Perturbing Networks*

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Abstract

Semi–definite positive Schrödinger operators on finite connected networks are particular examples of a general class of self–adjoint operators called elliptic operators. Any elliptic operator defines an automorphism on the subspace orthogonal to the eigenfunctions associated with the lowest eigenvalue, whose inverse is called *orthogonal Green operator*. Our aim is to study the effect of a perturbation of an elliptic operator on its orthogonal Green operator. The perturbation here considered is performed by adding a self–adjoint and positive semi–definite operator.

We show that Schrödinger operators on networks that are obtained by adding weighted edges to a given network can be seen as perturbations of the Schrödinger operators on the original network. Therefore, we can compute the Green function, the effective resistances and the Kirchhoff index of a perturbed network in terms of the corresponding ones on the original network. We apply the obtained results to the study of perturbations of a weighted Star, which includes as particular cases the Wheel and Fan networks.

1 Introduction

The Sherman–Morrison–Woodbury formulas compute the inverse of the perturbation of an invertible matrix through a small rank matrix in terms of the inverse of the original matrix. Slight modifications allow us to extend these formulas by replacing the inverse with the Moore–Penrose inverse, if necessary. Since its original formulation in the late forties, this problem has attracted the attention of many authors, [8, 9, 11]. The so–called Sherman–Morrison formula displays the particular case in which the perturbation has rank one. Iterating this formula one obtains the general case. However, the iteration of the Sherman–Morrison formula usually leads to complicated and somewhat unpleasant expressions.

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Recently, Y. Yang and D.J. Klein, [13], use the Sherman–Morrison formula to obtain a recursive procedure for the computation of effective resistances in a perturbed network in terms of the effective resistances of the original network. The motivation of the use of this technique is based on P. Chebotarev and E.V. Shamis' study of the so–called adjusted forest distances on a network, [6]. The adjusted forest distance coincides with the generalized effective resistance associated with a positive semi–definite Schrödinger operator on the network that was introduced by the authors in [2, 3]. Therefore, our aim here is to analyze the effect of a perturbation of the network on the generalized effective resistance. Actually, we compute the effective resistance of perturbed networks throughout Sherman–Morrison–Woodbury like–formulas, instead of using the Sherman–Morrison formula recursively.

A positive semi–definite Schrödinger operator on a perturbed network appears as a perturbation of a positive semi–definite Schrödinger operator on the original network. Therefore, we start our analysis by dealing with perturbations in a broader framework: we define discrete elliptic operators as the discrete counterpart of elliptic partial differential operators. We show that each operator in this class has a singular elliptic operator associated that we call the orthogonal Green operator. We perturb the elliptic operator with a positive semi–definite and self–adjoint operator; or equivalently, with a sum of projectors. Then, we obtain the orthogonal Green operator associated with the perturbed elliptic operator in terms of the orthogonal Green operator associated with the initial one.

The application of the general results to the case of perturbed positive semi–definite Schrödinger operators is carried out in Section 4. We extend the definition of effective resistance between any pair of vertices, associated with the given Schrödinger operators, to a similar concept between two pairs of vertices that we call pairwise effective resistance. It turns out to be the key for expressing the effective resistances in the perturbed network in terms of the effective resistances in the original network.

Finally, the last section is devoted to computing the orthogonal Green function, the effective resistances and the Kirchhoff index of the networks obtained by adding edges between consecutive vertices in the weighted Start network. The Sherman–Morrison–Woodbury like–formulas involved are related with Jacobi matrices. We invert these matrices by means of the resolvent kernels of suitable self–adjoint boundary value problems on a path. The comparison of our results with those previously obtained for very particular cases leads to somehow surprising identities relating trigonometric expressions and Fibonacci numbers.

2 Preliminaries

Given a finite set V, the space of real valued functions on V is denoted by $\mathcal{C}(V)$ and for any $x \in V$, $\varepsilon_x \in \mathcal{C}(V)$ stands for the Dirac function at x. The standard inner product on $\mathcal{C}(V)$ is denoted by $\langle \cdot, \cdot \rangle$; that is, $\langle u, v \rangle = \sum\limits_{x \in V} u(x)\, v(x)$ for each $u, v \in \mathcal{C}(V)$. A unitary and positive function is called a *weight* and we denote by $\Omega(V)$ the set of weights. Given $a \geq 0$ the expression $a^\dagger \geq 0$ means either a^{-1} when $a > 0$ or 0 when $a = 0$.

If $\mathcal{K}$ is an endomorphism of $\mathcal{C}(V)$, it is called *self–adjoint* when $\langle \mathcal{K}(u), v\rangle = \langle u, \mathcal{K}(v)\rangle$, for any $u, v \in \mathcal{C}(V)$. Moreover, $\mathcal{K}$ is called *positive semi–definite* when $\langle \mathcal{K}(u), u\rangle \geq 0$ for any $u \in \mathcal{C}(V)$ and *positive definite* when $\langle \mathcal{K}(u), u\rangle > 0$ for any non–null $u \in \mathcal{C}(V)$. A self–adjoint operator $\mathcal{K}$ is named *elliptic* if it is positive semi–definite and its lowest eigenvalue, λ, is simple. Therefore, there exists a unique, up to sign, unitary function $\omega \in \mathcal{C}(V)$ satisfying $\mathcal{K}(\omega) = \lambda\omega$ and then $\mathcal{K}$ is named (λ, ω)*–elliptic*. Clearly, a (λ, ω)–elliptic operator is singular iff $\lambda = 0$.

A function $K\colon V \times V \longrightarrow \mathbb{R}$ is called *a kernel on* V and determines an endomorphism of $\mathcal{C}(V)$ by assigning to any $u \in \mathcal{C}(V)$ the function $\mathcal{K}(u) = \sum\limits_{y\in V} K(\cdot, y)\, u(y)$. Conversely, each endomorphism of $\mathcal{C}(V)$ is determined by the kernel given by $K(x, y) = \langle \mathcal{K}(\varepsilon_y), \varepsilon_x\rangle$ for any $x, y \in V$. Therefore, an endomorphism $\mathcal{K}$ is self–adjoint iff its kernel is a symmetric function.

Given $\sigma, \tau \in \mathcal{C}(V)$, we denote by $\mathcal{P}_{\sigma,\tau}$ the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{P}_{\sigma,\tau}(u) = \langle \tau, u\rangle\, \sigma$ and hence, its kernel is $(\sigma \otimes \tau)(x, y) = \sigma(x)\tau(y)$. In particular, when $\omega \neq 0$ the endomorphism $\mathcal{P}_{\omega,\omega}$ is denoted simply by $\mathcal{P}_\omega$. The operators $\mathcal{P}_{\sigma,\tau}$ are generically named *Projectors*, since when $\langle \sigma, \tau\rangle = 1$, the subspaces $\tau^\perp$ and $\operatorname{span}\{\sigma\}$ are complementary and $\mathcal{P}_{\sigma,\tau}$ assigns to any $u \in \mathcal{C}(V)$ its projection on σ along τ.

Given $\lambda \geq 0$, $\omega \in \Omega(V)$ and $\mathcal{F}$ a (λ, ω)–elliptic operator, we shall be concerned with the so–called *Poisson equation for* $\mathcal{F}$ on V

$$\textit{Given } f \in \mathcal{C}(V) \textit{ find } u \in \mathcal{C}(V) \textit{ such that } \mathcal{F}(u) = f. \tag{1}$$

The general result about the resolubility of the Poisson equation is given in the following well–know result, where we use of the common terminology in Operator Theory.

Proposition 2.1 *Given $\lambda \geq 0$ and $\omega \in \Omega(V)$ any (λ, ω)–elliptic operator is an automorphism of $\omega^\perp$.*

Proof. Consider $\mathcal{F}$ a (λ, ω)–elliptic operator. If $\mathcal{F}(u) = f$, then

$$\langle f, \omega\rangle = \langle \mathcal{F}(u), \omega\rangle = \langle u, \mathcal{F}(\omega)\rangle = \lambda\langle u, \omega\rangle.$$

When $\mathcal{F}$ is non–singular, the above identity implies that $\mathcal{F}$ is an automorphism of $\omega^\perp$.

When $\mathcal{F}$ is singular, the result is nothing else but the *Fredholm Alternative*: the Poisson equation with data $f \in \mathcal{C}(V)$ has solution iff $f \in \omega^\perp$ and moreover the solution is unique up to a multiple of ω. In particular, there exists a unique solution belonging to $\omega^\perp$. $\square$

The inverse of a (λ, ω)–elliptic operator $\mathcal{F}$ on $\omega^\perp$ is called *orthogonal Green operator* and it is denoted by $\mathcal{G}$. We can extend the orthogonal Green operator to $\mathcal{C}(V)$ by assigning to any $f \in \mathcal{C}(V)$, the unique solution of the Poisson equation $\mathcal{F}(u) = f - \mathcal{P}_\omega(f)$.

Proposition 2.2 *The orthogonal Green operator $\mathcal{G}$ of a (λ,ω)–elliptic operator $\mathcal{F}$ is a singular elliptic operator satisfying $\mathcal{G}(\omega)=0$;* i.e., *it is a $(0,\omega)$–elliptic operator, satisfying*

$$\mathcal{G}\circ\mathcal{F}=\mathcal{F}\circ\mathcal{G}=\mathcal{I}-\mathcal{P}_\omega.$$

Moreover, when $\mathcal{F}$ is non–singular, then $\mathcal{F}^{-1}=\mathcal{G}+\lambda^{-1}\mathcal{P}_\omega$.

When $\mathcal{F}$ is a non–singular (λ,ω)–elliptic operator, $\mathcal{F}^{-1}$ is usually named *Green operator*. This is the main reason to introduce the terminology orthogonal Green operator, so we can distinguish between both of them. In the singular case, the kernel associated with the orthogonal Green operator corresponds to the Moore–Penrose inverse of the kernel associated with $\mathcal{F}$.

In any case, the kernel of the orthogonal Green operator is called *orthogonal Green kernel* and denoted generically by G.

3 Perturbation of elliptic operators

We study here the perturbations of a fixed (λ,ω)–elliptic operator $\mathcal{F}$ due to the addition of a self–adjoint and positive semi–definite operator, or equivalently a sum of projections. Specifically, we consider non–null functions $\sigma_1,\dots,\sigma_k\in\mathcal{C}(V)$, the associated self–adjoint projections $\mathcal{P}_{\sigma_j}$, $j=1,\dots,k$ and the operator

$$\mathcal{H}=\mathcal{F}+\sum_{j=1}^{k}\mathcal{P}_{\sigma_j}, \tag{2}$$

which is called *perturbation of $\mathcal{F}$ by $\sigma_1,\dots,\sigma_k$*.

Lemma 3.1 *If $\mathcal{H}$ is the perturbation of $\mathcal{F}$ by $\sigma_1,\dots,\sigma_k$, then it is a self–adjoint and positive semi–definite operator. Moreover, if $\lambda_{\mathcal{H}}$ is the lowest eigenvalue of $\mathcal{H}$ then $\lambda_{\mathcal{H}}\ge\lambda$ and $\lambda_{\mathcal{H}}=\lambda$ iff $\mathcal{H}$ is (λ,ω)–elliptic and this occurs iff $\sigma_j\in\omega^\perp$ for any $j=1,\dots,k$.*

Proof. Clearly $\mathcal{H}$ is self–adjoint and positive semi–definite. Given a unitary $u\in\mathcal{C}(V)$, then

$$\langle\mathcal{H}(u),u\rangle=\langle\mathcal{F}(u),u\rangle+\sum_{j=1}^{k}\langle\sigma_j,u\rangle^2\ge\lambda,$$

which implies that $\lambda_{\mathcal{H}}\ge\lambda$. Moreover, $\lambda_{\mathcal{H}}=\lambda$ iff $\langle\mathcal{F}(u),u\rangle=\lambda$ and $\sum\limits_{j=1}^{k}\langle\sigma_j,u\rangle^2=0$ for any $j=1,\dots,k$. The first identity implies that $u=\pm\omega$ and the second one that $\sigma_j\in\omega^\perp$ for any $j=1,\dots,k$. $\square$

In the sequel, we denote by A the matrix $\mathsf{A} = \mathsf{I} + \big(\langle \mathcal{G}(\sigma_j), \sigma_i\rangle\big)$. In addition, for any $\lambda \geq 0$ and any $\mathsf{v} \in \mathbb{R}^k$ we denote by $\mathsf{A}_{\lambda,\mathsf{v}}$ the matrix defined as

$$\mathsf{A}_{\lambda,\mathsf{v}} = \begin{bmatrix} \mathsf{A} & -\mathsf{v} \\ \mathsf{v}^\top & \lambda \end{bmatrix}. \tag{3}$$

Lemma 3.2 *The matrix A is invertible and the matrix $\mathsf{A}_{\lambda,\mathsf{v}}$ is invertible except when $\lambda = 0$ and $\mathsf{v} = 0$ simultaneously. In any case, the Moore–Penrose inverse of $\mathsf{A}_{\lambda,\mathsf{v}}$ is*

$$(\mathsf{A}_{\lambda,\mathsf{v}})^\dagger = \begin{bmatrix} \mathsf{A}^{-1} & 0 \\ 0^\top & 0 \end{bmatrix} - \big[\lambda + \langle \mathsf{A}^{-1}\mathsf{v}, \mathsf{v}\rangle\big]^\dagger \begin{bmatrix} \mathsf{A}^{-1}\mathsf{v} \otimes \mathsf{A}^{-1}\mathsf{v} & -\mathsf{A}^{-1}\mathsf{v} \\ (\mathsf{A}^{-1}\mathsf{v})^\top & -1 \end{bmatrix}.$$

Proof. Clearly A is symmetric and positive definite and hence invertible.

On the other hand, if $a \in \mathbb{R}$ and $\mathsf{a} \in \mathbb{R}^k$ satisfy $\begin{bmatrix} \mathsf{A} & -\mathsf{v} \\ \mathsf{v}^\top & \lambda \end{bmatrix} \begin{bmatrix} \mathsf{a} \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $\mathsf{A}\mathsf{a} = a\mathsf{v}$ and $a\lambda + \langle \mathsf{v}, \mathsf{a}\rangle = 0$. Therefore, $\langle \mathsf{A}\mathsf{a}, \mathsf{a}\rangle = a\langle \mathsf{v}, \mathsf{a}\rangle = -a^2\lambda \leq 0$, which implies that $a^2\lambda = 0$ and $\mathsf{a} = 0$, because A is positive definite. Finally, $a = 0$ except when $\lambda = 0$ and $\mathsf{v} = 0$ simultaneously.

A direct verification shows that the given expression is the Moore–Penrose inverse of $\mathsf{A}_{\lambda,\mathsf{v}}$. □

Our objective is to obtain, in terms of the orthogonal Green operator of $\mathcal{F}$, either the orthogonal Green operator of $\mathcal{H}$ when it is a (λ, ω)–elliptic operator or its inverse. To do this we denote by (b_{ij}) the matrix A^{-1}.

We start with the case in which $\mathcal{H}$ is a (λ, ω)–elliptic operator, where we take into account the result of Lemma 3.1.

Theorem 3.3 *Assume that $\sigma_j \in \omega^\perp$ for any $j = 1, \dots, k$ and consider $\mathcal{G}^{\mathcal{H}}$ the orthogonal Green operator of $\mathcal{H}$. Then,*

$$\mathcal{G}^{\mathcal{H}} = \mathcal{G} - \sum_{i,j=1}^{k} b_{ij} \mathcal{P}_{\mathcal{G}(\sigma_i), \mathcal{G}(\sigma_j)}.$$

Proof. Given $f \in \omega^\perp$, consider the Poisson equation

$$\mathcal{H}(u) = f \quad \text{on} \quad V.$$

Then,

$$\mathcal{F}(u) = f - \sum_{j=1}^{k} \mathcal{P}_{\sigma_j}(u) = f - \sum_{j=1}^{k} \langle \sigma_j, u\rangle \sigma_j.$$

and clearly, the function on the right hand side in the last identity is in $\omega^\perp$. Therefore, the unique solution of the Poisson equation in $\omega^\perp$ is given by

$$u = \mathcal{G}(f) - \sum_{j=1}^{k} \langle \sigma_j, u\rangle \mathcal{G}(\sigma_j).$$

Multiplying by σ_i, $i = 1\dots,k$, we get that

$$\langle \sigma_i, u\rangle + \sum_{j=1}^{k} \langle \sigma_j, u\rangle \langle \mathcal{G}(\sigma_j), \sigma_i\rangle = \langle f, \mathcal{G}(\sigma_i)\rangle.$$

The coefficient matrix of the above system is A and hence applying Lemma 3.2 we obtain that

$$\langle \sigma_j, u\rangle = \sum_{i=1}^{k} b_{ji} \langle f, \mathcal{G}(\sigma_i)\rangle,$$

and then,

$$u = \mathcal{G}(f) - \sum_{i,j=1}^{k} b_{ij} \langle f, \mathcal{G}(\sigma_i)\rangle \mathcal{G}(\sigma_j) = \mathcal{G}(f) - \sum_{i,j=1}^{k} b_{ij} \mathcal{P}_{\mathcal{G}(\sigma_j),\mathcal{G}(\sigma_i)}(f). \qquad \square$$

Corollary 3.4 *If $\sigma_j \in \omega^\perp$ for any $j = 1,\dots,k$ and consider $\mathcal{G}^{\mathcal{H}}$ the orthogonal Green operator of $\mathcal{H}$, then*

$$\mathsf{A}^{-1} = \mathsf{I} - \big(\langle \mathcal{G}^{\mathcal{H}}(\sigma_j), \sigma_i\rangle\big).$$

In particular, $0 < b_{jj} < 1$ for any $j = 1,\dots,k$.

Proof. First, observe that the identity $\mathsf{I} = \mathsf{A}^{-1}\mathsf{A} = \mathsf{A}^{-1} + \mathsf{A}^{-1}\big(\langle \mathcal{G}(\sigma_j), \sigma_i\rangle\big)$ holds, which implies that for any $\ell, i = 1,\dots,k$

$$\sum_{j=1}^{k} b_{\ell j} \langle \mathcal{G}(\sigma_j), \sigma_i\rangle = \varepsilon_\ell(i) - b_{\ell i}.$$

Applying now the above Theorem we obtain that

$$\begin{aligned}
\mathcal{G}^{\mathcal{H}}(\sigma_\ell) &= \mathcal{G}(\sigma_\ell) - \sum_{i,j=1}^{k} b_{ij} \mathcal{G}(\sigma_i) \langle \mathcal{G}(\sigma_j), \sigma_\ell\rangle = \mathcal{G}(\sigma_\ell) - \sum_{i=1}^{k} \mathcal{G}(\sigma_i) \sum_{j=1}^{k} b_{ij} \langle \mathcal{G}(\sigma_j), \sigma_\ell\rangle \\
&= \mathcal{G}(\sigma_\ell) - \sum_{i=1}^{k} \mathcal{G}(\sigma_i)\big(\varepsilon_i(\ell) - b_{i\ell}\big) = \sum_{i=1}^{k} \mathcal{G}(\sigma_i) b_{i\ell}
\end{aligned}$$

and hence,

$$\langle \mathcal{G}^{\mathcal{H}}(\sigma_\ell), \sigma_r\rangle = \sum_{i=1}^{k} \langle \mathcal{G}(\sigma_i), \sigma_r\rangle b_{i\ell} = \varepsilon_\ell(r) - b_{\ell r}$$

and the first claim follows.

Finally, given $j = 1,\dots,k$, we have that $b_{jj} > 0$ because A^{-1} is positive definite and the positive semi–definiteness of $\mathcal{G}^{\mathcal{H}}$ implies that $b_{jj} \le 1$. In addition, $b_{jj} < 1$ because $\mathcal{G}^{\mathcal{H}}$ is positive definite on $\omega^{\perp}$. $\square$

We tackle now the case in which $\lambda_{\mathcal{H}} > \lambda$. Recall that from Lemma 3.1, this hypothesis implies that $\mathcal{H}$ is positive definite, and hence invertible, but non necessarily elliptic. Therefore, there is not guarantee that the orthogonal Green operator for $\mathcal{H}$ exists, and hence we are concerned in obtaining $\mathcal{H}^{-1}$ in terms of the orthogonal Green operator of $\mathcal{F}$.

Theorem 3.5 *If $\sigma_j \notin \omega^{\perp}$ for some $j = 1,\dots,k$, then the operator $\mathcal{H}$ is positive definite and*

$$\mathcal{H}^{-1} = \mathcal{G} - \sum_{j,\ell=1}^{k} \hat{b}_{j\ell}\mathcal{P}_{\mathcal{G}(\sigma_j),\mathcal{G}(\sigma_\ell)} + \sum_{j=1}^{k} \hat{b}_{jk+1}\big[\mathcal{P}_{\mathcal{G}(\sigma_j),\omega} - \mathcal{P}_{\omega,\mathcal{G}(\sigma_j)}\big] + \hat{b}_{k+1k+1}\mathcal{P}_{\omega},$$

where $\hat{b}_{k+1k+1} = \Big[\lambda + \sum\limits_{rs=1}^{k} b_{r,s}\langle\sigma_r,\omega\rangle\langle\sigma_s,\omega\rangle\Big]^{-1}$ *and for any $i, j = 1,\dots,k$,*

$$\hat{b}_{jk+1} = \hat{b}_{k+1k+1}\sum_{r=1}^{k} b_{jr}\langle\sigma_r,\omega\rangle \quad \textit{and} \quad \hat{b}_{j\ell} = b_{j\ell} - \hat{b}_{k+1k+1}\Big(\sum_{r=1}^{k} b_{jr}\langle\sigma_r,\omega\rangle\Big)\Big(\sum_{r=1}^{k} b_{\ell r}\langle\sigma_r,\omega\rangle\Big).$$

Proof. Given $f \in \mathcal{C}(V)$, if we consider the Poisson equation $\mathcal{H}(u) = f$ on V, then

$$\mathcal{F}(u) = f - \sum_{j=1}^{k}\mathcal{P}_{\sigma_j}(u) = f - \sum_{j=1}^{k}\langle\sigma_j,u\rangle\sigma_j$$

and hence,

$$\sum_{j=1}^{k}\langle\sigma_j,u\rangle\langle\omega,\sigma_j\rangle + \lambda\langle u,\omega\rangle = \langle f,\omega\rangle.$$

On the other hand,

$$\mathcal{F}(u - \langle u,\omega\rangle\omega) = \mathcal{F}(u) - \lambda\langle u,\omega\rangle\omega = f - \sum_{j=1}^{k}\langle\sigma_j,u\rangle\sigma_j - \lambda\langle u,\omega\rangle\omega,$$

which implies

$$u = \mathcal{G}(f) - \sum_{j=1}^{k}\langle\sigma_j,u\rangle\mathcal{G}(\sigma_j) + \langle u,\omega\rangle\omega.$$

Multiplying by σ_i, $i = 1\ldots,k$, we obtain the following linear system

$$\langle\sigma_i,u\rangle + \sum_{j=1}^{k}\langle\sigma_j,u\rangle\langle\mathcal{G}(\sigma_j),\sigma_i\rangle - \langle u,\omega\rangle\langle\sigma_i,\omega\rangle = \langle f,\mathcal{G}(\sigma_i)\rangle, \quad i=1,\ldots,k,$$
$$\sum_{j=1}^{k}\langle\sigma_j,u\rangle\langle\omega,\sigma_j\rangle + \lambda\langle u,\omega\rangle = \langle f,\omega\rangle,$$

where the unknowns are $\langle\sigma_i,u\rangle$, $i=1,\ldots,k$, and $\langle u,\omega\rangle$.

If we consider $\mathsf{v} = (\langle\sigma_1,\omega\rangle,\ldots,\langle\sigma_k,\omega\rangle)^\top$, then $\mathsf{v}\neq\mathsf{0}$ and the coefficient matrix of the above system is $\mathsf{A}_{\lambda,\mathsf{v}}$. Therefore, applying Lemma 3.2

$$u = \mathcal{G}(f) - \sum_{j,\ell=1}^{k}\hat{b}_{j\ell}\langle f,\mathcal{G}(\sigma_\ell)\rangle\mathcal{G}(\sigma_j) + \sum_{j=1}^{k}\hat{b}_{jk+1}\langle f,\omega\rangle\mathcal{G}(\sigma_j) + \omega\left[\sum_{j=1}^{k}\hat{b}_{k+1j}\langle f,\mathcal{G}(\sigma_j)\rangle + \hat{b}_{k+1k+1}\langle f,\omega\rangle\right]$$

where $\left(\hat{b}_{ij}\right) = \begin{bmatrix}\mathsf{A}^{-1} & 0\\ \mathsf{0}^\top & 0\end{bmatrix} - \dfrac{1}{\lambda+\langle\mathsf{A}^{-1}\mathsf{v},\mathsf{v}\rangle}\begin{bmatrix}\mathsf{A}^{-1}\mathsf{v}\otimes\mathsf{A}^{-1}\mathsf{v} & -\mathsf{A}^{-1}\mathsf{v}\\ (\mathsf{A}^{-1}\mathsf{v})^\top & -1\end{bmatrix}$. □

Next we specify the simple case in which the perturbation is due to only one projector; or equivalenty, to a symmetric matrix of rank 1. This is a well–known result, see for instance [9, 11].

Corollary 3.6 *Consider $\sigma\in\mathcal{C}(V)$ and $\mathcal{H}_\sigma = \mathcal{F}+\mathcal{P}_\sigma$. Then, when $\langle\sigma,\omega\rangle = 0$*

$$\mathcal{G}^{\mathcal{H}} = \mathcal{G} - \frac{1}{1+\langle\mathcal{G}(\sigma),\sigma\rangle}\mathcal{P}_{\mathcal{G}(\sigma)}$$

whereas, when $\langle\sigma,\omega\rangle\neq 0$

$$\mathcal{H}^{-1} = \mathcal{G} - \frac{1}{\lambda(1+\langle\mathcal{G}(\sigma),\sigma\rangle)+\langle\sigma,\omega\rangle^2}\Big(\lambda\mathcal{P}_{\mathcal{G}(\sigma)} - \langle\sigma,\omega\rangle\big(\mathcal{P}_{\mathcal{G}(\sigma),\omega} - \mathcal{P}_{\omega,\mathcal{G}(\sigma)}\big) - (1+\langle\mathcal{G}(\sigma),\sigma\rangle)\mathcal{P}_\omega\Big).$$

We remark that the result in Theorem 3.3 can be obtained by applying k times the above Corollary. For instance, if $\sigma_\ell\in\omega^\perp$ for any $\ell=1,\ldots,k$, and we define the (λ,ω)–elliptic operator $\mathcal{H}_\ell = \mathcal{F} + \sum_{j=1}^{\ell}\mathcal{P}_{\sigma_j}$ and consider $\mathcal{G}_\ell$ the orthogonal Green operator of $\mathcal{H}_\ell$, then

$$\mathcal{G}_{\ell+1} = \mathcal{G}_\ell - \frac{1}{1+\langle\mathcal{G}_\ell(\sigma_{\ell+1}),\sigma_{\ell+1}\rangle}\mathcal{P}_{\mathcal{G}_\ell(\sigma_{\ell+1})},$$

because $\mathcal{H}_{\ell+1} = \mathcal{H}_\ell + \mathcal{P}_{\sigma_{\ell+1}}$.

4 Perturbation of elliptic Schrödinger operators

In this section we consider fixed the connected finite network $\Gamma = (V, E, c)$ on the vertex set V whose conductance is the symmetric kernel $c\colon V \times V \longrightarrow [0, +\infty)$ satisfying that $c(x, x) = 0$ for any $x \in V$ and moreover, x is adjacent to y iff $c(x, y) > 0$. We call Γ the *base network*.

Given $\omega \in \Omega(V)$ a weight on V for any pair (x, y) we call *ω–dipole between x and y* the function $\tau_{xy} = \dfrac{\varepsilon_x}{\omega(x)} - \dfrac{\varepsilon_y}{\omega(y)}$. Clearly, $\tau_{xy} = -\tau_{yx}$ and moreover, $\tau_{xy} = 0$ iff $x = y$.

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y)\,\big(u(x) - u(y)\big), \quad x \in V. \tag{4}$$

It is well–known, that the Laplacian is a singular elliptic operator on $\mathcal{C}(V)$ and moreover $\mathcal{L}(u) = 0$ iff u is a constant function.

Given $q \in \mathcal{C}(V)$, the *Schrödinger operator* on Γ with *potential* q is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$.

If $\omega \in \Omega(V)$ is a *weight on V*, the function $q_\omega = -\dfrac{1}{\omega}\,\mathcal{L}(\omega)$ is named *the potential determined by ω*.

It is well–known that the Schrödinger operator $\mathcal{L}_q$ is elliptic iff there exist unique $\omega \in \Omega(V)$ and $\lambda \geq 0$ such that $q = q_\omega + \lambda$, see for instance [2, 3]. Moreover, $\mathcal{L}_q$ is singular iff $\lambda = 0$ and then, $\mathcal{L}_{q_\omega}(v) = 0$ iff $v = a\omega$, $a \in \mathbb{R}$. Equivalently, the Schrödinger operator $\mathcal{L}_q$ is (λ, ω)–elliptic iff $q = q_\omega + \lambda$. In this case, we denote by $\mathcal{G}_{\lambda,\omega}$ the orthogonal Green operator and we also denote by $G_{\lambda,\omega}$ the kernel of $\mathcal{G}_{\lambda,\omega}$. In the sequel, we consider fixed the value $\lambda \geq 0$, the weight $\omega \in \Omega(V)$ and the corresponding Schrödinger operator $\mathcal{L}_q$, where $q = q_\omega + \lambda$.

Given $x, y \in V$, we call *effective resistance between x and y, with respect to λ and ω*, or simply effective resistance between x and y, the value

$$R_{\lambda,\omega}(x, y) = \langle \mathcal{G}_{\lambda,\omega}(\tau_{xy}), \tau_{xy} \rangle = \frac{G_{\lambda,\omega}(x, x)}{\omega(x)^2} + \frac{G_{\lambda,\omega}(y, y)}{\omega(y)^2} - 2\frac{G_{\lambda,\omega}(x, y)}{\omega(x)\omega(y)}. \tag{5}$$

The kernel $R_{\lambda,\omega}\colon V \times V \longrightarrow \mathbb{R}$ is symmetric, non–negative and, in addition, $R_{\lambda,\omega}(x, y) = 0$ iff $\tau_{xy} = a\omega$, which is equivalent to be $a = 0$ and hence to be $x = y$. In addition, it is well–known that the matrix $\big(R_{\lambda,\omega}(x, y)\big)$ is non–singular, see [2, Proposition 4.3], and moreover for any $x, y, z \in V$ it is satisfied the *triangular inequality*

$$R_{\lambda,\omega}(x, y) \leq R_{\lambda,\omega}(x, z) + R_{\lambda,\omega}(z, y) \tag{6}$$

and the equality holds iff $\lambda = 0$ and, in addition, z separates x and y, see [2, Corollary 4.4]. We must notice that when $\lambda = 0$ and ω is the constant weight the effective resistance $R_{\lambda,\omega}$ coincides, up to a normalization factor, with the classical effective resistance.

Analogously, given $x \in V$, we define the *total resistance at x, with respect to λ and ω*, or simply the total resistance at x as the positive value $r_{\lambda,\omega}(x) = \dfrac{G_{\lambda,\omega}(x,x)}{\omega(x)^2}$, that generalizes the notion of status of a vertex introduced in [10], see also [2]. The *Kirchhoff Index, with respect to λ and ω*, or simply the Kirchhoff Index of Γ is the value

$$\mathsf{k}(\lambda,\omega) = \sum_{x \in V} G_{\lambda,\omega}(x,x) = \frac{1}{2} \sum_{x,y \in V} R_{\lambda,\omega}(x,y)\omega^2(x)\omega^2(y). \tag{7}$$

Moreover for any $x \in V$, it is satisfied that

$$\mathsf{k}(\lambda,\omega) = \sum_{y \in V} R_{\lambda,\omega}(x,y)\omega^2(y) - r_{\lambda,\omega}(x) \quad \text{and} \quad |r_{\lambda,\omega}(x) - r_{\lambda,\omega}(y)| < R_{\lambda,\omega}(x,y), \quad y \neq x, \tag{8}$$

see [2, 3] for details.

Next, we extend the notion of effective resistance between vertices to pairs of vertices. So, given two pairs $(x,y),(\hat{x},\hat{y}) \in V \times V$, we define the *pairwise effective resistance* between the pairs (x,y) and $(\hat{x},\hat{y})$ as the value

$$R_{\lambda,\omega}(x,y;\hat{x},\hat{y}) = \langle \mathcal{G}_{\lambda,\omega}(\tau_{xy}), \tau_{\hat{x}\hat{y}} \rangle. \tag{9}$$

Therefore, for any $x,y,\hat{x},\hat{y} \in V$ we have

$$R_{\lambda,\omega}(x,y;\hat{x},\hat{y}) = \frac{1}{2}\Big(R_{\lambda,\omega}(x,\hat{y}) + R_{\lambda,\omega}(y,\hat{x}) - R_{\lambda,\omega}(x,\hat{x}) - R_{\lambda,\omega}(y,\hat{y}) \Big). \tag{10}$$

The triangular inequality (6) implies that for any $x,y,\hat{x},\hat{y} \in V$

$$0 \leq R_{\lambda,\omega}(x,y;\hat{x},y), R_{\lambda,\omega}(x,y;x,\hat{y}) \quad \text{and} \quad \big|R_{\lambda,\omega}(x,y;\hat{x},\hat{y})\big| \leq \min\big\{R_{\lambda,\omega}(x,y), R_{\lambda,\omega}(\hat{x},\hat{y})\big\}.$$

If we consider an orientation on the graph, ϑ, and for any edge $e \in E$ we denote the *tail* by $t(e)$ and the *head* by $h(e)$, we can generalize the notion of effective resistance between vertices to an analogue value between edges. Specifically, if given $e \in E$ we define $\tau_e = \tau_{h(e)t(e)}$, we call *edge effective resistance* between e and $\hat{e}$, with respect to ϑ, λ and ω, or simply the edge effective resistance between e and $\hat{e}$, the value

$$R^{\vartheta}_{\lambda,\omega}(e,\hat{e}) = \langle \mathcal{G}_{\lambda,\omega}(\tau_e), \tau_{\hat{e}} \rangle \tag{11}$$

Clearly, the kernel $R^{\vartheta}_{\lambda,\omega} \colon E \times E \longrightarrow \mathbb{R}$ is symmetric and positive on the diagonal. In fact, we have the identity

$$R^{\vartheta}_{\lambda,\omega}(e,\hat{e}) = R_{\lambda,\omega}\big(h(e),t(e);h(\hat{e}),t(\hat{e})\big) \tag{12}$$

which in particular implies that $R^{\vartheta}_{\lambda,\omega}(e,e) = R_{\lambda,\omega}\big(h(e),t(e)\big)$. Therefore, the matrix $\big(R^{\vartheta}_{\lambda,\omega}(e,\hat{e})\big)$ is positive semi–definite. Moreover, it is singular except when the subjacent graph is a tree, since the dipoles associated with the edges of any cycle in Γ are always linearly dependent. Newly, as a consequence of the triangular inequality for the effective resistance between vertices, we get that

$$\big|R^{\vartheta}_{\lambda,\omega}(e,\hat{e})\big| \le \min\big\{R^{\vartheta}_{\lambda,\omega}(e,e), R^{\vartheta}_{\lambda,\omega}(\hat{e},\hat{e})\big\}, \quad \text{for any } e,\hat{e}\in E.$$

Our next aim is to analyze the elliptic Schrödinger operators associated with a perturbation of the conductance. Specifically, we consider $\epsilon\colon V\times V \longrightarrow [0,+\infty)$ a symmetric function and denote by $\mathcal{L}^{\epsilon}$ the combinatorial Laplacian associated with the perturbed conductance $c+\epsilon$. In addition, for any $\omega\in\Omega(V)$ we denote by q^{ϵ}_{ω} the potential $-\omega^{-1}\mathcal{L}^{\epsilon}(\omega)$. Futhermore, if $q\in\mathcal{C}(V)$ is such that $\mathcal{L}^{\epsilon}_q$ is elliptic, then we denote by $\mathcal{G}^{\epsilon}_q$ the orthogonal Green operator of $\mathcal{L}^{\epsilon}_q$. Moreover, given $\lambda\ge 0$ and $\omega\in\Omega(V)$ we denote by $\mathcal{G}^{\epsilon}_{\lambda,\omega}$ the orthogonal Green operator and by $G^{\epsilon}_{\lambda,\omega}$ its corresponding kernel.

If we consider $E^{\epsilon}=\big\{(x,y):\epsilon(x,y)>0\big\}$, then the new network whose conductance is $c+\epsilon$ is $\Gamma^{\epsilon}=(V,E\cup E^{\epsilon},c+\epsilon)$. Therefore, we can understand the perturbed network as a new network built from the base network Γ by introducing new edges and/or by increasing the conductance of some old edges. Observe that Γ^{ϵ} is also connected.

Now we show that elliptic Schrödinger operators on the perturbed network can be seen as perturbations, in the sense of the above section, of elliptic Schrödinger operators on the base network. For this purpose, for any $e\in E^{\epsilon}$ we consider the positive value and the function

$$\rho(e)=\sqrt{\epsilon\big(h(e),t(e)\big)\omega\big(h(e)\big)\omega\big(t(e)\big)} \quad \text{and} \quad \sigma_e=\rho(e)\tau_e. \tag{13}$$

Proposition 4.1 *The following identity holds:*

$$\mathcal{L}^{\epsilon}_{q^{\epsilon}_{\omega}}=\mathcal{L}_{q_{\omega}}+\sum_{e\in E^{\epsilon}}\mathcal{P}_{\sigma_e}.$$

Proof. First, observe that the combinatorial Laplacian for the perturbed conductance can be expressed as

$$\mathcal{L}^{\epsilon}=\mathcal{L}+\sum_{e\in E^{\epsilon}}\mathcal{P}_{\gamma_e},$$

where $\gamma_e=\sqrt{\epsilon\big(h(e),t(e)\big)}\big(\varepsilon_{h(e)}-\varepsilon_{t(e)}\big)$. Therefore, we get that

$$q^{\epsilon}_{\omega}=-\omega^{-1}\mathcal{L}^{\epsilon}(\omega)=-\omega^{-1}\mathcal{L}(\omega)-\frac{1}{\omega}\sum_{e\in E^{\epsilon}}\mathcal{P}_{\gamma_e}(\omega),$$

and hence,

$$\mathcal{L}^{\epsilon}_{q^{\epsilon}_{\omega}}=\mathcal{L}_{q_{\omega}}+\sum_{e\in E^{\epsilon}}\Big[\mathcal{P}_{\gamma_e}-\frac{1}{\omega}\mathcal{P}_{\gamma_e}(\omega)\Big].$$

The result follows taking into account that given $e \in E^\epsilon$, if $h(e) = x$ and $t(e) = y$, then for any $u \in \mathcal{C}(V)$,

$$\begin{aligned}
\mathcal{P}_{\gamma_e}(u) - \frac{u}{\omega}\mathcal{P}_{\gamma_e}(\omega) &= \epsilon(x,y)\big(\varepsilon_x - \varepsilon_y\big)\Big[u(x) - u(y) - \frac{u}{\omega}\big(\omega(x) - \omega(y)\big)\Big] \\
&= \epsilon(x,y)\left[\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right]\Big(\omega(y)\varepsilon_x - \omega(x)\varepsilon_y\Big) \\
&= \epsilon(x,y)\omega(x)\omega(y)\left[\frac{\varepsilon_x}{\omega(x)} - \frac{\varepsilon_y}{\omega(y)}\right]\left[\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right] = \mathcal{P}_{\sigma_e}(u).
\end{aligned}$$

□

Now we can establish the claimed result about Schrödinger operators on the perturbed network and their Green operators. According with the results of the above section, the matrix

$$\mathsf{A} = \mathsf{I} + \big(\langle \mathcal{G}_{\lambda,\omega}(\sigma_e), \sigma_{\hat{e}}\rangle\big)$$

is invertible and we denote by $\big(b(e,\hat{e})\big)$ its inverse.

Theorem 4.2 *Given $q \in \mathcal{C}(V)$, then $\mathcal{L}_q^\epsilon$, the Schrödinger operator on the perturbed network, is (λ,ω)–elliptic; that is, $q = q_\omega^\epsilon + \lambda$ iff*

$$\mathcal{L}_q^\epsilon = \mathcal{L}_p + \sum_{e \in E^\epsilon} \mathcal{P}_{\sigma_e},$$

where $p = q_\omega + \lambda$; that is, iff it is a perturbation of the (λ,ω)–elliptic Schrödinger operator on the base network. Moreover, $\mathcal{L}_q^\epsilon$ is singular iff $\mathcal{L}_p$ is; and this occurs iff $\lambda = 0$. In any case, singular or not,

$$\mathcal{G}_{\lambda,\omega}^\epsilon = \mathcal{G}_{\lambda,\omega} - \sum_{e,\hat{e} \in E^\epsilon} b(e,\hat{e})\mathcal{P}_{\mathcal{G}_{\lambda,\omega}(\sigma_e),\mathcal{G}_{\lambda,\omega}(\sigma_{\hat{e}})}.$$

In particular, $\mathcal{L}_q^\epsilon \to \mathcal{L}_p$ and $\mathcal{G}_{\lambda,\omega}^\epsilon \to \mathcal{G}_{\lambda,\omega}$ when $\epsilon \to 0$.

Proof. Taking into account the characterization of elliptic Schrödinger operators, then the first part is a straightforward consequence of Proposition 4.1. In addition, bearing in mind that $\sigma_e \in \omega^\perp$, for any $e \in E^\epsilon$, the expression of $\mathcal{G}_{\lambda,\omega}^\epsilon$ is also a straightforward application of Theorem 3.3. □

Corollary 4.3 *The orthogonal Green operator for any elliptic Schrödinger operator on a network is a perturbation of the orthogonal Green operator of a Schrödinger operator on a spanning tree of the network.*

Now we analyze the effect of a perturbation on the effective resistances.

Corollary 4.4 *If $R^{\epsilon}_{\lambda,\omega}$ is the effective resistance on the perturbed network, then*

$$R^{\epsilon}_{\lambda,\omega}(x,y) = R_{\lambda,\omega}(x,y) - \sum_{e,\hat{e}\in E^{\epsilon}} b(e,\hat{e})\rho(e)\rho(\hat{e})R_{\lambda,\omega}(h(e),t(e);x,y)R_{\lambda,\omega}(h(\hat{e}),t(\hat{e});x,y), \quad x,y\in V$$

and hence, $R^{\epsilon}_{\lambda,\omega} \to R_{\lambda,\omega}$ when $\epsilon \to 0$.

In particular, when $\epsilon(x,y)>0$, then $R^{\epsilon}_{\lambda,\omega}(x,y) = \dfrac{1-b(e,e)}{\epsilon(x,y)\omega(x)\omega(y)}$, where $e\in E^{\epsilon}$ is the edge whose extremes are x and y.

Proof. The first identity follows from the identity (5) and Theorem 4.2

$$\begin{aligned} R^{\epsilon}_{\lambda,\omega}(x,y) &= R_{\lambda,\omega}(x,y) - \sum_{e,\hat{e}\in E^{\epsilon}} b(e,\hat{e})\langle \mathcal{G}_{\lambda,\omega}(\sigma_{\hat{e}}),\tau_{xy}\rangle\langle \mathcal{G}_{\lambda,\omega}(\sigma_{e}),\tau_{xy}\rangle \\ &= R_{\lambda,\omega}(x,y) - \sum_{e,\hat{e}\in E^{\epsilon}} b(e,\hat{e})\rho(e)\rho(\hat{e})R_{\lambda,\omega}(h(\hat{e}),t(\hat{e});x,y)R_{\lambda,\omega}(h(e),t(e);x,y). \end{aligned}$$

On the other hand, when $\epsilon(x,y)>0$, from Corollary 3.4 we get

$$\rho(e)^2 R^{\epsilon}_{\lambda,\omega}(x,y) = \langle \mathcal{G}^{\epsilon}_{p}(\tau_e),\tau_e\rangle = 1-b(e,e)$$

and the result follows. $\square$

Observe that the effective resistance of the perturbed network decrease with respect to the effective resistance of the original network. Moreover, the effective resistance between x and y does not change iff $R_{\lambda,\omega}(h(e),y)+R_{\lambda,\omega}(t(e),x) = R_{\lambda,\omega}(h(e),x)+R_{\lambda,\omega}(t(e),y)$, for any edge $e\in E^{\epsilon}$. Therefore, if $\epsilon(x,y)>0$, $R^{\epsilon}(x,y)<R(x,y)$.

In the following result we get an explicit expression for the Kirchhoff Index of the perturbed network in terms of the Kirchhoff Index of the original network, the effective resistances and the total resistances of the vertices involved in the perturbation.

Corollary 4.5

$$\begin{aligned} \mathsf{k}^{\epsilon}(\lambda,\omega) &= \mathsf{k}(\lambda,\omega) + \frac{1}{4}\sum_{e,\hat{e}\in E^{\epsilon}} b(e,\hat{e})\rho(e)\rho(\hat{e})\big[r_{\lambda,\omega}(t(\hat{e}))-r_{\lambda,\omega}(h(\hat{e}))\big]\big[r_{\lambda,\omega}(t(e))-r_{\lambda,\omega}(h(e))\big] \\ &\quad - \frac{1}{4}\sum_{x\in V}\omega^2(x)\sum_{e,\hat{e}\in E^{\epsilon}} b(e,\hat{e})\rho(e)\rho(\hat{e})\big[R_{\lambda,\omega}(t(\hat{e}),x)-R_{\lambda,\omega}(h(\hat{e}),x)\big]\big[R_{\lambda,\omega}(t(e),x)-R_{\lambda,\omega}(h(e),x)\big]. \end{aligned}$$

Proof. If we denote by $a(e,x) = \big[R_{\lambda,\omega}(t(e),x)-R_{\lambda,\omega}(h(e),x)\big]\omega(x)$ and $b(x)=\omega(x)$, we can apply the Binet–Cauchy Identity as follows

$$
\begin{aligned}
\mathsf{k}^\epsilon(\lambda,\omega) &= \mathsf{k}(\lambda,\omega) - \frac{1}{2}\sum_{e,\hat{e}\in E^\epsilon} b(e,\hat{e})\rho(e)\rho(\hat{e}) \sum_{x,y\in V} R_{\lambda,\omega}(h(\hat{e}),t(\hat{e});x,y)R_{\lambda,\omega}(h(e),t(e);x,y)\omega^2(x)\omega^2(y)\\
&= \mathsf{k}(\lambda,\omega) - \frac{1}{8}\sum_{e,\hat{e}\in E^\epsilon} b(e,\hat{e})\rho(e)\rho(\hat{e}) \sum_{x,y\in V} \big[a(\hat{e},x)b(y)-a(\hat{e},y)b(x)\big]\big[a(e,x)b(y)-a(e,y)b(x)\big]\\
&= \mathsf{k}(\lambda,\omega) - \frac{1}{4}\sum_{e,\hat{e}\in E^\epsilon} b(e,\hat{e})\rho(e)\rho(\hat{e}) \Big[\sum_{x\in V} a(\hat{e},x)a(e,x) - \Big(\sum_{x\in V} a(\hat{e},x)\omega(x)\Big)\Big(\sum_{x\in V} a(e,x)\omega(x)\Big)\Big]\\
&= \mathsf{k}(\lambda,\omega)\\
&\quad - \frac{1}{4}\sum_{e,\hat{e}\in E^\epsilon} b(e,\hat{e})\rho(e)\rho(\hat{e})\Big[\sum_{x\in V} a(\hat{e},x)a(e,x) - \big[r_{\lambda,\omega}(t(\hat{e}))-r_{\lambda,\omega}(h(\hat{e}))\big]\big[r_{\lambda,\omega}(t(e))-r_{\lambda,\omega}(h(e))\big]\Big].
\end{aligned}
$$

□

We conclude this section by observing that, as in the perturbation of elliptic operators, the result in Theorem 4.2 can be obtained by perturbing the network edge to edge. Specifically, given $x, y \in V$ with $x \neq y$, then the elliptic Schrödinger operator with potential q on the network resulting by adding an edge e between vertices x and y with conductance ϵ, is given by $\mathcal{L}_q^\epsilon = \mathcal{L}_p + \mathcal{P}_\sigma$, where $\sigma = \sqrt{\epsilon\,\omega(x)\omega(y)}\Big(\dfrac{\varepsilon_x}{\omega(x)} - \dfrac{\varepsilon_y}{\omega(y)}\Big)$, $q = -\omega^{-1}\mathcal{L}^\epsilon(\omega)$, $p = -\omega^{-1}\mathcal{L}(\omega)$ and then,

$$
\mathcal{G}_{\lambda,\omega}^\epsilon = \mathcal{G}_{\lambda,\omega} - \frac{1}{1+\langle \mathcal{G}_{\lambda,\omega}(\sigma),\sigma\rangle}\mathcal{P}_{\mathcal{G}_{\lambda,\omega}(\sigma)}.
$$

Therefore,

$$
R_{\lambda,\omega}^\epsilon(\hat{x},\hat{y}) = R_{\lambda,\omega}(\hat{x},\hat{y}) - \frac{\epsilon\omega(x)\omega(y)R_{\lambda,\omega}(x,y;\hat{x},\hat{y})^2}{1+\epsilon\omega(x)\omega(y)R_{\lambda,\omega}(x,y)}, \qquad R_{\lambda,\omega}^\epsilon(x,y) = \frac{1}{\epsilon\omega(x)\omega(y) + \dfrac{1}{R_{\lambda,\omega}(x,y)}}
$$

and

$$
\begin{aligned}
\mathsf{k}^\epsilon(\lambda,\omega) &= \mathsf{k}(\lambda,\omega)\\
&\quad - \frac{\epsilon\omega(x)\omega(y)}{4(1+\epsilon\omega(x)\omega(y)R_{\lambda,\omega}(x,y))}\left(\sum_{z\in V}\omega^2(z)\big[R_{\lambda,\omega}(x,z)-R_{\lambda,\omega}(y,z)\big]^2 - \big[r_{\lambda,\omega}(x)-r_{\lambda,\omega}(y)\big]^2\right).
\end{aligned}
$$

When $\lambda = 0$ and ω is constant, the above results coincide with those obtained in [13].

5 From Star to Wheel

Let us consider the simplest tree; that is, the *Star network* S_n with $n+1$ vertices, $\{x_0, x_1, \dots, x_n\}$, and conductances $a_i = a(x_i, x_0) > 0$, $i = 1, \dots, n$. Moreover, let $\omega_i = \omega(x_i)$, $i = 0, \dots, n$ be a weight on S_n. Then, $q_\omega(x_i) = -a_i + \dfrac{a_i\omega_0}{\omega_i}$, for any $i = 1, \dots, n$. In addition, given $\lambda \geq 0$, and the potential $q = q_\omega + \lambda$ we also consider the corresponding positive semi–definite Schrödinger operator $\mathcal{L}_q$. Although the following expressions for the orthogonal Green function, effective resistances and Kirchhoff Index can be deduced from the results in [5, Proposition 3.1], we include the proofs here for completeness.

For the sake of simplicity we consider the following value

$$Q(\lambda,\omega)=\sum_{j=1}^{n}\frac{\omega_j^3}{\lambda\omega_j+a_j\omega_0}.$$

Lemma 5.1 *It is satisfied that* $0\le\lambda Q(\lambda,\omega)<1$.

Proof. It suffices to observe that the following identity holds

$$\frac{1}{\omega_0}\Big[1-\lambda Q(\lambda,\omega)\Big]=\omega_0+\sum_{j=1}^{n}\frac{a_j\omega_j^2}{\lambda\omega_j+a_j\omega_0}.$$

□

Proposition 5.2 *If* $f\in\omega^\perp$, *then*

$$\mathcal{G}_{\lambda,\omega}(f)(x_0)=-\frac{\omega_0}{1-\lambda Q(\lambda,\omega)}\sum_{j=1}^{n}\frac{f(x_j)\omega_j^2}{\lambda\omega_j+a_j\omega_0},$$

$$\mathcal{G}_{\lambda,\omega}(f)(x_i)=\frac{\omega_i}{\lambda\omega_i+a_i\omega_0}\left[f(x_i)-\frac{a_i\omega_0}{1-\lambda Q(\lambda,\omega)}\sum_{j=1}^{n}\frac{f(x_j)\omega_j^2}{\lambda\omega_j+a_j\omega_0}\right],\quad i=1,\dots,n.$$

Proof. If we consider $u=\mathcal{G}_{\lambda,\omega}(f)$, then, for any $i=1,\dots,n$ we get

$$a_i\big(u(x_i)-u_{\lambda,\omega}(x_0)\big)+q_iu(x_i)=f(x_i)$$

and hence,

$$u(x_i)=\frac{f(x_i)+a_iu(x_0)}{q_i+a_i}=\frac{\omega_i}{\lambda\omega_i+a_i\omega_0}\Big[f(x_i)+a_iu(x_0)\Big].$$

In addition, the condition $u\in\omega^\perp$ is equivalent to

$$0=\omega_0u(x_0)+\sum_{j=1}^{n}\frac{f(x_j)\omega_j^2}{\lambda\omega_j+a_j\omega_0}+u(x_0)\sum_{j=1}^{n}\frac{a_j\omega_j^2}{\lambda\omega_j+a_j\omega_0},$$

which from Lemma 5.1 implies that

$$u(x_0)=-\frac{\omega_0}{1-\lambda Q(\lambda,\omega)}\sum_{j=1}^{n}\frac{f(x_j)\omega_j^2}{\lambda\omega_j+a_j\omega_0}$$

and the result follows. □

Corollary 5.3 *The orthogonal Green function of the Star with respect to λ and ω is given by*

$$G_{\lambda,\omega}(x_0,x_0)=\frac{\omega_0^2 Q(\lambda,\omega)}{1-\lambda Q(\lambda,\omega)},\qquad G_{\lambda,\omega}(x_0,x_i)=\frac{a_i\omega_i\omega_0}{\lambda\omega_i+a_i\omega_0}\left[\frac{\omega_0 Q(\lambda,\omega)}{1-\lambda Q(\lambda,\omega)}-\frac{\omega_i}{a_i}\right],$$

$$G_{\lambda,\omega}(x_k,x_i)=\frac{a_i a_k\omega_i\omega_k\omega_0}{\big[\lambda\omega_i+a_i\omega_0\big]\big[\lambda\omega_k+a_k\omega_0\big]}\left[\frac{\omega_0 Q(\lambda,\omega)}{1-\lambda Q(\lambda,\omega)}-\frac{\omega_i}{a_i}-\frac{\omega_k}{a_k}\right]-\frac{\lambda\omega_i^2\omega_k^2}{\big[\lambda\omega_i+a_i\omega_0\big]\big[\lambda\omega_k+a_k\omega_0\big]},$$

$$G_{\lambda,\omega}(x_i,x_i)=\frac{a_i^2\omega_i^2\omega_0}{\big[\lambda\omega_i+a_i\omega_0\big]^2}\left[\frac{\omega_0 Q(\lambda,\omega)}{1-\lambda Q(\lambda,\omega)}-\frac{2\omega_i}{a_i}\right]-\frac{\lambda\omega_i^4}{\big[\lambda\omega_i+a_i\omega_0\big]^2}+\frac{\omega_i}{\lambda\omega_i+a_i\omega_0}.$$

where $i,k=1,\dots,n$ and $k\neq i$.

Corollary 5.4 *When $\lambda>0$, the kernel of the operator $\mathcal{L}_q^{-1}$ is given by*

$$K_{\lambda,\omega}(x_0,x_0)=\frac{\omega_0^2}{\lambda\big[1-\lambda Q(\lambda,\omega)\big]},\qquad K_{\lambda,\omega}(x_0,x_i)=\frac{a_i\omega_i\omega_0^2}{\lambda\big[1-\lambda Q(\lambda,\omega)\big]\big[\lambda\omega_i+a_i\omega_0\big]},$$

$$K_{\lambda,\omega}(x_k,x_i)=\frac{a_i a_k\omega_i\omega_k\omega_0^2}{\lambda\big[1-\lambda Q(\lambda,\omega)\big]\big[\lambda\omega_i+a_i\omega_0\big]\big[\lambda\omega_k+a_k\omega_0\big]},$$

$$K_{\lambda,\omega}(x_i,x_i)=\frac{a_i^2\omega_i^2\omega_0^2}{\lambda\big[1-\lambda Q(\lambda,\omega)\big]\big[\lambda\omega_i+a_i\omega_0\big]^2}+\frac{\omega_i}{\lambda\omega_i+a_i\omega_0},$$

where $i,k=1,\dots,n$ and $k\neq i$.

Corollary 5.5 *The effective resistance function with respect to λ and ω is given by*

$$R_{\lambda,\omega}(x_i,x_0)=\frac{1}{\omega_i[\lambda\omega_i+a_i\omega_0]}+\frac{\lambda\omega_i^2}{\big[1-\lambda Q(\lambda,\omega)\big]\big[\lambda\omega_i+a_i\omega_0\big]^2},$$

$$R_{\lambda,\omega}(x_i,x_k)=\frac{1}{\omega_i[\lambda\omega_i+a_i\omega_0]}+\frac{1}{\omega_k[\lambda\omega_k+a_k\omega_0]}+\frac{\lambda\big[a_i\omega_k-a_k\omega_i\big]^2\omega_0^2}{\big[1-\lambda Q(\lambda,\omega)\big]\big[\lambda\omega_i+a_i\omega_0\big]^2\big[\lambda\omega_k+a_k\omega_0\big]^2},$$

where $i,k=1,\dots,n$ and $k\neq i$.

Corollary 5.6 *The pairwise resistance function with respect to λ and ω is given by*

$$R_{\lambda,\omega}(x_i,x_j;x_k,x_\ell)=\frac{\lambda\omega_0^2(a_i\omega_j-a_j\omega_i)(a_k\omega_\ell-a_\ell\omega_k)}{\big[1-\lambda Q(\lambda,\omega)\big]\big[\lambda\omega_i+a_i\omega_0\big]\big[\lambda\omega_j+a_j\omega_0\big]\big[\lambda\omega_k+a_k\omega_0\big]\big[\lambda\omega_\ell+a_\ell\omega_0\big]},$$

$$R_{\lambda,\omega}(x_i,x_j;x_i,x_k)=\frac{1}{\omega_i[\lambda\omega_i+a_i\omega_0]}+\frac{\lambda\omega_0^2(a_i\omega_j-a_j\omega_i)(a_i\omega_k-a_k\omega_i)}{\big[1-\lambda Q(\lambda,\omega)\big]\big[\lambda\omega_i+a_i\omega_0\big]^2\big[\lambda\omega_j+a_j\omega_0\big]\big[\lambda\omega_k+a_k\omega_0\big]},$$

$$R_{\lambda,\omega}(x_0,x_i;x_k,x_\ell)=\frac{\lambda\omega_0\omega_i(a_k\omega_\ell-a_\ell\omega_k)}{\big[1-\lambda Q(\lambda,\omega)\big]\big[\lambda\omega_i+a_i\omega_0\big]\big[\lambda\omega_k+a_k\omega_0\big]\big[\lambda\omega_\ell+a_\ell\omega_0\big]},$$

$$R_{\lambda,\omega}(x_0,x_i;x_0,x_\ell)=\frac{\lambda\omega_i\omega_\ell}{\big[1-\lambda Q(\lambda,\omega)\big]\big[\lambda\omega_i+a_i\omega_0\big]\big[\lambda\omega_\ell+a_\ell\omega_0\big]},$$

$$R_{\lambda,\omega}(x_0,x_i;x_i,x_\ell)=\frac{-1}{\omega_i[\lambda\omega_i+a_i\omega_0]}+\frac{\lambda\omega_0\omega_i(a_i\omega_\ell-a_\ell\omega_i)}{\big[1-\lambda Q(\lambda,\omega)\big]\big[\lambda\omega_i+a_i\omega_0\big]^2\big[\lambda\omega_\ell+a_\ell\omega_0\big]},$$

where $i, j, k, \ell = 1, \dots, n$ and are mutually different.

In this work we are interested on perturbations of the Star S_n by adding edges only between consecutive vertices. To study this kind of perturbations it is useful to define $x_{n+1} = x_1$, $a_{n+1} = a_1$, $\omega_{n+1} = \omega_1$ and the values $\alpha = \dfrac{\lambda\omega_0^2}{1-\lambda Q(\lambda,\omega)}$, $\gamma_i = \dfrac{1}{\omega_i[\lambda\omega_i + a_i\omega_0]}$, $i = 1, \dots, n+1$ and $r_i = \omega_i\omega_{i+1}\gamma_i\gamma_{i+1}[a_i\omega_{i+1} - a_{i+1}\omega_i]$, $i = 1, \dots, n$. Observe that $\gamma_{n+1} = \gamma_1$ and moreover $r_n = \dfrac{a_n\omega_1 - a_1\omega_n}{[\lambda\omega_n + a_n\omega_0][\lambda\omega_1 + a_1\omega_0]}$. According with Corollary 5.6 we get

$$\begin{aligned}
&R_{\lambda,\omega}(x_i, x_{i+1}; x_k, x_{k+1}) = \alpha r_i r_k, && i, k = 1\dots, n,\ 1 < |i-k| < n-1,\\
&R_{\lambda,\omega}(x_{i-1}, x_i; x_i, x_{i+1}) = -\gamma_i + \alpha r_{i-1} r_i, && i = 2, \dots, n,\\
&R_{\lambda,\omega}(x_i, x_{i+1}; x_i, x_{i+1}) = \gamma_i + \gamma_{i+1} + \alpha r_i^2, && i = 1, \dots, n,\\
&R_{\lambda,\omega}(x_n, x_1; x_1, x_2) = -\gamma_1 + \alpha r_n r_1.
\end{aligned}$$

We consider the *m–blade Fan network* on $1 \le m \le n$ blades; that is, the perturbation of the Star S_n by adding m edges with conductances $c_1, \cdots, c_m > 0$ between consecutive vertices. In particular, when $m = n-1$ we have the standard Fan network, whereas when $m = n$ we have the so–called Wheel network; see [3, 13].

If we denote by $\rho_j = \sqrt{c_j\omega_j\omega_{j+1}}$, then $\sigma_j = \rho_j\Big(\dfrac{\varepsilon_{x_{j+1}}}{\omega_{j+1}} - \dfrac{\varepsilon_{x_j}}{\omega_j}\Big)$, for $j = 1, \dots, m$. Therefore,

$$\mathsf{A} = \mathsf{I} + \big(\langle \mathcal{G}_p(\sigma_j), \sigma_k\rangle\big) = \mathsf{I} + \Big(\rho_j\rho_k R_{\lambda,\omega}(x_{j+1}, x_j; x_{k+1}, x_k)\Big) = \mathsf{T} + \alpha \mathsf{r} \otimes \mathsf{r},$$

where $\mathsf{r} = (\rho_1 r_1, \dots, \rho_m r_m)^t$ and T is a matrix that will be described below. Therefore, according to the *Sherman–Morrison formula*, see [11]

$$\mathsf{A}^{-1} = \mathsf{T}^{-1} - \frac{\alpha}{1 + \alpha\langle \mathsf{T}^{-1}\mathsf{r}, \mathsf{r}\rangle}(\mathsf{T}^{-1}\mathsf{r}) \otimes (\mathsf{T}^{-1}\mathsf{r}).$$

When $m = 1, 2$, the computation of T^{-1} is straightforward. If we assume $m < n$, then

$$\mathsf{T} = \begin{bmatrix}
1+\rho_1^2(\gamma_1+\gamma_2) & -\rho_1\rho_2\gamma_2 & 0 & \cdots & 0 & 0\\
-\rho_1\rho_2\gamma_2 & 1+\rho_2^2(\gamma_2+\gamma_3) & -\rho_2\rho_3\gamma_3 & \cdots & 0 & 0\\
0 & -\rho_2\rho_3\gamma_3 & 1+\rho_3^2(\gamma_3+\gamma_4) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 1+\rho_{m-1}^2(\gamma_{m-1}+\gamma_m) & -\rho_{m-1}\rho_m\gamma_m\\
0 & 0 & 0 & \cdots & -\rho_{m-1}\rho_m\gamma_m & 1+\rho_m^2(\gamma_m+\gamma_{m+1})
\end{bmatrix}. \tag{14}$$

When $m \ge 3$, T is a tridiagonal matrix and hence, to obtain its inverse we can apply the results involving this class of matrices, see for instance [12]. In addition, we can also apply the usual techniques for discrete boundary value problems, see [4, 7]. Specifically, we have the following result expressing the entries of T^{-1} in terms of two solutions of a difference equation.

Proposition 5.7 *Consider* $\{u_j\}_{j=1}^m$ *and* $\{v_j\}_{j=1}^m$ *the solutions of the difference equation*

$$\big(1+\rho_k^2(\gamma_k+\gamma_{k+1})\big)z_k - z_{k+1}\rho_k\rho_{k+1}\gamma_{k+1} - z_{k-1}\rho_{k-1}\rho_k\gamma_k = 0, \quad k=2,\dots,m-1,$$

characterized by satisfying the initial conditions $u_1=\rho_1\rho_2\gamma_2$, $u_2 = 1+\rho_1^2(\gamma_1+\gamma_2)$ *and the final conditions* $v_{m-1} = 1+\rho_m^2(\gamma_m+\gamma_{m+1})$, $v_m = \rho_{m-1}\rho_m\gamma_m$, *respectively. Then,* $\rho_1\rho_2\gamma_2 v_2 \neq \big(1+\rho_1^2(\gamma_1+\gamma_2)\big)v_1$ *and moreover the* (j,k)*–entry of* T^{-1} *is*

$$\mathsf{b}_{jk} = \frac{u_{\min\{j,k\}}v_{\max\{j,k\}}}{\rho_1\rho_2\gamma_2\big(1+(\rho_1^2(\gamma_1+\gamma_2))v_1 - \rho_1\rho_2\gamma_2 v_2\big)}, \qquad j,k=1,\dots,m.$$

This method can be generalized to the case of $(m_1,\dots,m_s)$–*blade Fan network* that is, the perturbation of the Star S_n by adding $m_1+\cdots+m_s = m \le n-1$ edges with conductances $c_1,\cdots,c_m>0$ in such a way that the edges corresponding to $c_1,\dots,c_{m_1}$ are consecutive, the edge corresponding to c_{m_1} is not incident with the one corresponding to c_{m_1+1}, the edges corresponding to $c_{m_1+1},\dots,c_{m_1+m_2}$ are consecutive and so on. Newly, $\mathsf{A}=\mathsf{T}+\alpha\mathsf{r}\otimes\mathsf{r}$, where T is a block–diagonal matrix with s–blocks and for any $j=1,\dots,s$ the block T_j is like (14) and has size m_j. In particular, if $m_j=1$ for $j=\lfloor\frac{n}{2}\rfloor$, matrix T is diagonal.

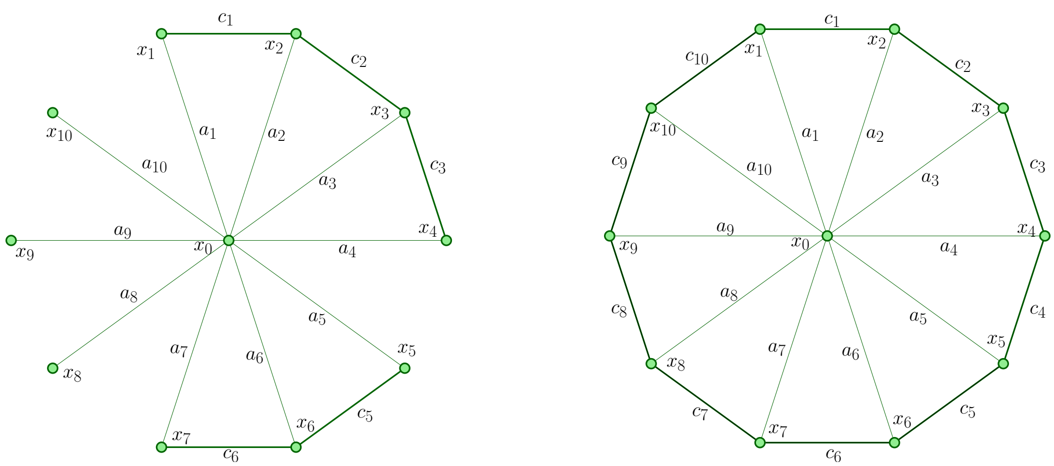

Figure 1: A $(3,2)$–blade Fan (left) and a Wheel (right) networks

Finally, we consider the case $m=n$; that is, the *Wheel network*. Then,

$$\mathsf{T} = \begin{bmatrix} 1+\rho_1^2(\gamma_1+\gamma_2) & -\rho_1\rho_2\gamma_2 & 0 & \cdots & 0 & -\rho_1\rho_n\gamma_1 \\ -\rho_1\rho_2\gamma_2 & 1+\rho_2^2(\gamma_2+\gamma_3) & -\rho_2\rho_3\gamma_3 & \cdots & 0 & 0 \\ 0 & -\rho_2\rho_3\gamma_3 & 1+\rho_3^2(\gamma_3+\gamma_4) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\rho_{n-1}^2(\gamma_{n-1}+\gamma_n) & -\rho_{n-1}\rho_n\gamma_n \\ -\rho_1\rho_n\gamma_1 & 0 & 0 & \cdots & -\rho_{n-1}\rho_n\gamma_n & 1+\rho_n^2(\gamma_n+\gamma_1) \end{bmatrix}$$

and hence T is a periodic Jacobi matrix. To obtain T^{-1}, we newly apply the usual techniques for discrete boundary value problems.

Proposition 5.8 *Consider* $\{u_j^1\}_{j=1}^n$, $\{u_j^2\}_{j=1}^n$, $\{v_j^1\}_{j=1}^n$ *and* $\{v_j^2\}_{j=1}^n$ *the solutions of the difference equation*

$$\big(1+\rho_k^2(\gamma_k+\gamma_{k+1})\big)z_k - z_{k+1}\rho_k\rho_{k+1}\gamma_{k+1} - z_{k-1}\rho_{k-1}\rho_k\gamma_k = 0, \quad k=2,\dots,n-1,$$

characterized by satisfying the initial conditions $u_1^1=\rho_1\rho_2\gamma_2$, $u_2^1=1+\rho_1^2(\gamma_1+\gamma_2)$, $v_1^1=0$, $v_2^1=\rho_1\rho_n\gamma_1$ *and the final conditions* $u_{n-1}^2=\rho_1\rho_n\gamma_1$, $u_n^2=0$, $v_{n-1}^2=1+\rho_n^2(\gamma_1+\gamma_n)$, $v_n^2=\rho_{n-1}\rho_n\gamma_n$, *respectively. Let now* $\{u_j\}_{j=1}^s$ *and* $\{v_j\}_{j=1}^s$ *be defined as*

$$u_j=\rho_{n-1}\rho_n\gamma_n u_j^1+\rho_1\rho_2\gamma_2 u_j^2 \quad \text{and} \quad v_j=\rho_{n-1}\rho_n\gamma_n v_j^1+\rho_1\rho_2\gamma_2 v_j^2 \quad j=1,\dots,n$$

Then, $u_1=v_n$, $u_1v_2\neq u_2v_1$ *and moreover the* (j,k)*–entry of* T^{-1} *is*

$$\begin{aligned}\mathsf{b}_{jk} &= \frac{1}{\rho_1\rho_2\gamma_2(u_2v_1-u_1v_2)}\,u_{\min\{j,k\}}v_{\max\{j,k\}} + \frac{\rho_{n-1}\rho_n^2\gamma_1\gamma_n u_1}{\rho_2\gamma_2(u_2v_1-u_1v_2)^2}\Big[v_ju_k+u_jv_k\Big]\\ &- \frac{\rho_{n-1}\rho_n\gamma_n}{\rho_1\rho_2\gamma_2(u_2v_1-u_1v_2)^2}\Big[\big((1+\rho_1^2(\gamma_1+\gamma_2)u_1-\rho_1\rho_2\gamma_2u_2\big)v_jv_k\\ &\qquad +\big((1+\rho_n^2(\gamma_1+\gamma_n)v_n-\rho_{n-1}\rho_n\gamma_nv_{n-1}\big)u_ju_k\Big].\end{aligned}$$

6 The constant case

In order to illustrate the above results, let us consider the Star network with constant conductances and weights. So, let S_n be the Star graph with $n+1$ vertices labeled as $\{x_0,x_1,\dots,x_n\}$, conductances $a_i=a>0$ and weight $\omega_i=w$, $i=1,\dots,n$. Hence, $\omega_0=w_0=\sqrt{1-nw^2}$ and $q_\omega(x_i)=q_w=a\Big(-1+\dfrac{w_0}{w}\Big)$, for any $i=1,\dots,n$.

Corollary 6.1 *The orthogonal Green function of the Star, with respect to* λ *and* ω *is given by*

$$G_{\lambda,\omega}(x_0,x_0)=\frac{nw^3w_0}{\lambda ww_0+a}, \qquad G_{\lambda,\omega}(x_i,x_i)=-\frac{w^3}{(\lambda w+aw_0)}\left(\frac{aw_0^2}{\lambda ww_0+a}+1\right)+\frac{w}{\lambda w+aw_0},$$

$$G_{\lambda,\omega}(x_0,x_i)=-\frac{w^2w_0^2}{\lambda ww_0+a}, \qquad G_{\lambda,\omega}(x_k,x_i)=-\frac{w^3}{(\lambda w+aw_0)}\left(\frac{aw_0^2}{\lambda ww_0+a}+1\right),$$

where $i,k=1,\dots,n$ *and* $k\neq i$*. Therefore, the Kirchhoff index for the Star is*

$$\mathsf{k}(\lambda,\omega)=\frac{nw\big(\lambda ww_0+a(1-w^2)\big)}{(\lambda w+aw_0)(\lambda ww_0+a)}.$$

Corollary 6.2 *When* $\lambda>0$*, the kernel of the operator* $\mathcal{L}_q^{-1}$ *is given by*

$$K_{\lambda,\omega}(x_0,x_0)=\frac{w_0(\lambda w+aw_0)}{\lambda(\lambda ww_0+a)}, \qquad K_{\lambda,\omega}(x_i,x_i)=\frac{a^2w^2w_0}{\lambda(\lambda ww_0+a)(\lambda w+aw_0)}+\frac{w}{\lambda w+aw_0},$$

$$K_{\lambda,\omega}(x_0,x_i)=\frac{aww_0}{\lambda(\lambda ww_0+a)}, \qquad K_{\lambda,\omega}(x_k,x_i)=\frac{a^2w^2w_0}{\lambda(\lambda ww_0+a)(\lambda w+aw_0)},$$

where $i,k=1,\dots,n$ *and* $k\neq i$.

In this case, $\gamma_i = \gamma = \dfrac{1}{w[\lambda w + a w_0]}$, $i = 1, \dots, n+1$, $r_i = 0$, $\rho_i = \rho = w\sqrt{c}$, and moreover $\sigma_i = \sqrt{c}\big(\varepsilon_{x_{i+1}} - \varepsilon_{x_i}\big)$, $i = 1, \dots, n$.

If we consider the *m–blade Fan network* on $1 \le m \le n-1$, then

$$\mathsf{A} = \begin{bmatrix} 1+2\rho^2\gamma & -\rho^2\gamma & 0 & \cdots & 0 & 0 \\ -\rho^2\gamma & 1+2\rho^2\gamma & -\rho^2\gamma & \cdots & 0 & 0 \\ 0 & -\rho^2\gamma & 1+2\rho^2\gamma & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2\rho^2\gamma & -\rho^2\gamma \\ 0 & 0 & 0 & \cdots & -\rho^2\gamma & 1+2\rho^2\gamma \end{bmatrix}. \tag{15}$$

To invert matrix A, the corresponding difference equation is

$$2pz_k - z_{k+1} - z_{k-1} = 0, \quad k = 2, \dots, m-1,$$

where $p = 1 + \dfrac{1}{2\rho^2\gamma}$, whose solutions are related with Chebyshev polynomials, [4]. Applying Proposition 5.7, we get that

$$\mathsf{b}_{jk} = \frac{2(p-1)U_{\min\{j,k\}-1}(p)U_{m-\max\{j,k\}}(p)}{U_m(p)} = \frac{T_{m+1-|k-j|}(p) - T_{m+1-k-j}(p)}{(p+1)U_m(p)}, \qquad j,k = 1, \dots, m,$$

where T_k, U_k are the first and second kind Chebyshev polynomials. Then, the perturbed Green function is

$$G^{\varepsilon}_{\lambda,\omega}(x_0,x_0) = \frac{nw^3w_0}{\lambda w w_0 + a}, \qquad G^{\varepsilon}_{\lambda,\omega}(x_0,x_i) = -\frac{w^2w_0^2}{\lambda w w_0 + a}, \qquad i = 1, \dots, n$$

$$G^{\varepsilon}_{\lambda,\omega}(x_s,x_i) = -\frac{w^3}{(\lambda w + a w_0)}\left(\frac{a w_0^2}{\lambda w w_0 + a} + 1\right) + \frac{2cw^2\big[T_{m+1-|s-i|}(p) + T_{m+2-s-i}(p)\big]}{(\lambda w + a w_0)(4cw + \lambda w + a w_0)U_m(p)}, \qquad s,i = 1, \dots, m+1,$$

$$G^{\varepsilon}_{\lambda,\omega}(x_s,x_i) = -\frac{w^3}{(\lambda w + a w_0)}\left(\frac{a w_0^2}{\lambda w w_0 + a} + 1\right) + \frac{w\,\delta_{si}}{\lambda w + a w_0}, \qquad \text{otherwise.}$$

Therefore, the effective resistances between vertices of the m–blade Fan network are given by

$$R^{\varepsilon}_{\lambda,\omega}(x_0,x_i)=\frac{\lambda w^2}{w_0(\lambda w w_0+a)(\lambda w+aw_0)}+\frac{2c\big[T_{m+1}(p)+T_{m+2-2i}(p)\big]}{(\lambda w+aw_0)(4cw+\lambda w+aw_0)U_m(p)},\quad i=1,\dots,m+1,$$

$$R^{\varepsilon}_{\lambda,\omega}(x_0,x_i)=\frac{\lambda w^2}{w_0(\lambda w w_0+a)(\lambda w+aw_0)}+\frac{1}{w(\lambda w+aw_0)},\qquad i=m+2,\dots,n,$$

$$R^{\varepsilon}_{\lambda,\omega}(x_i,x_j)=\frac{4c\big[T_{m+1}(p)+T_{m+2-i-j}(p)(T_{|i-j|}(p)-1)-T_{n+1-|i-j|}(p)\big]}{(\lambda w+aw_0)(4cw+\lambda w+aw_0)U_m(p)},\quad i,j=1,\dots,m+1,$$

$$R^{\varepsilon}_{\lambda,\omega}(x_i,x_j)=\frac{2}{w(\lambda w+aw_0)},\qquad i,j=m+2,\dots,n,$$

$$R^{\varepsilon}_{\lambda,\omega}(x_i,x_j)=\frac{2c\big[T_{m+1}(p)+T_{m+2-2i}(p)\big]}{(\lambda w+aw_0)(4cw+\lambda w+aw_0)U_m(p)}+\frac{1}{w(\lambda w+aw_0)},\quad i=1,\dots,m+1,\ j=m+2,\dots,n.$$

Moreover, the Kirchhoff index of the m–blade Fan network is

$$\begin{aligned}\mathsf{k}^{\varepsilon}(\lambda,\omega)&=\mathsf{k}(\lambda,\omega)-\frac{(m+1)\omega}{\lambda\omega+a\omega_0}+\frac{2cw^2\big[(m+1)T_{m+1}(p)+U_m(p)\big]}{(\lambda w+aw_0)(4c\omega+\lambda\omega+a\omega_0)U_m(p)}\\&=\mathsf{k}(\lambda,\omega)-\frac{mw}{\lambda w+aw_0}+\frac{\big[(m+1)T_{m+1}(p)-pU_m(p)\big]}{2c(p^2-1)U_m(p)}\\&=\frac{w\big((n-m)(\lambda w w_0+a)-aw^2\big)}{(\lambda w+aw_0)(\lambda w w_0+a)}+\sum_{i=1}^{m}\frac{w}{\lambda w+aw_0+4cw\sin^2\big(\frac{i\pi}{2(m+1)}\big)}.\end{aligned}$$

The last equality follows taking into account that

$$\frac{U_m'(x)}{U_m(x)}=\frac{(m+1)T_{m+1}(x)-xU_m(x)}{(x^2-1)U_m(x)}=\sum_{i=1}^{m}\frac{1}{x-\cos\big(\frac{i\pi}{m+1}\big)}.$$

When $m=n-1$, these expressions coincide with those obtained in [5] using a different approach, considering the Fan as the join network of a singleton with a path.

Finally, we consider the case $m = n$; that is, the *Wheel network*. Then,

$$\mathsf{A} = \begin{bmatrix} 1+2\rho^2\gamma & -\rho^2\gamma & 0 & \cdots & 0 & -\rho^2\gamma \\ -\rho^2\gamma & 1+2\rho^2\gamma & -\rho^2\gamma & \cdots & 0 & 0 \\ 0 & -\rho^2\gamma & 1+2\rho^2\gamma & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2\rho^2\gamma & -\rho^2\gamma \\ -\rho^2\gamma & 0 & 0 & \cdots & -\rho^2\gamma & 1+2\rho^2\gamma \end{bmatrix}. \tag{16}$$

To invert matrix A, we apply Proposition 5.8. So, defining the sequence $u_j = U_{j-1}(p)+U_{n-j-1}(p)$, $j = 1,\dots,n$, then

$$\mathsf{b}_{jk} = \frac{p-1}{T_n(p)-1}\, u_{\min\{j,k\}}u_{\max\{j,k\}-1} + \frac{(p-1)}{2\big(T_n(p)-1\big)^2}\Big[u_1\big(u_{j-1}u_k+u_ju_{k-1}\big)-u_0\big(u_{j-1}u_{k-1}+u_ju_k\big)\Big].$$

By properly using the Chebyshev Polynomials properties we get the equivalent expression

$$\mathsf{b}_{jk} = \frac{(p-1)\big(U_{n-1-|k-j|}(p)+U_{|k-j|-1}(p)\big)}{T_n(p)-1},$$

that coincides with the one obtained by some of this authors in [5]. Then, the perturbed Green function is

$$G^{\varepsilon}_{\lambda,\omega}(x_0,x_0) = \frac{nw^3w_0}{\lambda ww_0+a}, \quad G^{\varepsilon}_{\lambda,\omega}(x_0,x_i) = -\frac{w^2w_0^2}{\lambda ww_0+a}, \qquad i = 1,\dots,n,$$

$$G^{\varepsilon}_{\lambda,\omega}(x_s,x_i) = -\frac{w^3}{(\lambda w+aw_0)}\left(\frac{aw_0^2}{\lambda ww_0+a}+1\right) + \frac{U_{n-1-|i-s|}(p)+U_{|i-s|-1}(p)}{2c\big(T_n(p)-1\big)}, \qquad s,i = 1,\dots,n.$$

Therefore, the effective resistances are

$$R^{\varepsilon}_{\lambda,\omega}(x_0,x_i) = \frac{\lambda w^2}{w_0(\lambda ww_0+a)(\lambda w+aw_0)} + \frac{U_{n-1}(p)}{2cw^2(T_n(p)-1)}, \qquad i = 1,\dots,n,$$

$$R^{\varepsilon}_{\lambda,\omega}(x_i,x_j) = \frac{1}{cw^2\big(T_n(p)-1\big)}\Big(U_{n-1}(p)-U_{n-1-|i-j|}(p)-U_{|i-j|-1}(p)\Big), \qquad i,j = 1,\dots,n.$$

Moreover, the Kirchhoff index of the Wheel network is

$$\begin{aligned}\mathsf{k}^{\varepsilon}(\lambda,\omega) &= \mathsf{k}(\lambda,\omega) - \frac{nw}{\lambda w+aw_0} + \frac{nU_{n-1}(p)}{2c\big(T_n(p)-1\big)} \\ &= -\frac{naw^3}{(\lambda w+aw_0)(\lambda ww_0+a)} + \sum_{k=0}^{n-1}\frac{w}{\lambda w+aw_0+4cw\sin^2\left(\frac{k\pi}{n}\right)} \\ &= \frac{ww_0}{\lambda ww_0+a} + \sum_{k=1}^{n-1}\frac{w}{\lambda w+aw_0+4cw\sin^2\left(\frac{k\pi}{n}\right)},\end{aligned}$$

where we have taken into account that $T_n'(x) = nU_{n-1}(x)$ and hence $\dfrac{nU_{n-1}(x)}{T_n(x)-1} = \displaystyle\sum_{i=0}^{n-1} \frac{1}{x - \cos\left(\frac{2i\pi}{n}\right)}$.

For the standard Wheel and Fan; *i.e.*, $\lambda = 0$, $c = 1$ and $w = w_0 = (\sqrt{n+1})^{-1}$, an expression for the effective resistance was given in terms of the generalized Fibonacci numbers, G_k, in [1], see also [13].

Comparing the expressions for the effective resistances on the standard Fan where $a = 1$, we get

$$\frac{F_{2(n-i)+1}F_{2i-1}}{F_{2n}} = \frac{2\big(T_n(\frac{3}{2}) + T_{n+1-2i}(\frac{3}{2})\big)}{5U_{n-1}(\frac{3}{2})}$$

where $i = 1, \ldots, n$ and

$$\begin{aligned}&\frac{F_{2(n-j)+1}\big(F_{2j-1} - F_{2i-1}\big) + F_{2i-1}\big(F_{2(n-i)+1} - F_{2(n-j)+1}\big)}{F_{2n}}\\ &= \frac{4}{5U_{n-1}(\frac{3}{2})}\Big(T_n(\tfrac{3}{2}) + T_{n+1-i-j}(\tfrac{3}{2})\big(T_{|i-j|}(\tfrac{3}{2}) - 1\big) - T_{n-|i-j|}(\tfrac{3}{2})\Big),\end{aligned}$$

where $i, j = 1, \ldots, n$. The above equalities could have been obtained taking into account that $F_{2k+1} = V_k(\frac{3}{2})$, where $V_k(p) = U_k(p) - U_{k-1}(p)$ is the third kind Chebyshev polynomial.

On the other hand, comparing both expressions in the Wheel case we get the following nice identities

$$\begin{aligned}\frac{G_n^2}{G_{2n} - 2G_n} &= \frac{U_{n-1}(p)}{2c\big(T_n(p) - 1\big)} = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{a + 4\sin^2\left(\frac{k\pi}{n}\right)},\\ \frac{G_n^2}{G_{2n} - 2G_n}\left(2 - \frac{G_{2|i-j|}}{G_{|i-j|}}\right) &= \frac{U_{n-1}(p) - U_{n-1-|i-j|}(p) - U_{|i-j|-1}(p)}{c\big(T_n(p) - 1\big)}.\end{aligned}$$

These equalities could have been obtained taking into account that $G_k = U_{k-1}(p)$, for $p = 1 + \dfrac{a}{2}$. When $a = 1$ this equality becomes $F_{2k} = U_{k-1}(\frac{3}{2})$, where F_{2k} denotes the $2k$–th Fibonacci number.

From the expression for the Kirchhoff index we get that

$$\frac{G_n^2}{G_{2n} - 2G_n}\left(n^2 - \sum_{k=1}^{n-1} \frac{(n-k)G_{2k}}{G_k}\right) = \frac{1}{a} + \sum_{k=1}^{n-1} \frac{n+1}{a + 4\sin^2\left(\frac{k\pi}{n}\right)},$$

and therefore, the following sum rule for generalized Fibonacci numbers holds

$$\sum_{k=1}^{n-1} \frac{(n-k)G_{2k}}{G_k} = \left(\sum_{k=0}^{n-1} \frac{a}{a + 4\sin^2\left(\frac{k\pi}{n}\right)}\right)^{-1}\left(\sum_{k=1}^{n-1} \frac{4n\sin^2\left(\frac{k\pi}{n}\right)}{a + 4\sin^2\left(\frac{k\pi}{n}\right)}\right).$$

In particular,

$$\sum_{k=1}^{n-1} \frac{(n-k)F_{4k}}{F_{2k}} = \left(\sum_{k=0}^{n-1} \frac{1}{1 + 4\sin^2\left(\frac{k\pi}{n}\right)} \right)^{-1} \left(\sum_{k=1}^{n-1} \frac{4n\sin^2\left(\frac{k\pi}{n}\right)}{1 + 4\sin^2\left(\frac{k\pi}{n}\right)} \right).$$

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