On the smallest trees with the same restricted U-polynomial and the rooted U-polynomial

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Abstract

In this article, we construct explicit examples of pairs of non-isomorphic trees with the same restricted U-polynomial for every k; by this we mean that the polynomials agree on terms with degree at most $k + 1$. The main tool for this construction is a generalization of the U-polynomial to rooted graphs, which we introduce and study in this article. Most notably we show that rooted trees can be reconstructed from its rooted U-polynomial.

1. Introduction

The chromatic symmetric function [16] and the U-polynomial [12] are powerful graph invariants as they generalize many other invariants like, for instance, the chromatic polynomial, the matching polynomial and the Tutte polynomial. It follows easily from the definitions that the chromatic symmetric function and the U-polynomial are equivalent when restricted to trees, and there are examples of non-isomorphic graphs with cycles having the same U-polynomial (see [5] for examples of graphs with the same polychromate and [14, 11] for the equivalence between the polychromate and the U-polynomial) and also the same is true for the chromatic symmetric function (see [16]) . However, it is an open question to know whether there exist non-isomorphic trees with the same chromatic symmetric function (or, equivalently, the same U-polynomial). This problem is referred in the literature as *Stanley's tree isomorphism problem*. The negative answer to the latter question, that is, the assertion that two trees that have the same chromatic symmetric function must be isomorphic, is sometimes referred to in the literature as the *tree distinguishing conjecture*. This conjecture has been so far verified for trees up to 29 vertices [8] and also for some classes of trees, most notably caterpillars [2, 9] and spiders [10].

A natural approach consists in defining a truncation of the U-polynomial, and then search for non-isomorphic trees with the same truncation. A study

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of these examples could help to better understand the picture for solving the tree distinguishing conjecture. To be more precise, suppose that T is a tree with n vertices. Recall that a partition λ of n is a sequence $\lambda_1, \lambda_2, \ldots, \lambda_l$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. Recall that $U(T)$ can be expanded as

$$U(T) = \sum_\lambda c_\lambda \mathbf{x}_\lambda, \tag{1}$$

where the sum is over all partitions λ of n, $\mathbf{x}_\lambda = x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_l}$ and the c_λ are non-negative integer coefficients (for details of this expansion see Section 2). In a previous work [1], the authors studied the U_k-polynomial defined by restricting the sum in (1) to the partitions of length smaller or equal than $k + 1$, and then showed the existence of non-isomorphic trees with the same U_k-polynomial for every k. This result is based on a remarkable connection between the U-polynomial of a special class of trees and the Prouhet-Tarry-Escott problem in number theory. Although the Prouhet-Tarry-Escott problem is known to have solutions for every k, in general it is difficult to find explicit solutions, specially if k is large. Hence, it was difficult to use this result to find explicit examples of trees with the same U_k-polynomial.

The main result of this paper is to give an explicit and simple construction of non-isomorphic trees with the same U_k-polynomial for every k. It turns out that for $k = 2, 3, 4$ our examples coincide with the minimal examples already found by Smith, Smith and Tian [15]. This leads us to conjecture that for every k our construction yields the smallest non-isomorphic trees with the same U_k-polynomial. We also observe that if this conjecture is true, then the tree distinguishing conjecture is true.

To prove our main result, we first introduce and study a generalization of the U-polynomial to rooted graphs, which we call the rooted U-polynomial or U^r-polynomial. As it is the case for several invariants of rooted graphs, the rooted U-polynomial distinguishes rooted trees up to isomorphism. Under the correct interpretation, it can also be seen as a generalization of the pointed chromatic symmetric function introduced in [13] (see Remark 6). The key fact for us is that the rooted U-polynomial exhibits simple product formulas when applied to some joinings of rooted graphs. These formulas together with some non-commutativity is what allows our constructions to work.

Very recently, another natural truncation for the U-polynomial was considered in [8]; They restrict the range of the sum in (1) to partitions whose parts are smaller or equal than l. They also verified that trees up to 29 vertices are distinguished by their truncation with $l = 3$ and proposed the conjecture that actually $l = 3$ suffices to distinguish all trees.

This paper is organized as follows. In Section 2, we introduce the rooted U-polynomial and prove our main product formulas. In Section 3, we show that the rooted U-polynomial distinguishes rooted trees up to isomorphism. In Section 4, we recall the definition of the U_k-polynomial and prove our main result.

2. The rooted U-polynomial

We give the definition of the U-polynomial first introduced by Noble and Welsh [12]. We consider graphs where we allow loops and parallel edges.

Let $G = (V, E)$ be a graph. Given $A \subseteq E$, the restriction $G|_A$ of G to A is the subgraph of G obtained from G after deleting every edge that is not contained in A (but keeping all the vertices). The *rank* of A is defined as $r(A) = |V| - k(G|_A)$, where $k(G|_A)$ is the number of connected components of $G|_A$. The *partition induced by A*, denoted by $\lambda(A)$, is the partition of $|V|$ whose parts are the numbers of vertices of the connected components of $G|_A$.

Let y be an indeterminate and $\mathbf{x} = x_1, x_2, \ldots$ be an infinite set of commuting indeterminates that commute with y. Given an integer partition $\lambda = \lambda_1, \lambda_2, \cdots, \lambda_l$, define $\mathbf{x}_\lambda := x_{\lambda_1} \cdots x_{\lambda_l}$. The *$U$-polynomial* of a graph G is defined as

$$U(G; \mathbf{x}, y) = \sum_{A \subseteq E} \mathbf{x}_{\lambda(A)} (y-1)^{|A|-r(A)}.$$

Now we recall the definition of the W-polynomial for weighted graphs, from which the U-polynomial is a specialization. A *weighted graph* is a pair (G, ω) where G is a graph and $\omega : V(G) \to \mathbb{P}$ is a function. We say that $\omega(v)$ is the weight of the vertex v. Given a weighted graph (G, ω) and an edge e, the graph $(G - e, \omega)$ is defined by deleting the edge e and leaving ω unchanged. If e is not a loop, then the graph $(G/e, \omega)$ is defined by first deleting e and then by identifying the vertices u and u' incident to e into a new vertex v. We set $\omega(v) = \omega(u) + \omega(u')$ and leave all other weights unchanged.

The W-polynomial of a weighted graph (G, ω) is defined by the following properties:

1. If e is not a loop, then $W(G, \omega)$ satisfies

$$W(G, \omega) = W(G - e, \omega) + W(G/e, \omega); \qquad (2)$$

2. If e is a loop, then
$$W(G, \omega) = yW(G - e, \omega);$$

3. If G consists only of isolated vertices $v_1, \ldots, v_n$ with weights $\omega_1, \ldots, \omega_n$, then
$$W(G, \omega) = x_{\omega_1} \cdots x_{\omega_n}.$$

In [12], it is proven that the W-polynomial is well-defined and that $U(G) = W(G, 1_G)$ where 1_G is the weight function assigning weight 1 to all vertices of G. The deletion-contraction formula is very powerful, but in this paper we will only use it in the beginning of the proof of Theorem 12 in Section 4.

A *rooted graph* is a pair (G, v_0), where G is a graph and v_0 is a vertex of G that we call the *root* of G. Given $A \subseteq E$, define $\lambda_r(A)$ to be the number of vertices of the component of $G|_A$ that contains the root v_0, and $\lambda_-(A)$ to be the partition induced by the number of vertices of all the other components. The *rooted U-polynomial* is

$$U^r(G, v_0; \mathbf{x}, y, z) = \sum_{A \subseteq E} \mathbf{x}_{\lambda_-(A)} z^{\lambda_r(A)} (y-1)^{|A|-r(A)},$$
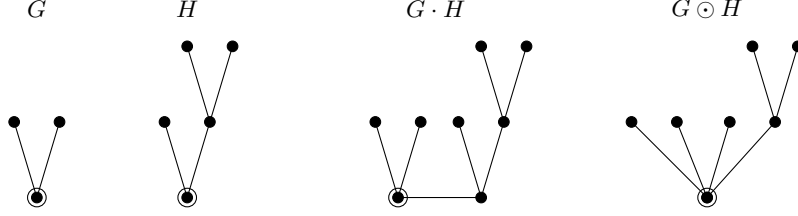
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Figure 1: Example of two rooted graphs G and H and their different products $G \cdot H$ and $G \odot H$.

where z is a new indeterminate that commutes with $x_1, x_2, \ldots$ and y. Observe that $\lambda_r(A) \geq 1$, which means that $z^{-1} U^r(G, v_0)$ is a polynomial. We often write G instead of (G, v_0) when v_0 is clear from the context, and so we will write $U^r(G)$ instead of $U^r(G, v_0)$. Also, if (G, v_0) is a rooted graph, we will write $U(G)$ for the U-polynomial of G (seen as an unrooted graph). If we compare $U^r(G)$ with $U(G)$, then we see that for each term of the form $\mathbf{x}_\lambda y^n z^m$ appearing in $U^r(G)$ there is a corresponding term of the form $\mathbf{x}_\lambda y^n x_m$ in $U(G)$. This motivates the following notation and lemma, whose proof follows directly from the latter observation.

Notation 1. *If $P(\mathbf{x}, y, z)$ is a polynomial, then $(P(\mathbf{x}, y, z))^*$ is the polynomial obtained by expanding P as a polynomial in z (with coefficients that are polynomials in $\mathbf{x}$ and y) and then substituting $z^n \mapsto x_n$ for every $n \in \mathbb{N}$. For instance, if $P(\mathbf{x}, y, z) = x_1 y z - x_2 x_3 z^3$, then $P(\mathbf{x}, y, z)^* = x_1^2 y - x_2 x_3^2$. Note that in general $(P(\mathbf{x}, y, z) Q(\mathbf{x}, y, z))^* \neq P(\mathbf{x}, y, z)^* Q(\mathbf{x}, y, z)^*$.*

Lemma 2. *For every graph G we have*

$$(U^r(G))^* = U(G).$$

Remark 3. *We could also define a rooted version of the W-polynomial, but we will not need this degree of generality for the purposes of this article.*

2.1. Joining of rooted graphs and product formulas

In this section we show two product formulas for the rooted U-polynomial. These will play a central role in the proofs of the results in the following sections. Let (G, v) and (H, v') be two rooted graphs. Define $G \odot H$ to be the rooted graph obtained after first taking the disjoint union of G and H and then by identifying v and v'. We refer to $G \odot H$ as *the joining* of G and H. Note that from the definition it is clear that $G \odot H = H \odot G$. We also define $G \cdot H$ to be the rooted graph obtained after first taking the disjoint union of G and H, then adding an edge between v and v' and finally declaring v as the root of the resulting graph. Since we made a choice for the root, in general $G \cdot H$ and $H \cdot G$ are isomorphic as unrooted graphs, but not as rooted graphs.

Lemma 4. *Let G and H be two rooted graphs. We have*

$$U^r(G \odot H) = \frac{1}{z}U^r(G)U^r(H). \tag{3}$$

Proof. By substituting the definition of U^r to G and H in the r.h.s. of (3)

$$\sum_{A_G \subseteq E(G)} \sum_{A_H \subseteq E(H)} \mathbf{x}_{\lambda_-(A_G) \cup \lambda_-(A_H)} z^{\lambda_r(A_G)+\lambda_r(A_H)-1}(y-1)^{|A_G|+|A_H|-r(A_G)-r(A_H)}. \tag{4}$$

Given $A_G \subseteq E(G)$ and $A_H \subseteq E(H)$, set $A = A_G \cup A_H$. By the definition of the joining, there is a set $A' \subseteq E(G \odot H)$ corresponding to A such that $\lambda_-(A') = \lambda_-(A_G) \cup \lambda_-(A_H)$ and $\lambda_r(A) = \lambda_r(A_G) + \lambda_r(A_H) - 1$. From these equations, one checks that $r(A) = r(A_G)+r(A_H)$. Plugging these relations into (4) and then rearranging the sum yields $U^r(G \odot H)$ and the conclusion now follows. $\square$

Lemma 5. *Let G and H be two rooted graphs. Then we have*

$$U^r(G \cdot H) = U^r(G)(U^r(H) + U(H)). \tag{5}$$

Proof. By definition, $E(G \cdot H) = E(G) \cup E(H) \cup \{e\}$, where e is the edge joining the roots of G and H. Thus, given $A \subseteq E(G \cdot H)$, we can write it as $A = A_G \cup A_H \cup F$ where $A_G \subseteq E(G)$, $A_H \subseteq E(H)$ and F is either empty or $\{e\}$. Let δ_F equal to one if $F = \{e\}$ and zero otherwise. The following relations are easy to check:

$$\lambda_-(A) = \begin{cases} \lambda_-(A_G) \cup \lambda(A_H), & \text{if } F = \emptyset, \\ \lambda_-(A_G) \cup \lambda_-(A_H), & \text{otherwise;} \end{cases}$$
$$\lambda_r(A) = \lambda_r(A_G) + \lambda_r(A_H)\delta_F;$$
$$r(A) = r(A_G) + r(A_H) + \delta_F;$$
$$|A| = |A_G| + |A_H| + \delta_F.$$

Now replacing the expansions of $U^r(G)$, $U^r(H)$ and $U(H)$ into the r.h.s. of (5) yields

$$\sum_{A_G \subseteq E(G), A_H \subseteq E(H)} \mathbf{x}_{\lambda_-(A_G) \cup \lambda_-(A_H)} z^{\lambda_r(A_G)+\lambda_r(A_H)}(y-1)^{|A_G|-r(A_G)+|A_H|-r(A_H)}$$
$$+ \sum_{A_G \subseteq E(G), A_H \subseteq E(H)} \mathbf{x}_{\lambda_-(A_G) \cup \lambda(A_H)} z^{\lambda_r(A_G)}(y-1)^{|A_G|-r(A_G)+|A_H|-r(A_H)}$$

Using the previous relations we can simplify the last equation to

$$\sum_{A=A_G \cup A_H \cup \{e\}} \mathbf{x}_{\lambda_-(A)} z^{\lambda_r(A)}(y-1)^{|A|-r(A)}$$
$$+ \sum_{A=A_G \cup A_H} \mathbf{x}_{\lambda_-(A)} z^{\lambda_r(A)}(y-1)^{|A|-r(A)},$$

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where in both sums A_G ranges over all subsets of $E(G)$ and A_H ranges over all subsets of $E(H)$. Finally, we can combine the sums to get $U^r(G \cdot H)$, which finishes the proof. $\qquad \square$

Remark 6. *By [12, Theorem 6.1]), we know that the chromatic symmetric function of a graph can be recovered from the U-polynomial by*

$$X(G) = (-1)^{|V(G)|}U(G; x_i = -p_i, y = 0).$$

In [13], Pawlowski introduced a pointed chromatic symmetric function $X_{G,v}$, which is related to our rooted U-polynomial by

$$X(G, v) = (-1)^{|V(G)|}\frac{1}{z}U^r(G, v; x_i = -p_i, y = 0).$$

Our Lemma 4 is equivalent to Proposition 3.4 in [13].

3. The rooted U-polynomial distinguishes rooted trees

In this section we will show that the rooted U-polynomial distinguishes rooted trees up to isomorphism. Similar results for other invariants of rooted trees appear in [4, 6, 7]. The proof given here follows closely the one in [6] but one can also adapt the proof of [4]. Before stating the result we need the two following lemmas.

Lemma 7. *Let (T, v) be a rooted tree. Then, the number of vertices of T and the degree of v can be recognized from $U^r(T)$.*

Proof. It is easy to see that $U^r(T) = z^n + q(z)$ where $q(z)$ is a polynomial in z of degree less than n with coefficients in $\mathbb{Z}[y, \mathbf{x}]$ and n is the number of vertices of T. Hence, to recognize the number of vertices of T, it suffices to take the term of the form z^j with the largest exponent in $U^r(T)$ and this exponent is the number of vertices. To recognize the degree of v, observe that a term of $U^r(T)$ has the form zx_λ for some λ corresponding to A if and only if no edge of A is incident with v. In particular, the unique term of this form with the smallest total degree corresponds to $A = E \setminus I(v)$ where $I(v)$ denotes the set of edges that are incident with v and in fact the term is $zx_{n_1}x_{n_2}\ldots x_{n_d}$ where $n_1, n_2, \ldots, n_d$ are the number of vertices in each connected component of $T - v$. Since each connected component is connected to v by an edge, this means that the degree of v is equal to d. $\qquad \square$

Lemma 8. *Let (T, v) be a rooted tree with n vertices. Let $d = \deg_T(v)$, so that T can be expressed uniquely (up to reordering) as $T_1 \odot \cdots \odot T_d$, where each T_i is a tree rooted at v and $\deg_{T_i}(v) = 1$ for all i. Then, the unique factorization of $\frac{1}{z}U^r(T, v)$ into irreducibles in $\mathbb{Z}[z, x_1, x_2, \ldots x_{n-1}]$ is*

$$\frac{1}{z}U^r(T) = \frac{1}{z}U^r(T_1)\frac{1}{z}U^r(T_2)\ldots\frac{1}{z}U^r(T_d).$$

In particular, this means that $\frac{1}{z}U^r(T)$ is irreducible when $d = 1$.

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Proof. We first show that if $d = 1$ then $\frac{1}{z}U^r(T, v)$ is irreducible. Denote by e the unique edge of T that is incident with v. It is easy to check that $\lambda_r(A) \geq 1$ for all $A \subseteq E$ and that, if $A = E - e$, then $\lambda(A) = (n - 1, 1)$. Consequently,

$$\frac{1}{z}U^r(T, v) = x_{n-1} + \sum_{A \subseteq E, A \neq E - e} \mathbf{x}_{\lambda_-(A)} z^{\lambda_r(A) - 1}$$

where the second sum does not depend on x_{n-1}. Thus $\frac{1}{z}U^r(T, v)$ is a monic polynomial of degree one with respect to the x_{n-1} variable, thence irreducible.

Now we suppose that $d > 1$. Since $(T, v) = T_1 \odot T_2 \odot \cdots \odot T_d$, it follows from Lemma 4 that

$$\frac{1}{z}U^r(T) = \frac{1}{z}U^r(T_1)\frac{1}{z}U^r(T_2)\ldots\frac{1}{z}U^r(T_l).$$

By the first part of the proof, each factor $\frac{1}{z}U^r(T)$ in the r.h.s. of the last equation is irreducible. Since $\mathbb{Z}[z, x_1, x_2, \ldots, x_n]$ is a unique factorization domain, the conclusion now follows. $\square$

We say that a rooted tree (T, v) can be reconstructed from its U^r-polynomial if we can determine (T, v) up to rooted isomorphism from $U^r(T, v)$. We show the following result.

Theorem 9. *Every rooted tree can be reconstructed from its U^r-polynomial.*

Proof. By Lemma 7 we can recognize the number of vertices of a rooted tree from its U^r-polynomial. Thus, we proceed by induction on the number of vertices. For the base case, there is only one tree with 1 vertex, hence the assertion is trivially true. Now suppose that all rooted trees with $n-1$ vertices can be reconstructed from their U^r-polynomials and let $U^r(T, v)$ be the U^r-polynomial of some unknown tree (T, v) with n vertices. Again by Lemma 7 we can determine the degree d of v from $U^r(T)$. We distinguish two cases:

- $d = 1$: In this case, let $T' = T - v$, rooted at the unique vertex of T adjacent to v. This means that $T = \bullet \cdot T'$ where $\bullet$ is the rooted tree with only one vertex. From Lemma 5 it follows that

$$U^r(T) = z(U^r(T') + U(T')) = (\frac{1}{z}U^r(T'))z^2 + U(T')z.$$

Since the variable z does not appear in $U(T')$, we can determine $U^r(T')$ from $U^r(T)$ by collecting all the terms in the expansion of $U^r(T)$ that are divisible by z^2 and then dividing them by z. Since T' has $k - 1$ vertices, by the induction hypothesis, we can reconstruct T' and hence the equality $T = \bullet \cdot T'$ allows us to reconstruct T.

- $d > 1$: In this case, by Lemma 8, the unique factorization into irreducibles in $\mathbb{Z}[z, x_1, \ldots, x_n]$ is given by

$$\frac{1}{z}U^r(T) = \frac{1}{z}U^r(T_1)\frac{1}{z}U^r(T_2)\cdots\frac{1}{z}U^r(T_d),$$

where T can be written in a unique way (up to reordering) as

$$T = T_1 \odot \cdots \odot T_d \tag{6}$$

and each T_i is a tree rooted at v with $\deg_{T_i}(v) = 1$. This implies that by factoring $\frac{1}{z}U^r(T)$, we can compute all the polynomials $\frac{1}{z}U^r(T_i)$ up to reordering. It is easy to see that every T_i has at most $n-1$ vertices. Thus, by the induction hypothesis we can reconstruct each T_i and the reconstruction of T now follows from (6).

$\square$

Corollary 10. *The U^r-polynomial distinguishes trees up to rooted isomorphism.*

Example 1. Suppose $U^r(T,v) = x_1^5 z + 3\,x_1^4 z^2 + 4\,x_1^3 z^3 + 4\,x_1^2 z^4 + 3\,x_1 z^5 + z^6 + 2\,x_1^3 x_2 z + 5\,x_1^2 x_2 z^2 + 4\,x_1 x_2 z^3 + x_2 z^4 + x_1^2 x_3 z + 2\,x_1 x_3 z^2 + x_3 z^3$. From the term z^6, we know that T has 6 vertices. The terms of the form $z\mathbf{x}_\lambda$ are $x_1^5 z + 2x_1^3 x_2 z + x_1^2 x_3 z$. Thus, the degree of v is 3. Moreover, if we factorize $\frac{1}{z}U^r(T,v)$ into irreducible factors we obtain

$$\frac{1}{z}U^r(T,v) = \left(x_1^3 + x_1^2 z + x_1 z^2 + z^3 + 2\,x_1 x_2 + x_2 z + x_3\right)(x_1 + z)(x_1 + z).$$

This means that

$$
\begin{array}{rcl}
U_r(T_1, v_1) & = & x_1^3 z + x_1^2 z^2 + x_1 z^3 + z^4 + 2\,x_1 x_2 z + x_2 z^2 + x_3 z, \\
U_r(T_2, v_2) & = & x_1 z + z^2, \\
U_r(T_3, v_3) & = & x_1 z + z^2.
\end{array}
$$

From the terms z^4 and $x_3 z$ in $U^r(T_1)$ it is easy to see that T_1 has 4 vertices and v_1 has degree 1. Hence, $T_1 = \bullet \cdot T_1'$, where

$$U^r(T_1') = \frac{1}{z}\left(x_1^2 z^2 + x_2 z^2 + x_1 z^3 + z^4\right) = x_1^2 z + x_2 z + x_1 z^2 + z^3.$$

Similarly $T_1' = \bullet \cdot T_1''$, where

$$U^r(T_1'') = \frac{1}{z}\left(x_1 z^2 + z^3\right) = x_1 z + z^2.$$

From this, it is not difficult to see that T_2, T_3 and T_1'' are rooted isomorphic to $\bullet \cdot \bullet$. Finally, we have

$$T = (\bullet \cdot (\bullet \cdot (\bullet \cdot \bullet))) \odot (\bullet \cdot \bullet) \odot (\bullet \cdot \bullet).$$
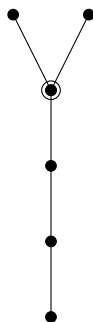
Figure 2: The reconstructed tree from Example 1.

4. The restricted U-polynomial

Let T be a tree with n vertices. We have that $r(A) = |A|$ for every $A \subseteq E(T)$. Hence, $U(T)$ and $U^r(T)$ (if T is rooted) do not depend on y. Given an integer k, the U_k-polynomial of T is defined by

$$U_k(T; \mathbf{x}) = \sum_{A \subseteq E, |E \setminus A| \leq k} \mathbf{x}_{\lambda(A)}.$$

Observe that since T is a tree, every term in $U_k(T)$ has degree at most $k + 1$ and that restricting the terms in the expansion of $U(T)$ to those of degree at most $k + 1$ yields $U_k(T)$. As noted in the introduction, it is proved in [1] that for every integer k there are non-isomorphic trees T and T' that have the same U_k-polynomial but distinct U_{k+1}-polynomial. However, the trees found in [1] are not explicit. In this section, with the help of the tools developed in previous sections, we will explicitly construct such trees.

We start by defining two sequences of rooted trees. Let us denote the path on three vertices, rooted at the central vertex, by A_0 and the path on three vertices, rooted at one of the leaves, by B_0. The trees A_k and B_k for $k \in \mathbb{N}$ are defined inductively as follows:

$$A_k := A_{k-1} \cdot B_{k-1} \quad \text{and} \quad B_k := B_{k-1} \cdot A_{k-1}. \tag{7}$$

We first observe that A_0 and B_0 are isomorphic as unrooted trees but not isomorphic as rooted trees, which means that they have different U^r. In fact, a direct calculation shows that

$$\Delta_0 := U^r(A_0) - U^r(B_0) = x_1 z^2 - x_2 z.$$

By applying Lemma 4 we deduce:

Proposition 11. *For all $k \in \mathbb{N}$, the trees A_k and B_k are isomorphic but not rooted-isomorphic. Moreover, we have*

$$U^r(A_k) - U^r(B_k) = \Delta_0 P_k, \tag{8}$$
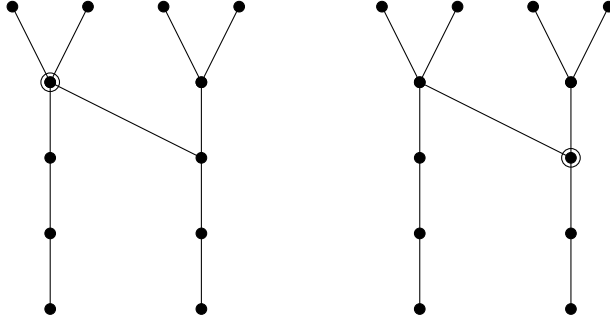
where $P_k := U(A_0)U(A_1) \cdots U(A_{k-1})$.

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Figure 3: The rooted trees A_2 and B_2

Proof. The proof is done by induction. The basis step is clear from the definition of Δ_0. For the induction step, we assume that for a given k, the graphs A_{k-1} and B_{k-1} are isomorphic and that $U^r(A_{k-1}) - U^r(B_{k-1}) = \Delta_0 P_{k-1}$. From (7), it is easy to see that A_k and B_k are isomorphic as unrooted trees. Also, combining (7) with (5) we get

$$U^r(A_k) = U^r(A_{k-1})(U^r(B_{k-1}) + U(B_{k-1})).$$

Similarly for B_k we get

$$U^r(B_k) = U^r(B_{k-1})(U^r(A_{k-1}) + U(A_{k-1})).$$

Subtracting these two equations, using that $U(A_{k-1}) = U(B_{k-1})$ and plugging the induction hypothesis yields

$$U^r(A_k) - U^r(B_k) = U(A_{k-1})\big(U^r(A_{k-1}) - U^r(B_{k-1})\big) = U(A_{k-1})P_{k-1}\Delta_0 = P_k\Delta_0.$$

Hence, by induction, (8) holds for every k. To finish the proof, notice that since A_k and B_k have distinct U^r, they are not rooted-isomorphic by Theorem 9. $\square$

Observe that all the terms of P_k have degree at least k. Now we can state our main result.

Theorem 12. *Given $k, l \in \mathbb{N}$, let*

$$Y_{k,l} = (A_k \odot A_l) \cdot (B_k \odot B_l) \quad and \quad Z_{k,l} = (A_l \odot B_k) \cdot (B_l \odot A_k).$$

Then the graphs $Y_{k,l}$ and $Z_{k,l}$ (seen as unrooted trees) are not isomorphic, have the same U_{k+l+2}-polynomial and distinct U_{k+l+3}-polynomial.

Before giving the proof, we need the following lemma, which is a corollary of Lemma 4 and Proposition 11.

Lemma 13. *Let T be a rooted tree and i an integer. Then*

$$U(A_i \odot T) - U(B_i \odot T) = P_i \mathcal{D}(T),$$

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where

$$\mathcal{D}(T) = x_1(zU^r(T))^* - x_2U(T). \tag{9}$$

In particular all the terms in $\mathcal{D}(T)$ have degree at least 2.

Proof. By Lemma 4, we have

$$U^r(A_i \odot T) - U^r(B_i \odot T) = z^{-1}U^r(T)\big(U^r(A_i) - U^r(B_i)\big).$$

Applying Proposition 11 to the last term yields

$$U^r(A_i \odot T) - U^r(B_i \odot T) = P_iU^r(T)\frac{\Delta_0}{z}.$$

The conclusion now follows by taking the specialization $z^n \to x_n$ in the last equation to obtain (note that P_i does not depend on z)

$$U(A_i \odot T) - U(B_i \odot T) = P_i\left[U^r(T)(x_1z - x_2)\right]^* = P_i\mathcal{D}(T).$$

$\square$

Proof of Theorem 12. We start by applying the deletion-contraction formula (2) to the edges corresponding to the $\cdot$ operation in the definitions of $Y_{k,l}$ and $Z_{k,l}$; it is easy to see that

$$U(Y_{k,l}) - U(Z_{k,l}) = U(A_k \odot A_l)U(B_k \odot B_l) - U(A_l \odot B_k)U(B_l \odot A_k), \quad (10)$$

since after contracting the respective edges we get isomorphic weighted trees.

We apply Lemma 13 twice, to $T = A_k$ and $i = l$ first, and then to $T = B_k$ and $i = l$, and replace the terms $U(A_k \odot A_l)$ and $U(A_l \odot B_k)$ in (10). Recalling that $\odot$ is commutative and after some cancellations, we obtain

$$U(Y_{k,l}) - U(Z_{k,l}) = P_l\Big(\mathcal{D}(A_k)U(B_k \odot B_l) - \mathcal{D}(B_k)U(B_l \odot A_k)\Big).$$

We use Lemma 13 once more, with $T = B_l$ and $i = k$, to arrive at

$$U(Y_{k,l}) - U(Z_{k,l}) = P_l\Big(\big(\mathcal{D}(A_k) - \mathcal{D}(B_k)\big)U(B_l \odot A_k) - \mathcal{D}(A_k)\mathcal{D}(B_l)P_k\Big) \quad (11)$$

Using (9) and Proposition 11 we get

$$\mathcal{D}(A_k) - \mathcal{D}(B_k) = x_1P_k(z\Delta_0)^* = x_1(x_1x_3 - x_2^2)P_k,$$

and substituting this into (11) yields

$$U(Y_{k,l}) - U(Z_{k,l}) = P_lP_k\Big((x_1^2x_3 - x_1x_2^2)U(B_l \odot A_k) - \mathcal{D}(A_k)\mathcal{D}(B_l)\Big). \quad (12)$$

This implies that all the terms that appear in the difference have degree at least $l + k + 4$. Hence $Y_{k,l}$ and $Z_{k,l}$ have the same U_{k+l+2}-polynomial. To see that

they have distinct U_{k+l+3}-polynomial, from (12) we can deduce that the only terms of degree $l + k + 4$ come from terms of degree 4 in the difference

$$\Big((x_1^2 x_3 - x_1 x_2^2)U(B_l \odot A_k) - \mathcal{D}(A_k)\mathcal{D}(B_l)\Big).$$

An explicit computation of these terms yields

$$(x_1^2 x_3 - x_1 x_2^2)x_{n(l)+n(k)-1} - (x_1 x_{n(k)+1} - x_2 x_{n(k)})(x_1 x_{n(l)+1} - x_2 x_{n(l)}),$$

where $n(k)$ is the number of vertices of A_k (and also B_k). From this last equation, the conclusion follows. $\qquad\square$

We may consider the following quantity:

$$\Phi(m) := \min\{l : \exists \text{ non-isomorphic trees } H, G \text{ with } l \text{ vertices s.t. } U_m(H) = U_m(G)\}.$$

Proposition 14. *The quantity $\Phi(m)$ is finite and weakly-increasing on m. Moreover, the following bound holds*

$$\Phi(m) \leq \begin{cases} 6 \cdot 2^{\frac{m}{2}} - 2, & \text{if } m \text{ is even} \\ 6 \cdot 3 \cdot 2^{\lfloor \frac{m}{2} \rfloor - 1} - 2, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. By Theorem 12, we see that $\Phi(m) \leq |Y_{k,l}|$ for all (k,l) such that $k+l+2 = m$. It is easy to check that $|A_i| = |B_i| = 3 \cdot 2^i$ for all i. Thus,

$$|Y_{k,l}| = 2(|A_k| + |B_l| - 1) = 6(2^k + 2^l) - 2 \quad \text{for all } (k,l).$$

If $m = k+l+2$ is fixed, then we see that $|Y_{k,l}|$ is minimized when $k = l = \frac{m}{2} - 1$ if m is even and otherwise is minimized when $k = \lfloor \frac{m}{2} \rfloor$ and $l = \lfloor \frac{m}{2} \rfloor - 1$. Replacing the values of k and l yields the desired inequality. $\qquad\square$

Observe that when $(k,l) \in \{(0,0), (1,0), (1,1)\}$ (respectively), the graphs $Y_{k,l}$ and $Z_{k,l}$ are the smallest examples of non-isomorphic trees with the same U_m for $m \in \{2, 3, 4\}$ (respectively). This fact was verified computationally in [15]. This leads us to make the following conjecture

Conjecture 15. *If m is even, then $Y_{m/2-1,m/2-1}$ and $Z_{m/2-1,m/2-1}$ are the smallest non-isomorphic trees with the same U_m-polynomial and if m is odd, then the same is true for $Y_{\lfloor m/2 \rfloor, \lfloor m/2 \rfloor - 1}$ and $Z_{\lfloor m/2 \rfloor, \lfloor m/2 \rfloor - 1}$. In other words,*

$$\Phi(m) = \begin{cases} 6 \cdot 2^{\frac{m}{2}} - 2, & \text{if } m \text{ is even} \\ 6 \cdot 3 \cdot 2^{\lfloor \frac{m}{2} \rfloor - 1} - 2, & \text{if } m \text{ is odd.} \end{cases}$$

The following proposition relates Φ with the tree distinguishing conjecture.

Proposition 16. *The following assertions are true:*

a) *For every m, the chromatic symmetric function distinguishes trees with at most $\Phi(m) - 1$ vertices.*

b) *The chromatic symmetric function distinguishes all trees if and only if*

$$\lim_{m} \Phi(m) = \infty.$$

Proof. To show a), observe that the existence of non-isomorphic trees T and T' with fewer than $\Phi(m)$ vertices with the same U-polynomial contradicts the definition of $\Phi(m)$. To see b), if $\lim_{m} \Phi(m) = \infty$, then by a) the chromatic symmetric function distinguishes all (finite) trees. For the converse, if $\lim_{m} \Phi$ is finite, then $\Phi(m)$ is uniformly bounded by N. Take $m > N$, by the definition of Φ there exist two trees T, T' with less than N vertices with the same U_m-polynomial. This implies that T and T' have the same U-polynomial since $m > N$. This finishes the proof of b). $\qquad\square$

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