+T_2} A(t)dt - \frac{1}{T_2}\int_a^{a+T_2} A_{n_0}(t)dt| + |\frac{1}{T_2}\int_a^{a+T_2} A_{n_0}(t)dt - \frac{1}{T_1}\int_a^{a+T_1} A_{n_0}(t)dt| +$$

$$|\frac{1}{T_1}\int_a^{a+T_1} A_{n_0}(t)dt - \frac{1}{T_1}\int_a^{a+T_1} A(t)dt| \leq$$

$$\frac{1}{T_2}\int_a^{a+T_2} |A(t) - A_{n_0}(t)|dt + |\frac{1}{T_2}\int_a^{a+T_2} A_{n_0}(t)dt - \frac{1}{T_1}\int_a^{a+T_1} A_{n_0}(t)dt| +$$

$$|\frac{1}{T_1}\int_a^{a+T_1} |A(t) - A_{n_0}(t)|dt \leq 3\varepsilon$$

Using Cauchy Criterion we conclude that there exists

$$\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt = \mathbf{A} \in L(\mathbb{X}),\ \forall a \in \mathbb{R}$$

This implies that $A \in \mathcal{GAP}$. □

Definition 3.8. *For $A \in \mathcal{GAP}$ we define the mean value of A as:*

$$\mathcal{M}(A) \doteq \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt \in L(\mathbb{X}).$$

Lemma 3.9. *The function $\mathcal{M} : \mathcal{GAP} \to L(\mathbb{X})$ is an uniformly continuous function.*

Proof: Let $A, B \in \mathcal{GAP}$. Then

$$|\mathcal{M}(A) - \mathcal{M}(B)| = |\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt - \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} B(t)dt| =$$

$$|\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} [A(t) - B(t)]dt| \leq \sup_{t\in\mathbb{R}} |A(t) - B(t)| = \|A - B\|.$$

□

Let $\mathcal{O} = \{A \in \ \mathcal{GAP} : \mathcal{M}(A) = \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} A(t)dt = 0,\ \forall\ a \in \mathbb{R}\}$

Corollary 3.10. *$\mathcal{O}$ is a closed subspace of $\mathcal{GAP}$.*

Proof: Since $\mathcal{M}(A)$ is a continuous function, the set $\mathcal{O} = \mathcal{M}^{-1}\{\{0\}\}$ is closed set.

Corollary 3.11. *Any function $A \in \mathcal{GAP}$ can be written as $A = A_0 + B$, where $A_0 = \mathcal{M}(A)$ and $B \in \mathcal{O}$.*

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The next theorem shows that stability is preserved if the linear perturbation has sufficiently large frequency:

Theorem 3.12. *Let $A,\ B:\ \mathbb{R}\to\ L(\mathbb{X})$ be continuous functions such that $\|A(t)\|\leq M$ and $\|B(t)\|\leq M$ for every $t\in\mathbb{R}$. Suppose that $B(t)$ is a generalized almost periodic function with mean value zero ($\mathcal{GAP}$). Consider the equations:*

$$\dot{x}=A(t)x \tag{3}$$

$$\dot{x}=A(t)x+B(\omega t)x \tag{4}$$

Let $T(t,s)$ the evolution operator of (3). Suppose that $\|T(t,s)\|\leq Ke^{-\alpha(t-s)}$ for $t\geq s,\ t,s\in\ \mathbb{R}$, where $\alpha>0$ and $K>1$. Then there exists $\tilde{K}$ and $\omega_0>0$ such that for $\omega>\omega_0$

$$|S_\omega(t,s)|\leq\tilde{K}e^{\frac{-\alpha}{2}(t-s)},\ t\geq s,$$

where $S_\omega(t,s)$ indicates the evolution operator of (4).

Proof:

We are going to show that for any $h>0$, $\delta>0$ there exists $\omega_0=\omega_0(h,\delta)>0$ such that if $\omega>\omega_0$ then

$$|\int_{t_1}^{t_2}B(\omega t)dt|\leq\delta \text{ for } |t_2-t_1|\leq h.$$

Let us consider first the case $|t_2-t_1|\leq\frac{\delta}{M}$. Since $|B(t)|\leq M$ for every $t\in\mathbb{R}$, we have

$$|\int_{t_1}^{t_2}B(\omega t)dt|\leq|\int_{t_1}^{t_2}|B(\omega t)|dt|\leq M|t_2-t_1|\leq M\frac{\delta}{M}=\delta.$$

To complete the proof we consider now the case $h\geq|t_2-t_1|\geq\frac{\delta}{M}$.

Since $B(t)$ has mean value zero, there exists $T_0=T_0(\frac{\delta}{h})>0$ such that

$$T\geq T_0\ \Rightarrow|\frac{1}{T}\int_s^{s+T}B(t)dt|\leq\frac{\delta}{h}\text{ for all }s\in\mathbb{R}.$$

By a change of variables,

$$\int_{t_1}^{t_2}B(\omega t)dt=\frac{1}{\omega}\int_{\omega t_1}^{\omega t_2}B(u)du$$

and so for $\frac{\delta}{M}\leq|t_2-t_1|\leq h$

$$|\int_{t_1}^{t_2}B(\omega t)dt|=\frac{1}{\omega|t_2-t_1|}|\int_{\omega t_1}^{\omega t_2}B(u)du||t_2-t_1|\leq\frac{1}{|\omega t_2-\omega t_1|}|\int_{\omega t_1}^{\omega t_2}B(u)du|\ h.$$

If we take $\omega_0\doteq\frac{MT_0}{\delta}$ we have for $\omega\geq\omega_0$

$$|\omega t_2-\omega t_1|\geq\omega_0|t_2-t_1|\geq\frac{MT_0}{\delta}\frac{\delta}{M}=T_0$$

Therefore,

$$|\int_{t_1}^{t_2}B(\omega t)dt|=\frac{1}{|\omega t_2-\omega t_1|}|\int_{\omega t_1}^{\omega t_2}B(u)du|\ h\leq\frac{\delta}{h}\ h=\delta$$

The result follows from Theorem 3.12 for δ sufficiently small. □

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Consider now $A \in \mathcal{GAP}$. Then we have $A(t) = A_0 + B(t)$, where $A_0 = \mathcal{M}(A)$ and $\mathcal{M}(B) = 0$. We suppose that $|A_0| \leq M$ and $|B(t)| \leq M$ for every $t \in \mathbb{R}$. Consider the equations:

$$\dot{x} = A_0 x \tag{5}$$

$$\dot{x} = A_0 x + B(\omega t)x \tag{6}$$

Let $T(t) \doteq e^{A_0 t}$ be the semigroup generated by (5) and $S_\omega(t,s)$ be the evolution operator of (6).

As a consequence of Theorem 3.12 it follows that if $\sigma(A_0) \subset \{\lambda \in \mathbb{C} : Re(\lambda) < -\alpha\}$ we willl have:

Corollary 3.13. *Suppose* $|T(t)| \leq Ke^{-\alpha t}$ *for* $t \geq 0$, $K \geq 1$. *Then there exists* $\widetilde{\alpha} < \alpha$, $\widetilde{K} > K$, $\omega_0 > 0$, *such that for* $\omega > \omega_0$ *we have*

$$S_\omega(t,s) \leq \tilde{K} e^{-\widetilde{\alpha}(t-s)}, \forall t \geq s.$$

4. **An infinite dimensional example.** In this section we will construct a true infinite dimension example to apply the results of the previous section. We are going to use some results of the paper Rodrigues and Solà-Morales [19]. Consider the space $\mathbb{X} = \ell_2$. We consider the operator $J \in \mathcal{L}(X)$ given by the infinite dimensional Jordan matrix:

$$J := \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{pmatrix} \tag{7}$$

As it is proved in Rodrigues and Solà-Morales [19] the spectrum of J is the closed unity circle of the complex plane. Now we take $0 < a < 1$ and we define the operator:

$$L := \begin{pmatrix} a & 0 \\ 0 & \nu J + aI \end{pmatrix} = aI + \nu \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix} = a\left(I - \left(\frac{-\nu}{a}\right)\begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}\right) \tag{8}$$

If we let

$$D = \left(\frac{-\nu}{a}\right)\begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}$$

we have that

$$L = a(I - D).$$

From the same paper above it follows that the spectrum of L is the closed disc $B_\nu(a)$ with center in a and radius ν. Then we take $0 < \nu < \min\{a, 1-a\}$

Then we let $A := \log L = (\log a)I + \log(I - D)$.

But

$$\log(I - D) = -(D + \frac{D^2}{2} + \cdots \frac{D^n}{n} \cdots).$$

Therefore

$$\|\log(I - D)\| \leq \frac{\nu}{a} + \frac{(\frac{\nu}{a})^2}{2} + \cdots \frac{(\frac{\nu}{a})^n}{n} + \cdots = -\log(1 - \frac{\nu}{a})$$

Let $\nu > 0$ sufficiently small such that $0 < -\log(1 - \frac{\nu}{a}) < \frac{a}{2}$.

Then it follows that

$$\|e^{At}\| \leq e^{(-at - \log(1-\frac{\nu}{a})t)} \leq e^{-\frac{a}{2}t}, \quad t \geq 0$$

In the space $\mathbb{X} = \ell_2$. We consider the operator $A \in \mathcal{L}(X)$ given above.

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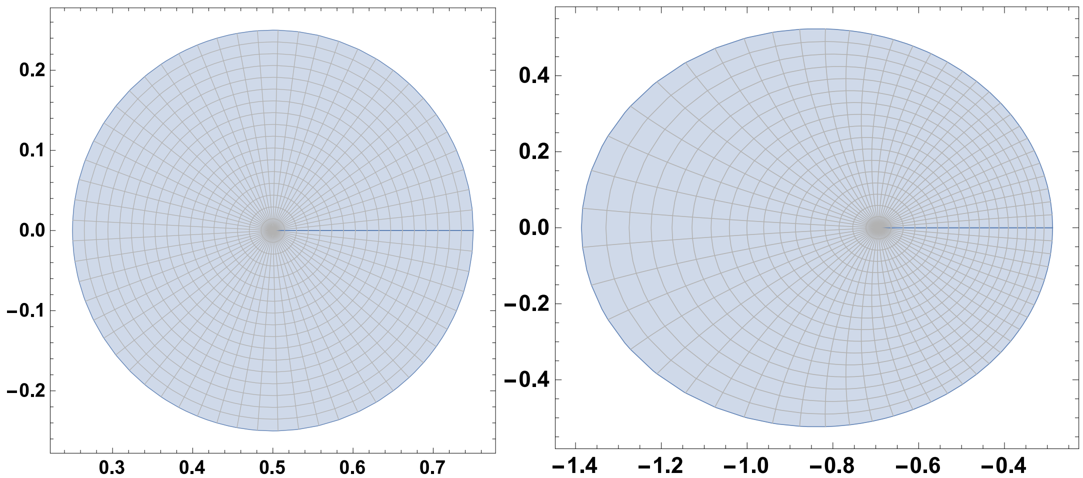

FIGURE 1. **Left**: The spectrum of L given by $\sigma(L) = B_\nu(a)$. **Right**: The spectrum of A given by $\sigma(A) = \log(\sigma(L))$, with $a = 1/2$ and $\nu = 1/4$.

Corollary 4.1. *Consider now the systems:*

$$\dot{x} = Ax \tag{9}$$

$$\dot{y} = Ay + B(\omega t)y \tag{10}$$

where $B \in \mathcal{GAP}$ with meanvalue zero. Let $M > 0$ be such that $|A| \leq M$ and $\sup_{t\in\mathbb{R}} |B(t)| \leq M$.

Let $S_\omega(t,s) = Y(t)Y^{-1}(s)$ be the evolution operator associated to to system (10), where $Y(t)$ is the solution with initial condition $Y(0) = I$, where I indicates the Identity operator.

Then there exists $\tilde{K}$, $\tilde{\alpha}$ and $\omega_0 > 0$ such that for $\omega > \omega_0$

$$|S_\omega(t,s)| \leq \tilde{K}e^{-\tilde{\alpha}(t-s)},\ t \geq s,$$

Proof: Follows from Theorem 3.12 . □

Next we will present a simple example where the perturbation $B(t)$ belongs to $\mathcal{GAP}$ but it is not almost periodic.

Example 4.2. *Let $b : \mathbb{R} \to \mathbb{R}$ be uniformly continuous, bounded with mean value zero. Let*

$$B(t) \doteq \begin{pmatrix} 0 & 0 & 0 & \cdots \\ b(t) & 0 & 0 & \cdots \\ 0 & b(t) & 0 & \cdots \\ 0 & 0 & b(t) & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{pmatrix} = b(t) \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{pmatrix} \tag{11}$$

Then $B \in \mathcal{GAP}$ and has mean value zero. Let $d(t) \doteq \sqrt{1-t^2}$ if $-1 \leq t \leq 1$, $d(t) = 0$ if $t \in (\infty, 0) \cup (1, \infty)$. In the special case that we take $b(t) \doteq d(t) + \cos t$, $B(t)$ is not almost periodic.

Therefore we can apply Corollary 4.1 if we take $b(\omega t) = d(\omega t) + \cos(\omega t)$ and then we can take $B(\omega t)$ as above.

5. **A case where the infinitesimal generator is unbounded.** Consider the equations:

$$\dot{x} = Ax \tag{12}$$

$$\dot{y} = Ay + B(t)y \tag{13}$$

We suppose that $\mathcal{D}$ is dense in $\mathbb{X}$ and $A : \mathcal{D} \to \mathbb{X}$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $T(t)$, such that $|T(t)| \leq Ke^{\alpha t},\ t \geq 0,\ K \geq 1, \alpha \in \mathbb{R}$.

Now we will analyse some smallness conditions on the perturbation $B(t)$, such that the equation 13 is also asymptotically stable in the case $\alpha < 0$. The case when $B(t)$ is uniformly small is studied in Kloeden-Rodrigues [10] without leaving the continuous case. Similar results are obtained by Carvalho et all [1], but they first find the result for the discrete case.

Similar results to the next theorem are treated by Carvalho et all [1] and Dalekii-Krein [4] but they use the stronger assumption that $\int_\tau^t |B(t)|$ is small, with the norm inside the integral and in the first one they prove via a discretiztion method. Similar results are obtained by Henry [8] in Thorem7.6.11, pag. 238, where he also consider first the discrete case, and requires that $B(t)$ is uniformly small and integrally small.

Our result is an extension of a classical result of Coppel [3] for the infinite dimensional case, and A being an unbounded operator.

We will follow the steps of Theorem 2.1 where we imposed that $|B(t)| \leq M$ for every $t \in \mathbb{R}$ and that $|\int_t^u B(\tau)d\tau| \leq \delta$ for $t \leq u \leq t+h$. We also assume that the range of $B(t)$ is contained in the domain of A.

Theorem 5.1. *We assume besides the above assumptions on A and $T(t)$, that $B: \mathbb{R} \to L(\mathbb{X})$ is a continuous function and such that for each $t \in \mathbb{R}$ $AB(t)$ is a bounded operator and $B(t)A$ can be extended to the whole space as a bounded operator. For each $t \in \mathbb{R}$ let $C_t(u) \doteq \int_t^u B(\tau)d\tau$, for $|t-u| \leq h$, where h is a positive real number. We suppose that there are positive numbers M and δ such that*

$$|C_t(u)B(u)| \leq M\delta,\ |C_t(u)A| \leq M\delta, \text{ and } |AC_t(u)| \leq M\delta,\ \text{for } |u-t| \leq h.$$

Let $S(t,s)$ be the evolution operator associated to system 13. Then

$$\|S(t,s)\| \leq (1+\delta)Ke^{\beta(t-s)} \text{ for } t \geq s,\ t,s \in \mathbb{R},\ \text{where } \beta = \alpha + 3MK\delta + \frac{\log((1+\delta)K)}{h}.$$

If α is negative, h is sufficiently large and δ sufficiently small in such a way that $\beta < 0$ then it follows that system (13) is asymptotically stable.

Proof: The proof follows the ideas of (2.1). By the variation of constants formula

$$S(t,s) = T(t-s) + \int_s^t T(t-u)B(u)S(u,s)du,\ t \geq s.$$

$$\int_s^t T(t-u)B(u)S(u,s)du = \int_s^t T(t-u)\frac{d}{du}\int_t^u B(\tau)d\tau S(u,s)du = \int_s^t T(t-u)\frac{d}{du}C_t(u)S(u,s)du$$

Taking derivatives,

$$\frac{d}{du}[T(t-u)C_t(u)S(u,s)] =$$

$$-T(t-u)AC_t(u)S(u,s) + T(t-u)B(u)S(u,s) + T(t-u)C_t(u)(A+B(u))S(u,s)$$

Integrating the above equation, we obtain

$$\int_s^t \frac{d}{du}[T(t-u)C_t(u)S(u,s)]du = -\int_s^t T(t-u)AC_t(u)S(u,s)du +$$
$$\int_s^t T(t-u)B(u)S(u,s)du + \int_s^t T(t-u)C_t(u)AS(u,s)du + \int_s^t T(t-u)C_t(u)B(u)S(u,s)du$$

And so,

$$-T(t-s)C_t(s) = -\int_s^t T(t-u)AC_t(u)S(u,s)du +$$
$$\int_s^t T(t-u)B(u)S(u,s)du + \int_s^t T(t-u)C_t(u)AS(u,s)du + \int_s^t T(t-u)C_t(u)B(u)S(u,s)du$$

Therefore,

$$\int_s^t T(t-u)B(u)S(u,s)du = -T(t-s)C_t(s) + \int_s^t T(t-u)AC_t(u)S(u,s)du$$
$$-\int_s^t T(t-u)C(_t u)AS(u,s)du - \int_s^t T(t-u)C_t(u)B(u)S(u,s)du.$$

Therefore,

$$S(t,s) = T(t-s) + \int_s^t T(t-u)B(u)S(u,s)du =$$
$$T(t-s)(I - C_t(s)) + \int_s^t T(t-u)AC_t(u)S(u,s)du$$
$$-\int_s^t T(t-u)C_t(u)AS(u,s)du - \int_s^t T(t-u)C_t(u)B(u)S(u,s)du.$$

We first suppose that $s \leq t \leq s+h$ and estimate $|S(t,s)|$.
If $0 \leq |u-t| \leq h$ then

$$|C_t(u)B(u)| \leq |\int_t^u B(\tau)d\tau B(u)| \leq M\delta,\ |C_t(u)A| \leq M\delta \text{ and } |AC_t(u)| \leq M\delta$$

Therefore,

$$|S(t,s)| \leq K(1+\delta)e^{\alpha(t-s)} + 3MK\delta \int_s^t e^{-\alpha(t-u)}|S(u,s)|du.$$

and so using Gronwall's inequality it follows that in an arbitrary interval of length h, say for $s \leq t \leq s+h$ we have

$$|S(t,s)| \leq K(1+\delta)e^{\alpha(t-s)}e^{3MK\delta(t-s)} = K(1+\delta)e^{(\alpha+3MK\delta)(t-s)}$$

For $t \geq s$ there exists $n \in \mathbb{N}$, $n = n(t,s)$ such that $s+nh \leq t \leq s+(n+1)h$ and so

$$|S(t,s+nh)| \leq K(1+\delta)e^{(\alpha+3MK\delta)(t-s-nh)}.$$

We are going to prove by induction that for $s+nh \leq t \leq s+(n+1)h$

$$|S(t,s)| \leq [K(1+\delta)]^{n+1}e^{(\alpha+3KM\delta)(t-s)}$$

The case $n=0$ has already been proved.
But $S(s+nh,s) = S(s+nh, s+(n-1)h)\cdots S(s+h,s)$ and so

$$|S(s+nh,s)| \leq [K(1+\delta)]^n e^{(\alpha+3KM\delta)nh}$$

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Therefore for $s+nh \le t \le s+(n+1)h$

$$|S(t,s)| \le |S(t,s+nh)||S(s+nh,s)| \le$$

$$K(1+\delta)e^{(-\alpha-3KM\delta)(t-s-nh)}[K(1+\delta)]^n e^{(\alpha+3KM\delta)nh} = [K(1+\delta)]^{n+1}e^{(\alpha+3KM\delta)(t-s)}$$

Therefore for $s+nh \le t \le s+(n+1)h$ we have

$$|S(t,s)| \le [K(1+\delta)]^{n+1}e^{(\alpha+3KM\delta)(t-s)}.$$

Let $\gamma \doteq \frac{\ln((1+\delta)K)}{h}$. Since $t \ge s+nh$, we have

$$[(1+\delta)K]^n = e^{\gamma nh} \le e^{\gamma(t-s)}.$$

Therefore,

$$|S(t,s)| \le K(1+\delta)e^{(\alpha-+3KM\delta+\frac{\ln((1+\delta)K)}{h})(t-s)} \quad \square$$

6. Applications of Section 5.

Consider the following result from Henry [8] pg. 30.

Theorem 6.1. *Suppose A is a closed operator in the Banach space $\mathbb{X}$ and suppose that σ_1 is a bounded spectral set of A, and $\sigma_2 = \sigma(A) - \sigma_1$ so $\sigma_2 \cup \{\infty\}$ is another spectral set. Let E_1, E_2 be the projections associated with these spectral sets, and $X_j = E_j(\mathbb{X})$, $j = 1, 2$. Then $\mathbb{X} = X_1 \oplus X_2$, the X_j are invariant under A, and if A_j is the restriction of A to X_j, then*

$$A_1 : X_1 \to X_1 \text{ is bounded}, \sigma(A_1) = \sigma_1, \quad \mathcal{D}(A_2) = \mathcal{D}(A) \cap X_2 \text{ and } \sigma(A_2) = \sigma_2.$$

With our techniques what we can get is the next result:

Theorem 6.2. *Let h and δ be positive real numbers.*

Suppose that $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ a generator of a C_0-semigroup $T(t)$, $t \ge 0$, $B(t) \in L(\mathbb{X})$ and $|B(t)| \le M$ for every $t \in \mathbb{R}$. Suppose we can decompose $\sigma(A) \doteq \sigma_1 \cup \sigma_2$, where σ_1 is a bounded spectral set and $\sigma_2 = \sigma(A) - \sigma_1$ so $\sigma_2 \cup \{\infty\}$ is another spectral set. Suppose there is a smooth curve Γ, oriented positively, that contains σ_1 in its interior and σ_2 is in the exterior of Γ. Consider the projection $P_1 \doteq \frac{-1}{2\pi i}\oint_\Gamma (\lambda - A)^{-1} d\lambda$ that projects $\mathbb{X}$ in the subspace X_1 associated to the spectral set σ_1. Let $P_2 \doteq I - P_1$. $|T(t)P_1| \le Ke^{-\alpha t}$ and $|T(t)P_2| \le Ke^{-\mu t}$, for $t \ge 0$, where $\mu > \alpha$. Then AP_1 is a bounded operator and $P_1 A = AP_1$ and so $P_1 A$ is also a bounded operator.

The above decomposition is chosen in such a way that $|P_2 B(t)| \le M\delta$ for every $t \in \mathbb{R}$.

In analogy with the bounded case if $C_t(u) \doteq \int_t^u B(\tau)d\tau$, we suppose that

$$|P_1 C_t(u) B| \le M\delta, \ |P_1 A C_t(u)| \le M\delta \text{ and } |P_1 C_t(u) A| \le M\delta, \text{ for } t \le u \le t+h. \tag{14}$$

Consider the equations:

$$\dot{x} = Ax \tag{15}$$

$$\dot{y} = Ay + B(t)y \tag{16}$$

If the above assumptions are satisfied if δ is sufficiently small, h is sufficiently large and (15) is asymptotically stable then system (16) is also asymptotically stable.

Proof: The proof follows the ideas of Theorem 5.1.

Remark 6.3. *The decomposition $\sigma(A) = \sigma_1 \cup \sigma_2$ and the smallness conditions (6.2) are satisfied if A is at least a sectorial operator and if $B(t)$ comutes with P_1.*

7. Stabilising unstable systems under small periodic perturbations, with large period.

In contrast with the results of the previous sections on the robustness of the stability, this section and the next one are devoted to show, by means of examples, that instability is not so difficult to break.

The next example is in $\mathbb{X} = \mathbb{R}^2$ and it shows that it is possible to stabilise an unstable system under a small (in mean value) periodic perturbation.

Let $0 < \alpha < \beta$ and $\delta < T$. Let

$$A \doteq \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}, \quad R \doteq \begin{pmatrix} 0 & \frac{\pi}{2\delta} \\ -\frac{\pi}{2\delta} & 0 \end{pmatrix}$$

Let $D(t)$ the T-periodic operator given by

$$D(t) = -A + R, \ T - \delta \leq t < T, \ D(t) = 0 \ t \in \mathbb{R} - [T - \delta, T). \tag{17}$$

Consider the systems:

$$\dot{x} = Ax \tag{18}$$

$$\dot{y} = Ay + D(t)y \tag{19}$$

First we observe that $\lim_{T\to\infty} \frac{1}{T}\int_0^T D(s)ds = 0$, that is $B(t)$ has zero mean value, but has large period.

Next we are going to prove, using Floquet Theorem that system (19) is uniformly asymptotically stable.

For the sistem $\dot{x} = A(t)x$, where $A(t)$ is continuous and T-periodic, will use Floquet Theorem even if $A(t)$ is not continuous, according to the comment in [7] page 118.

Consider the matrix solution $X(t)$ of (18) such that $X(0) = I$ the identity matrix. Then it is given by

$$X(t) = e^{At} = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{-\beta t} \end{pmatrix}$$

If we let $R \doteq \begin{pmatrix} 0 & \frac{\pi}{2\delta} \\ -\frac{\pi}{2\delta} & 0 \end{pmatrix}$ then we have the rotation matrix:

$$e^{Rt} = \begin{pmatrix} \cos(\frac{\pi t}{2\delta}) & \sin(\frac{\pi t}{2\delta}) \\ -\sin(\frac{\pi t}{2\delta}) & \cos(\frac{\pi t}{2\delta}) \end{pmatrix}$$

Since $X(T-\delta) = e^{A(t-\delta)} = \begin{pmatrix} e^{\alpha(T-\delta)} & 0 \\ 0 & e^{-\beta(T-\delta)} \end{pmatrix}$, the fundamental matrix $Y(t)$ of $\dot{y} = (A + D(t))y$, such that $Y(0) = I$ will be given by

$$Y(t) = e^{At} \text{ for } 0 \leq t < T - \delta, \quad Y(t) = e^{R(t-(T-\delta))}e^{A(T-\delta)} = e^{R(t-T)}e^{R\delta}e^{A(T-\delta)},$$

for $T - \delta \leq t < T$.

Then the monodromy matrix will be

$$Y(T) = e^{R\delta}e^{A(T-\delta)} = \begin{pmatrix} \cos(\frac{\pi\delta}{2\delta}) & \sin(\frac{\pi\delta}{2\delta}) \\ -\sin(\frac{\pi\delta}{2\delta}) & \cos(\frac{\pi\delta}{2\delta}) \end{pmatrix} \begin{pmatrix} e^{\alpha(T-\delta)} & 0 \\ 0 & e^{-\beta(T-\delta)} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{\alpha(T-\delta)} & 0 \\ 0 & e^{-\beta(T-\delta)} \end{pmatrix} = \begin{pmatrix} 0 & e^{-\beta(T-\delta)} \\ -e^{\alpha(T-\delta)} & 0 \end{pmatrix}$$

Now we can find the eigenvalues of the monodromy $Y(T)$ and they will be the caracteristic multipliers of (19)

$$Y(T) - \lambda I = \begin{pmatrix} -\lambda & e^{-\beta(T-\delta)} \\ -e^{\alpha(T-\delta)} & -\lambda \end{pmatrix}.$$

The caracteristic polynomial is given by $p(\lambda) \doteq \lambda^2 + e^{(\alpha-\beta)(T-\delta)}$. Since $\beta > \alpha$ this implies that

$$|\lambda| = \sqrt{e^{(\alpha-\beta)(T-\delta)}} < 1.$$

Therefore $\dot{y} = (A + D(t))y$ is uniformly asymptotic stable.

8. **Stabilizing Unstable Linear ODE in Infinite Dimensions.**

There is a classical example in Operator Theory due to S. Kakutani of a bounded operator in an infinite-dimensional Hilbert space whose spectrum shrinks drastically from a disk to a single point under an arbitrarily small bounded perturbation. The example can be found in [13] (p. 282) and [6] (p. 248) and it is also described in [21], where the present authors recently used it to build an example of the possibility of nonlinear stabilization of an unstable linear map under Fréchet differentiability hypotheses. It is also briefly described below. The purpose of the present section is, by means of two examples, to use the ideas of Kakutani's example to show this drastic stabilization in linear ordinary differential equations in infinite dimensional Hilbert spaces, of the form

$$\dot{x}(t) = Ax(t) + B(t)x(t), \tag{20}$$

when the system $\dot{x}(t) = Ax(t)$ is unstable and the perturbation $B(t)$ is small in some senses. Roughly speaking, we could say that the examples of this section show that while stability is a robust feature, instability does not need to be so.

Let us describe briefly the example of Kakutani with the notations and choices of [21]. In a real separable Hilbert space H with a Hilbert orthonormal basis $(e_n)_{n\geq 1}$ a weighted shift operator $W \in \mathcal{L}(H)$ is a bounded linear operator defined by the relations $We_i = \alpha_i e_{i+1}$ for a bounded sequence of real numbers $(\alpha_n)_{n\geq 1}$. One readily sees that

$$\|W\| = \sup\{|\alpha_n|\} \quad \text{and} \quad \|W^k\| = \sup\{|\alpha_n \alpha_{n+1} \cdots \alpha_{n+k-1}|\}. \tag{21}$$

We choose first the sequence $\varepsilon_m = M/K^{m-1}$ for some $M > 0$ and some $K > 1$, and define a weighted shift W_ε by $\alpha_n = \varepsilon_m$ if $n = 2^{(m-1)}(2\ell + 1)$, where ℓ is a non-negative integer. This sophisticated way of distributing the numbers ε_m into a sequence α_n makes a number ε_m to appear for the first time in the α_n sequence at the position $n = 2^{(m-1)}$ and from that position onwards to appear periodically, infinitely many times, with a period of 2^m.

Then, one also defines the weighted shifts L_m by a sequence of weights α_n that are all of them equal to zero, except at the positions $n = 2^{(m-1)}(2\ell + 1)$, where ℓ is a non-negative integer, where $\alpha_n = \varepsilon_m$. With this choice, the operator $W_\varepsilon - L_m$ is also a weighted shift, and it has zeroes along its sequence of weights, distributed each 2^m places, and starting at the $2^{(m-1)}$ position. This means, according to 21, that $W_\varepsilon - L_m$ is nilpotent of index 2^m, $(W_\varepsilon - L_m)^{2^m} = 0$. Consequently, its spectral radius $\rho(W_\varepsilon - L_m) = 0$. One can also obtain, after some work, that $\rho(W_\varepsilon) = M/K$ and that the spectrum $\sigma(W_\varepsilon)$ is the whole disk of radius M/K centered at zero. Concerning the norms, by using (21) one gets that $\|W_\varepsilon\| = M$ and $\| - L_m\| = \varepsilon_m$.

In this way, Kakutani's example shows the existence of a bounded linear operator W_ε with positive spectral radius that is approximated, in the operator norm, by a sequence $W_\varepsilon - L_m$ of operators whose spectrum reduces to the single point 0.

Our first example of translation of these ideas to (20) is very simple. Let us choose a number R and the previous numbers M and K in such a way that $0 < R - M/K < R < 1 < R + M/K$ and with these choices define the new operator $T = R\,I + W_\varepsilon$, where I is the identity operator. The spectrum of T is a disk of radius M/K centered at the point R. This spectrum intersects the exterior of the unit circle and lies entirely in the half-plane $Re\,z \geq R - M/K > 0$. Because of this last property, the operator $A \doteq \log(T)$ can be defined, and by the Spectral Mapping Theorem

$$\|e^{tA}\| \geq \rho(e^{tA}) = e^{t\log(R+M/K)}, \tag{22}$$

which is unstable since $R + M/K > 1$.

We construct now the sequence of operators $S_m = R\,I + W_\varepsilon - L_m$. All of these operators have their spectra reduced to the single point $z = R$, and these operators converge in the operator norm to $T = R\,I + W_\varepsilon$, which spectrum is the disk of radius M/K centered at $z = R$. If we take now $A_m = \log(R\,I + W_\varepsilon - L_m)$, we again have that the sequence A_m tends to $A = \log(T)$ as $m \to \infty$ in the operator norm, by the continuity of the logarithm. Also, by the properties of the exponential, perhaps by using adapted norms, for all $\delta > 0$ and all m, there exists a number $D_{m,\delta}$ such that

$$\|e^{tA_m}\| \leq D_{m,\delta}\; e^{t(\log(R)+\delta)}, \tag{23}$$

which implies stability since $\log(R) < 0$, and δ can be chosen small enough.

In this way we have perturbed an autonomous unstable system $\dot{x}(t) = Ax(t)$ to a new autonomous system $\dot{x}(t) = Ax(t) + (A_m - A)x(t)$, with a perturbation that can be taken as small as we wish in the operator norm, and the new system is asymptotically stable.

This example deserves to be commented in relation of Theorem 4 of [10] (p. 2704). According to that theorem, if an equation $\dot{x}(t) = A(t)x(t)$ exhibits an exponential dichotomy with nontrivial stable and an unstable part (which in particular means that it is unstable), then a new system $\dot{x}(t) = A(t)x(t) + B(t)x(t)$ will exhibit a similar dichotomy (which means that it is also unstable) if $\sup\{\|B(t)\|; t \in \mathbb{R}\}$ is sufficiently small, and if some compactness conditions are met, that are automatically satisfied in our case since B does not depend on t. This robustness of the instability is broken in our example, since the spectrum of A is a connected set that has points both in $Re\,z < 0$ and in $Re\,z > 0$, but it is not possible to divide it into two spectral sets by the vertical line $Re\,z = 0$. This is something very typical from infinite dimensional functional analysis, that cannot be expected in finite dimensions.

Our second example, also based on Kakutani's construction, starts with the same system $\dot{x}(t) = Ax(t)$ as above, with $A = \log(R\,I + W_\varepsilon)$ and W_ε, with the relations $0 < R - M/K < R < 1 < R + M/K$, whose instability is expressed by the inequality (22) above. We want to add to it now a time-dependent perturbation $B(t)$, depending continuously on $t \geq 0$ such that $\sup\{\|B(t)\|; t \in [0,\infty)\}$ can be taken as small as we wish, but with the novelty that $\lim_{t\to\infty} \|B(t)\| = 0$. Despite this, we want to obtain a system $\dot{x}(t) = Ax(t) + B(t)x(t)$ that will be stable.

Let us name B_m the operators $A_m - A$ considered above. Let us say again that $\|B_m\| \to 0$ as $m \to \infty$ and that the spectra $\sigma(A + B_m) = \sigma(A_m) = \{\log R\}$. Let us fix now one value of $\delta > 0$ in (23) such that if we define $\omega = -\log(R) - \delta$ we still have $\omega > 0$. For example, $\delta = -\frac{1}{2}\log R$. If we write D_m for $D_{m,\delta}$ in (23) we will have $\|e^{tA_m}\| \le D_m e^{-\omega t}$. We do not expect the sequence D_m to be bounded as $m \to \infty$. Let us choose an index $m_0 \ge 1$ and define

$$B(t) = \begin{cases} B_{m_0+k}, & \text{for } t_k \le t \le t_{k+1}-1, \\ (t_{k+1}-t)B_{m_0+k} + (t-t_{k+1}+1)B_{m_0+k+1} & \text{for } t_{k+1}-1 \le t \le t_{k+1}, \end{cases} \tag{24}$$

for an increasing sequence t_k with $t_0 = 0$ and $t_k + 1 < t_{k+1}$, to be defined later. It is clear that $B(t)$ is a continuous function from $[0,\infty)$ to $\mathcal{L}(H)$. Since $\|B_m\| \to 0$ it is clear that

$$E_{m_0} \doteq \sup\{\|B_m\|; m \ge m_0\} \to 0 \text{ as } m_0 \to \infty.$$

Therefore, $\|B(t)\| \le E_{m_0}$ for all $t \ge 0$, and this can be made as small as we like by choosing m_0 sufficiently large.

In order to define the sequence $(t_k)_{k\ge 0}$ let us now bound the solutions of

$$\begin{cases} \dot{x}(t) = Ax(t) + B(t)x(t), \\ x(0) = x_0. \end{cases} \tag{25}$$

For t between t_k and $t_{k+1}-1$ we will have $A + B(t) = A_{m_0+k}$ and, because of (23),

$$\|x(t)\| \le \|x(t_k)\| D_{m_0+k} e^{-\omega(t-t_k)}.$$

To fix ideas, let us start with $k = 0$. For $0 = t_0 \le t \le t_1 - 1$ we can write $\|x(t)\| \le D_{m_0} e^{-\omega t}\|x(0)\|$. Then, for $t_1 - 1 \le t \le t_1$ we can broadly bound as

$$\|x(t)\| \le e^{(t-t_1+1)(\|A\|+E_{m_0})}\|x(t_1-1)\| \le e^{(\|A\|+E_{m_0})}\|x(t_1-1)\|,$$

and, putting the two parts together

$$\|x(t)\| \le e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega t}\|x(0)\|, \tag{26}$$

which obviously implies the weaker bound

$$\|x(t)\| \le e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t}\|x(0)\|, \tag{27}$$

both for $0 \le t \le t_1$. Then, we continue with $t_1 \le t \le t_2 - 1$, and for this range of t we have $A + B(t) = A_{m_0+1}$ and

$$\|x(t)\| \le D_{m_0+1} e^{-\omega(t-t_1)}\|x(t_1)\|,$$

and, as before,

$$\|x(t)\| \le e^{(\|A\|+E_{m_0})} D_{m_0+1} e^{-\omega(t-t_1)}\|x(t_1)\|,$$

now for the whole $t_1 \le t \le t_2$. Putting this together with (26) we get, again for $t_1 \le t \le t_2$,

$$\|x(t)\| \le e^{(\|A\|+E_{m_0})} D_{m_0+1} e^{-\omega(t-t_1)} e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega t_1}\|x(0)\|,$$

that we can write again as

$$\|x(t)\| \le e^{(\|A\|+E_{m_0})} D_{m_0+1} e^{-\omega(t-t_1)} e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t_1} e^{-\frac{1}{2}\omega t_1}\|x(0)\|,$$

and at this point we see that we can choose t_1 large enough in such a way that

$$e^{(\|A\|+E_{m_0})} D_{m_0+1} e^{-\frac{1}{2}\omega t_1} \le 1.$$

With this choice we get

$$\|x(t)\| \le e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega(t-t_1)} e^{-\frac{1}{2}\omega t_1}\|x(0)\|, \tag{28}$$

for $t_1 \leq t \leq t_2$, which will be needed in the next interval, and also deduce, together with (27) the weaker but more global bound

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t} \|x(0)\|, \tag{29}$$

now for all t such that $0 \leq t \leq t_2$.

Now we proceed inductively. Suppose that along the interval $t_{k-1} \leq t \leq t_k$, where t_k is still to be chosen, we have obtained, as in (28), the bound

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega(t-t_{k-1})} e^{-\frac{1}{2}\omega t_{k-1}} \|x(0)\|, \tag{30}$$

for $t_{k-1} \leq t \leq t_k$, and the weaker inequality

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t} \|x(0)\|, \tag{31}$$

for $0 \leq t \leq t_k$. Then we analyze for $t_k \leq t \leq t_{k+1}$ and obtain that

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0+k} e^{-w(t-t_k)} e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega(t_k-t_{k-1})} e^{-\frac{1}{2}\omega t_{k-1}} \|x(0)\|.$$

Then we choose t_k in such a way that

$$e^{(\|A\|+E_{m_0})} D_{m_0+k} e^{-\frac{1}{2}\omega(t_k-t_{k-1})} \leq 1,$$

and obtain

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\omega(t-t_k)} e^{-\frac{1}{2}\omega t_k} \|x(0)\|, \tag{32}$$

for $t_k \leq t \leq t_{k+1}$, and the weaker inequality

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t} \|x(0)\|, \tag{33}$$

for $0 \leq t \leq t_{k+1}$.

With these choices of the t_k one can make $k \to \infty$ and obtain the final bound

$$\|x(t)\| \leq e^{(\|A\|+E_{m_0})} D_{m_0} e^{-\frac{1}{2}\omega t} \|x(0)\|, \tag{34}$$

for all $t \geq 0$, that proves the exponential asymptotic stability of the solutions of (25).

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