On the number of coloured triangulations of d-manifolds

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Abstract

We give superexponential lower and upper bounds on the number of coloured d-dimensional triangulations whose underlying space is a manifold, when the number of simplices goes to infinity and $d \geq 3$ is fixed. In the special case of dimension 3, the lower and upper bounds match up to exponential factors, and we show that there are $2^{\Theta(n)} n^{\frac{n}{6}}$ coloured triangulations of 3-manifolds with n tetrahedra. Our results imply that random triangulations of 3-manifolds have $o(n)$ vertices.

Our upper bounds apply in particular to coloured d-spheres for which they seem to be the best known bounds in any dimension $d \geq 3$, even though it is often conjectured that exponential bounds hold in this case.

We also ask a related question on regular edge-coloured graphs having the property that each 3-coloured component is planar, which is of independent interest.

1 Introduction and main results

A famous question, sometimes attributed to Gromov [16, 13] but going back at least to Durhuus and Jónsson [11], asks whether for any dimension $d \geq 2$ the number of inequivalent triangulations of the d-sphere by n unlabelled simplices is bounded by K^n for some constant $K = K(d)$. In dimension $d = 2$, it is not difficult to see that the answer is yes by noticing that each planar triangulation can be encoded by a spanning tree and a parenthesis word (one can also use the explicit formula due to Tutte [20]). In dimension $d \geq 3$, this question is open. In the pioneering paper [11], Durhuus and Jónsson introduced a subclass of triangulated spheres called locally constructible, or LC, and showed that their number is exponentially bounded. They conjectured that all 3-spheres are LC, but this was disproved many years later by Benedetti and Ziegler in [6]. Other subclasses of spheres with exponential growth have been considered [15, 5, 12, 9, 1], but the question remains wide open.

A motivation for the study of triangulations comes from the discretization of space in quantum gravity, see [3, 17]. Recently there has been a renewed interest in this topic via coloured tensor models, which are a higher dimensional generalization of matrix integrals, see [14, 8]. The objects

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that naturally arise in these models are *coloured triangulations*: roughly speaking, a triangulation is *coloured* if the vertices are coloured with colours from 1 to $d+1$, and all colours appear in each d-simplex. These triangulations are defined by gluing in pairs the facets ($(d-1)$-dimensional faces) in a family of n abstract coloured d-simplices, and need not be simplicial – precise definitions are given below. Because each triangulation can be made coloured by an appropriate barycentric subdivision that multiplies the number of simplices by a factor that only depends on d, the answer to the "Gromov question" is the same for coloured and uncoloured triangulations. In this paper we will work only with coloured triangulations, that are nicer and more natural as combinatorial objects.

In [16], Rivasseau showed that the number of coloured triangulations of the d-sphere with n *labelled* simplices grows at most like $n^{\left(\frac{d-1}{2}+\frac{1}{d}\right)n}$ up to exponential factors of the form K^n. This bound improves the trivial bound $n^{\left(\frac{d+1}{2}\right)n}$ which counts *all* complexes obtained by arbitrary gluings of n labelled d-simplices along their facets, up to exponential factors. Equivalently there are at most $n^{\left(\frac{d-3}{2}+\frac{1}{d}\right)n}$ inequivalent (unlabelled) triangulations of the d-sphere with n d-simplices up to exponential factors, which as far as we know was prior to this work the best known upper bound in the direction of Gromov's question.

This naturally raises the question of improving further the constants driving this superexponential growth. In this paper we address this question, but under a weaker topological constraint: instead of considering d-spheres, we consider triangulations whose underlying space is a d-manifold. Since spheres are manifolds, the superexponential upper bounds that we obtain for d-manifolds apply in particular to d-spheres, and in fact they improve the ones of [16]. We also give superexponential lower bounds obtained by explicit constructions (of course these lower bounds do not apply to d-spheres). Our lower and upper superexponential bounds match in dimension $d=3$, but a gap remains in higher dimension. Our main result, Theorem 1 below, summarizes these results.

1.1 Notation

We will mostly be interested in the superexponential growth of the sequences we consider, *i.e.* we will often disregard factors of the form K^n, and for this we will use the following notation

$$f(n) \preceq g(n) \text{ iff } \exists K>0 \text{ such that for } n \text{ large enough, } f(n) \leq K^n g(n) \quad (1)$$

$$f(n) \asymp g(n) \text{ if } f(n) \preceq g(n) \text{ and } g(n) \preceq f(n). \quad (2)$$

Moreover, in all asymptotic statements in this paper, it is implicitely assumed that d is constant and n goes to infinity with $(d+1)n$ being even.

1.2 Main results

For $d \geq 2, n \geq 2$, we let $M_d(n)$ be the number of coloured d-dimensional triangulations of orientable manifolds, with n d-simplices labelled from 1 to n (see formal definition in Section 2.1).

Theorem 1 (Main results)**.** *For all $d \geq 3$ with $d \neq 4$ we have*

$$n^{\frac{n}{2d}} \preceq \frac{1}{n!} M_d(n) \preceq n^{\left(\frac{1}{6}+(d-3)\frac{3}{20}\right)n}, \quad (3)$$

and for $d=4$ we have

$$n^{\frac{n}{8}} \preceq \frac{1}{n!} M_4(n) \preceq n^{\frac{n}{3}}.$$

Since the upper and lower bounds in (3) match in dimension $d = 3$, we obtain:

Corollary 2. *The number of coloured triangulations of 3-manifolds with n labelled tetrahedra satisfies*

$$\frac{1}{n!} M_3(n) \asymp n^{\frac{n}{6}}.$$

These results raise the question of determining the constant

$$\alpha_d := \limsup_{n\to\infty} \frac{\log(M_d(n)/n!)}{n \log n}.$$

Our main theorem shows that $\alpha_3 = 1/6$, $\alpha_4 \in [\frac{1}{8}, \frac{1}{3}]$, and for $d \geq 5$

$$\frac{1}{2d} \leq \alpha_d \leq \frac{1}{6} + (d-3)\frac{3}{20}, \tag{4}$$

which leaves an important gap especially for large d. We believe that the lower bound is closer to the truth, and we conjecture that α_d goes to zero when d goes to infinity.

1.3 Additional results, related work and comments

In the uncoloured setting, the fact that the number of triangulated 3-manifolds with n unlabelled tetrahedra grows *at least* as $n^{\epsilon n}$ for explicit positive values of ϵ was proved in [2] (see also [4]). Their construction is based on Heegard gluings of high genus triangulations and inspired our general lower bound construction. Using the distribution of short cycles in the configuration model [21], in [10] it was proved that the probability the underlying space of a 3-dimensional triangulation with n unlabelled tetrahedra is a 3-manifold is $o(1)$. It may be possible to make the argument in [10] quantitative in order to obtain an upper bound for the number triangulated 3-manifolds of the form $n^{(1-\epsilon)n}$ for an explicit $\epsilon > 0$. However, the uncoloured case seems combinatorially more involved than the one we address here, and this would probably not lead to a sharp value of ϵ as the one we have here in the coloured case.

We have also mentioned Rivasseau's result that applies to homology d-spheres [16]. Our results also apply to this case. See Theorem 8 on Section 3.2 for our most general upper bounds. In particular, note that for homology d-spheres of dimension $d = 4$, we obtain a slightly better upper bound than the one we have for $M_4(n)$.

Finally, it is natural to ask if our enumerative results have probabilistic consequences, especially in dimension $d = 3$ where our upper and lower bounds match. Because they match only up to exponential factors, the only events that there is hope to control are the ones with exponentially small probability. Indeed we can show, as a corollary of our proofs:

Theorem 3. *Let X_n be a coloured 3-dimensional triangulation with n labelled tetrahedra whose underlying space is a 3-manifold, chosen uniformly at random. Then X_n has $O(\frac{n}{\log n})$ vertices with high probability.*

More precisely, if V_n is the number of vertices of X_n, then for all $c > 0$ there exists K such that

$$\mathbf{Pr}\left(V_n \leq \frac{Kn}{\log n}\right) \geq 1 - e^{-cn}.$$

It is natural to expect that the number of vertices is in fact $O(\log n)$ with high probability as in the 2-dimensional case, but our techniques do not enable to prove it.

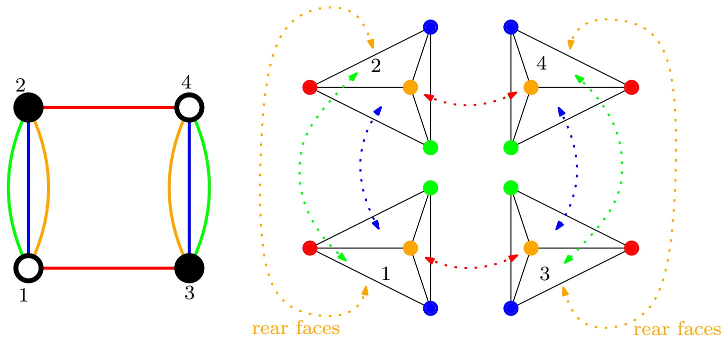

Figure 1: A $(d+1)$-colourful graph G with $d = 3$ and $n = 4$, and the corresponding cell complex. The resulting space $|X(G)|$ is a 3-sphere – an easy way to see that is to perform first the six "non-red" gluings, which clearly gives two disjoint 3-balls, that are then glued along their boundary via the "red" gluings. Note that the complex $X(G)$ is *not* simplicial, since the two tetrahedra on each side share the same set of vertices. Since the sphere is a manifold, this object contributes to the number $M_3(4)$.

2 Colourful graphs and triangulations

2.1 Definitions

Definition 1. For $d \geq 1$, a *(d+1)-colourful graph* of order n is a bipartite $(d+1)$-regular multigraph on $[1..n]$, equipped with a colouring of its edges with colours in $[1..d+1]$, such that each vertex is incident to all colours.

We equip the colourful graph with a colouring of its vertex set, where one part of the bipartition is assigned colour *white* and the other one, colour *black*.

Given a $(d+1)$-colourful graph G, we can construct a topological object as follows. For each $v \in [1..n]$, we consider an abstract d-simplex, and we colour its vertices from 1 to $(d+1)$. Apart from these colours, the vertices of each simplex are unlabelled. We see each simplex as a solid body being a copy of the regular d-simplex $\{x_1 + \cdots + x_{d+1} = 1, x_i \geq 0\}$ equipped with its Euclidean topology. Now for each edge $e = \{u, v\}$ of G of colour i, we consider the unique facet in each of the two simplices corresponding to u and v, whose vertices are coloured by $[1..d+1] \setminus \{i\}$. We glue these two facets together according to the unique isometric gluing that preserves colours, and we repeat this procedure for each edge of G.

We call $X(G)$ the corresponding cell complex. Complexes obtained in this way are called *labelled d-dimensional coloured triangulations*. We denote by $|X(G)|$ the resulting topological space, and we observe that it is orientable because G is bipartite. Note that one can recover the graph G from the triangulation $X(G)$ by taking its dual graph, in which vertices correspond to highest dimensional cells and edges to $(d-1)$-dimensional incidences between these cells.

While for $d = 2$ the space $|X(G)|$ is always a manifold, we emphasize that if $d \geq 3$ this is not always the case.

Definition 2. We let $M_d(n)$ be the number of $(d+1)$-colourful graphs G on $[1..n]$ such that $|X(G)|$ is a d-manifold.

As in the introduction, $M_d(n)$ can also be defined as the number of coloured d-dimensional triangulations with n d-simplices labelled from 1 to n whose underlying space is an orientable manifold.

2.2 Homology and residues

Let G be a $(d+1)$-colourful graph and $\mathcal{I} \subseteq [1..d+1]$. We let $G_{\mathcal{I}}$ be the graph on the same vertex set as G, keeping only the edges whose colour is in $\mathcal{I}$. Up to relabelling colours keeping their relative natural order, $G_{\mathcal{I}}$ is an $|\mathcal{I}|$-colourful graph associated with some coloured triangulation $X(G_{\mathcal{I}})$ of dimension $|\mathcal{I}| - 1$.

It is easy to see that connected components of $G_{\mathcal{I}}$ are in bijection with $(d+1-|\mathcal{I}|)$-dimensional cells of $X(G)$ whose colours do not belong to $\mathcal{I}$. For a proof, see [7] in which these connected components are called *residues* of G. For example, for each $i \in [1..d+1]$, vertices of $X(G)$ of colour i are in bijection with connected components of $G_{\mathcal{I}}$ for $\mathcal{I} = [1..d+1] \setminus \{i\}$.

Recall that a *rational homology sphere* of dimension d is a cell complex which has the same homology groups over the field of rationals as a d-sphere. Equivalently, all its Betti numbers are zero, except the 0-th and d-th ones that are equal to one. Each integral homology sphere is a rational homology sphere, but the converse is not true in general.

Proposition 4. *Let G be a $(d+1)$-colourful graph, and consider the three following properties*

(i) For any $I \subseteq [1..d+1]$ such that $1 \leq |\mathcal{I}| \leq d$, $|X(G_{\mathcal{I}})|$ is a disjoint union of spheres (of dimension $|\mathcal{I}| - 1$);

(ii) $|X(G)|$ is a d-manifold;

(iii) For any $I \subseteq [1..d+1]$ such that $1 \leq |\mathcal{I}| \leq d$, and each connected component H of $G_{\mathcal{I}}$, the space $|X(H)|$ is a rational homology sphere (of dimension $|\mathcal{I}| - 1$).

Then one has: $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Proof. Let H be a connected component of $G_{\mathcal{I}}$, and let σ be the $\mathcal{I}^c$-coloured cell of $X(G)$ corresponding to H. Let b_σ be the barycentre of σ, which is a vertex in the barycentric subdivision $X(G)'$ of $X(G)$. It is easy to check (see [7, Proposition 2.5] and the sentence before it for a proof) that the topological join $|X(H)| * (\delta\sigma)'$ is the link of b_σ in $X(G)'$ (in the case where $|\mathcal{I}| = d$, i.e. σ is a 0-simplex, we conventionally understand this join as being just $|X(H)|$; in the case where $|\mathcal{I}| = 1$, i.e. H is just one edge, we conventionally understand $|X(H)|$ as the disjoint union of two isolated vertices).

Therefore the implication $(i) \Rightarrow (ii)$ is clear. Let us prove $(ii) \Rightarrow (iii)$. Let $U = |X(H)|$ and $V = (\delta\sigma)'$, we have the isomorphism

$$\tilde{H}_k(U \times V) \equiv \tilde{H}_{k+1}(U * V) \oplus \tilde{H}_k(U) \oplus \tilde{H}_k(V),$$

where $k \geq 0$ and $\tilde{H}_k$ is the k-th reduced homology group over the rationals. By taking ranks, multiplying by x^k and summing over k, we get

$$P_{U\times V}(x) - 1 = x^{-1}\big(P_{U*V}(x) - P_{U*V}(0)\big) + P_U(x) - 1 + P_V(x) - 1,$$

where P_X denotes the Poincaré polynomial of the topological space X. Now if $|X(G)|$ is a manifold, so is $|X(G)'|$, and since $X(G)'$ is a *simplicial* complex, the link $U * V$ is a homology sphere, see e.g. [18, Prop. 5.2.4]. We thus have $P_{U*V}(x) = 1 + x^{d-1}$, and since $P_{U\times V} = P_U P_V$ we finally obtain

$$(P_U(x) - 1)(P_V(x) - 1) = x^{d-2}.$$

Since $P_U = P_{|X(H)|}$ has degree $|\mathcal{I}| - 1$, it follows that $P_U(x) - 1 = x^{|\mathcal{I}|-1}$, i.e. $|X(H)|$ is a rational homology sphere. □

Remark 1. In the special case $|\mathcal{I}| = 3$, say $\mathcal{I} = \{i, j, k\}$ with $i < j < k$, the graph $G_{\mathcal{I}}$ is a 3-edge coloured bipartite cubic graph. We can define an embedding of $G_{\mathcal{I}}$ in an orientable surface by defining the clockwise order of edges to be (i, j, k) around white vertices and (i, k, j) around black vertices. This enables us to view the graph $G_{\mathcal{I}}$ as an *embedded* graph, and it is easy to see that it is the dual of the complex $X(G_{\mathcal{I}})$, which is a surface triangulation. In particular, if $|X(G)|$ is a manifold, then for any $\mathcal{I}$ with $|\mathcal{I}| = 3$ the complex $X(G_{\mathcal{I}})$ is a disjoint union of spherical triangulations by Proposition 4, and each connected component of the embedded graph $G_{\mathcal{I}}$ is a plane graph – with the canonical embedding just defined.

2.3 A question about 3-planar colourful graphs

Let $\mathcal{H}_d(n)$ be the class of $(d + 1)$-colourful graphs on $[1..n]$ having the following property:

(P): For any subset of colours $\mathcal{I} \subseteq [1..d + 1]$ such that $|\mathcal{I}| = 3$, the embedded graph $G_{\mathcal{I}}$ is planar.

From Remark 1, colourful graphs associated to manifolds satisfy property (P), and this fact is important in the proof of our upper bounds.

We do not expect $H_d(n) := |\mathcal{H}_n(d)|$ and $M_d(n)$ to have the same superexponential growth, and in fact the proof of our main upper bound uses more than property (P). However determining the growth of $H_d(n)$ is a purely graph-theoretic question of independent interest. Let

$$\beta_d := \limsup_{n\to\infty} \frac{\log(H_d(n)/n!)}{n \log n}.$$

Our proofs show that $\beta_3 = \frac{1}{6}$ and that for $d \geq 4$

$$\frac{1}{6} \leq \beta_d \leq \frac{d-2}{6}. \tag{5}$$

Determining β_d is an interesting problem on its own. Moreover, since $\alpha_d \leq \beta_d$, a substantial improvement of the upper bound on β_d, would lead to an improvement of our main result. The problem of determining β_d seems much more tractable *a priori* than the one of determining α_d. On the other hand, we believe that α_d and β_d have different asymptotic behaviour, so it is unlikely that one can obtain a tight upper bound on α_d using β_d, for $d \geq 4$.

3 Upper bounds

3.1 Main lemmas

We now fix a $(d + 1)$-colourful graph G on $[1..n]$. For $\mathcal{J} \subseteq [1..d + 1]$ we let $\kappa_{\mathcal{J}}$ be the number of connected components of the graph $G_{\mathcal{J}}$. For small values of $|\mathcal{J}|$ we drop brackets in the notation, for example $\kappa_{i,j} = \kappa_{\{i,j\}}$.

For $\mathcal{I} \subseteq [1..d + 1]$ with $|\mathcal{I}| = \hat{d} + 1$ and for $r \in [0..\hat{d} + 1]$, we also let

$$\kappa_{\mathcal{I}}^{(r)} := \sum_{\substack{\mathcal{J}\subseteq\mathcal{I} \\ |\mathcal{J}|=r}} \kappa_{\mathcal{J}} .$$

Note that $\kappa_{\mathcal{I}}^{(r)}$ is the number of cells of $X(G_{\mathcal{I}})$ of dimension $(\hat{d}-r)$. In particular, $\kappa_{\mathcal{I}}^{(0)} = n$ is the number of $\hat{d}$-simplices. Note also that, because $G_{\mathcal{I}}$ is $(\hat{d}+1)$-regular, we have $\kappa_{\mathcal{I}}^{(1)} = \frac{\hat{d}+1}{2}n$.

Moreover, if $\hat{d}$ is even and if $|X(G_{\mathcal{I}})|$ is a disjoint union of rational homology $\hat{d}$-spheres, the Euler-Poincaré formula states that

$$\sum_{r=0}^{\hat{d}} (-1)^r \kappa_{\mathcal{I}}^{(r)} = 2\kappa_{\mathcal{I}}^{(\hat{d}+1)} \tag{6}$$

Lemma 5. *Let $\mathcal{I} \subseteq [1..d+1]$ with $|\mathcal{I}| = 3$ such that $|X(G_{\mathcal{I}})|$ is a disjoint union of 2-spheres. Then there exist distinct $i, j \in \mathcal{I}$ such that*

$$\kappa_{i,j} - \kappa_{\mathcal{I}} \leq \frac{n}{6}. \tag{7}$$

Proof. The Euler-Poincaré formula (6) for $\hat{d} = 2$ implies

$$\kappa_{\mathcal{I}}^{(0)} + \kappa_{\mathcal{I}}^{(2)} = \kappa_{\mathcal{I}}^{(1)} + 2\kappa_{\mathcal{I}}^{(3)} \ .$$

Since $\kappa_{\mathcal{I}}^{(0)} = n$ and $\kappa_{\mathcal{I}}^{(1)} = 3n/2$, we obtain

$$\kappa_{\mathcal{I}}^{(2)} = 2\kappa_{\mathcal{I}}^{(3)} + \frac{n}{2} = 2\kappa_{\mathcal{I}} + \frac{n}{2} \ .$$

By averaging over pairs of colours in $\mathcal{I}$, there exist distinct $i, j \in \mathcal{I}$ such that

$$\kappa_{i,j} \leq \frac{2\kappa_{\mathcal{I}}}{3} + \frac{n}{6} \leq \kappa_{\mathcal{I}} + \frac{n}{6} \ . \tag{8}$$

□

Lemma 6. *Let $\mathcal{I} \subseteq [1..d+1]$ with $|\mathcal{I}| = 5$ such that $|X(G_{\mathcal{I}})|$ is a disjoint union of rational homology 4-spheres. Then there exist distinct $i, j, k \in \mathcal{I}$ such that*

$$\kappa_{i,j} - \kappa_{i,j,k} \leq \frac{3}{20}n. \tag{9}$$

Proof. The Euler-Poincaré formula (6) for $\hat{d} = 4$ implies

$$\kappa_{\mathcal{I}}^{(0)} + \kappa_{\mathcal{I}}^{(2)} + \kappa_{\mathcal{I}}^{(4)} = \kappa_{\mathcal{I}}^{(1)} + \kappa_{\mathcal{I}}^{(3)} + 2\kappa_{\mathcal{I}}^{(5)} \ ,$$

Since by deleting a colour the number of connected components can only increase, we have $\kappa_{\mathcal{I}}^{(4)} \geq 5\kappa_{\mathcal{I}}^{(5)}$ and in particular $\kappa_{\mathcal{I}}^{(4)} \geq 2\kappa_{\mathcal{I}}^{(5)}$. Since $\kappa_{\mathcal{I}}^{(0)} = n$ and $\kappa_{\mathcal{I}}^{(1)} = 5n/2$, we obtain

$$\kappa_{\mathcal{I}}^{(2)} \leq \kappa_{\mathcal{I}}^{(3)} + \frac{3n}{2} \ ,$$

or equivalently,

$$\frac{1}{3} \sum_{i,j,k \in \mathcal{I}} (\kappa_{i,j} + \kappa_{j,k} + \kappa_{k,i}) \leq \sum_{i,j,k \in \mathcal{I}} \left(\kappa_{i,j,k} + \frac{3n}{20} \right) \ .$$

Thus, there exists a triple of distinct $i, j, k \in \mathcal{I}$ satisfying

$$\kappa_{i,j} + \kappa_{j,k} + \kappa_{k.i} \leq 3\kappa_{i,j,k} + \frac{9n}{20} .$$

Therefore, up to relabelling i, j, k, so that $\kappa_{i,j}$ is the smallest term in the left-hand side, we obtain

$$\kappa_{i,j} \leq \kappa_{i,j,k} + \frac{3n}{20} . \qquad \square$$

Remark 2. It is natural to expect that by using the Euler-Poincaré formula for rational homology spheres of higher dimensions one could obtain variants of Lemmas 5 and 6 with gradual improvements of the constants $\frac{1}{6}, \frac{3}{20}$ as the dimension gets higher. However this does not seem to be the case – at least not without new ideas. Similarly, we have not been able to obtain any improvement by looking at the whole set of Dehn-Sommerville equations rather than only the Euler-Poincaré formula, in any dimension.

Lemma 7. *Let C be a 2-colourful graph on $[1..n]$ with $\kappa_{1,2} = c$ connected components. Then the number of 3-colourful planar graphs G on $[1..n]$ with k components such that $G_{\{1,2\}} = C$, is at most $2^{O(n)} n^{c-k}$, uniformly in c and k.*

Proof. We will bound the number of graphs G satisfying the required properties by showing how to construct such graphs in two steps. In Step 1, we bound the number of ways to construct a minimal subgraph H of G that contains C and has the same connected components as G (hence k connected components). Then in Step 2, we bound the number of ways to extend H to G, preserving planarity. We let $\ell_1, \dots, \ell_c$ be the lengths of the $\{1, 2\}$-cycles in C.

Step 1. The subgraph H consists of C together with $k - c$ extra edges that connect components of C together. We encode H using a labelled plane forest F on $[1..c]$ with k components, a binary string w_1 of length n and a string $w_2 \in [1..\ell_1] \times \cdots \times [1..\ell_c]$. The vertices of the forest correspond to the cycles of C, say ordered by increasing minimum vertex, and the edges of the forest determine which cycles are connected together with edges of colour 3 in H.

We use w_1 and w_2 to specify the attachment of the edges of colour 3 between cycles as follows. We explore F component by component using a clockwise DFS, using the minimum vertex yet unexplored as root for each new component. At the same time, we add edges of colour 3 to C as follows. The word w_1 indicates which vertices of $[1..n]$ are adjacent to an edge of colour 3 in H. The word w_2 specifies the vertex to which an edge of colour 3 connects the first time a cycle is visited in the DFS exploration. Note that once we have attached the first edge of colour 3 to a cycle, this fixes the order in which the other edges of colour 3 appear along the same cycle.

Clearly, given the choice of a plane forest and two words, there is at most one way to connect the cycles in C with edges of colour 3 that is compatible with them.

To bound from above the number of ways to construct H, we bound the total number of encodings. There are at most 2^n choices for w_1 and $\prod_{i=1}^{c} \ell_i$ choices for w_2. From the arithmetic-geometric mean inequality and since $\sum_{j=1}^{c} \ell_j = n$, it follows that

$$\prod_{i=1}^{c} \ell_i \leq \left(\frac{\sum_{j=1}^{c} \ell_j}{c} \right)^c = \left(\frac{n}{c}\right)^c \leq \binom{n}{c} \leq 2^n .$$

Moreover, the number of labelled plane forests with m vertices and k components is at most $\frac{m!}{k!}\binom{2m-k}{m}$. Indeed, an unlabelled plane forest with k ordered components and m vertices can be encoded by its Lukaciewicz path (see e.g. [19, Chap. 5]) which is a path with $m-k$ up-steps and m down steps. The number of inequivalent ways to label the vertices is at most $m!$, and once vertices are labelled all the $k!$ possible orderings of components are inequivalent.

Since $m!/k! = m(m-1)\dots(k+1) \le m^{m-k}$, this shows that the number of choices for H is at most $2^{3n}n^{c-k}$ and concludes the discussion on Step 1.

Step 2. Given the choice of H, we now bound the number of ways to extend it to a 3-colourful planar graph G with the same number of components. We first add a new half-edge to each vertex of H not already incident to an edge of colour 3. The edges of $G-H$ can be seen as a perfect matching on these half-edges. Now, because G is planar and has the same number of components as H, the edges of this perfect matching form a non-crossing arch system around each face of G. The matching of these half-edges can therefore be encoded by a well-formed parenthesis word of length equal to the number of half-edges. This shows that the number of ways to construct G from H is at most 2^{2n}.

The lemma follows from the bounds obtained in Step 1 and Step 2. □

3.2 Induction on dimension and proof of upper bounds in Theorem 1

The upper bounds in Theorem 1 easily follow from the implication $(ii) \Rightarrow (iii)$ in Proposition 4 together with Lemmas 5, 6 and 7, as we now show. In fact, we are going to show a more general result. For $d \ge 2, n \ge 2$, we let $\mathcal{N}_d(n)$ be the set of $(d+1)$-colourful graphs on $[1..n]$ having the property that for each $\mathcal{I} \subseteq [1..d]$ with $|\mathcal{I}| \in \{3,5\}$ and $|\mathcal{I}| \le d$, the space $|X(G_{\mathcal{I}})|$ is a disjoint union of rational homology spheres. We also let $\mathcal{S}_d(n)$ be the set of graphs such that $|X(G)|$ itself is a rational homology sphere. We let $N_d(n) = |\mathcal{N}_d(n)|$ and $S_d(n) = |\mathcal{S}_d(n)|$. As $M_d(n) \le N_d(n)$, the upper bounds in Theorem 1 follow directly from the next result.

Theorem 8. *For all $d \ge 3$ with $d \ne 4$ we have*

$$\frac{1}{n!}N_d(n) \preceq n^{\left(\frac{1}{6}+(d-3)\frac{3}{20}\right)n}, \tag{10}$$

and for $d = 4$ we have

$$\frac{1}{n!}N_4(n) \preceq n^{\frac{n}{3}}. \tag{11}$$

Moreover for any $d \ge 3$ we have

$$\frac{1}{n!}S_d(n) \preceq n^{\left(\frac{1}{6}+(d-3)\frac{3}{20}\right)n}. \tag{12}$$

We first state a lemma.

Lemma 9. *Suppose we are given, for each $d \ge 2$ and $n \ge 2$, a family $\mathcal{A}_d(n)$ of $(d+1)$-colourful graphs on $[1..n]$ such that:*

(a) For any $d \ge 3$, any $G \in \mathcal{A}_d(n)$, and any colour $i \in [1..d+1]$, the d-colourful graph obtained from G by removing all edges of colour i belongs to $\mathcal{A}_{d-1}(n)$;

(b) For each $\mathcal{I} \subseteq [1..d]$ with $|\mathcal{I}| = 3$, the embedded graph $G_{\mathcal{I}}$ is plane;

(c) *There exists a sequence* $(a_d)_{d\geq 3}$ *such that for any* $d \geq 3$ *and any* $G \in \mathcal{A}_d(n)$*, there exists a triple of distinct colours* $i, j, k \in [1..d+1]$ *such that* $\kappa_{i,j} - \kappa_{i,j,k} \leq a_d n$*.*

Then, we have

$$\frac{1}{n!}|\mathcal{A}_d(n)| \preceq n^{(a_3+a_4+\cdots+a_d)n}$$

Proof of Lemma 9. The proof proceeds by induction on $d \geq 2$.

For $d = 2$, Property (b) implies that G is planar. The number of planar cubic multigraphs with n labelled vertices is[1] at most $n!K^n$ for some constant K, and for each graph, there is an exponential number of colourings of the edges using 3 colours. It follows that $|\mathcal{A}_2(n)| \preceq n^n$.

Let $d \geq 3$, and let $G \in \mathcal{A}_d(n)$. Take any triple of colours i, j, k as in Property (c). The graph G is the union of the graph $G_{[1..d+1]\setminus\{k\}}$ and the graph $G_{\{i,j,k\}}$. By induction and Property (a), there are at most $n^{(a_3+a_4+\cdots+a_{d-1})n}$ choices for the first graph. By Property (b), the second graph is planar, so by Lemma 7 there at most $n^{a_d n}$ ways to choose it once the edges of colour i and j have been placed. □

Proof of Theorem 8 and of (5)*.* We apply Lemma 9 with several different sequences $\mathcal{A}_d(n)$ depending on which bound we want to obtain.

We first choose choose $\mathcal{A}_d(n) = \mathcal{N}_d(n)$ for $d \neq 4$ and $\mathcal{A}_4(n) = \mathcal{S}_4(n)$. This sequence satisfies Property (a) and (b) by definition. We can then take $a_3 = \frac{1}{6}$ by Lemma 5, and $a_d = \frac{3}{20}$ for $d \geq 4$ by Lemma 6. This implies both (10) and the case $d = 4$ of (12).

If we choose $\mathcal{A}_d(n) = \mathcal{N}_d(n)$ for all d, we can take $a_3 = a_4 = \frac{1}{6}$ and $a_d = \frac{3}{20}$ for $d \geq 5$, which gives a less good bound in general but is the best we can do for $d = 4$, proving (11).

If we choose $\mathcal{A}_d(n) = \mathcal{H}_d(n)$ for all d, then by Lemma 5 we can take $a_d = \frac{1}{6}$ for all d, hence proving (5).

It only remains to prove (12) for $d \neq 4$. But this is a direct consequence of (10) since $\mathcal{S}_d(n) \subseteq \mathcal{N}_d(n)$.

□

4 Lower bounds

The goal of this section is to construct many triangulated d-dimensional manifolds. We will use the implication $(i) \Rightarrow (ii)$ in Proposition 4 to ensure that the complexes we construct are manifolds. In order to ensure that the residues are spheres, our strategy will be to create them in such a way they satisfy the stronger property of being *locally constructible*, which is the following property introduced by Durhuus and Jónsson [11].

Definition 3. For $d \geq 2$, let X be a pure d-dimensional triangulated pseudomanifold with n simplices. A *local construction* for X is a sequence $X_1, \ldots, X_\ell$ with $\ell = (d+1)n/2 + 1$ such that X_i is a d-dimensional triangulated pseudomanifold for each i, and

(1) X_1 is a simplex;

[1] The number of unlabelled embedded planar multigraphs with $3n$ edges is bounded by K^n for some $K > 0$, which follows either by a spanning tree argument or by exact counting formulas [20] as recalled in the introduction. Here since we have n vertices, the labelling multiplies by at most $n!$.

(2) if $i \leq n-1$, then X_{i+1} is obtained from X_i by gluing a new d-simplex to X_i alongside one of the $(d-1)$-simplices of ∂X_i;

(3) if $i \geq n$, then X_{i+1} is obtained from X_i by identifying a pair of $(d-1)$-simplices of ∂X_i whose intersection contains a $(d-2)$-simplex;

(4) $X_\ell = X$.

We say that X is *locally constructible* (LC) if it has a local construction.

In [11], the authors proved that any LC manifold is a sphere. In our context, this can be reformulated as

Lemma 10. *Let H be a $(k+1)$-colourful connected graph. Suppose that there exists a spanning tree T of H, and a sequence of edges $e_1, \dots, e_s$ such that $E(H-T) = \{e_1, \dots, e_s\}$ and e_j belongs to a bicoloured cycle in H_j, where $H_j = ([1..n], E_j)$ with $E_j = E(T) \cup \{e_1, \dots, e_j\}$. Then $|X(H)|$ is a sphere.*

Proof. Let $f_1, \dots, f_{n-1}$ be the sequence of the edges of T traversed in a DFS exploration of it starting at 1. Then, the sequence of edges $f_1, \dots, f_{n-1}, e_1, \dots, e_s$ yields a local construction for $X(H)$, by letting X_1 be the k-simplex corresponding to vertex 1 and X_{i+1} be obtained from X_i by gluing the two facets of the k-simplices corresponding to u and v that do not contain the colour of uv, where uv is the i-th edge in the sequence. The condition that $f_1, \dots, f_{n-1}$ is a DFS order implies (b) and the condition that e_j belongs to a bicoloured cycle in H_j implies (c). It follows that $X(H)$ is LC, and by the result of Durhuus and Jónsson in [11] (see also [6]), $|X(H)|$ is a k-sphere. □

We will use this lemma to prove our lower bounds.

Proof of lower bounds in Theorem 1. Let $k \in \mathbb{N}$. Consider the multigraph $G_0 = (V_0, E_0)$ defined as follows. Let $A = \{a_1, \dots, a_{kd}\}$, $A' = \{a'_1, \dots, a'_{kd}\}$, $B = \{b_1, \dots, b_{kd}\}$, $B' = \{b'_1, \dots, b'_{kd}\}$ and $V_0 = A \cup A' \cup B \cup B'$, thus G_0 has order $n = 4kd$. The edge multiset is defined as

$$
\begin{aligned}
&E_P = \{x_i x_{i+1} : x \in \{a, a', b, b'\}, i \in [1..kd-1]\} \cup \{a_1 a'_1, b_1 b'_1\},\\
&E_V = \{e_i(j) = a_i b_i : i \in [1..kd], j \in [1..d], i \notin \{j, j+1\} \,(\mathrm{mod}\ d)\} \cup \{e_{kd}(1) = a_{kd} b_{kd}\},\\
&E'_V = \{e'_i(j) = a'_i b'_i : i \in [1..kd], j \in [1..d], i \notin \{j, j+1\} \,(\mathrm{mod}\ d)\} \cup \{e'_{kd}(1) = a'_{kd} b'_{kd}\},\\
&E_R = \{e_i(d+1) = a_i b_i : i \in [1..kd], i \neq 0 \,(\mathrm{mod}\ d)\},\\
&E'_R = \{e'_i(d+1) = a'_i b'_i : i \in [1..kd], i \neq 0 \,(\mathrm{mod}\ d)\},\\
&E_0 = E_P \cup E_V \cup E'_V \cup E_R \cup E'_R \,.
\end{aligned}
$$

We colour the edges $x_i x_{i+1} \in E_P$ with colour $i+1 \,(\mathrm{mod}\ d)$, the edges $a_1 a'_1$ and $b_1 b'_1$ with colour 1, and the edges $e_i(j)$ and $e'_i(j)$ with colour j.

We will use the graph G_0 as a basis to construct many $(d+1)$-colourful graphs. Observe that the vertices x_i with $x \in \{a, a', b, b'\}$ have degree d if $i = 0 \,(\mathrm{mod}\ d)$ and degree $d+1$ otherwise.

Let σ, τ be permutations of length k. The graph $G = G(\sigma, \tau)$ is constructed from G_0 by adding the edges $a_{id} a'_{\sigma(i)d}$ and $b_{id} b'_{\tau(i)d}$ with colour $d+1$. Observe that G is a $(d+1)$-colourful graph (see Figure 2). Note that the coloured automorphism group of G (automorphisms of G that preserve

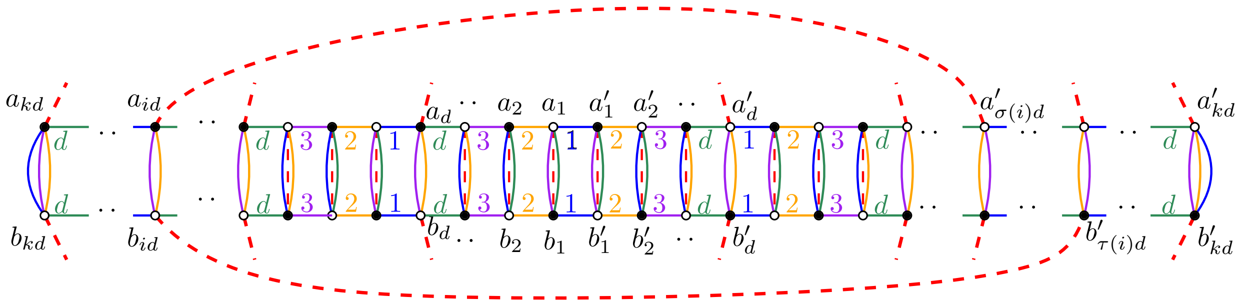

Figure 2: The lower bound construction, pictured here for $d = 4$. The colour in $[1..d]$ of horizontal edges is indicated, and edges of colour $d+1$ are represented with dashed lines.

the edge-colouring) has size at most 4. Therefore, there are at least $(k!)^2 n!/4 \asymp n^{n/2d} n!$ labelled $(d+1)$-colourful graphs of this form.

It remains to show that $|X(G)|$ is a d-manifold. Let x be a vertex of $X(G)$, we will show that the link of x is a $(d-1)$-sphere using Lemma 10. For $i \in [1..d+1]$, let G^i be the subgraph of G where edges of colour i have been deleted. If x has colour i in $X(G)$, then the link of x is homeomorphic to $|X(H)|$, where H is a connected component of G^i. It suffices to prove that $|X(H)|$ is a sphere.

Consider first the case $i = d+1$. Since G^{d+1} is connected, $H = G^{d+1}$. Let T be the spanning tree containing the edges $b_i b_{i+1}$ and $b'_i b'_{i+1}$ for $i \in [1..kd-1]$, the edge $b_1 b'_1$, the edges $e_i(1)$ and $e'_i(1)$ and for $i \notin \{0,1\} \pmod d$ and the edges $e_i(3)$ and $e'_i(3)$ and for $i \in \{0,1\} \pmod d$ (see Figure 3). Consider the following sequence of edges of $G^{d+1} - T$. First add the edges in $E_V \cup E'_V \setminus E(T)$, every added edge is a multiedge of the current graph and thus it belongs to a bicoloured cycle of length two. Then add the edges in $E_P \setminus E(T)$, when adding the edge $a_i a_{i+1}$, it will close a bicoloured cycle containing the edges $e_i(1)$, $b_i b_{i+1}$ and $e_{i+1}(1)$, and similarly for $a'_i a'_{i+1}$. By Lemma 10, it follows that $|X(G^{d+1})|$ is a $(d-1)$-sphere.

For each $i \in [1..d]$, the argument to show that $|X(H)|$ is a sphere is analogous, so we provide the proof for $i = 3$. The graph G^3 consists of a set of *gadgets* of size $2d$ together with a central gadget that includes the vertices a_1, a'_1, b_1, b'_1 (see Figure 4 left). Each non-central gadget is connected to two other gadgets through edges of colour $d+1$ that attach to vertices x_{id} for $x \in \{a, a', b, b'\}$ and $i \in [1..kd]$. Thus, H is a cycle of gadgets (see Figure 4 right). Let T be the spanning tree of H that contains a spanning tree for each gadget (similarly as in the case $i = d+1$) and all edges of colour $d+1$ between the gadgets but one. Consider the following sequence of edges of $H - T$. Firstly, add the edges in $E_V \cup E'_V \setminus E(T)$ and then the edges in $E_P \setminus E(T)$. As before, every edge closes a bicoloured cycle when added. Finally, add the last edge in colour $d+1$ between the gadgets. This edge closes a bicoloured cycle with colours $d+1$ and 2 (see Figure 4 right). By Lemma 10, it follows that $|X(H)|$ is a $(d-1)$-sphere.

Since the $(d-1)$-dimensional links of $X(G)$ are spheres, any smaller link also is. It follows that $|X(G)|$ is a d-manifold and we conclude that $M_d(n) \succeq n^{n/2d} n!$.

□

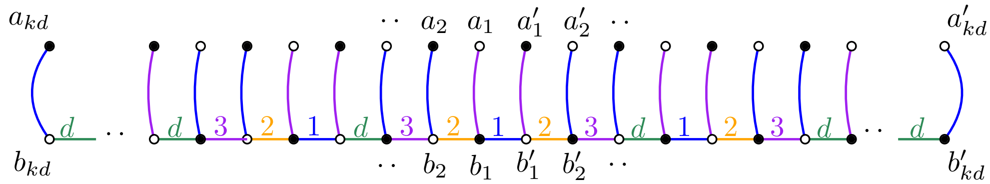

Figure 3: The initial spanning tree for the LC construction, in the case when colour $d+1$ is removed from the graph of Figure 2. Missing vertical edges can be added respecting the LC-rules since multiedges are bicoloured cycles. Once vertical edges have been added, each upper horizontal edge can be added since it creates a bicoloured cycle of length 4 with its lower counterpart.

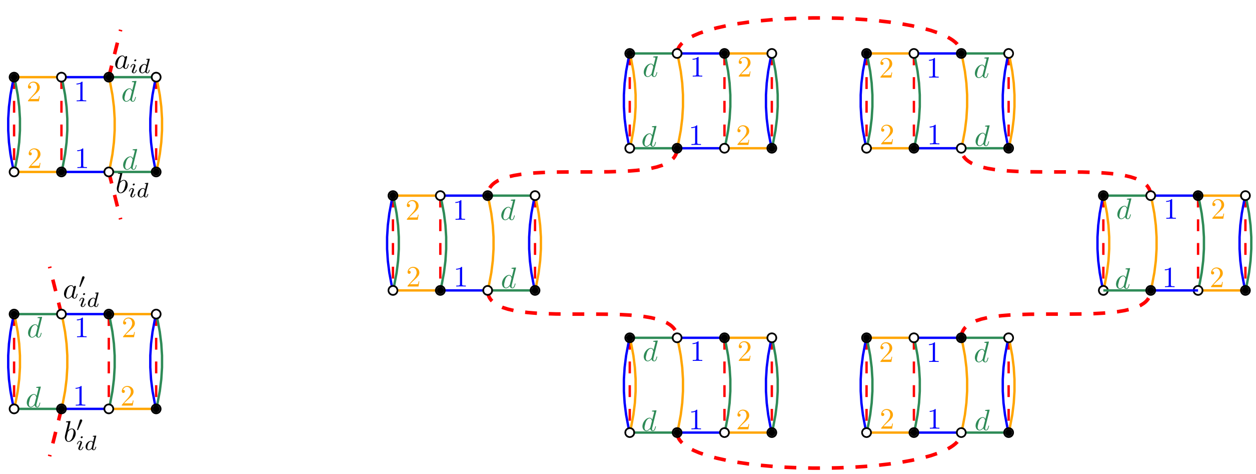

Figure 4: Left: Gadgets obtained after removing all edges of colour i for an $i \in [1..d]$ (here $i=3$). Each of them has two "outgoing" half-edges of colour $d+1$. Right: After removing all edges of colour i, each connected component is a "cycle of gadgets". It is easy to see that the associated complex is LC – for the original spanning tree, we remove one of the edges of colour $d+1$ on the cycle, and in each gadget we choose a spanning tree similar to the one of Figure 3.

5 Remaining proofs

It remains to prove Theorem 3, and the lower bound in (5). Both proofs are simple variants of the previous ones.

Proof of Theorem 3. The theorem directly follows from the two following claims:

(a) for any $c > 0$ there exist $K > 0$ such that the number of coloured triangulations of 3-manifolds with n labelled tetrahedra and with more that $\frac{Kn}{\log n}$ vertices is at most $2^{-cn} n^{\frac{7n}{6}}$;

(b) there exists $c_0 > 0$ such that the number of coloured triangulations of 3-manifolds with n labelled tetrahedra and with less than $\frac{n}{\log n}$ vertices is at least $2^{-c_0 n} n^{\frac{7n}{6}}$.

We first prove claim (a). Let $K > 0$ (to be chosen later) and let G be a 4-colourful graph such that $X(G)$ has more than $\frac{Kn}{\log n}$ vertices and its underlying space is a 3-manifold. Since vertices of $X(G)$ are in bijection with connected components of 3-coloured subgraphs of G, there exist distinct colours i, j, k such that $\kappa_{i,j,k} \geq \frac{Kn}{4 \log n}$. By (8) with $\mathcal{I} = \{i, j, k\}$ and up to relabelling i, j, k, we have

$$\kappa_{i,j} - \kappa_{i,j,k} \leq \frac{n}{6} - \frac{1}{3}\kappa_{i,j,k} \leq \frac{n}{6} - \frac{Kn}{12 \log n}.$$

We can upper bound the number of ways to construct such a graph G as in the proof of our main theorem. We first choose the planar graph $G_{\{i,j,\ell\}}$ where $\ell \in [1..4] \setminus \mathcal{I}$. There are at most n^n ways to do it. As in Lemma 7, once this graph has been chosen there are at most $2^{5n} n^{\kappa_{i,j} - \kappa_{i,j,k}}$ ways to place the edges of colour k. We thus have at most $2^{(5-K/12)n} n^{\frac{7n}{6}}$ choices of graphs in total, which is less than $2^{-cn} n^{\frac{7n}{6}}$ provided we take $K = K(c)$ large enough.

We now prove claim (b) by following the construction given in the proof of the lower bound of Theorem 1 (see Section 4). The 4-coloured graph constructed in that proof depends on two permutations σ and τ that describe the incidences of the edges of colour $d + 1$. By construction, $\kappa_{1,2,3} = 1$, and for any $\mathcal{I} \subseteq [1..4]$ with $|\mathcal{I}| = 3$ and $\mathcal{I} \ni 4$, $\kappa_{\mathcal{I}}$ is the number of cycles of the permutation $\sigma\tau^{-1}$ (or this number plus one, for choices of colours that involve the central gadget). If σ and τ are chosen uniformly at random, the expected number of cycles of $\sigma\tau^{-1}$ is $O(\log n)$. With positive probability, the number of vertices in $X(G)$ is $O(\log n)$, which is smaller than $\frac{n}{\log n}$ for n large enough. The claim follows since there are $2^{\Omega(n)} n^{7n/6}$ such graphs G.

□

Proof of the lower bound in (5). The construction is similar to the one of Section 4 and we will only sketch it. Let G be a graph obtained for $d = 3$ (see Figure 2). For $d \geq 4$, we obtain a $(d+1)$-colourful graph adding edges $a_i b_i$ and $a'_i b'_i$ for every $i \in [1..kd]$ and every colour in $\{5, \ldots, d+1\}$. Note that there are $M_3(n) \succeq n^{n/6} n!$ such graphs. Using a similar analysis as the one in Section 4, it is easy to see that for any set of colours $\mathcal{I}$ of size 3, the graph $G_{\mathcal{I}}$ is planar.

□

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