An online optimization-based procedure for the assignment of airplane seats

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Jordi Castro · **Fernando Sarachaga**

Abstract Due to the large number of air flights these days, all procedures involved in their operational management should be carefully optimized. This work presents a novel approach to the seat assignment problem, which focuses on deciding where to seat the passengers of different online purchases. This problem is currently solved by most airlines with a set of simple pre-defined rules that do not take into account future sales. Instead, the approach in this work is based on solving an integer multicommodity network flow problem, where different commodities are associated with expected future demands of different types of passengers. One feature of the developed optimization model is that it has to be solved online (that is, in real-time), thus it must be both effective and fast, which prevented the use of more sophisticated (but also more time consuming, as it was experimentally observed) models based on stochastic programming. Using a real database of flights by Vueling Airlines S.A., we generated a set of synthetic online purchases simulating a pseudo-real flight. Applying our approach to this synthetic data, we observed that (1) the optimization model could be satisfactorily solved in real-time using the state-of-the-art CPLEX solver; (2) and the seat assignment obtained was of higher quality than that obtained by the simple pre-defined rules used by airlines.

Keywords OR in airlines · seat assignment · network optimization · multicommodity flows · integer programming

Mathematics Subject Classification (2000) 90C90 · 90C10 · 90C35 · 90B10

1 Introduction

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Passenger air traffic is currently experiencing a significant growth. The number of seats offered over a recent ten-year period increased 48% all over the world (this increment was 40% in Europe), from 3156M seats in 2005 to 4665M in 2015. Companies expect a growth rate of 5% per year, such that the expected number of seats will be around 6000M by 2020. The number of airplanes needed to satisfy this demand is thus also growing, and the current number of 33M take-offs is expected to be around 38.5M by 2020. Due to these orders of magnitude, it is instrumental for both airlines and airports to be highly efficient in any procedure that involves the operational management of flights.

In this work we focus on the optimal assignment of airplane seats. Without loss of generality, we will focus on the seat map of an Airbus A320, although our approach is valid for any aircraft. Indeed, the Airbus A320 was the airplane that had the most take-offs in 2015: approximately 5.5M, representing 17% of the world total.

Optimal seat assignment is a highly strategic service, which involves three different agents—passengers, airlines, and airports—each of them with different interests:

– For passengers flying in groups, one of the most valued features is having all group members seated together. In addition, passengers appreciate either having the option of selecting the seat (premium seats including a fee) or knowing the assigned seat after the purchase. Airlines are thus forced to manage several strategies within the same seat map.
– For airlines, carrying out good seat assignment is required not only for customer satisfaction, but also from an operational point of view: fast boarding reduces the time the airplane is on the ground, which affects the company's competitiveness.
– For airport authorities good assignment has a double benefit, since reducing the boarding time will lead to: (i) an increase in the number of take-offs, and (ii) passengers potentially spending more time in airport shopping areas.

Several other airline operations have been extensively applied in the past, including fleet assignment (i.e., optimal assignment of aircraft to routes; see, for instance, Sherali et al (2006) for a survey and references therein); crew scheduling (i.e, assignment of crews to aircraft, see Gopalakrishnan and Johnson (2005) for a survey and references therein); ground staff management (e.g., one airline implemented the approach of Felici and Gentile (2004)); air traffic management (to avoid airport congestion and reduce delays) (Agustín et al, 2012a,b; Dembo et al, 1989); and aircraft conflict resolution (to guarantee safe distances between aircraft to avoid collisions) (Alonso-Ayuso et al, 2016a,b). A general survey on operations research for the airline industry can be found in Yu and Thengwall (2002), which deals with additional topics such as revenue management and irregular operations.

Most of the literature on airline operations deals with the seat allocation problem instead of seat assignment, which is the purpose of this work. Seat allocation—either nested or nonnested (Yu and Thengwall, 2002)—seeks the optimal number of seats to be offered for each fare class in order to maximize revenue management. Some early approaches to seat allocation considered network optimization models (Dror et al, 1988; Glover et al, 1982), where variables are associated with classes of passengers (not groups on a flight, as we will do). Probabilistic models for seat allocation using estimates of uncertain demand were introduced, for instance, in Belobaba (1989);

Brumelle and McGill (1993); Sawaki (1989). A more recent probabilistic model considering replenishment for lower fares was presented in Sato and Sawaki (2009). A different but related problem was addressed in Lee and Hersh (1993): this work formulated a model for deciding whether a booking request for seats in a certain booking class should be accepted or denied. The approach of Tajima and Misono (1999) formulated a set packing integer problem to fill the aircraft while considering groups of passengers so that members of the group are seated as close to each other as possible (as our approach will do); this work used real data by Japan Airlines, but assumed no stochasticity. We also remark that optimization procedures have been used to decide the seat allocation in parliaments (Hales and García, 2019).

This work presents a novel approach to the seat assignment problem, which consists of deciding where to seat the different passengers or groups of passengers in an aircraft, according to different characteristics such as fare class, number of passengers in a group, or sales channel. This problem is currently being solved by companies such as Vueling Airlines S.A. through a set of pre-defined rules that look for (using a greedy search) a set of seats that have some characteristics. In this work we propose an optimization model that represents the aircraft topology as a network and the groups of passengers as different commodities. This model maximizes benefits and at the same time tries to satisfy two main goals:

- to avoid separating members of the same group;
- to assign seats according to the fare class.

Compared to current models in use (for instance, at Vueling Airlines), this new approach improves on the following aspects:

- Expected demand is considered. The current rules in use are the same for all flights, independently of the expected demand. In this new approach the demand for different fares (economy and business) is estimated from historical data for similar flights.
- Cooperative solution. The current rules assign seats considering only the current passenger or group of passengers, independently of future sales. The new model forecasts future sales for different fares (economy and business), and assign seats accordingly, thus increasing revenues. This approach could be formulated as a stochastic optimization model; however, since the different optimization problems (one per online sale) must be solved in real-time—thus, quickly—we will consider a deterministic version. In other words, only the most probable future scenario (that associated with the expected demand) will be considered. Some results with a tentative stochastic optimization model will be provided, showing that is computationally expensive for an online system.
- Quantitative comparison. The new model is based on optimizing an objective function, and thus allows a quantitative comparison between different seat assignments. This was not possible with the rules-based approach.

The structure of this document is as follows. Section 2 outlines the overall ticket sales procedure, in which the optimization model's solution for the seat assignment is one of the crucial steps. This mixed integer linear optimization model is described in Section 3, which also presents a stochastic optimization model of this problem. Finally, Section 4 provides computational results that show the effectiveness of the new

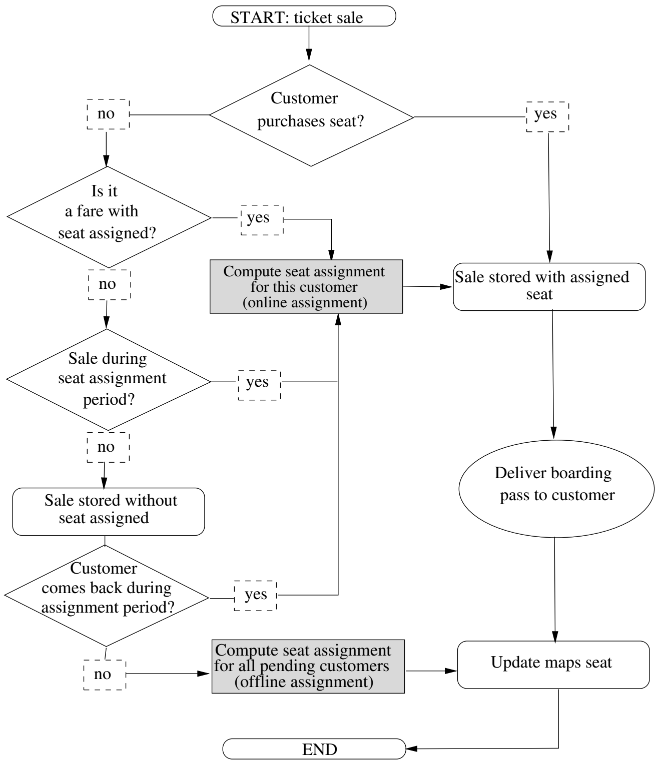

Fig. 1 Scheme of the overall ticket sales and seat assignment procedure

approach in terms of both computation time and revenues. The instances considered come from a synthetic pseudo-real flight, which was generated by Vueling Airlines from its extensive (and confidential) database; future demands were also estimated from this database by the company. Computational results are also reported for the stochastic version of the model, showing that the resulting stochastic optimization problem is computationally too expensive for an online system.

2 The overall ticket sales procedure

In this work we consider a realistic modern scenario, in which a significant percentage of sales is performed online by customers, who would ideally like to know the seats assigned by the company immediately after their purchase, or at least as soon as possible. In this scenario, seats are assigned without knowing the future sales, which is the opposite of the simpler scenario where seats are assigned just before flight departure, and after all purchases have been made.

The overall ticket sales procedure being considered is shown in Figure 1. The realistic assumptions performed for this procedure are the following:

– Customers may pay to choose their seat among those currently available.

- Only customers paying a premium fare will know their assigned seat immediately after the purchase, independently of the date of purchase.
- When a purchase is made after some deadline (usually a date close to departure), all customers can know their assigned seats. Customers who make the purchase before this deadline can return to the system for their seat assignments.
- When the customer has a seat, the airline delivers the boarding pass.
- All pending customers that did not return to the system for a seat will have one assigned by the airline before departure; customers are notified of their seats at check-in.

The company must assign seats in two steps of the procedure, which correspond to the grey boxes in Figure 1. In the first step (upper grey box), seats are assigned only for the current online purchase; this will be named "online assignment", and it must be performed before knowing the future uncertain demand. In the second step (lower grey box), all pending customers are assigned seats without expecting new purchases (except for the last minute ones); this will be called "offline assignment". These two steps are embedded in an optimization model, which is described in the next section.

3 The mathematical optimization model

The seat map of the aircraft is represented by a directed graph $G = (A,N)$, where A and N are the set of arcs and nodes, respectively. For an aircraft of n seats, the set of nodes is defined as $N = \{0\} \cup \{n+1\} \cup I \cup J'$, where $I = \{1,2,\ldots,n\}$ and $J' = \{1',2',\ldots,n'\}$. Nodes $i \in I$ and $j' \in J'$ are associated with seats; nodes 0 and $n+1$ will be used as initial and final nodes in the model. Four types of arcs will be considered, i.e., $A = O \cup F \cup S \cup D$ where:

- $O = \{(0,i), i \in I\}$: arcs from the initial node to each seat.
- $F = \{(j',n+1), j' \in J'\}$: arcs from each seat to the final node.
- $S = \{(i, j'), i \in I, j' \in J' : i = j\}$: arcs associated with seats. Flow through these arcs means that the associated seat has been assigned.
- $D = \{(j',i), j' \in J', i \in I : j > i\}$: arcs connecting different seats. These arcs are needed to select seats for different passengers in the same group. In principle, all seats are connected to each other, so we could use $j \neq i$ in the definition of set D. However, to avoid symmetries in the solution it is preferable to consider only half of the arcs, so that $j > i$. This resulted in becoming instrumental in efficiently solving the optimization problems.

Figure 2 shows the graph for a hypothetical aircraft with only four seats, indicating the different types of arcs. For an Airbus A320—the plane considered for the computational results—the number of seats is $n = 180$ (distributed in 30 rows of 6 seats each). This graph representation is valid, however, for any aircraft.

The optimization problem will consider a set $K = \{1,\ldots,\kappa\}$ of κ different purchases or groups of passengers to be assigned. All variables will be replicated according to K, and then—as will be shown—the resulting model will be an integer multicommodity flow problem with side constraints. Commodity $K \ni k = 1$ corresponds to the current sale (i.e., the group who is currently purchasing the online

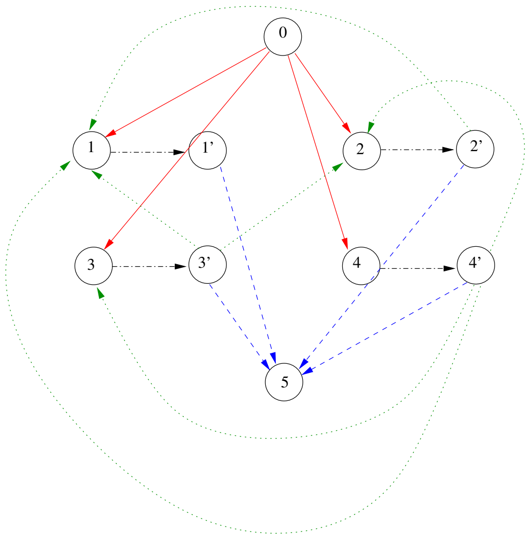

Fig. 2 Graph associated with a small aircraft with only four seats. Continuous red lines correspond to arcs O; dashed blue lines to arcs F; dashed-dotted black lines to arcs S; dotted green lines to arcs D.

tickets). The rest of commodities $K \ni k > 1$ represent the number of expected future seats to be purchased by different classes of passengers (economy, business, etc). All these commodities (the current group and the expected future groups) compete for a seat in the plane, that is, the capacity of the arc in S representing a seat is one, and the sum of the binary flows for all $k \in K$ traversing this arc must be less than or equal to one (which are the usual mutual capacity constraints in multicommodity flow models, defined in below equations (1c)). In this way, the current decision is taken by considering the expected future demands for economy/business seats, thus obtaining better assignments for the company. In other words, some "better" seats may not be assigned to the current purchase, with the expectation that these seats will be bought in the future by a business passenger.

We will deal with two different cases. When computing the assignment for the current purchase (either a passenger or a group of passengers), which corresponds to the online assignment of the upper grey box of Figure 1, κ will take a value of $\{1,3,4,5\}$. When $\kappa = 1$ we consider only the current purchase; this gives rise to the simplest optimization problem. When $\kappa = 3$, we consider the current purchase and two additional groups (possibly with a large number of passengers) which represent all the upcoming economy- and business-class purchases. Similarly, we also consider the cases in which $\kappa = 4$ and $\kappa = 5$. When $\kappa = 4$, in addition to the economy and business groups, we also include a "top-business" segment, i.e., premium business passengers that frequently flight with the company and have additional benefits (such as priority boarding, better offline seat assignment, priority in reallocation in case of flight cancellation, etc.). On average, it is estimated that the number of top-business passengers represents 10% of all business class purchases. When $\kappa = 5$, we also add to the previous groups a "top-economy" group (with benefits similar to those for top-business); it has been estimated that the size of the top-economy group represents 50% of all economy passengers. The resulting optimization problems will thus have a multicommodity flow-like structure, with κ equal to either 1 (this case being single-commodity), 3, 4 or 5 commodities.

On the other hand, when we are close to flight departure, seats must be assigned for all pending groups of customers; this is the offline assignment of the lower grey box in Figure 1. In this case κ (usually $\kappa > 1$) is the number of pending groups, and future expected purchases are not considered.

In addition to the previous graph G and set K, the optimization model also requires the following parameters:

- $p_k > 0, k \in K$: number of passengers in group k. In the online assignment, p_1 is the real number of passengers for the current sale; $p_k, k > 1$, is an estimate of the future sales for each group (economy, business, top-economy, and top-business, as explained above). In the offline assignment, p_k is the real number of passengers in pending group k.
- $a_i \in \{0,1\}, i \in I$: availability of seat i. If a_i is 1, this seat is available; if it is 0, it has already been assigned to a previous customer. Parameters a_i are updated after each purchase.
- $c^O_{ik}, (0,i) \in O, k \in K$: cost associated with arcs in O, which are the arcs that start the assignment for group k. This is the only cost whose values differ according to the type of passenger, thus allowing the control and creation of different environments on the plane according to different fares.
- $c^S_{ii'k}, (i,i') \in S, k \in K$: cost associated with arcs in S, for group k. This is the cost of selecting a particular seat i. This cost is useful for reserving some seats for future customers who can afford to pay for them, or to give more importance to, say, window or aisle seats.
- $c^D_{j'ik}, (j',i) \in D, k \in K$: cost associated with arcs in D for group k. This cost is instrumental if $p_k > 1$ (that is, if they have to seat more than one passenger for group k), since it controls the penalization between far away seats. Contiguous seats i and j have a small cost $c^D_{j'ik}$, and the cost increases with the "distance" between i and j. It is worth noting that D includes only arcs from seat j to seat i if $j > i$ (i.e., if the row of seat j is posterior to the row of seat i). This means that the solution of the optimization problem (associated with flows in the graph G) will assign passengers of the same group in a non-increasing sequence of rows, thus avoiding equivalent solutions with different orders (symmetric solutions).
- $w^O_k, k \in K$: weighting factor for costs c^O_{ik} for different groups k. For instance, by setting $w^O_1 < w^O_k$, for $k > 1$, in the online assignment (upper grey box in Figure 1), we give priority to future purchases.
- $w^D_k, k \in K$: weighting factor for costs $c^D_{j'ik}$ for different groups k. For instance, by setting $w^D_1 > w^D_k$, for $k > 1$ in the online assignment (upper grey box in Figure 1), we give preference to seating together passengers of the online purchase.

It is worth remarking that the costs for arcs in O and S could be adjusted to deal with other considerations than just the comfort of passengers, such as, for instance, the location of the gravity center of the aircraft (which, for safety reasons, must be within some predefined range). Another option to fully control the location of the gravity center would be to include extra side constraints to the below model, at the expense of complicating the optimization procedure.

The variables of the optimization problem (all of them binary) are unit flows through the arcs in Figure 2, replicated for each group (commodity). The purpose is to send a unit flow from the initial to the final node, traversing as many seat arcs as there are passengers with each commodity. The variables are:

- $o_{ik} \in \{0,1\}, (0,i) \in O, k \in K$: flows through arcs in O. They start the assignment of seats for group k.
- $f_{j'k} \in \{0,1\}, (j', n+1) \in F, k \in K$: flows through arcs in F. They end the assignment of seats for group k.
- $s_{ii'k} \in \{0,1\}, (i,i') \in S, k \in K$: flows through arcs in S. If a unit flow traverses arc $s_{ii'k}$, seat i is selected for group k.
- $d_{j'ik} \in \{0,1\}, (j',i) \in D, k \in K$: flows through arcs in D. A unit flow traversing arc $d_{j'ik}$ means that, if a member of the group k is at seat j, the next member will be at seat i.

The optimization problem to be solved is:

$$
\begin{aligned}
\min \quad & \sum_{k\in K} \left(w_k^O \sum_{i\in I} c_{ik}^O o_{ik} + w_k^D \sum_{(j',i)\in D} c_{j'ik}^D d_{j'ik} + \sum_{(i,i')\in S} c_{ii'k}^S s_{ii'k} \right) && \text{(1a)}\\
\text{s. t.} \quad & \sum_{(i,i')\in S} s_{ii'k} = p_k \qquad k \in K && \text{(1b)}\\
& \sum_{k\in K} s_{ii'k} \le 1 \qquad (i,i') \in S && \text{(1c)}\\
& s_{ii'k} \le a_i \qquad i \in I, k \in K && \text{(1d)}\\
& \sum_{i\in I} o_{ik} = 1 \qquad k \in K && \text{(1e)}\\
& \sum_{j'\in J'} f_{j'k} = 1 \qquad k \in K && \text{(1f)}\\
& s_{ii'k} = o_{ik} + \sum_{(j',i)\in D} d_{j'ik} \qquad i \in I, k \in K && \text{(1g)}\\
& s_{jj'k} = f_{j'k} + \sum_{(j',i)\in D} d_{j'ik} \qquad j' \in J', k \in K && \text{(1h)}\\
& \sum_{(j',i)\in D} d_{j'ik} = p_k - 1 \qquad k \in K && \text{(1i)}\\
& o_{ik} \in \{0,1\}, f_{j'k} \in \{0,1\} \qquad i \in I, j' \in J', k \in K && \text{(1j)}\\
& s_{ii'k} \in \{0,1\}, d_{j'ik} \in \{0,1\} \qquad (i,i') \in S, (j',i) \in D, k \in K. && \text{(1k)}
\end{aligned}
$$

The objective function (1a) minimizes the cost of the assignment (equivalently, it maximizes the company revenue and benefit to passengers). Constraints (1b) guarantee that the right number of seats is selected for each group. Constraints (1c) impose at most one passenger per seat. Constraints (1d) prevent using previously assigned seats. Constraints (1e) inject a unit flow at node 0 of graph G, for each commodity;

Table 1 Time (in seconds) to solve 10 instances with different values of p_1, with or without constraint (1i)

	Value of p_1									
	1	1	1	1	1	3	1	1	2	2
With (1i)	12.9	19.0	10.1	7.1	7.4	19.5	5.4	4.2	5.1	7.0
Without (1i)	22.5	29.4	21.5	20.5	20.2	16.4	16.0	15.3	7.0	7.4

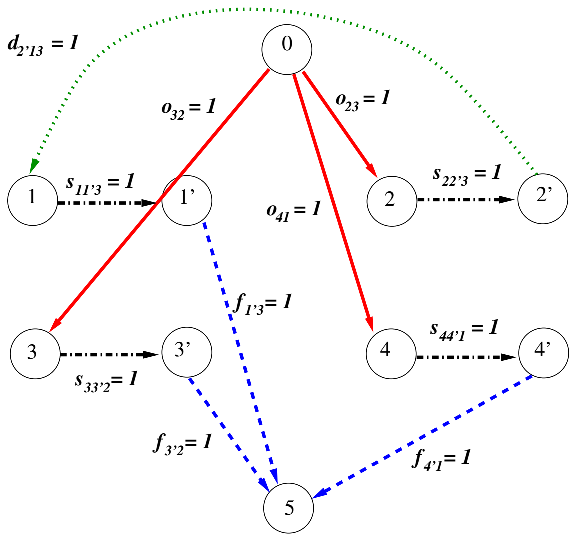

Fig. 3 Solution for the small example of Figure 2 considering a set K with $\kappa = 3$ commodities: commodity 1 is the current online economy sale with one passenger; commodity 2 is the expected future demand of economy seats ($p_2 = 1$ in this example); and commodity 3 is the expected future demand of business seats ($p_3 = 2$ in this example).

these unit flows exit the graph at node $(n+1)$ according to constraints (1f). Constraints (1g) and (1h) are the balance equations at nodes of I and J', respectively. Constraints (1i) guarantee that the unit flow circulating through the network traverses $p_k - 1$ arcs of D. Constraints (1i) are indeed redundant, but when $p_k = 1$, arcs $d_{j'ik}$ are set to 0 during the preprocessing. This resulted in being very useful, as shown in Table 1. This table reports the time (in seconds) for the solution of 10 different instances with values of p_1 in $\{1,2,3\}$. Clearly, when $p_1 = 1$ the times are 2-3 times less with constraints (1i). For $p_1 > 1$ constraints (1i) did not significantly affect the performance of the model, thus they will be kept for all the instances.

Figure 3 illustrates the solution for the small example of Figure 2, considering a set K with $\kappa = 3$ commodities or groups. The first group corresponds to the current online purchase with $p_1 = 1$ (one passenger); commodity 2 is the expected demand of economy seats, with $p_2 = 1$ (one expected passenger); and commodity 3 is the future demand for business seats, with with $p_3 = 2$ (two expected passengers). The arcs of Figure 3 are associated to the only variables with value one in the optimal solution, the rest having a zero value. It is observed that the current purchase is assigned to seat 4, while seat 3 is assigned to the future expected economy purchase, and the contiguous seats 1 and 2 of the first row are reserved for the expected future economy purchases.

3.1 The stochastic optimization model

The optimization problem (1) can be made stochastic by considering that $(p_k, k > 1)$ is a random vector with a certain distribution. A two-stage stochastic model considers two types of variables: the first-stage variables are the decisions to be made before the realization of future random events; and the second-stage variables, which are decided after the outcome of random events. The different realizations of the random events are associated to the future demands for all groups of passengers $k > 1$, and are represented by a set of scenarios $L = \{1,\dots,\lambda\}$, λ being the number of realizations of the random vector. First-stage variables correspond to group $k = 1$ (the group of the known current sale), while second-stage variables are associated to the (stochastic) demand for groups $k > 1$ ("business", "top-business", "economy", "top-economy"). To simplify the notation and the model, parameter p_1 (the real—and deterministic—number of passengers of the current sale) will also be included in the random vector, replicating the same value for all the scenarios. Accordingly, the first-stage variables (those associated to $k = 1$) will be replicated by scenarios. Since they must have the same value for all the scenarios (because they are the "today" decisions, thus unique), they will be forced to be equal by a set of constraints which are named "nonanticipativity constraints" in stochastic optimization. With these premises, the only changes in the definition of parameters and variables respect to Model (1) are:

- $p_{kl}, k \in K, l \in L$: number of passengers in group k under scenario l. Parameters p_{1l} are equal for every scenario l, since they correspond to the real (and thus deterministic) number of passengers for the current sale.
- $\beta_l \geq 0, l \in L$: probability of each scenario, such that $\sum_{l \in L} \beta_l = 1$.
- $o_{ikl} \in \{0,1\}, (0,i) \in O, k \in K, l \in L$: flows through arcs in O. They start the assignment of seats for group k under scenario l.
- $f_{j'kl} \in \{0,1\}, (j',n+1) \in F, k \in K, l \in L$: flows through arcs in F. They end the assignment of seats for group k under scenario l.
- $s_{ii'kl} \in \{0,1\}, (i,i') \in S, k \in K, l \in L$: flows through arcs in S. If a unit flow traverses arc $s_{ii'kl}$, seat i is selected for group k under scenario l.
- $d_{j'ikl} \in \{0,1\}, (j',i) \in D, k \in K, l \in L$: flows through arcs in D. A unit flow traversing arc $d_{j'ikl}$ means that, if a member of the group k is at seat j, the next member will be at seat i, under scenario L.

The two-stage stochastic optimization model is:

$$
\begin{aligned}
\min \quad & \sum_{l\in L}\left(\beta_l \sum_{k\in K}\left(w_k^O \sum_{i\in I} c_{ik}^O o_{ikl} + w_k^D \sum_{(j',i)\in D} c_{j'ik}^D d_{j'ikl} + \sum_{(i,i')\in S} c_{ii'k}^S s_{ii'kl}\right)\right) && \text{(2a)}\\
\text{s. t.} \quad & \sum_{(i,i')\in S} s_{ii'kl} = p_{kl} \qquad k\in K, l\in L && \text{(2b)}\\
& \sum_{k\in K} s_{ii'kl} \le 1 \qquad (i,i')\in S, l\in L && \text{(2c)}\\
& s_{ii'kl} \le a_i \qquad i\in I, k\in K, l\in L && \text{(2d)}\\
& \sum_{i\in I} o_{ikl} = 1 \qquad k\in K, l\in L && \text{(2e)}\\
& \sum_{j'\in J'} f_{j'kl} = 1 \qquad k\in K, l\in L && \text{(2f)}\\
& s_{ii'kl} = o_{ikl} + \sum_{(j',i)\in D} d_{j'ikl} \qquad i\in I, k\in K, l\in L && \text{(2g)}\\
& s_{jj'kl} = f_{j'kl} + \sum_{(j',i)\in D} d_{j'ikl} \qquad j'\in J', k\in K, l\in L && \text{(2h)}\\
& \sum_{(j',i)\in D} d_{j'ikl} = p_{kl} - 1 \qquad k\in K, l\in L && \text{(2i)}\\
& o_{ikl}\in\{0,1\}, f_{j'kl}\in\{0,1\} \qquad i\in I, j'\in J', k\in K, l\in L && \text{(2j)}\\
& s_{ii'kl}\in\{0,1\}, d_{j'ikl}\in\{0,1\} \qquad (i,i')\in S, (j',i)\in D, k\in K, l\in L && \text{(2k)}\\
& o_{i1l} = o_{i1(l+1)} \qquad i\in I, l\in L\setminus\{\lambda\} && \text{(2l)}\\
& f_{j'1l} = f_{j'1(l+1)} \qquad j'\in J', l\in L\setminus\{\lambda\} && \text{(2m)}\\
& s_{ii'1l} = s_{ii'1(l+1)} \qquad (i,i')\in S, l\in L\setminus\{\lambda\} && \text{(2n)}\\
& d_{j'i1l} = d_{j'i1(l+1)} \qquad (j',i)\in D, l\in L\setminus\{\lambda\}. && \text{(2o)}
\end{aligned}
$$

Equations (2a)–(2k) are the stochastic versions of (1a)–(1k), and they have the same meaning. Equations (2l)–(2o) are the nonanticipativity constraints for first-stage variables.

We remark that this stochastic model is purely academic and experimental and it was developed independently of any airline.

4 Computational results

Next two subsections present computational results for two different scenarios. In Subsection 4.1 a pseudo-real flight was synthesized from the (confidential) Vueling Airlines database. Subsection 4.2 presents results for instances generated as variations of the previous realistic flight.

4.1 Results for pseudo-real synthetic flight

Based on the real database of flights by Vueling Airlines, we simulated all the purchases and expected future demands for a pseudo-real flight. For reasons of confidentiality, we will only provide an overview of the procedure—which was fully performed by Vueling Airlines—for the generation of this pseudo-real flight.

From the real database, a sample of 19799 flights was extracted for one year. These flights were clustered in 144 groups, according to factors such as the day of the week of the flight, time of the flight, and route. Each of the 144 groups has a distribution of economy, business, top-economy and top-business passengers. A particular type of flight (of the 144 available) was selected, and all the sales in this group were considered; this amounted to 67782 sales. From these sales, values p_k, $k > 1$, (i.e., the expected number of passengers for each category on some particular flight) were estimated by computing confidence intervals for the number of economy and business passengers, and the expected number of passengers was the upper limit of this interval. Note that using the upper limit of the interval is the worst case for our approach based on (1), since it corresponds to a larger load factor for the flight, and thus the optimization problems involve more variables and constraints. From these data it was also estimated that the number of top-business and top-economy passengers were, respectively, 10% and 50% of the number of business and economy passengers. The “synthetic” pseudo-real flight was finally generated from the above mentioned 67782 sales (that is, a synthetic sequence of sales, each belonging to a particular group—economy, business, etc.).

Once the pseudo-real flight was generated, the aircraft was filled according to the procedure in Figure 1, using the optimization Model (1) to solve the several online and offline seat assignments. The seat map configuration that was considered is shown in Figure 4.a, which corresponds to an Airbus A320 (30 rows, 6 seats each, separated by one aisle) and where different types of seats are marked with colors according to the legend. It is worth noting that row 13, which is removed by many airlines (including Vueling) due to superstitious reasons, is considered in Figure 4: without loss of generality, our model uses the natural order of rows, not the numbering used by airlines. The cost scheme considered below is consistent with the commercial policy of companies such as Vueling Airlines. Costs c^O_{ik} and $c^S_{ii'k}$ are given in Figure 4.b. Costs c^O_{ik} are equal for seats $i \in I$ in the same row, but they change with the type of purchase $k \in K$ (top-business, business, top-economy, economy), as was stated in Section 3. Costs $c^S_{ii'k}$ are the same for all commodities $k \in K$, but change with seats $(i,i') \in S$. For the other cost coefficients in the objective function we used:

- The weighting factors w^O_k were 1 if $k = 1$, and 1.5 for $k > 1$, thus giving priority to expected future purchases.
- The weighting factors w^D_k were 1 if $k = 1$, and 0.5 for $k > 1$, in an attempt to contiguously seat all the passengers of the online purchase.
- Costs $c^D_{j'ik}$ depend only on j' and i, and they are the same for all $k \in K$. They were computed as: $c^D_{j'ik} = 1n_H + 1.5n_V$, where n_H and n_V denote the horizontal (i.e., within row) and vertical (i.e., between rows) “moves” to reach seat i from seat j.

Type of seat

- First Class
- Reserved for upgrades
- Top within class
- Priority /XL
- Reserved for operations/promotions

	A	B	C	D	E	F
1	1	2	3	4	5	6
2	7	8	9	10	11	12
3	13	14	15	16	17	18
4	19	20	21	22	23	24
5	25	26	27	28	29	30
6	31	32	33	34	35	36
7	37	38	39	40	41	42
8	43	44	45	46	47	48
9	49	50	51	52	53	54
10	55	56	57	58	59	60
11	61	62	63	64	65	66
12	67	68	69	70	71	72
13	73	74	75	76	77	78
14	79	80	81	82	83	84
15	85	86	87	88	89	90
16	91	92	93	94	95	96
17	97	98	99	100	101	102
18	103	104	105	106	107	108
19	109	110	111	112	113	114
20	115	116	117	118	119	120
21	121	122	123	124	125	126
22	127	128	129	130	131	132
23	133	134	135	136	137	138
24	139	140	141	142	143	144
25	145	146	147	148	149	150
26	151	152	153	154	155	156
27	157	158	159	160	161	162
28	163	164	165	166	167	168
29	169	170	171	172	173	174
30	175	176	177	178	179	180

(a)

	Top-Business	Business	Top-Economy	Economy	A	B	C	D	E	F
1	50	50	50	50	10	10	10	10	10	10
2	15	15	29	29	5	5	5	5	5	5
3	10	10	28	28	5	5	5	5	5	5
4	3	8	27	27	2.5	2.5	2.5	2.5	2.5	2.5
5	2	7	26	26	1	1	1	1	1	1
6	1	6	23	23	0.1	0.1	0.1	0.1	0.1	0.1
7	4	5	22	22	0.1	0.1	0.1	0.1	0.1	0.1
8	5	4	21	21	0.1	0.1	0.1	0.1	0.1	0.1
9	6	3	20	20	0.1	0.1	0.1	0.1	0.1	0.1
10	7	2	19	19	0.1	0.1	0.1	0.1	0.1	0.1
11	8	1	18	18	0.1	0.1	0.1	0.1	0.1	0.1
12	15	15	25	25	5	5	5	5	5	5
13	10	10	24	24	5	5	5	5	5	5
14	9	9	17	17	1	1	1	1	1	1
15	10	10	13	13	0.1	0.1	0.1	0.1	0.1	0.1
16	11	11	12	6	0.1	0.1	0.1	0.1	0.1	0.1
17	12	12	11	5	0.1	0.1	0.1	0.1	0.1	0.1
18	13	13	10	4	0.1	0.1	0.1	0.1	0.1	0.1
19	14	14	9	3	0.1	0.1	0.1	0.1	0.1	0.1
20	15	15	8	2	0.1	0.1	0.1	0.1	0.1	0.1
21	16	16	7	1	0.1	0.1	0.1	0.1	0.1	0.1
22	17	17	6	12	0.1	0.1	0.1	0.1	0.1	0.1
23	18	18	5	11	0.1	0.1	0.1	0.1	0.1	0.1
24	19	19	4	10	0.1	0.1	0.1	0.1	0.1	0.1
25	20	20	3	9	0.1	0.1	0.1	0.1	0.1	0.1
26	21	21	2	8	0.1	0.1	0.1	0.1	0.1	0.1
27	22	22	1	7	0.1	0.1	0.1	0.1	0.1	0.1
28	23	23	14	14	0.1	0.1	0.1	0.1	0.1	0.1
29	24	24	15	15	0.1	0.1	0.1	0.1	0.1	0.1
30	25	25	16	16	0.1	0.1	0.1	0.1	0.1	0.1

(b)

Fig. 4 (a) Seats map of the Airbus A320. (b) Costs c^O_{ik} for arcs O and different types of passengers (top-business, business, top-economy, economy), and $c^S_{ii'k}$ for arcs S, that were used in the computational results.

All the executions in this Section were carried out on a DELL PowerEdge 6950 server with 4 AMD Opteron 8222 CPUs at 3.0GHz and 64 GB of RAM, using a Linux operating system. It is worth noting that this hardware, from the year 2007, is capable of approximately 360 Megaflops (millions of floating point operations per second), while current processors (as the one used for the results of below Subsection 4.2) have about 4300 Megaflops. (We remark that, since the overall performance does not depend only on the processor Megaflops, but also on the parallelism capabilities, amount of memory, etc., speedups of 4300/360 $\approx$ 12 are hardly obtained). The results in this section were obtained (by the second author) at an early stage of the work on a computer which is no longer available; the results in below Subsection 4.2 were computed (by the first author) later. Replicating the first set of results in the new computer is not immediate, since some some seats were assigned by the optimization procedure, but others were simulated to be purchased by the passenger using some rules. This explains why two different computers were used. Anyway, we can practically consider that the times provided in this section could be roughly divided by 10 in a modern hardware.

The optimization model was implemented in AMPL, and it was solved with CPLEX 12.5. Since this procedure is to be used online, we set a time limit of 30 seconds for finding the solution to each integer optimization problem (1). When this time limit is exhausted, the current incumbent (best seat assignment found so far) is provided as a solution. An incumbent was always found within this time limit, and in most cases, the optimal solution was found in much less than 30 seconds.

Table 2 shows the results for $\kappa = 1$ and $\kappa = 3$. For $\kappa = 1$ we have only one commodity when solving the optimization model, which corresponds to the current online purchase. For $\kappa = 3$, the three commodities are the current online purchase and the two expected numbers of future business and economy passengers. Each row in Table 2 simulates a particular online purchase. On this flight we have 78 sequential online purchases, while the last row of the table corresponds to the offline assignment of passengers without a seat. For each online purchase, the table shows: the sequential purchase number (column "Num. sale"); the number of available seats when the purchase is completed (column "Available seats"); the number of passengers in this group (column "p_1"); the type of group (column "Group type"); the number of variables in problem (1) (columns "Num. var"); the number of MIP simplex iterations performed (columns "MIP iter."); the number of branch-and-bound nodes (columns "B&B nodes"); the elapsed time in seconds (columns "total time"); the optimality gap achieved (columns "gap%"); the objective function achieved ("Obj. f."); and, for $\kappa = 3$, the contribution of $k = 1$ (the group making the online purchase) to the objective function, so we can compare the solutions of $\kappa = 1$ and $\kappa = 3$ in terms of objective functions. (Note that when $\kappa = 3$ the sum in the objective function (1a) considers two more terms for $k = 2$ and $k = 3$, such that the objective function value is much larger than when $\kappa = 1$. To compare the quality of the seats assignment for the current online purchase when $\kappa = 1$ and $\kappa = 3$ we must focus only on the term for $k = 1$ in the objective function, and this is why this term is provided in Table 2, and the rest of similar tables of the paper.) As we can see, the two possible types of groups in Table 2 are business and economy. Those in italics correspond to passengers that paid for choosing their seats. In these cases, there is no need to solve the optimization problem (1); this situation is marked with † in Table 2. For $\kappa = 1$ and $p_1 = 1$, the solution of (1) is trivial, and CPLEX preprocessing finds the solutions without requiring any optimization, as is clearly marked with ‡ in Table 2. Finally, we note that the last row of Table 2 does not provide the value of the objective function, since it corresponds to the offline assignment.

Table 2: Results for $\kappa = 1$ and $\kappa = 3$

Num. sale	Available seats	p_1	Group type	$\kappa=1$ Num. var.	MIP iter.	B&B nodes	total time	gap%	Obj. f.	$\kappa=3$ Num. var.	MIP iter.	B&B nodes	total time	gap%	Obj. f.	Obj. f. $k=1$
1	180	1	Business			‡			1.1	33576	4013	34	12.85	0.04%	320.9	1.1
2	179	1	*Economy*			†							†			
3	178	1	Business			‡			1.1	32748	17264	730	18.95	0.00%	316.8	1.1
4	177	1	Business			‡			1.1	32387	7977	565	10.07	4.49%	314.7	1.1
5	176	2	*Economy*			†							†			
6	174	1	*Economy*			†							†			
7	173	2	*Economy*			†							†			
8	171	2	*Business*			†							†			
9	169	1	Business			‡			1.1	29571	4929	232	7.13	2.67%	286.3	1.1
10	168	1	Business			‡			1.1	29228	7442	341	7.36	0.04%	284.2	1.1
11	167	3	Business	14360	274	0	1.05	0.00%	4.3	43080	11626	312	19.52	0.90%	276.9	14.3
12	164	1	Business			‡			1.1	27876	1401	0	5.36	0.00%	262.8	2.1
13	163	1	Business			‡			2.1	27543	2981	19	4.19	1.31%	260.5	2.1
14	162	2	Business	13525	250	0	0.75	0.00%	3.2	40575	3567	14	5.14	0.00%	258.4	4.2
15	160	2	Business	13198	128	0	0.85	0.00%	4.2	39594	6027	108	7.06	1.63%	255.2	5.2
16	158	1	*Economy*			†							†			
17	157	2	*Economy*			†							†			
18	155	2	*Economy*			†							†			
19	153	1	*Economy*			†							†			
20	152	2	Business	11930	162	0	0.88	0.00%	4.2	35790	6141	126	7.77	0.29%	241.4	5.2
21	150	1	Business			‡			4.1	23396	2511	23	5.88	0.03%	236.2	2.1
22	149	1	Business			‡			4.1	23091	2235	15	3.60	0.15%	234.6	1.1
23	148	1	Business			‡			4.1	22788	2348	17	3.59	0.15%	234.5	3.1
24	147	1	Business			‡			4.1	22487	2344	13	3.17	3.21%	232.9	2.1
25	146	1	Business			‡			4.1	22188	1261	0	5.20	0.00%	232.8	4.1
26	145	2	*Economy*			†							†			
27	143	1	*Economy*			†							†			
28	142	2	*Economy*			†							†			
29	140	2	Business	10148	202	0	0.50	0.00%	5.2	30444	40291	5021	11.60	1.12%	221.9	13.2
30	138	2	Business	9865	217	0	0.93	0.00%	6.2	29595	3378	56	5.36	2.88%	210.7	6.3

Table 2 Results for $\kappa = 1$ and $\kappa = 3$ (continued)

				$\kappa = 1$						$\kappa = 3$						
Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	Obj. f.	Num. var.	MIP iter.	B&B nodes	total time	gap%	Obj. f.	Obj. f. $k = 1$
31	136	1	Business			‡			4.1	19308	2274	7	3.30	3.03%	206.5	4.1
32	135	1	*Economy*			†							†			
33	134	2	*Economy*			†							†			
34	132	1	*Business*			†							†			
35	131	1	Business			‡			5.1	17943	2348	119	2.7	3.06%	194.6	2.7
36	130	1	*Economy*			†							†			
37	129	1	*Economy*			†							†			
38	128	2	Economy			‡			1.1	25530	2599	4	2.92	2.42%	183.8	3.2
39	126	1	Business			‡			5.1	16628	1785	4	2.15	2.46%	183.1	3.1
40	125	1	Business			‡			5.1	16371	2058	4	2.06	0.02%	183	5.1
41	124	1	Economy			‡			1.1	16116	2029	4	1.65	2.48%	176.9	1.1
42	123	1	Business			‡			5.1	15863	1957	4	2.24	2.53%	178.8	5.1
43	122	1	Economy			‡			1.1	15612	1703	4	2.45	2.54%	172.7	1.1
44	121	1	*Economy*			†							†			
45	120	2	Economy			‡			1.2	22494	3537	159	5.82	0.43%	175.5	4.2
46	118	1	Business			‡			6.1	14628	2136	23	2.49	2.20%	172.8	10
47	117	2	*Economy*			†							†			
48	115	2	*Business*			†							†			
49	113	1	Business			‡			6.1	13443	859	0	1.74	0.00%	151.1	5.1
50	112	1	*Economy*			†							†			
51	111	2	*Business*			†							†			
52	109	4	*Economy*			†							†			
53	105	1	*Economy*			†							†			
54	104	1	Business			‡			6.1	11436	1053	0	1.58	0.10%	127.9	5.1
55	103	1	Economy			‡			2.1	11223	746	0	1.35	0.00%	121.8	1.1
56	102	1	*Business*			†							†			
57	101	1	Economy			‡			2.1	10803	704	0	1.41	0.00%	117.7	2.1
58	100	1	Business			‡			6.1	10596	736	0	1.57	0.00%	118.6	5.1
59	99	1	Business			‡			6.1	10391	742	0	1.17	0.00%	117	4.1
60	98	1	*Business*			†							†			

Table 2 Results for $\kappa = 1$ and $\kappa = 3$ (continued)

				$\kappa = 1$						$\kappa = 3$						
Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	Obj. f.	Num. var.	MIP iter.	B&B nodes	total time	gap%	Obj. f.	Obj. f. $k = 1$
61	97	1	*Economy*			†							†			
62	96	1	*Economy*			†							†			
63	95	1	*Economy*			†							†			
64	94	1	*Economy*			†							†			
65	93	1	*Economy*			†							†			
66	92	2	Economy	4651	146	0	0.36	0.00%	3.2	13380	718	0	1.49	0.00%	98.9	4.2
67	90	1	Economy			‡			2.1	8636	627	0	0.87	0.00%	95.2	1.1
68	89	1	Economy			‡			2.1	8451	463	0	0.81	0.00%	95.1	3.1
69	88	1	*Economy*			†							†			
70	87	2	Business	4273	176	0	0.50	0.00%	9.6	12000	2796	860	1.60	0.82%	94.9	8.2
71	85	1	Business			‡			8	7731	470	0	1.60	0.00%	91.2	5.1
72	84	1	Business			‡			8	7556	556	0	1.24	0.00%	90.6	8
73	83	1	Economy			‡			2.1	7383	344	0	0.90	0.00%	83.6	3.1
74	82	1	*Economy*			†							†			
75	81	2	Economy	3736	151	0	0.45	0.00%	4.2	10443	506	0	0.93	0.00%	79.4	5.2
76	79	1	Business			‡			8	6711	364	0	0.72	0.00%	79.2	6.1
77	78	1	*Economy*			†							†			
78	77	3	*Economy*			†				5842	324	0	0.65	0.00%	68.5	10.3
79	74	28*	Economy	6448	5429	180	4.45	0.00%	3.1	22292	72864	6540	10.33	0.80%	124.4	§

† Customer purchased seat, no seat assignment required
‡ For $\kappa = 1$ and $p_1 = 1$ only preprocessing is needed (no optimization performed)
* Number of passengers in offline assignment
§ Not applicable for last offline assignment

Table 3: Results for $\kappa = 4$ and $\kappa = 5$

			$\kappa = 4$								$\kappa = 5$							
Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	Obj. f.	Obj. f. $k=1$	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	Obj. f.	Obj. f. $k=1$
1	180	1	Business	50124	7253	19	29.09	0.09%	316.90	1.1	Business	66772	10911	0	29.09	0.05%	312.90	1.1
2	179	1	*Economy*				†				*Top-Econ*				†			
3	178	1	Business	49033	34606	735	29.24	0.06%	314.80	1.1	Business	65318	4507	12	29.24	0.14%	336.50	1.1
4	177	1	Business	48492	18271	660	29.18	4.21%	315.90	1.1	Business	64597	10986	44	29.27	9.24%	318.50	1.1
5	176	2	*Economy*				†				*Top-Econ*				†			
6	174	1	*Economy*				†				*Top-Econ*				†			
7	173	2	*Economy*				†				*Economy*				†			
8	171	2	*Business*				†				*Business*				†			
9	169	1	Top-Busn	44789	45527	1870	29.48	10.72%	301.08	1.1	Top-Busn	59662	6946	5	29.48	12.88%	312.60	1.1
10	168	1	Business	44272	6550	102	15.14	0.01%	298.50	1.1	Business	58973	7606	64	29.01	0.08%	281.70	1.1
11	167	3	Business	58120	26192	889	29.24	7.42%	285.60	14.3	Business	72650	26977	462	29.03	6.45%	268.90	14.3
12	164	1	Business	42234	2012	0	10.67	0.00%	260.60	1.1	Business	56257	16035	198	17.62	1.51%	253.60	1.1
13	163	1	Business	41732	3784	15	13.34	0.00%	258.00	1.1	Business	55588	8358	64	17.97	0.18%	252.20	2.1
14	162	2	Business	54760	6306	55	15.04	0.00%	256.90	4.2	Business	68450	10055	231	20.27	0.16%	250.40	4.2
15	160	2	Business	53444	4058	12	6.26	0.03%	252.70	4.2	Business	66805	15363	189	28.89	8.98%	265.60	15
16	158	1	*Economy*				†				*Economy*				†			
17	157	2	*Economy*				†				*Economy*				†			
18	155	2	*Economy*				†				*Top-Econ*				†			
19	153	1	*Economy*				†				*Economy*				†			
20	152	2	Business	48340	6802	93	10.08	3.25%	239.90	5.2	Business	60425	25337	1364	22.92	0.77%	229.00	4.2
21	150	1	Business	35479	3215	13	9.83	0.03%	234.70	2.1	Business	47225	4923	17	12.70	0.04%	227.40	1.1
22	149	1	Business	35019	1910	0	7.43	0.00%	233.60	3.1	Business	46642	9970	355	12.17	4.01%	227.30	3.1
23	148	1	Top-Busn	34562	4311	17	5.25	0.15%	230.50	1.1	Top-Busn	46033	4534	22	10.88	3.44%	221.20	4.5
24	147	1	Business	34108	4213	15	5.93	0.15%	230.40	3.1	Business	45428	3526	16	8.15	1.69%	217.70	3.1
25	146	1	Business	33657	3226	25	8.99	3.30%	228.30	3.1	Business	44827	3661	5	8.21	1.76%	215.60	3.1
26	145	2	*Economy*				†				*Top-Econ*				†			
27	143	1	*Economy*				†				*Top-Econ*				†			
28	142	2	*Economy*				†				*Top-Econ*				†			
29	140	2	Business	31014	9920	319	9.98	3.35%	246.30	6.2	Business	51455	6212	122	9.64	0.74%	205.50	13
30	138	2	Business	40592	10901	321	11.10	0.39%	226.10	13	Business	50030	35522	748	12.42	1.37%	193.50	5.2
31	136	1	Business	29733	4214	234	4.62	2.86%	200.10	2.1	Business	39037	6269	14	6.11	2.27%	189.80	2.1
32	135	1	*Economy*				†				*Top-Econ*				†			
33	134	2	*Economy*				†				*Economy*				†			
34	132	1	*Business*				†				*Business*				†			
35	131	1	Business	27658	3351	150	4.30	2.88%	191.20	4.1	Business	36292	4725	76	4.70	0.44%	179.40	4.1
36	130	1	*Economy*				†				*Economy*				†			
37	129	1	*Economy*				†				*Economy*				†			
38	128	2	Economy	26449	2331	46	3.27	0.09%	181.30	2.1	Economy	43205	9596	389	9.80	2.69%	173.60	3.2
39	126	1	Business	26052	2314	30	3.32	0.41%	181.20	4.1	Business	33647	4933	85	7.14	0.59%	172.40	4.1
40	125	1	Business	25658	2848	92	3.67	0.04%	179.10	4.1	Business	33130	3452	169	5.06	0.02%	170.30	4.1
41	124	1	Economy	25267	2294	19	3.72	0.21%	174.00	1.1	Top-Econ	32617	5957	745	5.59	2.61%	165.20	1.1
42	123	1	Top-Busn	24879	3192	35	3.46	4.98%	172.90	1.1	Top-Busn	32108	5054	294	4.68	0.88%	164.10	1.1
43	122	1	Economy	24494	2285	38	2.77	3.90%	170.80	1.1	Top-Econ	31603	5276	376	6.23	2.88%	162.00	1.1
44	121	1	*Economy*				†				*Economy*				†			
45	120	2	Economy	31480	3366	108	4.28	4.09%	167.10	3.2	Top-Econ	38105	12017	1466	7.56	1.74%	164.80	3.2
46	118	1	Business	22984	2779	25	2.76	5.70%	166.90	5.1	Business	29623	9849	1091	7.24	4.99%	163.20	10.1
47	117	2	*Economy*				†				*Economy*				†			
48	115	2	*Top-Busn*				†				*Top-Busn*				†			

Table 3 Results for $\kappa = 4$ and $\kappa = 5$ (continued)

			$\kappa = 4$								$\kappa = 5$							
Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	Obj. f.	Obj. f. $k=1$	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	Obj. f.	Obj. f. $k=1$
49	113	1	Business	14148	1985	91	2.03	1.34%	155.10	6.1	Business	20809	1552	0	2.10	0.01%	143.40	4.1
50	112	1	*Economy*				†				*Top-Econ*				†			
51	111	2	*Business*				†				*Business*				†			
52	109	4	*Economy*				†				*Top-Econ*				†			
53	105	1	*Economy*				†				*Economy*				†			
54	104	1	Business	12087	1272	0	0	1.64%	127.90	3.1	Business	17424	1256	31	2.07	0.20%	122.70	4.1
55	103	1	Economy	11868	1162	18	1.46	3.16%	123.80	1.1	Top-Econ	17102	1224	21	1.99	2.78%	117.60	1.1
56	102	1	*Business*				†				*Business*				†			
57	101	1	Economy	11436	1102	19	1.44	3.27%	119.70	2.1	Economy	16467	1652	22	2.07	2.89%	113.40	1.1
58	100	1	Business	11223	1202	88	1.41	3.45%	120.60	5.1	Business	16154	1577	11	2.09	2.85%	112.70	3.1
59	99	1	Business	11012	1022	15	1.57	2.95%	118.50	8	Business	15844	1177	13	2.05	2.80%	113.7	5.1
60	98	1	*Business*				†				*Business*				†			
61	97	1	*Economy*				†				*Top-Econ*				†			
62	96	1	*Economy*				†				*Top-Econ*				†			
63	95	1	*Economy*				†				*Economy*				†			
64	94	1	*Economy*				†				*Economy*				†			
65	93	1	*Economy*				†				*Economy*				†			
66	92	2	Economy	9591	547	0	0.92	0.00%	94.00	3.2	Economy	18220	24771	2255	5.06	2.06%	107.00	4.2
67	90	1	Economy	9396	486	0	1.24	0.00%	89.80	1.1	Economy	13189	6111	605	2.99	11.50%	101.80	1.1
68	89	1	Economy	9203	487	0	1.00	0.00%	88.70	2.1	Economy	12909	5171	521	2.33	10.36%	100.70	2.1
69	88	1	*Economy*				†				*Top-Econ*				†			
70	87	2	Business	13098	519	0	1.38	0.23%	88.50	8.2	Business	16360	5375	437	2.56	12.23%	99.50	7.2
71	85	1	Business	8451	396	0	0.69	0.00%	83.80	4.1	Business	11819	4802	468	2.32	12.69%	95.80	4.1
72	84	1	Business	8268	381	0	0.92	0.00%	83.70	5.1	Business	11554	8079	879	3.02	11.67%	95.70	5.1
73	83	1	Economy	8087	361	0	0.93	0.00%	81.10	3.1	Economy	11292	5707	588	2.56	14.50%	90.60	2.1
74	82	1	*Economy*				†				*Top-Econ*				†			
75	81	2	Economy	11469	531	0	1.28	0.00%	76.90	5.2	Economy	14260	11780	1195	3.88	16.25%	87.40	5.2
76	79	1	Business	7556	336	0	0.69	0.11%	75.70	5.1	Business	10274	5885	563	2.25	15.16%	86.20	5.1
77	78	1	*Economy*				†				*Top-Econ*				†			
78	77	3	Business	5842	545	0	0.78	0.00%	73.70	14	Business	7860	5749	645	2.43	16.46%	88.50	10.3
79	74	28*	Economy	25177	35410	3537	10.92	2.38%	97.90	‡	Economy	22855	12358	1403	6.30	2.89%	77.90	§

† Customer purchased seat, no seat assignment required
* Number of passengers in offline assignment
§ Not applicable for last offline assignment

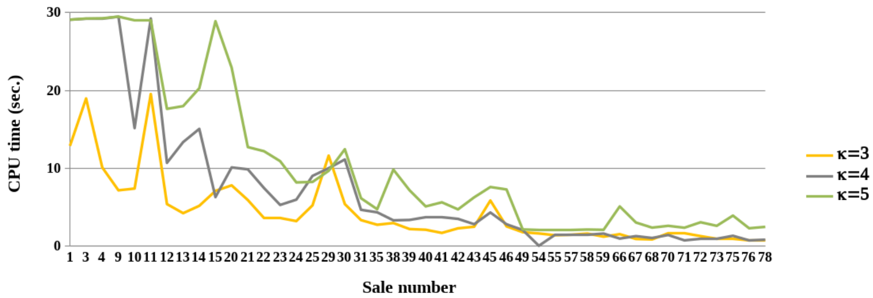

Fig. 5 Solution time of the optimization problem for each sale and $\kappa > 1$.

Table 3 shows the same information as Table 2, but for models with $\kappa = 4$ and $\kappa = 5$ which include "top-business" (both for $\kappa = 4$ and $\kappa = 5$) and a "top-economy" (only for $\kappa = 5$) customers. The meaning of the rows and columns is the same as in Table 2, and customers that paid for seat selection are also marked in italics. Note that the distribution of the type of groups is consistent with what was stated in previous Section 3: the numbers of "economy", "business", "top-economy" and "top-business" groups are, respectively, 21, 35, 19, 4. That is, "top-economy" (19) is approximately a 50% of all economy (19+21) groups; and "top-business" (4) is roughly a 10% of all business (35+4) groups.

From Tables 2 and 3 we can extract several conclusions. First, looking at the total time, the model with $\kappa = 1$ is much easier to solve than with $\kappa > 1$: all problems with $\kappa = 1$ were solved in less than one second. For $\kappa = 3$, optimal solutions were computed within the 30-second time limit, while this time limit was reached in the first sales with $\kappa = 4$ and $\kappa = 5$. This is clearly seen in Figure 5 which provides the solution time of the optimization problem for every sale and $\kappa > 1$. It is also observed that the optimization problems associated with the first sales are more time-consuming, since the number of available seats (thus the number of variables, number of constraints, and number of feasible solutions) is larger. On the other hand, optimization problems for the last sales are quickly solved for any κ. If we added more sales (beyond the 79 considered in the tables) to fill the aircraft, the extra optimization problems would be solved very quickly. For instance, we added six more sales to fill the plane, and the total solution time for all new six optimization problems was 3.5 seconds. The parameters that can significantly affect to the solution time are $p_k, k > 1$, that is, the future expected demand for each group ("economy", "business", "top-economy", "top-business"); these parameters appear in constraints (1b) and (1i). If a highly (fully) booked flight is expected then $\sum_{k>1} p_k$ would be close (equal) to 180, and the optimization problems should compute seat assignments for large groups (which in principle is more time consuming). In all the computational results of the paper the values considered for $p_k, k > 1$, were $p_2 = 80$, $p_3 = 44$, $p_4 = 9$ and $p_5 = 44$, such that $\sum_{k>1} p_k = 177$, so the flight is (practically) fully booked, which is the worst case for the optimization procedure.

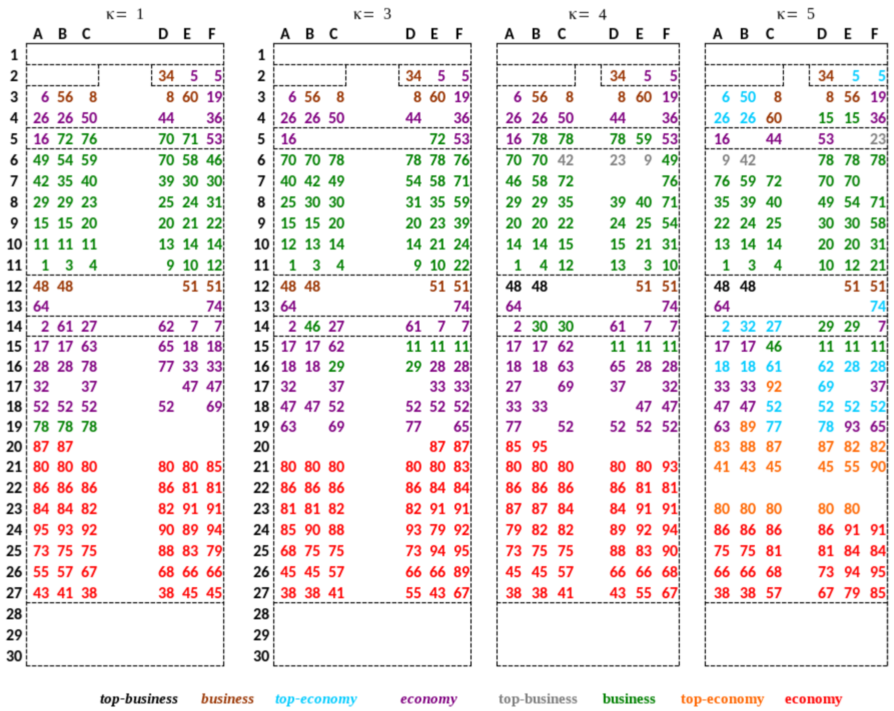

Fig. 6 Seat assignments obtained for $\kappa \in \{1,3,4,5\}$. Seat numbers correspond to sales and different types of groups are marked with different colors. Italic names in the legend refer to groups that paid for seat selection.

Since the results of Tables 2 and 3 were obtained using a (quite) old hardware, it would be worth knowing whether optimal solutions (or which gaps) could be found with a recent and faster computer (as the one used in next Subsection 4.2). This question can be answered by looking at the first rows of Tables 3 (for $\kappa = 5$) and 4–5: since no seat has been still purchased by any group of passengers, the rows correspond to the same optimization problem, which is one of the largest and most challenging of the paper. It can be seen that with the first old computer the time limit of 30 seconds is exhausted with a solution of gap 0.05%. On the other hand, with the faster computer an optimal solution of 0% gap is found in 9.58 seconds, and the first incumbent of gap 43.1% was obtained in 3.23 seconds. Then we can conclude that with a recent computer an optimal solution of 0% gap would be obtained for all the optimization problems of Tables 2 and 3 within the 30 seconds time limit.

The model with $\kappa = 1$ is similar to the current simple rules used by some airlines, so it is not surprising that it can be easily solved. However, rather than comparing the solution time, it is far more interesting to compare the quality of the seat assignments for the different κ, which is shown in Figure 6. The seat numbers in Figure 6 correspond to the sale number (i.e., passengers of n-th sale were assigned to seats numbered n). Different colors represent different types of groups: business, top-business, economy, top-economy. For each group, we have two possibilities, depending on whether they paid for seat selection (the name of the groups that paid are marked in italics in the legend). From Figure 6, we see that $\kappa = 1$ provided some bad assignments: for instance, (1) the two business passengers of sale 70 were assigned to different rows (5 and 6); and (2) the last business sale (number 78) was assigned to row 19. In addition, $\kappa = 1$ filled the first rows earlier than $\kappa > 1$, thus avoiding future sales able to pay for these seats. On the other hand, for $\kappa = 3$, $\kappa = 4$ and $\kappa = 5$ the model left some seats unassigned in, respectively, rows 5, 6 and 7, as it expected last-minute business sales, which is a good policy. However, for $\kappa = 5$ the model assigned business sale number 15 to the top first rows early.

4.2 Results for instances generated as variations of the realistic flight

Taking the realistic flight of previous section as a base case, alternative instances can be generated by changing some of its parameters. In particular, we generated two more flights, both for $\kappa = 5$, which is the most difficult situation from a computational point of view, and both considering all 79 sales of the realistic case. The first generated flight only differs from the base case in that none of the 79 sales purchased a seat, that is, 79 optimization problems had to be solved, one per sale, which is the worst case for our optimization approach. In the second generated flight, in addition to considering that no sale purchased a seat, we manually increased p_1, the number of passengers in the group (thus making the optimization problems harder). The differences in p_1 for the two generated flights can be seen in the respective columns p_1 of Tables 4 and 5. Although the assumptions for these two generated flights are not realistic, its main purpose is to validate the efficiency of the approach in a difficult scenario.

The results for each flight are reported, respectively, in Tables 4 and 5, whose columns have the same meaning than in previous tables. These runs were executed on a recent Fujitsu Primergy RX2530 M4 server with two 2.3 GHz Intel Xeon Gold 6140 CPUs (48 cores) and 500 Gigabytes of RAM, under a GNU/Linux operating system (opensuse 15.0), considerably faster than the hardware used for the base instance. As in previous runs, AMPL and CPLEX 12.5 were used. The time limit was set to 30 seconds and the optimality tolerance to 0.1%. We remark that, with this faster hardware, a solution with a gap less than or equal to the optimality tolerance was obtained for all the problems within the time limit. In addition, Tables 4 and 5 include two more columns than previous tables: column “1st inc time” shows the time needed to compute the first incumbent, while “1st inc gap%” provides the gap% of this incumbent. It is observed that, although a first solution is in general quickly obtained for this model, it is worth waiting some few seconds to get a much better seat assignment. At the end of the tables some summary statistics are provided: number of problems solved (i.e., solution gap is less than or equal to the optimality tolerance), average solution time (in seconds), average gap%, average solution time to first incumbent, and average gap% of first incumbent. From these numbers, it can be concluded that, in general, this model is efficient enough to be run online even on a difficult scenario where no passenger purchased a seat (i.e., the model is in charge of the whole seat assignment).

Table 4: Results for 1st generated file with $\kappa = 5$

Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	1st inc time	1st inc gap%	Obj. f.	Obj. f. $k=1$
1	180	1	Business	66772	19	0	9.58	0.0	3.23	43.1	365.45	1.10
2	179	1	Top-Econ	66043	20	0	5.96	0.0	2.99	40.0	364.35	1.10
3	178	1	Business	65318	19	0	6.14	0.0	1.45	40.6	363.25	1.10
4	177	1	Business	64597	19	0	6.38	0.0	3.31	41.8	362.15	1.10
5	176	2	Top-Econ	79630	23	0	6.30	0.0	3.43	41.8	361.05	2.20
6	174	1	Top-Econ	62458	18	0	5.81	0.0	1.23	42.1	358.85	1.10
7	173	2	Economy	76975	87	0	5.79	0.0	3.70	43.9	357.75	2.20
8	171	2	Business	75230	204	0	5.79	0.0	3.60	40.4	356.55	3.20
9	169	1	Top-Busn	2742	49452	390	9.78	0.1	2.88	37.8	353.60	1.10
10	168	1	Business	58288	21	0	8.22	0.1	1.31	36.6	353.50	1.10
11	167	3	Business	4468	841436	5298	25.47	0.1	3.50	43.9	354.15	4.80
12	164	1	Business	2119	3571	0	10.36	0.1	2.64	42.6	350.10	1.10
13	163	1	Business	54923	379	0	6.96	0.1	2.76	41.3	350.00	1.10
14	162	2	Business	67625	1851	0	9.47	0.0	1.51	42.5	350.90	4.20
15	160	2	Business	12299	1036449	8113	24.32	0.1	3.16	43.3	349.20	3.70
16	158	1	Economy	4025	85968	655	11.70	0.1	1.12	41.2	345.50	1.10
17	157	2	Economy	11703	129239	1182	15.26	0.1	3.13	40.0	345.40	3.20
18	155	2	Top-Econ	3815	4481	3	15.17	0.1	3.08	46.3	343.20	3.20
19	153	1	Economy	48493	986	0	7.23	0.1	1.01	42.0	341.00	2.10
20	152	2	Business	59650	6354	0	12.91	0.1	1.25	44.9	341.40	4.70
21	150	1	Business	46642	370	0	5.56	0.0	0.85	39.6	338.45	2.10
22	149	1	Business	12169	2866	0	6.64	0.0	0.93	42.8	338.35	2.10
23	148	1	Top-Busn	1875	3077	0	7.32	0.1	1.97	41.7	336.75	1.10
24	147	1	Business	44827	3777	0	3.41	0.0	0.85	41.8	337.65	3.10
25	146	1	Business	44230	348	0	3.01	0.0	1.95	42.8	337.30	3.10
26	145	2	Top-Econ	54365	5823	0	7.03	0.0	2.18	40.8	335.70	3.20
27	143	1	Top-Econ	42463	343	0	2.32	0.0	0.74	42.3	333.50	1.10
28	142	2	Top-Econ	18879	60037	580	11.15	0.1	1.91	41.7	334.15	2.70
29	140	2	Business	1813	6977	3	9.15	0.1	1.81	44.8	334.45	4.70
30	138	2	Business	7219	24496	335	7.29	0.1	0.86	42.1	332.75	5.20
31	136	1	Business	38480	4541	0	2.84	0.0	1.52	40.6	330.55	3.10
32	135	1	Top-Econ	37927	20	0	2.68	0.0	0.62	37.3	329.45	2.10
33	134	2	Economy	5854	11746	148	6.05	0.1	0.82	42.7	329.35	4.20
34	132	1	Business	36292	4914	0	5.01	0.0	1.30	36.5	328.15	3.10
35	131	1	Business	35755	4792	0	5.61	0.0	1.51	40.4	328.05	4.10
36	130	1	Economy	35222	3592	0	2.11	0.0	1.23	36.5	324.95	1.10
37	129	1	Economy	34693	3344	0	1.52	0.0	0.52	38.0	324.85	1.10
38	128	2	Economy	42550	6389	0	6.58	0.0	1.52	40.9	326.25	3.70
39	126	1	Business	19949	93901	1984	7.90	0.1	0.53	40.6	325.80	4.10
40	125	1	Business	6592	198530	4286	7.14	0.1	0.55	39.5	324.95	4.10
41	124	1	Top-Econ	3129	243023	3541	4.75	0.1	0.52	41.8	323.85	4.10
42	123	1	Top-Busn	1522	4944	3	2.98	0.0	0.48	42.1	320.00	1.10
43	122	1	Top-Econ	3017	53697	947	3.86	0.1	0.46	41.7	321.90	4.10
44	121	1	Economy	1480	5164	5	3.26	0.0	0.98	40.2	319.55	1.10
45	120	2	Top-Econ	19131	122817	2520	8.41	0.1	1.28	43.9	322.45	6.20
46	118	1	Business	29138	3092	0	2.91	0.0	1.04	39.1	320.25	5.10
47	117	2	Economy	19287	415725	8703	11.16	0.1	0.64	41.7	318.15	3.70
48	115	2	Top-Busn	20749	265704	5301	11.15	0.1	0.56	41.9	316.20	3.60
49	113	1	Business	2464	16606	180	4.50	0.1	2.00	17.3	316.60	4.10
50	112	1	Top-Econ	17992	295200	8459	8.65	0.1	0.53	45.1	316.50	4.10
51	111	2	Business	2331	52256	736	5.73	0.1	0.52	39.3	316.40	6.20

Table 4 Results for 1st generated file with $\kappa = 5$ (continued)

Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	1st inc time	1st inc gap%	Obj. f.	Obj. f. $k=1$
52	109	4	Top-Econ	8884	42408	566	4.30	0.1	0.48	42.1	314.70	8.90
53	105	1	Economy	23197	2474	0	1.25	0.0	1.12	0.0	308.55	2.10
54	104	1	Business	22768	3153	0	1.92	0.0	0.74	40.2	310.45	5.10
55	103	1	Top-Econ	22343	2206	0	1.82	0.0	1.01	17.4	309.35	4.10
56	102	1	Business	21922	2717	0	1.73	0.0	0.48	41.8	309.25	5.10
57	101	1	Economy	21505	2421	0	1.43	0.0	0.37	42.1	307.15	3.10
58	100	1	Business	3568	3736	0	2.09	0.1	0.71	17.9	309.05	5.10
59	99	1	Business	588	3582	0	2.19	0.0	1.05	38.2	309.70	8.00
60	98	1	Business	20278	3085	0	2.21	0.0	0.28	39.0	307.70	6.10
61	97	1	Top-Econ	8941	10543	250	4.05	0.1	0.31	43.0	306.60	5.10
62	96	1	Top-Econ	796	2937	0	1.71	0.1	0.85	17.0	306.50	5.10
63	95	1	Economy	956	3035	0	2.48	0.1	0.62	16.7	304.40	4.10
64	94	1	Economy	903	3501	1	2.46	0.1	0.90	36.5	303.30	4.10
65	93	1	Economy	4411	8079	202	2.37	0.0	0.32	40.3	302.20	4.10
66	92	2	Economy	859	3587	0	2.63	0.0	0.31	43.4	301.60	5.70
67	90	1	Economy	17182	2177	0	1.12	0.1	0.64	21.0	299.65	3.10
68	89	1	Economy	16813	2205	0	1.80	0.1	0.62	9.8	300.55	5.10
69	88	1	Top-Econ	16448	2167	0	1.22	0.1	0.88	39.6	300.45	6.10
70	87	2	Business	20000	2616	0	2.53	0.0	0.92	7.4	300.85	10.00
71	85	1	Business	316	2364	0	2.28	0.0	0.27	40.2	296.85	6.10
72	84	1	Business	468	3378	0	1.77	0.0	0.19	42.3	297.75	13.00
73	83	1	Economy	433	1938	0	0.86	0.0	0.78	0.0	288.75	4.10
74	82	1	Top-Econ	264	1571	0	1.00	0.0	0.90	3.6	286.15	6.10
75	81	2	Economy	17405	2077	0	1.28	0.0	0.89	7.7	284.05	6.20
76	79	1	Business	429	2060	0	1.20	0.0	0.40	39.0	284.85	8.00
77	78	1	Top-Econ	6294	3146	7	2.28	0.0	0.55	20.5	289.85	14.10
78	77	3	Business	1705	6325	207	1.15	0.1	0.68	1.8	282.75	25.00
79	74	28	Economy	3127	236	0	0.07	0.0	0.04	65.7	100.10	§

Number problems solved within opt. gap:	69
Average solution time (sec.):	5.56
Average gap%:	0.04
Average time to first incumbent (sec.):	1.28
Average gap% of first incumbent:	36.17

§ Not applicable for last offline assignment

Table 5: Results for 2nd generated file with $\kappa = 5$

Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	1st inc time	1st inc gap%	Obj. f.	Obj. f. $k=1$
1	180	1	Business	66772	19	0	9.62	0.0	3.23	43.1	365.45	1.10
2	179	2	Top-Econ	82330	21	0	10.21	0.0	4.08	40.6	364.35	2.20
3	177	1	Business	64597	19	0	6.25	0.0	3.11	39.6	362.15	1.10
4	176	1	Business	63880	19	0	7.09	0.0	3.32	37.7	361.05	1.10
5	175	3	Top-Econ	12137	470253	2117	20.12	0.1	1.55	43.3	360.45	3.80
6	172	1	Top-Econ	61052	263	0	4.66	0.0	3.00	39.7	357.65	1.10
7	171	3	Economy	8637	1513	6	10.49	0.0	3.46	42.5	356.80	3.30
8	168	3	Business	11590	11158	67	15.87	0.1	3.37	46.2	354.50	4.30
9	165	1	Top-Busn	4562	44	0	9.41	0.1	2.82	41.9	350.45	1.10
10	164	2	Business	9823	317962	1059	18.41	0.1	3.16	44.4	351.10	2.20
11	162	3	Business	4730	816	0	13.12	0.1	2.91	46.6	350.90	5.30
12	159	1	Business	52303	20	0	6.47	0.0	2.36	42.0	346.60	1.10
13	158	1	Business	1899	48492	307	8.20	0.0	2.70	39.1	347.25	2.10
14	157	2	Business	63575	1159	0	8.09	0.0	3.02	43.1	347.15	4.20
15	155	2	Business	4144	160198	1300	15.56	0.0	3.16	41.6	345.45	4.70

Table 5 Results for 2nd generated file with $\kappa = 5$ (continued)

Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	1st inc time	1st inc gap%	Obj. f.	Obj. f. $k = 1$
16	153	1	Economy	48493	23	0	5.12	0.0	2.28	43.5	341.75	1.10
17	152	2	Economy	59650	862	0	7.44	0.0	1.24	44.1	341.65	3.20
18	150	2	Top-Econ	3638	7149	40	11.45	0.0	1.22	46.1	339.45	3.20
19	148	1	Economy	45428	185	0	5.76	0.0	0.85	39.6	337.25	1.10
20	147	2	Business	25525	3238	0	11.26	0.0	1.09	45.3	338.65	4.70
21	145	1	Business	43637	4449	0	3.23	0.0	1.77	38.4	335.95	2.10
22	144	2	Business	30379	6311	4	11.18	0.1	1.04	44.7	337.10	5.20
23	142	1	Top-Busn	3948	32710	306	6.01	0.1	0.76	40.8	332.15	1.10
24	141	2	Business	8395	33525	228	12.06	0.1	2.09	45.5	334.55	4.70
25	139	1	Business	40163	3759	0	3.12	0.1	0.90	40.7	331.85	2.10
26	138	2	Top-Econ	1898	4106	10	8.78	0.1	1.11	42.6	332.25	3.20
27	136	1	Top-Econ	3143	27387	501	6.77	0.1	0.68	44.6	329.30	1.10
28	135	2	Top-Econ	10017	110962	2147	8.20	0.1	0.90	45.3	331.20	5.20
29	133	2	Business	5783	7525	94	4.45	0.1	0.90	42.4	328.50	4.70
30	131	2	Business	3603	41332	596	3.58	0.1	0.77	46.5	327.80	6.20
31	129	1	Business	34693	3252	0	2.27	0.1	2.30	0.8	324.35	3.10
32	128	1	Top-Econ	34168	3515	0	2.53	0.1	0.61	42.0	323.25	2.10
33	127	3	Economy	41900	4656	0	5.05	0.1	1.64	44.7	322.65	4.80
34	124	1	Business	2803	4446	0	2.19	0.1	0.51	41.2	320.85	3.10
35	123	2	Business	3356	172163	2471	6.24	0.1	1.30	45.4	322.25	6.70
36	121	1	Economy	30605	3162	0	2.94	0.0	2.20	4.0	317.30	1.10
37	120	1	Economy	8510	5820	0	5.32	0.0	0.49	43.2	318.20	2.10
38	119	3	Economy	1489	10775	89	5.84	0.1	0.56	45.4	318.85	5.80
39	116	1	Business	6064	103131	1551	5.29	0.1	0.64	42.6	317.05	4.10
40	115	1	Business	5557	150311	3124	4.03	0.1	1.46	3.9	316.95	5.10
41	114	4	Top-Econ	2466	10057	57	4.64	0.1	0.60	45.2	315.85	8.40
42	110	1	Top-Busn	1021	8180	50	2.76	0.0	0.53	39.8	307.95	1.10
43	109	1	Top-Econ	2280	7570	94	3.67	0.1	1.75	39.9	310.85	4.10
44	108	2	Economy	6732	111356	1684	4.91	0.1	0.48	41.9	309.25	4.70
45	106	2	Top-Econ	29405	314786	5158	11.36	0.0	0.40	42.8	310.55	5.70
46	104	1	Business	1406	8348	182	2.64	0.1	0.39	42.2	308.85	4.10
47	103	2	Economy	8599	412761	8608	6.74	0.1	0.28	42.5	307.75	3.70
48	101	2	Top-Busn	1801	29868	1117	3.58	0.1	1.33	38.9	306.05	5.00
49	99	1	Business	7672	367620	7616	6.97	0.1	0.32	37.3	305.30	5.10
50	98	1	Top-Econ	2129	98040	2930	2.31	0.1	1.08	4.5	305.20	5.10
51	97	2	Business	6271	203690	4979	5.85	0.0	0.74	39.0	306.35	6.70
52	95	4	Top-Econ	17441	54549	1381	7.42	0.0	1.22	17.5	302.65	9.40
53	91	1	Economy	440	2374	0	2.27	0.1	0.32	41.1	296.00	2.10
54	90	1	Business	1987	4318	49	2.47	0.1	0.21	40.0	300.15	8.00
55	89	1	Top-Econ	6292	4413	32	1.56	0.1	0.44	10.8	305.15	14.10
56	88	2	Business	1420	7038	164	2.98	0.1	0.33	42.1	297.30	12.00
57	86	1	Economy	397	1860	0	1.45	0.0	0.85	39.8	288.30	3.10
58	85	2	Business	517	3261	0	1.18	0.0	0.74	11.9	292.20	14.00
59	83	1	Business	1314	12231	802	1.19	0.1	0.60	4.1	285.20	6.10
60	82	2	Business	1976	5303	47	2.73	0.1	1.12	41.3	285.60	19.00
61	80	1	Top-Econ	6217	4005	31	1.99	0.0	0.54	12.9	279.60	14.10
62	79	1	Top-Econ	4869	3560	13	2.09	0.0	0.38	10.2	278.50	14.10
63	78	2	Economy	1189	2913	5	1.77	0.0	1.13	5.4	266.40	5.20
64	76	1	Economy	419	1542	0	0.55	0.0	0.49	0.6	265.20	5.10
65	75	1	Economy	12067	1311	0	0.88	0.0	0.54	19.3	264.10	5.10
66	74	2	Economy	209	1690	0	0.73	0.0	0.20	42.6	263.00	6.20
67	72	1	Economy	320	1315	0	0.78	0.0	0.57	14.0	261.55	4.10
68	71	1	Economy	333	1375	0	0.59	0.0	0.48	3.0	262.45	6.10
69	70	1	Top-Econ	3661	5985	251	1.91	0.0	0.28	11.1	267.10	6.10

	A	B	C	D	E	F
1						
2						
3	72	78	78	78		
4						
5	48	59		70	70	76
6	48	60	42	9	71	23
7	46	51	51	56	58	54
8	40	39	30	30	35	49
9	24	25	11	11	15	34
10	21	8	8	11	15	22
11	10	4	3	12	13	1
12						
13						
14						
15	79	79	79	79	79	79
16				79	79	79
17	68	75	75		66	79
18	63	64	65	73	66	79
19	47	33	33	38	57	67
20	47	17	17	38	19	53
21	36	7	7	37	16	44
22	74	69	79	52	52	52
23	45	45	61	62		52
24	50	20	41	43	55	29
25	31	20	14	14		29
26	32	18	18	26	26	28
27	27	5	5	2	6	28
28	79	79	79	79	79	77
29	79	79	79	79	79	79
30		79	79	79	79	79

Business Economy Top-Busn Top-Econ

	A	B	C	D	E	F
1						
2						
3	70	70		72	60	60
4	58	58		76		56
5	48	48	71	54		56
6	51	23	35	9	59	42
7	51	49	35	40	30	30
8	20	46	39	22	22	24
9	20	34	11	11	11	24
10	13	25	8	8	8	21
11	10	10	4	12	3	1
12						
13						
14		78	78	78		
15	79	79	79	79	79	79
16	68	75	75	79	79	79
17	64	65	66	66	79	
18	63	63	38	38	44	67
19	47	33	33	38	44	57
20	47	37	33	17	17	53
21	16	19	7	7	7	36
22	52	52	52	52		69
23	50	41	41	41	41	45
24	43	28	28	15	29	45
25	31	14	14	15	29	
26	32	18	18	5	26	26
27	2	2	6	5	5	27
28	74	74	73	62	61	55
29			79	79	79	77
30	79	79	79	79	79	79

Business Economy Top-Busn Top-Econ

Fig. 7 Seat assignments obtained for flights of Tables 4 (left picture) and 5 (right picture) . Seat numbers correspond to sales and different types of groups are marked with different colors.

Table 5 Results for 2nd generated file with $\kappa = 5$ (continued)

Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	1st inc time	1st inc gap%	Obj. f.	Obj. f. $k=1$
70	69	2	Business	5696	5373	60	1.82	0.1	0.33	39.8	268.00	19.00
71	67	1	Business	5333	5516	90	2.50	0.1	0.14	43.7	256.00	8.00
72	66	1	Business	1715	5541	153	1.20	0.1	0.26	11.7	255.75	13.00
73	65	1	Economy	6677	22105	776	2.31	0.1	0.12	43.7	251.00	14.10
74	64	2	Top-Econ	4965	10676	257	2.67	0.1	0.28	27.1	250.15	15.20
75	62	2	Economy	858	2095	3	0.82	0.1	0.44	21.3	239.95	7.20
76	60	1	Business	1064	2558	52	0.94	0.0	0.22	12.4	240.25	10.50
77	59	1	Top-Econ	5034	12243	979	1.32	0.1	0.40	20.2	243.75	15.10
78	58	3	Business	1560	4117	112	1.13	0.1	0.65	3.5	236.65	14.00
79	55	19	Economy	1799	104	0	0.06	0.0	0.02	68.8	88.70	§

Number problems solved within opt. gap:	67
Average solution time (sec.):	5.30
Average gap%:	0.05
Average time to first incumbent (sec.):	1.21
Average gap% of first incumbent:	33.78

§ Not applicable for last offline assignment

Figure 7 shows the resulting seat assignments for the flights of Tables 4 and 5. Some of the “best” rows (i.e., first two rows, and rows with extra space for legs) are empty because the weights used in the objective function give priority to future

	A	B	C	D	E	F
1						
2						
3		78	78	76	70	70
4			78			
5	48		58	59	60	
6	48	71	9	23	42	72
7	40	49	51	51	54	56
8	29	29	30	30	39	46
9	20	11	11	11	35	34
10	20	8	8	10	24	25
11	13	3	21	12	4	1
12						
13						
14	79					
15	79	79	79	79	79	79
16	75	75	79	79	79	79
17	66	66	73	79	79	79
18	64	47	47	65		79
19	63	37	38	38	68	67
20	44	33	17	17	53	57
21	19	33	36	16	7	7
22	62	52	52	74		79
23	61	45	52	52	55	79
24	50	45	43	41	15	69
25	31	22	14	14	15	
26	32	18	18	28	28	26
27	27	6	2	5	5	26
28	79	79	79	79	79	77
29	79	79	79	79	79	79
30						

Business Economy Top-Busn Top-Econ

	A	B	C	D	E	F
1						
2						
3	71	78	78	78		72
4				76		
5	56	48		60	70	70
6	42	48	9	58	59	23
7	40	49	51	51	54	46
8	29	29	35	39	30	30
9	15	24	25	11	11	20
10	15	21	8	8	11	20
11	4	13	10	12	1	3
12						
13						
14						
15	79	79	79	79	79	79
16	79	79	79	79	79	79
17	75	75		73	68	79
18	65	66	66	64	63	79
19	47	47	38	38	57	67
20	44	33	37	53	17	17
21	16	33	36	19	7	7
22	74	69	79	79	79	
23	61	52	52	52	52	62
24	41	50	43	45	45	55
25	31	22	14	14	34	
26	26	26	18	18	28	28
27	27	5	5	2	6	32
28	79	79	79	79	79	77
29	79	79	79	79	79	79
30						

Business Economy Top-Busn Top-Econ

Fig. 8 Sensitivity analysis: seat assignments obtained for the flight of Table 4 using two sets of small variations of the weights w_k^O, w_k^D and costs $c_{j'ik}^D$. These seat assignments should be compared with those of left picture of Figure 7.

(e.g., last minute) purchases. It is also observed that passengers of the same group are sometimes in different rows; this may correspond to either an optimal solution due to the current seats availability, or to a suboptimal one given the optimality tolerances considered. In a realistic situation some of the empty and assigned seats would have been previously purchased by some passengers, such that the final seat assignment could have been considerably different (and likely easier to fill).

We performed an empirical sensitivity analysis by re-running the optimization problems for the flight of Table 4 using two sets of small variations of the weights w_k^O and w_k^D, and the costs $c_{j'ik}^D$. The results are shown in Figure 8. For the optimizations of left picture of Figure 8 we considered $w_k^O = 1.4$ for $k > 1$ (instead of the default value 1.5), $w_k^D = 0.6$ for $k = 1$ (instead of 0.5) , and $c_{j'ik}^D = 1n_H + 1.4n_V$ (instead of $1n_H + 1.5n_V$); that is, we give (slightly) more priority to the current sale, and less importance to seating passengers of the same group in the same row. For the runs of the right picture of Figure 8 we used $w_k^O = 1.6$ for $k > 1$, $w_k^D = 0.4$ for $k = 1$, and $c_{j'ik}^D = 1n_H + 1.6n_V$, thus giving less priority to the current sale, but more importance to seating passengers of the same group in the same row. Comparing these seat assignments with those obtained in the left picture of Figure 7 with the default weights and costs, we observe slight variations. For instance, the three members of the "business" group 78 are seated in two rows in the left picture of Figure 8 (likely because it considers $1.4n_V$ instead of $1.5n_V$ in the definition of the costs), while they are seated in the same row in the other two situations. However, a similar distribution of the colors representing the different groups is observed in all the solutions, so it can be concluded that small variations may slightly affect individually to some passenger(s), but not significantly change the general map of seats. It is also worth remarking that the solutions times for the flights of Figure 8 (not provided here) were similar to those reported in Table 4 with the default weights.

4.3 Results for the stochastic optimization model

For the solution of Model (2), the probability distribution of p_k, $k > 1$, is needed, instead of just an estimation of their expected values. Since such a distribution was not provided for this work, we generated the set of scenarios as follows. Five different scenarios were considered, for "very low", "low", "medium", "high" and "very high" demands, with probabilities 0.05, 0.15, 0.5, 0.2 and 0.1, respectively. The values of p_k, $k > 1$, for the "very high demand" scenario were those used for the results of Tables 3–4 (indeed the values of p_k used in previous tables forecasted an almost fully booked flight, which is a worst-case situation for the optimization problem). The values of p_k for the other scenarios were obtained by reducing the number of future passenger for each group.

The resulting stochastic optimization problem had about five times the number of variables and constraints of the expected value deterministic model. For this reason the time limit was increased to 300 seconds, maintaining the optimality tolerance of 0.1%. We used the same hardware and same version of AMPL and CPLEX as for the instances of Tables 4 and 5. The results obtained with the stochastic model are reported in Table 6. Each row provides the information for each group sale, and the meaning of the columns is the same as in previous tables. The average solution time for all the 79 sales was 187.6 seconds, and in many cases the time limit of 300 seconds was reached, and in two of them with a very large (> 46%) gap; the average gap of the solutions was 1.93%. It can be concluded that this stochastic model, at least using a generic state-of-the-art 0-1 solver such as CPLEX, is not practical for an online system. In addition, the resulting seat assignments—shown in Figure 9—are not better than the ones computed by the deterministic model (although they are not totally comparable, since the deterministic model is not the expected value solution of the stochastic one).

Table 6: Results for the stochastic optimization model

Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	1st inc time	1st inc gap%	Obj. f.	Obj. f. $k = 1$
1	180	1	Business	333140	326794	663	299.40	6.0	30.17	53.0	287.82	1.10
2	179	1	Top-Econ	329499	398254	1476	300.78	3.3	27.91	53.5	278.85	1.10
3	178	1	Business	325878	535324	1635	300.99	2.4	28.01	53.6	275.20	1.10
4	177	1	Business	136084	753840	5710	299.03	0.1	29.01	54.5	268.10	1.10
5	176	2	Top-Econ	334446	295755	609	298.91	4.2	34.72	53.0	277.99	2.20

Table 6 Results for the stochastic optimization model (continued)

Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	1st inc time	1st inc gap%	Obj. f.	Obj. f. $k=1$
6	174	1	Top-Econ	311594	449514	1968	302.15	1.4	27.29	53.2	267.84	1.10
7	173	2	Economy	323295	123535	133	299.94	2.7	39.98	55.2	270.48	3.20
8	171	2	Business	315966	85251	66	298.57	43.2	29.98	52.6	461.07	6.20
9	169	1	Top-Busn	128500	703288	5310	182.75	0.1	22.83	53.8	259.74	1.10
10	168	1	Business	290768	1012247	5636	296.09	0.1	16.55	52.7	259.25	1.10
11	167	3	Business	301560	325799	2219	301.96	1.6	25.41	55.2	263.11	4.80
12	164	1	Business	277284	864199	5137	299.12	0.3	17.35	56.0	256.14	1.10
13	163	1	Business	91929	2414020	18900	281.41	0.1	21.04	54.1	255.69	1.10
14	162	2	Business	284025	92230	35	298.94	47.6	19.04	54.8	487.86	16.20
15	160	2	Business	181341	665730	5007	298.85	0.1	20.88	55.5	254.23	3.70
16	158	1	Economy	257658	1012081	7462	251.43	0.1	13.48	55.1	250.45	1.10
17	157	2	Economy	267015	894782	5316	298.88	0.2	16.44	56.0	250.85	3.20
18	155	2	Top-Econ	260358	540267	2462	301.40	0.4	20.04	54.3	248.86	3.20
19	153	1	Economy	87538	475088	4727	138.54	0.1	11.95	55.7	246.15	1.10
20	152	2	Business	250530	690558	2658	300.93	1.0	20.77	55.3	249.06	4.20
21	150	1	Business	232610	559615	4090	163.10	0.1	11.75	55.0	244.80	2.10
22	149	1	Business	229569	776226	7629	225.38	0.1	13.73	54.2	244.71	3.10
23	148	1	Top-Busn	226548	613598	6155	144.36	0.1	11.24	57.5	241.88	1.10
24	147	1	Business	223547	515816	4523	155.71	0.1	12.21	56.3	242.81	2.10
25	146	1	Business	54538	286446	4690	81.70	0.1	10.95	55.7	242.62	3.10
26	145	2	Top-Econ	228333	482331	4989	299.02	1.0	13.75	55.5	243.19	4.20
27	143	1	Top-Econ	211743	457841	3742	116.74	0.1	9.84	55.4	239.18	1.10
28	142	2	Top-Econ	219135	443028	4984	299.13	0.4	11.42	55.6	241.31	3.70
29	140	2	Business	213108	966756	11541	270.36	0.1	10.26	59.3	240.15	4.70
30	138	2	Business	207165	597628	7138	190.21	0.1	10.72	56.6	238.62	5.20
31	136	1	Business	191856	592763	5787	188.35	0.1	8.21	55.4	236.33	3.10
32	135	1	Top-Econ	189095	308122	4851	92.77	0.1	7.99	54.7	235.25	2.10
33	134	2	Economy	195531	570519	4815	169.69	0.1	9.41	56.2	234.80	2.70
34	132	1	Business	117096	606118	7054	143.54	0.1	8.24	56.2	234.93	3.10
35	131	1	Business	178251	795206	10503	164.55	0.1	7.88	56.4	234.23	4.10
36	130	1	Economy	50464	361110	4759	84.92	0.1	7.94	54.8	231.88	1.10
37	129	1	Economy	50811	292085	4508	59.07	0.1	8.01	55.1	232.28	2.10
38	128	2	Economy	178710	895714	12467	194.78	0.1	9.77	56.4	232.35	4.20
39	126	1	Business	165146	754395	11921	134.71	0.1	7.72	55.0	231.10	4.10
40	125	1	Business	50253	232947	4118	66.16	0.1	7.27	56.1	229.98	3.10
41	124	1	Top-Econ	46079	294079	4643	56.59	0.1	6.86	53.6	229.88	4.10
42	123	1	Top-Busn	157523	711023	10828	147.23	0.1	6.73	55.9	225.98	1.10
43	122	1	Top-Econ	35238	827424	9148	73.63	0.1	6.58	54.7	227.75	4.10
44	121	1	Economy	152541	530647	7212	109.59	0.1	6.50	57.8	225.62	2.10
45	120	2	Top-Econ	157458	425176	6688	102.56	0.1	12.40	55.1	226.99	5.70
46	118	1	Business	145218	411201	5215	100.25	0.1	5.77	54.2	225.37	5.10
47	117	2	Economy	149835	597604	9437	129.96	0.1	7.03	54.4	222.99	4.70
48	115	2	Top-Busn	138075	383540	5858	64.42	0.1	6.69	56.9	219.83	3.60
49	113	1	Business	78301	309861	4976	76.55	0.0	5.49	54.7	220.11	4.10
50	112	1	Top-Econ	124674	360801	5730	46.96	0.1	5.22	55.1	219.89	4.10
51	111	2	Business	128831	370876	5749	87.56	0.1	6.62	55.0	219.91	6.20
52	109	4	Top-Econ	124329	756805	14600	144.91	0.1	5.66	58.3	218.97	8.40
53	105	1	Economy	36127	201537	4780	37.32	0.1	5.23	55.7	213.32	2.10
54	104	1	Business	107862	591302	10322	86.21	0.1	5.49	52.9	215.24	5.10
55	103	1	Top-Econ	105846	500371	7291	87.07	0.1	4.02	55.7	212.81	4.10
56	102	1	Business	103849	851382	15061	99.29	0.1	5.01	53.7	212.78	5.10
57	101	1	Economy	101871	352512	4462	69.80	0.1	4.36	55.9	210.66	4.10
58	100	1	Business	58088	408899	5527	67.23	0.1	4.76	54.3	211.45	5.10
59	99	1	Business	97972	480273	8724	73.78	0.1	4.67	55.1	212.22	6.10

Table 6 Results for the stochastic optimization model (continued)

Num. sale	Available seats	p_1	Group type	Num. var.	MIP iter.	B&B nodes	total time	gap%	1st inc time	1st inc gap%	Obj. f.	Obj. f. $k=1$
60	98	1	Business	61311	250810	5223	58.42	0.1	4.34	56.0	212.12	8.00
61	97	1	Top-Econ	94149	636535	12352	94.99	0.0	4.09	54.4	211.76	5.10
62	96	1	Top-Econ	92266	523196	10457	92.59	0.1	4.00	52.6	213.61	6.10
63	95	1	Economy	90402	872313	14765	115.08	0.1	3.64	55.2	210.41	3.10
64	94	1	Economy	88557	680670	17113	103.41	0.1	4.35	20.8	210.28	4.10
65	93	1	Economy	86731	689753	12811	85.26	0.1	3.51	51.3	208.50	4.10
66	92	2	Economy	89292	867796	19081	111.53	0.1	7.16	54.3	207.92	6.20
67	90	1	Economy	81367	1662181	47685	299.76	1.0	4.82	53.8	207.49	3.10
68	89	1	Economy	79617	1569093	47405	299.75	0.2	4.78	51.2	208.39	4.10
69	88	1	Top-Econ	77886	1718722	46497	299.76	1.4	4.96	52.8	213.12	6.10
70	87	2	Business	80087	2399842	47435	298.71	2.4	3.33	52.2	213.42	10.00
71	85	1	Business	72807	2277205	47678	299.78	2.3	2.82	53.0	210.08	6.10
72	84	1	Business	71152	4223043	71058	301.88	1.9	3.11	53.8	210.38	8.00
73	83	1	Economy	69516	2435950	47478	299.79	1.0	2.44	56.0	206.03	5.10
74	82	1	Top-Econ	67899	2111231	45670	299.80	5.3	3.79	54.8	210.05	6.10
75	81	2	Economy	69701	3701879	44974	302.40	3.5	2.67	54.4	208.07	6.70
76	79	1	Business	63162	3960867	47269	299.81	4.6	3.82	53.2	208.50	10.50
77	78	1	Top-Econ	61621	2187618	46469	299.83	3.2	3.29	53.9	207.13	6.10
78	77	3	Business	63177	3007508	49757	299.01	5.7	2.41	58.6	207.87	17.50
79	74	28	Economy	4338	248	0	0.07	0.0	0.07	75.6	101.60	§

Number problems solved within gap:	1
Average solution time (sec.):	187.56
Average gap%p:	1.93
Average time to first incumbent (sec.):	10.88
Average gap%p of first incumbent:	54.71

§ Not applicable for last offline assignment

5 Conclusions

A new approach has been presented for the airplane seat assignment procedure. Unlike current methods used by airlines, that are based on simple rules, the new approach relies on a network optimization model, with either a single type or many types of passenger groups (the latter resulting in a multicommodity network flow model).

In general, multicommodity models for $\kappa > 1$ provided better assignments by considering (even in a simple way) expected future demands by types of passengers. In addition, by modifying the cost scheme in Figure 4.a we can easily tune the behaviour of the optimization procedure, thus making it a very flexible tool.

In this work we considered a tentative stochastic optimization model for this problem, but it resulted to be computationally too difficult for an online system (solutions took several minutes) using a generic solver such as CPLEX. The solution time could be reduced by using specialized methods and optimization packages for 0–1 stochastic optimization, including heuristics/metaheuristics/matheuristics. Exploring these alternative methods would be part of the future work.

	A	B	C	D	E	F
1						
2						
3						
4	76	78	78	78		
5	48	60	70	70	72	
6	48	59	23	9	42	71
7	46	58	56	54	51	51
8	30	30	35	8	39	49
9	15	22	34	11	25	40
10	15	24	8	11	11	21
11	10	13	1	3	4	12
12						
13						
14						
15	79	79	79	79	79	79
16					75	79
17	66	66	73		75	79
18	57	64	65	47	68	79
19	38	38	63	47	67	14
20	17	17	44	33	53	37
21	16	19	7	33	7	36
22	62	69	74		77	79
23	45	52	52	52	52	61
24	45	50	43	41	29	55
25	31	14	20	20	29	
26	32	18	18	26	28	26
27	27	5	5	2	6	28
28	79	79	79	79	79	79
29	79	79	79	79	79	79
30	79	79	79	79	79	79

Business Economy Top-Busn Top-Econ

Fig. 9 Seat assignments obtained for flights of Table 6. Seat numbers correspond to sales and different types of groups are marked with different colors.

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