A b-SYMPLECTIC SLICE THEOREM

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ABSTRACT. In this article, motivated by the study of symplectic structures on manifolds with boundary and the systematic study of b-symplectic manifolds started in [10], we prove a slice theorem for Lie group actions on b-symplectic manifolds.

1. INTRODUCTION

The linearization of Lie group actions for compact groups in a neighbourhood of a fixed point is due to Bochner [2]. It gives a precise description of the local normal form for a Lie group action in the neighbourhood of a fixed point. The next level of difficulty in describing group actions is in the neighbourhood of an invariant submanifold for the action: It was not until the work of Palais in the 60's that such a portrayal was achieved [23, 24]. The description of the normal form in this case is semilocal (in a neighbourhood of an orbit) and it is done in terms of the action of the group along the orbit and on the *linearization* of its (orthogonal) complement, which Palais denominated "slice".

The existence of slices re-conducts the computation of the orbits for the action in terms of its normal space on which it acts linearly. When additional geometrical structures are added into the picture, the existence of normal forms gains interest as it can often be adapted to the new ingredient (the geometric structure). This is the case of symplectic structures where Lie group actions are naturally related to the investigation of Hamiltonian symmetries.

In particular, symplectic slice theorems, give equivariant normal forms around orbits of symplectic group actions and become particularly handy when computing the orbits of fundamental vector fields of Hamiltonian actions. For example the Marle-Guillemin-Sternberg normal form [12], [17] or its generalisations, have been used extensively to study the local structure of symplectic manifolds with symmetries.

The purpose of this article is to extend these results to the singular set-up, more precisely to b-symplectic manifolds. In this singular framework the motivation to find equivariant normal forms comes from the study of symmetric manifolds with boundary. These singular symplectic structures have been intensely studied since their introduction in [10]. A study of their geometry in the presence of symmetries was initiated in [11] (see also [8]) which gave global results on the structure of b-symplectic manifolds with toric symmetries and also semilocal models.

In [14] two of the authors in this paper described integrable systems as b-cotangent lifts of rotations on a Liouville torus to its cotangent bundle. These can be understood as semilocal models for free actions of tori (associated to integrable systems). Motivated by these models in the integrable case, we plan to give linearized models for general actions of Lie groups in the language

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of the symplectic slice theorem. Many examples motivating the study of more general symmetries comes from the study of non-commutative integrable systems on (b-)symplectic manifolds as considered in [15] and [4].

These cotangent lifts appear naturally on the study of geodesic flows: A P-manifold is a Riemannian manifold M with all the geodesics closed. 2-dimensional examples of P-manifolds are Zoll and Tannery surfaces (see Chapter 4 in [3]). In this case the geodesics admit a common period (see Lemma 7.11 in [3]) inducing an S^1-action on M. In the same way that the standard cotangent lift induces a Hamiltonian action on T^*M we can use the twisted b-cotangent lift to obtain a b-Hamiltonian S^1-action on T^*M. In this case the action is given as a *twisted b-cotangent lift* which is a "linear model" of the b-Poisson structure parametrized by an additional invariant: a constant c. This constant is the *modular period* of the structure.

The b-symplectic slice theorem gives a normal form for a b-symplectic form via the symplectic slice theorem, which we recall here (for details on the constructions see [13] for the Hamiltonian case or [21] for the more general symplectic case):

Theorem 1. *Let (M, ω) be a symplectic manifold and let H be a Lie group acting properly and by symplectomorphisms on M. Let $z \in M$. Denote the isotropy group of z by H_z and the orbit of m by $\mathcal{O}_z^H$. Let V_z be the symplectic normal space*

$$V_z := (T_z\mathcal{O}_z^H)^\omega / \left((T_z\mathcal{O}_z^H)^\omega \cap T_z\mathcal{O}_z^H\right).$$

Let $\mathfrak{h}$ be the Lie algebra of H and consider the following subalgebra of $\mathfrak{h}$,

$$\mathfrak{k} := \left\{\eta \in \mathfrak{h} \,|\, \eta_M(z) \in (T_z\mathcal{O}_z^H)^\omega\right\}$$

where η_M is the generating vector field of η. Let $\mathfrak{i}$ be the Lie algebra of H_z and note that $\mathfrak{i} \subset \mathfrak{k}$. Denote by $\mathfrak{m}$ an Ad_{H_z}-invariant complement of $\mathfrak{i}$ in $\mathfrak{k}$. Then the twisted product

$$Y_z^H := H \times_{H_z} (\mathfrak{m}^* \times V_z) \tag{1}$$

is a symplectic H-space and can be chosen such that there is an H-invariant neighbourhood $\mathcal{U}$ of z in M, an H-invariant neighbourhood $\mathcal{U}'$ of $[e, 0, 0]$ in Y_z^H and an equivariant symplectomorphism $\phi : \mathcal{U} \to \mathcal{U}'$ satisfying $\phi(z) = [e, 0, 0]$. Equipping the bundle Y_z^H with coordinates $[k, \eta, v]$ for $k \in H, \eta \in \mathfrak{m}^*$ and $v \in V_z$, H acts on Y_z^H as $h \cdot [k, \eta, v] = [h \cdot k, \eta, v]$.

In the case that the action is symplectic, the symplectic form on the quotient bundle is called the MGS-normal form[1] and denoted by ω_{MGS}. In order to prove a b-symplectic analogue of Theorem 1, we show that b-symplectic manifolds equipped with b-symplectic actions transverse to the symplectic foliation possess a finite cover which is a product. The slice theorem then reduces to a "product slice theorem" modulo the action of a finite group. This linearized model has an additional invariant compared to the symplectic one: the modular period of the component of the critical hypersurface where the orbit lies. We will prove:

Theorem 2. *Let (M, ω, G) be a b-symplectic manifold together with an effective b-symplectic action by a compact connected Lie group G. Let Z be the critical set of the b-symplectic form. Assume that Z is compact and connected and that the orbits of G are transverse to the symplectic foliation of Z. Let $\mathcal{L}$ be a symplectic leaf of Z. Then*

(i) G is necessarily of the form $G = (S^1 \times H)/\mathbb{Z}_k$ where H is a compact, connected Lie group.
(ii) The action of G lifts to an action of the product group $\tilde{G} = S^1 \times H$ on a finite cover $\tilde{M}$ of a collar neighbourhood of Z, $\tilde{M} := (-\epsilon, \epsilon) \times \tilde{Z}$, $\tilde{Z} \cong S^1 \times \mathcal{L}$, where S^1 acts on $\tilde{Z}$ by translations on the S^1-factor and H by symplectomorphisms on the symplectic leaf $\mathcal{L}$.

[1]MGS for Marle [17] and Guillemin-Sternberg [13]

(iii) *Let $\tilde{z} \in \tilde{Z}$. Denote by $\mathcal{O}_{\tilde{z}}^{\tilde{G}}$ the orbit of $\tilde{z}$ in $\tilde{Z}$ under the action of $\tilde{G}$ and by Y_z^H the bundle of Theorem 1 associated to the action of H on $\mathcal{L}$. Then there is an equivariant b-symplectomorphism from a neighbourhood of the orbit $\mathcal{O}_{\tilde{z}}^{\tilde{G}} \cong S^1 \times \mathcal{O}_{\tilde{z}}^H$ to the zero section of the bundle $\tilde{E} = T^*S^1 \times Y_z^H$ where $\tilde{G}/H_{\tilde{z}}$ is embedded as the zero section and the b-symplectic form on $\tilde{E}$ is given by*

$$\tilde{\omega}_0 = \omega_{c'} + \omega_{MGS}.$$

Here $\omega_{c'}$ is the standard b-symplectic form of modular period $c' = kc$ on the manifold T^*S^1, see Equation (5), c is the modular period of Z and ω_{MGS} is the MGS normal form as given by Theorem 1.

(iv) *Let $\mathcal{O}_z$ be the orbit of $z \in Z$ under the action of G. There is an equivariant b-symplectomorphism from a neighbourhood of $\mathcal{O}_z$ to a neighbourhood of the zero section of the bundle $E = (T^*S^1 \times Y_z^H)/\mathbb{Z}_l$ where $\mathbb{Z}_l$ is a finite cyclic group acting by the cotangent lifted action on T^*S^1.*

An important step in the proof is the analysis of the group action along the critical set, Z which is naturally endowed with a cosymplectic structure. To achieve the proof we first analyse the consequences of a cosymplectic manifold having a group action transverse to the symplectic foliation. In particular, we prove that given a cosymplectic action of a group G then there are two distinct cases:

(1) G is a group isomorphic to the product of Lie groups $G = S^1 \times H$ or
(2) $G = (S^1 \times H)/\Gamma$ where where $\Gamma \subset S^1 \times H$ is of the form $\Gamma = \mathbb{Z}_l \times \mathbb{Z}_k$ and $\mathbb{Z}_k$ is a non-trivial cyclic subgroup of H.

We will prove that there is a finite covering $\tilde{Z}$ of the cosymplectic manifold Z which is trivial (in the sense that $\tilde{Z} \cong S^1 \times \mathcal{L}$) and this finite covering comes equipped with an $S^1 \times H$-action which projects to the action of G on Z. Examples of cosympletic manifolds with cosymplectic symmetries include in particular co-Kähler manifolds as discussed in [1], which inspired some techniques used here.

We remark that the aim here is to show the rigidity of b-symplectic group actions, for which the group action and symplectic form are completely determined in a neighbourhood of an orbit by the isotropy group and its representation on the symplectic normal space. Therefore Theorem 2 does not reference the traditional moment map sometimes given as part of the symplectic slice theorem, although there does exist a generalization of the moment map to the b-symplectic case which could, in theory, be used to extend the theorem further.

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2. Preliminiaries

2.1. **Introduction to b-symplectic geometry.** We briefly recall the basics of b-symplectic geometry, see [10] for details.

A b**-manifold** is a pair (M, Z) of an oriented manifold M and an oriented hypersurface $Z \subset M$. The hypersurface Z is called the *critical hypersurface*.

A b**-vector field** on a b-manifold (M, Z) is a vector field which is tangent to Z at every point $p \in Z$.

If a is a local defining function for the hypersurface Z on some open set $U \subset M$ and $(a, z_2, \ldots, z_n)$ is a chart on $\mathcal{U}$, then the set of b-vector fields on $\mathcal{U}$ is a free $C^\infty(U)$-module with basis

$$\left(a\frac{\partial}{\partial a}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n} \right). \tag{2}$$

The corresponding vector bundle, which exists by the Serre-Swan theorem [25], is the *b-tangent bundle*:

Definition 3. *The b-**tangent bundle**, ${}^bTM$, is the vector bundle whose sections are b-vector fields.*

The classical exterior derivative d on the complex of (smooth) k-forms extends to the complex of b-forms in a natural way. Any b-form of degree k can locally be written

$$\omega = \alpha \wedge \frac{da}{a} + \beta$$

where $\alpha \in \Omega^{k-1}, \beta \in \Omega^k$, a is a local defining function of Z and $\frac{da}{a}$ is the b-one-form dual to $a\frac{\partial}{\partial a}$ in a frame of the form (2). The exterior derivative of ω is then given by

$$d\omega := d\alpha \wedge \frac{da}{a} + d\beta.$$

Definition 4. *A b-symplectic form is a b-form of degree 2 which is closed and non-degenerate as a b-form.*

If Z is the critical hypersurface of a b-symplectic form, it can be shown that Z has a codimension-one foliation by symplectic leaves (see [9]). The hypersurface Z is then cosymplectic as studied in [16].

We recall the following notions for symplectic codimension one foliations given in [9]:

Definition 5. *Let $\mathcal{F}$ be a codimension one symplectic foliation of a manifold Z. A form $\alpha \in \Omega^1(Z)$ is a **defining one-form** of $\mathcal{F}$ if it is nowhere vanishing and $\iota_{\mathcal{L}}^*\alpha = 0$ for all leaves $\mathcal{L}$, where $\iota_{\mathcal{L}}$ is the inclusion $\mathcal{L} \hookrightarrow Z$, i.e. the kernel of α at any point $p \in Z$ is the tangent space of the leaf through p.*

*A form $\omega \in \Omega^2(Z)$ is a **defining two-form** of $\mathcal{F}$ if $\iota_{\mathcal{L}}^*\omega$ is the given symplectic form on each leaf of the foliation.*

If Z is the critical hypersurface of a b-symplectic manifold, then the defining one- and two-form of the induced symplectic foliation can be chosen to be *closed* [9]. Conversely, a manifold Z with a codimension one symplectic foliation that admits closed defining one- and two-form α resp. β can be extended to a b-symplectic manifold $M = Z \times \mathbb{R}$ with b-symplectic form

$$\omega = \pi_Z^*\alpha \wedge \frac{da}{a} + \pi_Z^*\beta.$$

where $\pi_Z : Z \times \mathbb{R} \to Z$ is the canonical projection and a the coordinate on $\mathbb{R}$.

b-Symplectic manifolds can also be viewed dually as a particular class of Poisson manifolds. As such they have a modular vector field:

Definition 6. *Let (M, Π) be a Poisson manifold equipped with a volume form Ω and for each $f \in C^\infty(M)$ denote by X_f the Hamiltonian vector field associated to f. Then the **modular vector field** of (M, Π) is the following derivation on $C^\infty(M)$:*

$$v_{mod} : C^\infty(M) \to \mathbb{R} : f \mapsto \frac{\mathcal{L}_{X_f}\Omega}{\Omega}.$$

It can be shown that the modular vector field is a Poisson vector field and that the modular vector fields associated to different volume forms only differ by a Hamiltonian vector field. Having chosen a modular vector field v_{mod}, we can choose defining one and two-forms of the symplectic foliation on Z uniquely by imposing

$$\alpha(v_{mod}) = 1 \text{ and } \iota_{v_{mod}}\omega = 0. \tag{3}$$

We will call defining one- and two-forms fulfilling this condition *the* defining one- and two-forms of the foliation. They are automatically closed [9].

Finally, we note that the flow of the modular vector field can be used to define the mapping torus structure of Z and define the *modular period* of the b-symplectic form as follows (cf. [9]):

Definition 7. *Let (M, Z) be a b-symplectic manifold and suppose that Z is compact and connected and that its symplectic foliation has a compact leaf $\mathcal{L}$. Then Z is a mapping torus. More precisely, taking any modular vector field v_{mod}, there exists a number $c > 0$ such that*

$$Z \cong \frac{[0, c] \times \mathcal{L}}{(0, x) \sim (c, \phi(x))}$$

where the time t-flow of v_{mod} corresponds to translation by t in the first coordinate. In particular, ϕ is the time c-flow of v_{mod}. The number $c > 0$ is called the ***modular period*** *of Z and does not depend on the choice of modular vector field v_{mod}.*

For notational convenience we will consider rather $t \in [0, 1]$. The modular vector field is then given by

$$v_{mod} = \frac{1}{c}\frac{\partial}{\partial t}$$

and the defining one-form is given by

$$\alpha = cdt.$$

The b-analogue of the Moser theorem for symplectic manifolds is proved in [10].

Theorem 8 (**b-Moser Theorem**). *Let ω_0 and ω_1 be two b-symplectic forms on (M, Z). If they induce on Z the same corank one Poisson structure and their modular vector fields differ on Z by a Hamiltonian vector field, then there exist neighbourhoods U_0, U_1 of Z in M and a diffeomorphism $\gamma : U_0 \to U_1$ such that $\gamma|_Z = id_Z$ and $\gamma^*\omega_1 = \omega_0$.*

The condition that ω_0 and ω_1 induce the same Poisson structure on Z and the same modular vector field (up to a Hamiltonian vector field) is equivalent to demanding that the induced symplectic foliations have the same defining one- and two-forms.

A consequence is the following semilocal model [10]:

Corollary 9 (Extension Theorem). *Let (M, Z) be a b-symplectic manifold where Z is compact and connected. Then there is a neighbourhood of Z in M that is b-symplectomorphic to the collar neighbourhood $Z \times (-\epsilon, \epsilon)$ with b-symplectic form*

$$\omega = \alpha \wedge \frac{da}{a} + \pi_Z^*\beta. \tag{4}$$

where α, β are the defining one- resp. two-forms on Z, a is the coordinate on the interval $(-\epsilon, \epsilon)$ and π_Z the projection of the collar to Z.

As noted in [11], by averaging the vector fields of the b-Moser theorem, given two b-symplectic forms invariant under a group action and b-symplectomorphic by the b-Moser theorem, we can choose the b-symplectomorphism to be equivariant with respect to the group action. In the special case where M is two-dimensional this yields the following semilocal normal form:

Proposition 10. *Let (M, Z) be a two-dimensional b-symplectic manifold with compact connected critical hypersurface Z and modular period $c > 0$. Then $Z \cong S^1$ and there exists a neighbourhood of Z which is b-symplectomorphic to $S^1 \times (-\epsilon, \epsilon)$ with b-symplectic form*

$$\omega_c := cdt \wedge \frac{da}{a}. \tag{5}$$

Here (t, a) are the standard coordinates on $S^1 \times \mathbb{R}$.

It will be convenient to view the b-symplectic manifold $S^1 \times (-\epsilon, \epsilon)$ as a neighbourhood of the zero section of the cotangent bundle $T^*S^1 \cong S^1 \times \mathbb{R}$ with b-symplectic form given by the formula in Equation (5). We also remark for future purposes that ω_c is clearly invariant under the cotangent lift of the action of S^1 on itself by translations.

2.2. **Transversally equivariant fibrations.** A bundle map $\pi : Z \to S^1$ is a *transversally equivariant fibration* if there is a smooth S^1 -action on Z such that the orbits of the action are transversal to the fibres of π and $\pi(t\cdot x)-\pi(x)$ depends on $t \in S^1$ only. The following is a specialization of a theorem by Sadowski which was applied to the case of co-Kähler manifolds in [1].

Theorem 11. *Let $Z \stackrel{\pi}{\to} S^1$ be a smooth bundle projection from a closed manifold Z to the circle. The following are equivalent:*

(1) $Z \stackrel{\pi}{\to} S^1$ is a mapping torus associated to a diffeomorphism of finite order
(2) The bundle map π is transversally equivariant with respect to an S^1-action on Z, $\rho : S^1 \times Z \to Z$.

Let $\mathcal{L}$ be the fibre of π. If the above conditions are satisfied then Z has a $\mathbb{Z}_k$-cover ($k \in \mathbb{N}$)

$$p : \tilde{Z} = S^1 \times \mathcal{L} \to Z$$

given by the action $(t, l) \mapsto \rho_t(l)$, where $\mathbb{Z}_k$ acts diagonally on $S^1 \times \mathcal{L}$ and by translations on S^1.

Explicitly, we can describe the $\mathbb{Z}_k$ action as follows: Consider the leaf-fixing subgroup of S^1,

$$\Gamma = \{s \in S^1 \ : \ \rho_s(\mathcal{L}) = \mathcal{L}\}. \tag{6}$$

Identifying $S^1 \cong \mathbb{R} \bmod 1$, the group Γ is of the form $\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\}$ for some $k \in \mathbb{N}$ and hence we can identify it with $\mathbb{Z}_k$ in the natural way. Then for $m \in \mathbb{Z}_k = \{0, 1, \ldots, k-1\}$, the action $\rho_{\frac{m}{k}}$ restricts to a leaf automorphism

$$\sigma_m : \mathcal{L} \to \mathcal{L}, \quad \sigma_m(l) = \rho_{\frac{m}{k}}(l). \tag{7}$$

The mapping torus Z is then the quotient of the cover $\tilde{Z}$ by the following action of $\mathbb{Z}_k$ on $\tilde{Z}$

$$\mu_m(t, l) = (t - \frac{m}{k}, \sigma_m(l)), \quad m \in \mathbb{Z}_k,\, (t, l) \in \tilde{Z} = S^1 \times \mathcal{L}. \tag{8}$$

From the condition of transverse equivariance, it is clear that ρ maps leaves to leaves. It induces an action on the base S^1 given by translations $t \mapsto t + ks$ and the equivariance condition reads

$$\pi(\rho_s(l)) = ks, \quad l \in \mathcal{L} := \pi^{-1}(\{0\}).$$

There is an associated S^1-action $\tilde{\rho}$ on the cover $\tilde{Z}$ given by

$$\tilde{\rho}_s(t, l) = (t + s, l), \quad s \in S^1, (t, l) \in S^1 \times \mathcal{L}. \tag{9}$$

The projection $\tilde{Z} \to Z$ is equivariant with respect to this action.

The existence of a finite trivializing cover of the critical hypersurface Z will play a crucial role in the b-symplectic slice theorem.

3. A TRIVIALIZING COVER FOR THE CRITICAL HYPERSURFACE

Now we consider (M, Z) a b-symplectic manifold. As we focus on a semi local result, we will assume $M \cong Z \times (-\epsilon, \epsilon)$ where the critical hypersurface Z is compact and connected with b-symplectic form given by Equation (4). On a semilocal level the last assumption is not an additional restriction, since as we have seen in the previous section, any b-symplectic manifold satisfying the previous conditions is of this form on a tubular neighbourhood of its critical hypersurface. We will further assume that Z has a compact leaf.

Definition 12. *A group action on a b-symplectic manifold is called* transverse *if it is transverse to the symplectic foliation of the critical hypersurface. If the action, restricted to the critical hypersurface, preserves the cosymplectic structure we will call the action* cosymplectic*. Finally, if the action preserves the b-symplectic form we will call the action* b-symplectic.

Cosymplectic and b-symplectic actions are special cases of Poisson actions, when considering the manifolds with the associated Poisson structures.

As cosymplectic actions are automatically transversely equivariant the next proposition follows directly from Theorem 11:

Proposition 13. *Let Z be a cosymplectic manifold and suppose Z has a transverse S^1-action preserving the cosymplectic structure. Then Z has a finite cover $\tilde{Z} := S^1 \times \mathcal{L}$, $\mathcal{L}$ a leaf of the foliation, equipped with an S^1 action given by translation in the first coordinate for which the projection $p : S^1 \times \mathcal{L} \to Z$ is equivariant.*

To get a cosymplectic structure on the cover, one simply lifts the associated defining one and two-forms.

Proposition 14. *In the setting of the previous proposition, the cosymplectic structure on Z is given by the quotient of a cosymplectic structure on $\tilde{Z} = S^1 \times \mathcal{L}$ by the action of a finite cyclic group $\mathbb{Z}_k$.*

Proof. Let $p : \tilde{Z} \to Z$ be the finite cover given by Proposition 13. Denote the one and two forms of the cosymplectic structure by α and β respectively. Then $\tilde{\beta} = p^*\beta$ and $\tilde{\alpha} = p^*\alpha$ can easily be shown to define a cosymplectic structure on $S^1 \times \mathcal{L}$ and by construction, the cosymplectic structure on the quotient agrees with the cosymplectic structure on Z. □

To extend this cover to a b-symplectic neighbourhood of Z we simply use the extension theorem (Corollary 9):

Corollary 15. *Let $M = Z \times (-\epsilon, \epsilon)$ come equipped with a transverse S^1-action preserving the b-symplectic form ω. Then the b-symplectic structure on M is b-symplectomorphic in a neighbourhood of Z to the quotient of a b-symplectic structure on $S^1 \times \mathcal{L} \times (-\epsilon, \epsilon)$ by the action of a finite cyclic group.*

Proof. As before let $p : \tilde{Z} \to Z$ be the finite cover. Let v_{mod} be some choice of modular vector field and denote the defining one and two-forms of Z fulfilling the condition in Equation (3) by α and β respectively. Denote by $\tilde{\alpha}, \tilde{\beta}$ the corresponding one and two forms defined in Proposition 14. By the extension theorem we can assume that the b-symplectic form on M is

$$\omega = \pi_Z^* \alpha \wedge \frac{da}{a} + \pi_Z^* \beta.$$

Let $\tilde{M} := \tilde{Z} \times (-\epsilon, \epsilon)$. Then we have a finite cover $p_M : \tilde{M} \to M$ for M given by the product map of the cover $p : \tilde{Z} \to Z$ and the identity on $(-\epsilon, \epsilon)$. Let $\pi_{\tilde{Z}} : \tilde{M} \to \tilde{Z}$ be the projection onto the first factor. Define for $a \in (-\epsilon, \epsilon)$ the b-symplectic form on $\tilde{M}$

$$\tilde{\omega} = \pi_{\tilde{Z}}^* \tilde{\alpha} \wedge \frac{da}{a} + \pi_{\tilde{Z}}^* \tilde{\beta}.$$

Then by construction $(p_M)^*\omega = \tilde{\omega}$. □

Remark 16. *Note that the modular period of the associated b-symplectic form on the $\mathbb{Z}_k$ cover is k times the modular period of the b-symplectic form on the base.*

Remark 17. *Similarly, any b-symplectic structure with defining one and two-forms $\tilde{\alpha}$ and $\tilde{\beta}$ equipped with a discrete b-symplectic group action gives a b-symplectic structure on the quotient. For such a group action there are well defined one and two-forms, α and β, on the base manifold defined by $p^*(\alpha) = \tilde{\alpha}$ and $p^*(\beta) = \tilde{\beta}$, where p is the projection to the quotient. Then α and β automatically fulfil the conditions to define a cosymplectic structure on the image of the critical hypersurface. As the group action is discrete, the quotient of the symplectic structure on leaves is likewise symplectic.*

4. THE b-SYMPLECTIC SLICE THEOREM FOR AN S^1-ACTION

First, we wish to simplify the expression of the b-symplectic form in the neighbourhood of an orbit. In the case that the leaf $\mathcal{L}$ is simply connected, the b-symplectic form has a particularly simple expression.

Proposition 18. *Let $M \cong Z \times (-\epsilon, \epsilon)$ be a b-symplectic manifold and suppose that Z is a product, $Z \cong S^1 \times \mathcal{L}$, $\mathcal{L}$ a leaf of the symplectic foliation. Suppose furthermore that $\mathcal{L}$ is simply connected. Then for a suitable defining function f of Z the b-symplectic form is given by*

$$\omega = c dt \wedge \frac{df}{f} + \pi_{\mathcal{L}}^*(\beta) \tag{10}$$

where t is the standard coordinate on S^1, β is the symplectic form on $\mathcal{L}$ and $\pi_{\mathcal{L}}$ is the projection $S^1 \times \mathcal{L} \to \mathcal{L}$.

Proof. A b-symplectic form on on $S^1 \times \mathcal{L} \times (-\epsilon, \epsilon)$ equipped with coordinates (t, l, a) can be written

$$\omega = c dt \wedge \frac{da}{a} + dt \wedge \eta + \pi_{\mathcal{L}}^*(\beta)$$

where β is the symplectic form on $\mathcal{L}$. Since $\mathcal{L}$ is simply connected, $\eta = dh$ for some $h \in C^\infty(M)$. The function $f = ae^h$ is then a defining function for Z and moreover

$$\frac{df}{f} = \frac{da}{a} + dh$$

Whence we have

$$\omega = c dt \wedge \frac{df}{f} + \pi_{\mathcal{L}}^*(\beta).$$

□

As in the symplectic slice theorem, the normal form of a b-symplectic form in the neighbourhood of an orbit is given by virtue of an equivariant Moser theorem. Equivariant b-Moser theorems for isotopic forms invariant under S^1-actions have been given in [11] and for more general groups in [20]. As we wish to compare b-symplectic forms in the neighbourhood of an orbit rather than on the whole of Z we require an equivariant b-Moser theorem of a slightly different nature:

Proposition 19. *Suppose that ω_1 and ω_0 are b-symplectic forms on M, invariant under an action of a group G on M which is transverse Poisson for ω_1 and ω_0. Denote by $\mathcal{O}_z$ the orbit of some $z \in Z$ and suppose that ω_1 and ω_0 coincide at z. Then ω_1 and ω_0 are equivariantly b-symplectomorphic in some neighbourhood $\mathcal{U}$ of $\mathcal{O}_z$.*

Proof. As the defining one and two-forms associated to ω_1 and ω_0 are invariant under the S^1 action, it follows that on $\mathcal{O}_z$ we have $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$. By the relative Poincaré lemma, in a contractible neighbourhood of $\mathcal{O}_z$ we have that $\alpha_0 - \alpha_1 = dg$, an exact one-form on $\mathcal{U}$ and similarly $\beta_0 - \beta_1 = d\eta$, an exact two-form on $\mathcal{U}$. Whence $\omega_0 - \omega_1 = d(-g\frac{df}{f} + \eta)$. Then $\omega_t = \omega_0 + (1-t)\omega_1$ is non degenerate on $\mathcal{O}_z$ and so on a neighbourhood of $\mathcal{O}_z$. We use this to define a b-vector field v_t by $\iota_{v_t}\omega_t = g\frac{df}{f} - \eta$. As v_t is zero on $\mathcal{O}_z$, the time-one flow exists in a neighbourhood of $\mathcal{O}_z$ and gives the required b-symplectomorphism. As both b-symplectic forms are invariant under the group action, we can choose the b-symplectomorphism to be equivariant. □

Theorem 20. *Let $M \cong (-\epsilon, \epsilon) \times Z$ be a b-symplectic manifold equipped with a b-symplectic form ω of modular period c and a transverse b-symplectic S^1-action. Let $z \in \mathcal{L} \subset Z$, let $\mathcal{O}_z$ be its orbit under the S^1 action, let $V := T_z\mathcal{L}$ and let $\mathbb{Z}_l$ be the isotropy group of z. Then there exists an S^1-equivariant neighbourhood $(-\epsilon, \epsilon) \times \mathcal{U}$ of $\mathcal{O}_z$ in M and an S^1-equivariant mapping*

$$\phi : (-\epsilon, \epsilon) \times \mathcal{U} \to (T^*S^1 \times V)/\mathbb{Z}_l \tag{11}$$

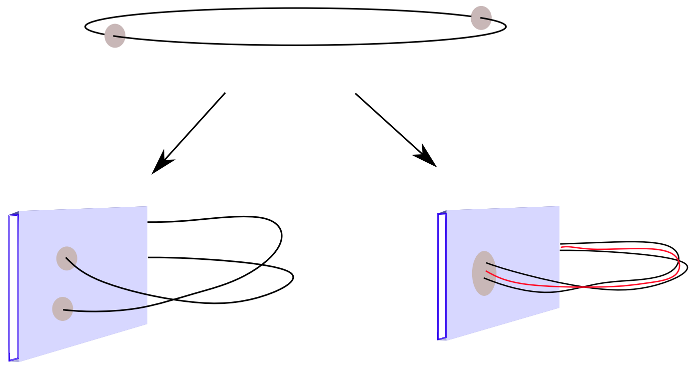

FIGURE 1. A scheme of the trivializing finite cover with a regular orbit ($\Gamma_z = 0$) in black and exceptional orbit ($\Gamma_z = \mathbb{Z}_2$) in red.

where $\mathcal{O}_z$ is embedded as the zero section of the bundle $S^1 \times_{\mathbb{Z}_l} \mathbb{R} \times V \cong (T^*S^1 \times V)/\mathbb{Z}_l$ and where the action of $\mathbb{Z}_l$ is given by the cotangent lifted action on T^*S^1 and by the isotropy representation on V.

Moreover, if we equip the bundle $T^*S^1 \times V$ with the b-symplectic form:

$$\tilde{\omega}_0 = \omega_{c'} + \omega_V$$

where $\omega_{c'}$ the b-symplectic normal form on T^*S^1 as given in Definition 5 with modular period $c' = kc$ and ω_V the linear symplectic form on V, and the quotient $(T^*S^1 \times V)/\mathbb{Z}_l$ with the quotient b-symplectic form ω_0 (see Remark 17) then the mapping becomes an equivariant b-symplectomorphism onto its image.

Proof. Let $z \in Z$ be a point in the critical set and $\mathcal{O}_z$ the orbit of z under the S^1 action ρ. Denote by Γ_z the isotropy group of z. Note that Γ_z is automatically a subgroup of $\mathbb{Z}_k$ and so $\Gamma_z \cong \mathbb{Z}_l$ for some l. By the slice theorem there exists a neighbourhood $\mathcal{U}$ of $\mathcal{O}_z$ in Z equivariantly diffeomorphic to a neighbourhood of the zero section of the vector bundle $S^1 \times_{\Gamma_z} T_zZ/T_z\mathcal{O}_z$, where S^1 acts on the latter according to $s \cdot [t, v] = [t + s, v]$. By choosing the invariant Riemannian metric in the proof of the slice theorem in such a way that $T_z\mathcal{L}$ is orthogonal to $T_z\mathcal{O}_z$, the equivariant diffeomorphism can be expressed

$$S^1 \times_{\Gamma_z} T_z\mathcal{L} \to \mathcal{U} : [t, v] \mapsto \rho_t(\exp_z v).$$

Denote by ψ the corresponding diffeomorphism on the neighbourhood $(-\epsilon, \epsilon) \times \mathcal{U}$ of $\mathcal{O}_z$ in M:

$$\psi : (-\epsilon, \epsilon) \times \mathcal{U} \to (-\epsilon, \epsilon) \times S^1 \times_{\Gamma_z} T_z\mathcal{L}.$$

Restricting the defining one and two forms of ω to $\mathcal{U}$, we have that $\mathcal{U}$ is a cosymplectic manifold with a cosymplectic S^1-action. The symplectic leaves of $\mathcal{U}$ are given by $\mathcal{L}_{\mathcal{U}} := \mathcal{U} \cap \mathcal{L}$ and the leaf fixing subgroup as defined by Equation (6) is Γ_z. By Proposition 13 there is a trivial Γ_z-cover $\tilde{\mathcal{U}} \cong S^1 \times \mathcal{L}_{\mathcal{U}}$ of $\mathcal{U}$. Then $\omega|_{(-\epsilon,\epsilon)\times\mathcal{U}}$ is the quotient of a unique b-symplectic form $\tilde{\omega}$ on $(-\epsilon, \epsilon) \times \tilde{\mathcal{U}}$ as given by Corollary 15. By Proposition 18 we may assume $\tilde{\omega}$ is of the form

$$\tilde{\omega} = ckdt \wedge \frac{da}{a} + \pi^*_{\mathcal{L}_{\mathcal{U}}}\beta$$

where $a \in (-\epsilon, \epsilon)$ and β is a symplectic two-form given on a leaf $\mathcal{U}_{\mathcal{L}}$. Consider the two form β_z on $T_z\mathcal{L}$. On $(-\epsilon, \epsilon) \times S^1 \times T_z\mathcal{L}$ define the b-symplectic form

$$\tilde{\omega}_0 = ckdt \wedge \frac{da}{a} + \beta_z.$$

Denote the quotient b-symplectic form on $((-\epsilon, \epsilon) \times S^1 \times T_z\mathcal{L}))/\Gamma_z$ given in Remark 15 by ω_0. Finally consider the b-symplectic form $\psi^*\omega_0$ on $(-\epsilon, \epsilon) \times \mathcal{U}$. This is a b-symplectic structure, invariant under the S^1 action agreeing with ω at z. By Theorem 19 there is an equivariant b-symplectomorphism φ

between neighborhoods of $\mathcal{O}_z$ such that $\varphi^*(\psi^*\omega_0) = \omega$. Making $\mathcal{U}$ smaller if necessary and setting $\phi = \psi \circ \varphi$ we obtain the b-symplectomorphism given in the statement of the theorem. □

Remark 21. *Note that the modular period of the form ω_0 is $\frac{k}{l}c$ where c is the modular period of the b-symplectic form. This is not necessarily the modular period of the original form ω.*

Example 22. *Consider the following symplectic mapping torus: take as a symplectic leaf a torus $\mathbb{T}^2$ with coordinates (φ, ψ), $\varphi, \psi \in \mathbb{R} \bmod 1$ equipped with the standard symplectic form and the holonomy map given by the diffeomorphism of $\mathbb{T}^2$ which descends from the diffeomorphism of $\mathbb{R}^2$ given by $\phi \in \mathrm{GL}(2, \mathbb{Z})$:*

$$\phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The mapping on $\mathbb{R}^2$ corresponds to rotation by $\frac{\pi}{2}$ and so we have $\phi^4 = Id$. Denote the mapping torus $Z = ([0, 1] \times \mathbb{T}^2)/(0, x) \sim (1, \phi(x))$.

Consider the following b-symplectic form on $(t, \varphi, \psi, s) \in Z \times S^1$:

$$\omega = dt \wedge \frac{ds}{\sin(s)} + \beta$$

where β is the standard symplectic form on $\mathbb{T}^2$. Consider the action of S^1 on $Z \times S^1$ given by translation in the t-coordinate. Then there is a neighbourhood of a regular orbit contained in Z which is equivariantly diffeomorphic to a neighbourhood of the zero section $(t, \boldsymbol{0})$ of $S^1 \times \mathbb{R}^3$ where S^1 acts by translations on the S^1 factor of $S^1 \times \mathbb{R}^3$. Moreover, there exist coordinates (t, x, y, a) on $S^1 \times \mathbb{R}^3$ so that the equivariant diffeomorphism becomes a symplectomorphism where $S^1 \times \mathbb{R}^3$ is equipped with the b-symplectic form

$$\omega = 4dt \wedge \frac{da}{a} + dx \wedge dy \tag{12}$$

On the critical set there is also the exceptional orbit at $\phi = \psi = 0$. In a neighbourhood of the singular orbit the b-sympletic form is the quotient of the b-symplectic structure (12) *given above where the group action $\sigma_n \in GL(2, \mathbb{Z})$ on the vector space (x, y) is given by*

$$\sigma_n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n.$$

Example 23. *We can find examples from integrable systems having a naturally associated S^1-action model with non-trivial isotropy group.*

Take $M = T^*S^1 \times \mathbb{R}^2$ endowed with coordinates (p, t, x, y) and b-symplectic form $\omega = \frac{1}{p} dp \wedge dt + dx \wedge dy$. Consider the b-integrable system on M given by $F = (\log(p), xy)$. This b-integrable system has hyperbolic singularities. Now let $\mathbb{Z}/2\mathbb{Z}$ act on M in the following way: $(-1) \cdot (p, t, x, y) = (p, t, -x', -y')$ observe that this action leaves the hyperbola $xy = cnst$ invariant and switches its branches. The action clearly preserves the b-integrable system and induces a new integrable system on the quotient space $M/\sim$. Observe that the first component of the integrable system naturally induces an S^1-action given by the b-symplectic vector field associated to the singular Hamiltonian function $\log(p)$ *(named as b-function, see* [11] *for a discussion). This circle action also descends to the quotient and the model for the circle action has non-trivial isotropy group of order two.*

This twisted hyperbolic case in b-symplectic manifolds is a reminiscent of the twisted hyperbolic construction in the symplectic case in [5] *and* [19][2] *and it is an invitation to study the invariants of a non-degenerate singularity of a b-symplectic manifold. This example can be extended to higher dimensions and the action of a $\mathbb{Z}/2\mathbb{Z}$ can be considered for every hyperbolic block added as long as the corank of the singularity is equal or bigger than one. The situation can be visualized using the* curled torus, *the picture below showing the structure of the set $p = 0, xy = 0$.*

[2]This example shows up in physical examples and corresponds to the 1:2 resonance (see for instance the example in page 32 of the monograph [6])

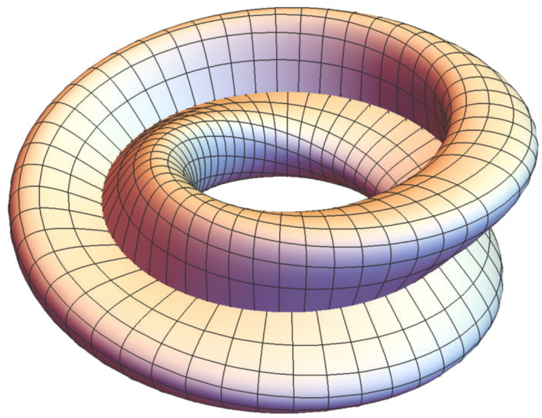

FIGURE 2. The Curled Torus. Source: Konstantinos Efstathiou.

5. ACTIONS OF COMPACT LIE GROUPS ON COSYMPLECTIC MANIFOLDS

We treat the case of more general group actions on a b-symplectic manifold close to the critical set. First, we prove that only groups of a particular form can act on a b-symplectic manifold. For now we will treat group actions on a mapping torus Z and then extend the results to a neighbourhood of the critical set.

In the following, we assume that the group G is compact and connected and acts on a mapping torus Z via a transverse, effective and foliation preserving action ρ. For a more general treatment of the lifting of group actions see [22].

Proposition 24. *Suppose an element $h \in G$ fixes a leaf of the mapping torus, $\rho_h(\mathcal{L}_0) = \mathcal{L}_0$. Then h fixes every leaf of the mapping torus.*

This is an easy consequence of the fact that compact connected subgroups of $\text{Diffeo}^+(S^1)$ are conjugate to $\text{SO}(2)$, which is itself a consequence of $\text{Diffeo}^+(S^1)$ having a unique maximal compact subgroup, see [7] for the case of orientation preserving homeomorphisms which can be adapted mutatis mutandis for the smooth case.

Proof. Let $\pi : Z \to S^1$ be the mapping torus projection. The action of the group G on a symplectic mapping torus Z induces an action of G on the base S^1 in the obvious fashion

$$\tau : G \times S^1 \to S^1 \tag{13}$$

$$(g, \pi(x)) \mapsto \pi(\rho_g(x)) =: \tau_g(x) \tag{14}$$

As G is compact and connected its image $\tau(G, \cdot)$ is a compact subgroup of $\text{Diffeo}^+(S^1)$, the group of orientation preserving diffeomorphisms of the circle. Whence $\tau(G, \cdot)$ is conjugate by some $w \in \text{Diffeo}^+(S^1)$ to $\text{SO}(2)$. Suppose $h \in G$ fixes a leaf $\mathcal{L}_0$. This corresponds to a fixed point of the induced action τ_h on S^1, and so a fixed point for $w\tau_h w^{-1} \in \text{SO}(2)$. Whence $w\tau_h w^{-1} = \text{Id}_{S^1}$ and so $\tau_h = \text{Id}_{S^1}$. This corresponds to h fixing all leaves of Z. □

It can be checked easily that this defines a subgroup of G. We call

$$H_0 = \{h \in G \,|\, \rho_h(\mathcal{L}_0) = \mathcal{L}_0\}$$

the *leaf preserving* subgroup of G.

Proposition 25. *Let G be a group acting in a transverse and foliation preserving manner on a symplectic mapping torus. Let H_0 be the leaf preserving subgroup of G. Then*
(i) H_0 is a normal subgroup of G.
(ii) H_0 is a closed Lie subgroup of G.
(iii) The codimension of H_0 in G is one.

Proof. (i) This follows immediately from the fact that for $h \in H_0$, $g \in G$ we have $\tau_{ghg^{-1}} = \tau_g \tau_h \tau_g^{-1} = \tau_g \tau_g^{-1} = \mathrm{Id}_{S^1}$, hence $ghg^{-1} \in H_0$.
(ii) Consider the projection

$$\Phi : G \to \mathrm{SO}(2)$$
$$\Phi(g) = w \tau_g w^{-1}$$

corresponding to the map from G to $\mathrm{SO}(2)$ given in Proposition 24. It is clear that the level set $\Phi^{-1}(\mathrm{Id})$ consists precisely of the elements of G which are leaf preserving. Hence $\Phi^{-1}(\mathrm{Id}) = H_0$ is a closed subgroup of G.
(iii) The codimension of H_0 is at most one since it is given as the level set $\Phi^{-1}(\mathrm{Id}) = H_0$. As G induces an action transverse to the foliation of Z it follows that the codimension of H_0 is exactly one.

□

Proposition 26. *The action of G on the mapping torus Z lifts to an action of a product group $\tilde{G} = S^1 \times H$ on a finite trivializing cover of Z where H is compact and connected. Moreover, G is necessarily of the form $G = (S^1 \times H)/\Gamma$ for a finite cyclic subgroup Γ (which might be trivial).*

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of H_0, the leaf preserving subgroup of G and consider a complementary ideal $\mathfrak{k}$ of $\mathfrak{h}$ in $\mathfrak{g}$ such that the subgroup $K = \exp(\mathfrak{k})$ is closed. K is then a one dimensional closed subgroup of the compact group G and so $K \cong S^1$. The action of K is transverse to the foliation and so by Proposition 13 there exists a finite trivializing cover $\tilde{Z} \cong S^1 \times \mathcal{L}$ of Z, such that Z is the quotient of $\tilde{Z}$ by the action of the leaf fixing subgroup $\Gamma' \cong \mathbb{Z}_k$ of K on $\tilde{Z}$ where Γ' acts as

$$\mu_m(t, l) = (t - \frac{m}{k}, \sigma_m(l)), \quad m \in \mathbb{Z}_k, (t, l) \in S^1 \times \mathcal{L} \tag{15}$$

and σ is the leaf automorphism induced by the leaf-fixing elements of K on $\mathcal{L}$. Denote $\exp(\mathfrak{h}) \subset G$ by H. Denote by $\tilde{G}$ the group $K \times H$. Then we have an action $\tilde{\rho}$ of $\tilde{G}$ on $\tilde{Z}$ given by

$$\tilde{\rho} : \tilde{G} \times \tilde{Z} \to \tilde{Z}, \quad \tilde{\rho}_{(s,h)}(t, l) = (t + s, \rho_h(l)).$$

Suppose $\sigma_m = \rho_h$ as an equality of maps on $\mathcal{L}$ for some $h \in H$, $m \in \Gamma' \backslash \{0\}$. As H is connected, $\sigma_1 = \rho_{h'}$ for some $h' \in H$. Whence, the action μ of Γ' on $\tilde{Z}$ is equivalent to the action $\tilde{\rho}$ of $\Gamma \subset \tilde{G}$ on $\tilde{Z}$ where Γ is the group

$$\Gamma = \left\{ \Big(-\frac{m}{k}, (h')^m \Big) \,\Big|\, m = 0, \ldots, k-1 \right\},$$

i.e. $\mu_m = \tilde{\rho}_{(-\frac{m}{k}, (h')^m)}$ for all $m \in \mathbb{Z}_k$ Letting $p_{\tilde{Z}}$ and $p_{\tilde{G}}$ denote the projections to $\tilde{Z}/\Gamma$ and $\tilde{G}/\Gamma$ respectively, we have a commutative diagram

$$\begin{array}{ccc} \tilde{G} \times \tilde{Z} & \xrightarrow{\tilde{\rho}} & \tilde{Z} \\ \Big\downarrow{\scriptstyle p_{\tilde{G}} \times p_{\tilde{Z}}} & & \Big\downarrow{\scriptstyle p_{\tilde{Z}}} \\ \tilde{G}/\Gamma \times Z & \xrightarrow{\rho} & Z \end{array}$$

By construction, the action of $\tilde{G}/\Gamma$ on Z and the action of G on Z possess the same fundamental vector fields. Moreover, the action of both groups is effective. Necessarily, then, $\tilde{G}/\Gamma = G$. Conversely, assume that $\sigma_1 \neq \rho_h$ for any $h \in H$.

Then $\exp(\mathfrak{k}) \cap \exp(\mathfrak{h}) = 0$ and so $G \cong K \times H$. The action $\tilde{\rho}$ of $\tilde{G} \cong G$ on $\tilde{Z}$ projects to an action of G on Z where the projection $p_{\tilde{G}}$ is given by quotienting the group by the subgroup $\Gamma = \Gamma' \times \{e_H\}$ and the projection $p_{\tilde{Z}}$ is given by the action μ of $\Gamma \cong \Gamma'$ in (15). □

Proposition 27. *Let $G = S^1 \times H$ be a product group acting on a mapping torus Z such that the S^1 factor acts transverse to the foliation. Let $z \in Z$ and denote by G_z the isotropy group of z. Then $G_z \cong \mathbb{Z}_l \times H_z$ where H_z the isotropy group of z under the H-action and $\mathbb{Z}_l$ is a cyclic subgroup and $\mathbb{Z}_l \times \{e_H\}$ acts as the identity on $\mathcal{O}_z^H$.*

Proof. Let $\mathcal{L}_0$ be a leaf of Z. Denote by $\mathcal{O}_z^H \subset \mathcal{L}_0$ the orbit of z under the action of $(0, H) \subset G$. Denote the subgroup $(S^1, e_H) \subset G$ by K. Let ρ^K be the action of K on Z. The leaf preserving subgroup of K can be identified with $\mathbb{Z}_k$; for $m \in \mathbb{Z}_k$, i.e. $\frac{m}{k} \in K$ is leaf-preserving, we either have $\rho^K_{\frac{m}{k}}(\mathcal{O}_z^H) \cap \mathcal{O}_z^H = \emptyset$ or $\rho^K_{\frac{m}{k}}(\mathcal{O}_z^H) \cap \mathcal{O}_z^H = \mathcal{O}_z^H$. Moreover elements $m \in \mathbb{Z}_k$ satisfying $\rho^K_{\frac{m}{k}}(\mathcal{O}_z^H) \cap \mathcal{O}_z^H = \mathcal{O}_z^H$ form a subgroup $\mathbb{Z}_l$ of $\mathbb{Z}_k$.

If $\rho^K_{\frac{m}{k}}(z) \notin \mathcal{O}_z^H$ for all $m \in \mathbb{Z}_k$ then $\mathbb{Z}_l = \{0\}$ and $G_z = \{0\} \times H_z$ where H_z is the isotropy group of z under the action of $(0, H)$. Alternatively suppose $\mathbb{Z}_l \neq \{0\}$, so that $\rho^K_{\frac{1}{l}}(z) = h \cdot z$ for some $h \in H$. If $h \neq e_H$, we can find a new $K' \subset G$ which acts as the identity on z as follows: let η in $\mathfrak{k}$ be such that $K = \exp(t\eta)$ where $t \in [0, 1)$. Let $\nu \in \mathfrak{h}$ be such that $\exp(\frac{1}{l}\nu) = h^{-1}$. Consider the subgroup

$$K' = \{\exp(t(\eta + \nu)) \,|\, t \in [0, 1)\}.$$

Then the isotropy group of z is of the form $\mathbb{Z}_l \times H_z$ where $\mathbb{Z}_l \cong \{\exp\left(\frac{n}{l}(\eta + \nu)\right) \,|\, n = 0, \ldots, l - 1\}$. □

6. A b-SYMPLECTIC SLICE THEOREM

Let $(M \cong (-\epsilon, \epsilon) \times Z, \omega)$ be a b-symplectic manifold together with an effective b-symplectic action by a compact connected Lie group G acting transversely to the symplectic leaves inside the critical hypersurface Z. First we will construct the b-symplectic models which will give us a normal form for the b-symplectic form about an orbit of G. By Proposition 26 there are two distinct cases

(1) G is a group isomorphic to the product of Lie groups $G = S^1 \times H$.
(2) $G = (S^1 \times H)/\Gamma$ where $\Gamma \cong \mathbb{Z}_l \times \mathbb{Z}_k \subset S^1 \times H$ and $\mathbb{Z}_k$ is a non-trivial cyclic subgroup of H.

Recall from Proposition 26 that there is a trivial finite cover $\tilde{Z} = S^1 \times \mathcal{L}$ of Z equipped with an $(S^1 \times H)$-action which projects to the action of G on Z.

Let $z \in \mathcal{L}_0$ be a point in a symplectic leaf of Z and consider the orbit $\mathcal{O}_z^H$ of z given by the group action of $H = \exp \mathfrak{h}$ on $\mathcal{L}_0$. Denote the isotropy group of z by H_z. By the symplectic slice theorem (Theorem 1), there is an H-equivariant neighbourhood $\mathcal{U}$ of $\mathcal{O}_z^H$ in $\mathcal{L}$ which is equivariantly symplectomorphic to a neighbourhood of the zero section of the vector bundle $Y_z^H = (H \times \mathfrak{m}^* \times V_z)/H_z$ with symplectic form ω_{MGS} as given by Theorem 1. Recall that $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{h}$, the vector space $V_z \subset T_z\mathcal{L}$ is the symplectic orthogonal $V_z = (T_m\mathcal{O}_z^H)^\omega / T_m\mathcal{O}_z^H$ and H_z acts on V_z by the isotropy representation.

Definition 28 (b-Symplectic models)**.** *Consider the b-symplectic form on $\tilde{E} = T^*S^1 \times (H \times_{H_z} \mathfrak{m}^* \times V_z)$ given by*

$$\tilde{\omega}_0 = \omega_{c'} + \omega_{MGS} \tag{16}$$

where $\omega_{c'}$ is the b-symplectic form on T^*S^1 of modular period $c' = ck$ given by Definition 5, and ω_{MGS} is the symplectic form on $Y_z^H = H \times_{H_z} \mathfrak{m}^* \times V_z$ given by the symplectic slice theorem (Theorem 1). Consider the quotient b-symplectic structure on $E = \tilde{E}/\mathbb{Z}_l$ where $\frac{m}{l} \in \mathbb{Z}_l$ acts on T^*S^1 as the cotangent lift of $\mathbb{Z}_l$ acting by translations on S^1 and acts on the factor $H \times_{H_z} \mathfrak{m}^* \times V_z$ equipped with the coordinates $[k, \eta, v]$ of Theorem 1 either by

(1) $\frac{m}{l} \cdot [k, \eta, v] = [k, \sigma^m(\eta), \sigma^m(v)]$ *for a linear symplectomorphism σ.*
(2) $\frac{m}{l} \cdot [k, \eta, v] = [h^m \cdot k, \eta, v]$ *where h is some element of H.*

*Then E has a unique b-symplectic structure such that the projection is a local b-symplectomorphism (see Remark 17). We call these normal forms **b-symplectic models with symplectic slice V_z and modular period** $c\frac{k}{l}$.*

We will now show that a neighborhood of an orbit of a G-action on a b-symplectic manifold with the properties stated in the beginning is b-symplectomorphic to a neighborhood of the zero section of one of the models above, completing the proof of Theorem 2, which we recall here in a succinct way:

Theorem 29. *Let G be a compact Lie group acting on a b-symplectic manifold (M, ω) transverse to the symplectic foliation. Suppose that the action of G is b-symplectic, effective and Hamiltonian when restricted to the symplectic leaves of Z. Let $\mathcal{O}_z$ be an orbit of the group action contained in the critical set of M. Then there is a neighbourhood $\mathcal{V}$ of $\mathcal{O}_z$ in M which is b-symplectomorphic to a neighbourhood of the zero section of a bundle given by the b-symplectic model E in Definition 28.*

Proof. By Proposition 26 there exists a finite cover $\tilde{Z} \cong S^1 \times \mathcal{L}$ which comes equipped with the action of a product group $\tilde{G} \cong S^1 \times H$ which covers the action of G on Z. Let $z \in \mathcal{L}_0 \subset Z$ and let $\tilde{z} \in \tilde{Z}$ be a point projecting to z. Denote by $\mathcal{O}_z^H$ the orbit of $\tilde{z}$ under the action of the subgroup H and by $\mathcal{O}_{\tilde{z}}^{\tilde{G}}$ the orbit of $\tilde{z}$ under the action of $\tilde{G} \cong S^1 \times H$ on the cover $\tilde{Z}$. Then an invariant open neighbourhood of $\mathcal{O}_{\tilde{z}}^{\tilde{G}}$ in $\tilde{M} \cong (-\epsilon, \epsilon) \times \tilde{Z}$ is of the form $\tilde{\mathcal{V}} = (-\epsilon, \epsilon) \times S^1 \times \mathcal{U}$ where $\mathcal{U}$ is an invariant open neighbourhood of $\mathcal{O}_z^H$ in $\mathcal{L}_0$. Recall that Z is the quotient of $\tilde{Z}$ by a cyclic subgroup Γ of $\tilde{G}$. By Proposition 26, we may assume that Γ is of the form

$$\Gamma = \left\{ \left(-\frac{m}{k}, h^m \right) \mid m = 0, \ldots, k-1 \right\}. \tag{17}$$

Let $\tilde{\omega}$ be the lift of ω to $\tilde{Z}$ as given by Proposition 13. By Theorem 18 we may assume that locally around $\tilde{z}$, $\tilde{\omega}$ is of the form

$$\tilde{\omega} = ckdt \wedge \frac{da}{a} + \beta$$

where β is the symplectic form on the leaf $\mathcal{L}_0$. Denote by H_z the isotropy group of z under the action of H. By the symplectic slice theorem, Theorem 1, a neighbourhood $\mathcal{U}$ of $\mathcal{O}_z^H$ with symplectic form β is equivariantly symplectomorphic to a neighbourhood of the zero section of the bundle $Y_z^H = H \times_{H_z} \mathfrak{m}^* \times V_z$ with symplectic form given by the MGS normal form.

Consider the vector bundle $\tilde{E} = T^*S^1 \times (H \times_{H_z} \mathfrak{m}^* \times V_z)$ with symplectic form given by $\tilde{\omega}_0$ in Equation (16), where c is the modular period of ω and k is the order of Γ. Let $\tilde{\psi}$ be the equivariant diffeomorphism form $\tilde{\mathcal{V}}$ to a neighborhood of the zero section in $\tilde{E}$ obtained from the slice theorem on $\mathcal{L}_0$. Then $\tilde{\psi}^*\tilde{\omega}_0$ is a b-symplectic form on a neighbourhood of $\mathcal{O}_{\tilde{z}}^{\tilde{G}}$ and, since $(\tilde{\psi}^*\tilde{\omega}_0)_{\tilde{z}} = \tilde{\omega}_{\tilde{z}}$, by the equivariant relative Moser Theorem, Theorem 19, and after making $\tilde{\mathcal{V}}$ smaller if necessary, we can conclude that there is an equivariant b-symplectomorphism from $\tilde{\mathcal{V}}$ to a neighbourhood of the zero section of $\tilde{E}$ equipped with the b-symplectic form $\tilde{\omega}_0$.

Denote by Γ_z the subgroup of Γ defined by $\{m \in \Gamma \mid \rho_m(\mathcal{O}_z^H) \cap \mathcal{O}_z^H = \mathcal{O}_z^H\}$, where ρ is the action of $\Gamma_z \subset \tilde{G}$ on Z equivariant with respect to the projection $\tilde{Z} \to Z$, as in Proposition 26.

Then Γ_z is a cyclic subgroup $\Gamma_z \cong \mathbb{Z}_l$ of Γ of the form

$$\Gamma_z = \left\{ \left(-\frac{m}{k}, (h')^m \right) \mid m = 0, \ldots, l-1 \right\}$$

for some $h' \in H$. Denote by $p_{\tilde{\mathcal{V}}}, p_{\tilde{E}}$ the projections to the quotients of $\tilde{\mathcal{V}}$ and $\tilde{E}$ by the action of Γ_z respectively. Define ψ by the condition that the following diagram commutes:

$$\begin{CD} \tilde{\mathcal{V}} @>{\tilde{\psi}}>> \tilde{E} \\ @V{p_{\tilde{\mathcal{V}}}}VV @VV{p_{\tilde{E}}}V \\ \tilde{\mathcal{V}}/\Gamma @>{\psi}>> E \end{CD}$$

First consider the case where $G \cong S^1 \times H$. We may assume by Proposition 27 that the action of Γ_z on the orbit $\mathcal{O}_z^H$ and so on the base of the bundle Y_z^H is trivial. Moreover it preserves the slice V_z and acts by linear symplectomorphisms and so ψ is b-symplectomorphism to Model (1) of Definition 28.

For $h \neq e_H$ in the group Γ in Equation 17 (that is, the case $G \cong (S^1 \times H)/\Gamma$ for Γ non trivial and $\Gamma_z \subset \Gamma$), the the action of Γ_z on Y_z^H is given by the symplectic slice theorem, Theorem 1, and the equivariant normal form is given by Model (2) of Definition 28. ☐

Example 30. *Let G be a compact Lie group with non-trivial centre. Let $\xi_1 \in \mathfrak{g}$ be a central element of the Lie algebra and $\xi_2, \ldots, \xi_n \in \mathfrak{g}$ be such that $\xi_1, \ldots, \xi_n$ form a basis of the Lie algebra. Denote by η_i the basis of the Lie algebra dual such that $(\eta_i, \xi_j) = \delta_{ij}$. Denote the associated invariant vector fields $L_{g*}\xi_i$ by v_i and $L_g^*\eta_i$ by m_i respectively. At each point $g \in G$ these give a basis for the tangent and cotangent spaces at g. Consider the singular 2 -form on T^*G*

$$\omega = \pi^* m_1 \wedge \frac{d\left(\lambda\left(v_1\right)\right)}{\lambda\left(v_1\right)} + \sum_{i=2}^{n} \pi^* m_i \wedge d\left(\lambda\left(v_i\right)\right) \tag{18}$$

where π the canonical projection $T^*G \to G$. It can be checked directly that ω is a b-symplectic form on T^*G invariant under the cotangent lifted action of G on T^*G. By Proposition 26, G has a finite cover of the form $S^1 \times H$. The b-symplectic model for the action of G on T^*G is given by $\tilde{E} = (T^*S^1 \times T^*H)/\mathbb{Z}_k$ where $\mathbb{Z}_k$ acts diagonally on $T^*S^1 \times T^*H$ by the cotangent lift of translations on S^1 and the b-symplectic form on $\tilde{E}$ is

$$\tilde{\omega}_0 = \omega_c + \omega_H \tag{19}$$

where

- *ω_c is the standard b-symplectic form of modular period c on the manifold T^*S^1, as given in Definition 5.*
- *ω_H is the canonical symplectic form on T^*H.*

Example 31. *Consider the symplectic mapping torus*

$$Z = \frac{[0,1] \times \mathcal{L}}{(0,l) \sim (1, \phi(l))} \tag{20}$$

where

- *$\mathcal{L} \cong S^2 \times S^2$, where S^2 is the two sphere equipped with the standard symplectic form and $\mathcal{L}$ is equipped with the product symplectic form.*
- *$\phi : \mathcal{L} \to \mathcal{L}$ is the diffeomorphism of $\mathcal{L}$ given by exchanging the S^2 factors of $\mathcal{L}$, i.e., $\phi(x,y) = (y,x)$.*

Consider the group $G = S^1 \times SO(3)$ where $SO(3)$ acts diagonally on the product $\mathcal{L} \cong S^2 \times S^2$ by rotation on each factor and S^1 acts by translations on the factor $[0,1]$ of the above mapping torus:

$$(s, A) \cdot (t, x, y) = (t + 2s, A \cdot x, A \cdot y), \qquad (s, A) \in S^1 \times SO(3),\ (t, x, y) \in [0, 1] \times \mathcal{L}.$$

Consider a point $z = (0, x, y) \in Z$ and the corresponding orbit $\mathcal{O}_z$ in the b-symplectic manifold $M = (-\epsilon, \epsilon) \times Z$. We distinguish three cases:

First, suppose $x \neq \pm y$. Then the action of $S^1 \times SO(3)$ on the orbit $\mathcal{O}_z$ is free. There is a neighbourhood a neighbourhood $\mathcal{V}$ of $\mathcal{O}_z$ equivariantly b-symplectomorphic to the zero section of the bundle $E = T^*\mathrm{S}^1 \times Y_z^{SO(3)}$, where E is equipped with the b-symplectic form

$$\tilde{\omega}_0 = \omega_2 + \omega_{MGS}$$

and ω_2 is the standard symplectic form of modular period 2 on T^*S^1 and ω_{MGS} is a symplectic form on $Y_z^{SO(3)}$ given by Theorem 1.

Now let $x = y$. Then z has isotropy group $\mathbb{Z}_2 \times SO(2)$. The associated b-symplectic model is given by $E = T^*\mathrm{S}^1 \times F$, where $F = SO(3) \times_{SO(2)} V$ is a bundle over the homogeneous space $SO(3)/SO(2) \cong S^2$, V a 2-dimensional vector space with Darboux symplectic form ω_V. The b-symplectic form on E is given by

$$\tilde{\omega}_0 = \omega_1 + 2\omega_{S^2} + \omega_V$$

where ω_1 is the standard symplectic form of modular period 1 on T^*S^1 and ω_{S^2} is the usual symplectic form on the sphere.

Finally, suppose $y = -x$. Let $\nu \in \mathfrak{k}$, $\mathfrak{k}$ the Lie algebra of S^1. Let $\exp(t\xi) \cong SO(2)$ be the 1-parameter subgroup of $SO(3)$ such that $g = exp(\xi)$ acts on S^2 by $g(x) = -x$ and $d\rho_g = -Id$. Then the subgroup $(exp(t\nu), exp(t\xi)) \hookrightarrow S^1 \times SO(3)$ acts as the identity on the orbit $\mathcal{O}_z \cong S^2$ and the b-symplectic model is given by the quotient bundle $E = T^*\mathrm{S}^1 \times (SO(3) \times_{SO(2)} V)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $u, v \in V$ by $(u, v) \to (-u, -v)$ and E is equipped with the b-symplectic form

$$\tilde{\omega}_0 = \omega_1 + 2\omega_{S^2} + \omega_V.$$

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