THE GEOMETRY AND TOPOLOGY OF CONTACT STRUCTURES WITH SINGULARITIES

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ABSTRACT. In this article we introduce and analyze in detail singular contact structures, with an emphasis on b^m-contact structures, which are tangent to a given smooth hypersurface Z and satisfy certain transversality conditions. These singular contact structures are determined by the kernel of non-smooth differential forms, called b^m-contact forms, having an associated critical hypersurface Z. We provide several constructions, prove local normal forms, and study the induced structure on the critical hypersurface. The topology of those are related to smooth contact structures through a desingularization technique. The problem of existence of b^m-contact structures on a given manifold is also tackled in this paper. We prove that a connected component of a convex hypersurface of a contact manifold can be realized as a connected component of the critical set of a b^m-contact structure. In particular, in the 3-dimensional case, this construction yields the existence of a generic set of surfaces Z such that the pair (M, Z) is a b^{2k}-contact manifold and Z is its critical hypersurface. As a consequence of the desingularization techniques in [GMW], we prove the existence of folded contact forms on any almost contact manifold.

1. INTRODUCTION

Contact manifolds have been known for a long time to be the odd-dimensional counterpart to symplectic manifolds. As opposed to symplectic manifolds, contact manifolds are in a way more flexible: Any odd dimensional manifold satisfying an algebraic-topological condition (more precisely almost contact) admits a contact structure (see [BEM] and [CPP]). The history of contact geometry dates back to Sophus Lie and the study of optics [Ge]. Contact and symplectic manifolds are closely related and come into the scene as a natural language associated to (Hamiltonian) dynamics.

In this alluring connection between symplectic and contact manifolds a natural aspect has been neglected: Can we consider *singular forms*? This is too wild as a question as singularities can be too complicated. However in the last years, a class of singular symplectic forms called b^m-symplectic forms has been widely explored by several authors [GMP, GMPS, KM]. In this article, we do a first step towards studying the geometry and topology of the odd-dimensional counterpart of those manifolds, called b^m-contact manifolds.

Those manifolds constitute a generalization of contact manifolds, where the non-integrability condition is satisfied on a dense subset, but is integrable on a hypersurface. Alternatively, this can be thought of as studying contact manifolds with boundary, where the distribution is tangent to the boundary.

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Those distributions are described as the kernel differential 1-forms satisfying the usual contact condition away from the hypersurface, but which are singular on that given hypersurface. The language of those non-smooth forms in the case of manifolds with boundary is not new. The notion of b-tangent bundle was already introduced by Melrose in [M] as a framework to study differential calculus on manifolds with boundary. Recently, it regained a lot of attention in the Poisson and symplectic setting. Indeed, in the foundational work of Radko [R], a certain type of Poisson structures on closed surfaces (*topologically stable Poisson surfaces*) are classified. Later, in [GMP], it was shown that those Poisson structures can be studied under the eyes of symplectic geometry associating a symplectic form over the db-cotangent bundle. Since then many efforts have been united to understand the local and global behavior of this generalization of symplectic manifolds, see for example [BDMOP, FMM, GL] and references therein. This article can be considered to be the first one to consider an odd-dimensional counterpart of the story.

The investigation of existence of contact structures on any odd-dimensional manifold has a particularly rich history and led to many important developments in the field. In this article we consider the singular analogue and provide an answer in our setting by narrowly linking the existence problem of singular contact structures to convex hypersurfaces in contact geometry, thereby shedding new light on the theory of convex surfaces initiated by [Gir1]. We also relate the topology of b^m-contact manifolds, depending on the parity of m, to the one of contact manifolds, or to the one of so called *folded contact*, which are a particular case of confoliations as studied in [ET].

Organization of this article:

We start by reviewing the basics of b-symplectic geometry in Section 2 by explaining in greater details the construction of the b-tangent bundle and the extension of the de Rham exterior derivative. We also include a selection of results in b-symplectic geometry that we use in this article. We then give the main definitions of this article, namely the one of b-contact manifolds. We prove local normal forms for b-contact forms in Section 4. We will see in Section 5 that the right framework to study those geometric structures is the one of Jacobi manifolds. The induced structure by the b-contact structure on the boundary is explained in Section 6. We continue by explaining the relation with b-symplectic geometry in Section 7. In Section 8 we introduce contact structures admitting more general singularities, as for example b^m-contact structures, but also structures dual to b-contact structures, that we call *folded contact* structures. The existence of singular contact structures on a prescribed manifold is dealt with in Sections 9 and 10: Namely, we explore the relation of b^m-contact manifolds to smooth contact structures following the techniques of [GMW] and proving existence theorems for b^m-contact structures on a given manifold. The constructions in Section 9 and 10 rely strongly on the existence of convex hypersurfaces on contact manifolds but also on the desingularization constructions in [GMW] and on new *singularization* techniques. We end the article with a discussion on open problems concerning the dynamical properties of the Reeb vector field associated to b^m-contact forms and possible applications to celestial mechanics. An appendix on the local analysis of Jacobi manifolds is included.

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2. Preliminaries: b-Symplectic survival kit

Let (M^n, Z) be a smooth manifold of dimension n with a hypersurface Z. In what follows, the hypersurface Z will be called *critical set*. Assume that there exist a global defining function for

Z, that is $f : M \to \mathbb{R}$ such that $Z = f^{-1}(0)$. A vector field is said to be a b-vector field[1] if it is everywhere tangent to the hypersurface Z. The space of b-vector fields is a Lie sub-algebra of the Lie algebra of vector fields on M. A natural question to ask is whether or not there exist a vector bundle such that its sections are given by the b-vector fields. A coordinate chart of a neighbourhood around a point $p \in Z$ is given by $\{(x_1, \dots, x_{n-1}, f)\}$ and the b-vector fields restricted to this neighbourhood form a locally free C^∞-module with basis

$$(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}, f\frac{\partial}{\partial f}).$$

By the Serre–Swan theorem [Sw], there exists an n-dimensional vector bundle which sections are given by the b-vector fields. We denote this vector bundle by ${}^bTM$, the *b-tangent bundle*. We now adopt the classical construction to obtain differential forms for this vector bundle. We denote the dual of this vector bundle by ${}^bT^*M := ({}^bTM)^*$ and call it the *b-cotangent bundle*. A b-form of degree k is the section of the kth exterior wedge product of the b-cotangent bundle: $\omega \in \Gamma\big(\Lambda^k({}^bT^*M)\big) := {}^b\Omega^k(M)$. To extend the de Rham differential to an exterior derivative for b-forms, we need a decomposition lemma.

Lemma 2.1 ([GMP]). *Let $\omega \in {}^b\Omega^k(M)$ be a b-form of degree k. Then ω decomposes as follows:*

$$\omega = \frac{df}{f} \wedge \alpha + \beta, \quad \alpha \in \Omega^{k-1}(M),\ \beta \in \Omega^k(M).$$

Equipped with this decomposition lemma, we extend the exterior derivative by putting

$$d\omega := \frac{df}{f} \wedge d\alpha + d\beta.$$

It is clear that this is indeed an extension of the usual exterior derivative and that $d^2 = 0$.

Definition 2.2. An even-dimensional b-manifold M^{2n} with a b-form $\omega \in {}^b\Omega^2(M)$ is b-symplectic if $d\omega = 0$ and $\omega^n \neq 0$ as element of $\Lambda^{2n}({}^bT^*M)$.

Outside of the critical set Z, we are dealing with symplectic manifolds. On the critical set, the local normal form of the b-symplectic form is given by the following theorem.

Theorem 2.3 (b-Darboux theorem, [GMP]). *Let ω be a b-symplectic form on (M^{2n}, Z). Let $p \in Z$. Then we can find a local coordinate chart $(x_1, y_1, \dots, x_n, y_n)$ centered at p such that hypersurface Z is locally defined by $y_1 = 0$ and*

$$\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

The b-Darboux theorem for b-symplectic forms has been proved using two different approaches. The first proof follows Moser path method, that can be adapted in the b-setting. Another way of proving it is to show that a b-form of degree 2 on a $2n$-dimensional b-manifold is b-symplectic if and only if its dual bi-vector field is a Poisson vector field Π whose maximal wedge product is transverse to the zero section of the vector bundle $\Lambda^{2n}({}^bTM)$, that is $\Pi^n \pitchfork 0$. A Poisson manifold satisfying this condition is called a *b-Poisson manifold*. Using the transversality condition in Weinstein's splitting theorem, one sees that the Poisson structure is of the form

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}. \tag{2.4}$$

[1]The letter *b* stands for *boundary* as introduced by [M].

Furthermore, Weinstein splitting theorem implies that the critical set of a b-symplectic manifold is a regular codimension one foliation of symplectic leaves. Even better, it is proved in [GMP] that the critical set is a cosymplectic manifold[2].

The relation of b^m-symplectic manifolds to symplectic manifolds and the less well-known folded symplectic manifolds was investigated in [GMW].

Theorem 2.5 ([GMW]). *Let ω be a b^m-symplectic structure on a manifold M and let Z be its critical hypersurface.*

- *If m is even, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If m is odd, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z.*

We say that (M, ω_ϵ) is the f_ϵ-desingularization of (M, ω). A direct consequence of this theorem is that any orientable manifold admitting a b^{2k}-symplectic structure admits a symplectic structure.

3. b-CONTACT MANIFOLDS

In this section we introduce the main objects of this article. Inspired by the definition of b-symplectic manifolds, we define the contact case as follows:

Definition 3.1. Let (M, Z) be a (2n+1)-dimensional b-manifold. A b-contact structure is the distribution given by the kernel of a one b-form $\xi = \ker \alpha \subset {}^bTM$, $\alpha \in {}^b\Omega^1(M)$, that satisfies $\alpha \wedge (d\alpha)^n \neq 0$ as a section of $\Lambda^{2n+1}({}^bT^*M)$. We say that α is a b-contact form and the pair (M, ξ) a b-contact manifold.

The hypersurface Z is called *critical hypersurface*. In what follows, we always assume that Z is non-empty. Away from the critical set Z the b-contact structure is a smooth contact structure. The former definition fits well with what is standard in contact geometry where coorientable contact manifolds are considered (i.e. there exists a defining contact form with kernel the given contact structure).

Example 3.2. *Let (M, Z) be a b-manifold of dimension n. Let $z, y_i, i = 2, \dots, n$ be the local coordinates for the manifold M on a neighbourhood of a point in Z, with Z defined locally by $z = 0$ and $x_i, i = 1, \dots, n$ be the fiber coordinates on ${}^bT^*M$, then the canonical Liouville one-form is given in these coordinates by*

$$x_1 \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i.$$

The bundle $\mathbb{R} \times {}^bT^*M$ is a b-contact manifold with b-contact structure defined as the kernel of the one-form

$$dt + x_1 \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

where t is the coordinate on $\mathbb{R}$. The critical set is given by $\tilde{Z} = Z \times \mathbb{R}$. Using the definition of the extended de Rham derivative, one checks that $\alpha \wedge (d\alpha)^n \neq 0$. Away from $\tilde{Z}$, $\xi = \ker \alpha$ is a non-integrable hyperplane field distribution, as in usual contact geometry. On the critical set however, ξ is tangent to $\tilde{Z}$. This comes from the definition of b-vector fields. Since the rank of ξ can drop by 1 on $\tilde{Z}$, we cannot say that ξ is a hyperplane field.

As we will see in the next example, the rank does not necessarily drop.

[2] A cosymplectic manifold is manifold M^{2n+1} together with a closed one-form η and a closed two-form ω such that $\eta \wedge \omega^n$ is a volume form.

Example 3.3. *Let us take $\mathbb{R}^{2n+1}$ with coordinates $(z, x_1, \ldots, x_n, y_1, \ldots, y_n)$. We consider the distribution of the kernel of $\alpha = \frac{dz}{z} + \sum_{i=1}^n x_i dy_i$. The critical set is given by $z = 0$ and the rank does not drop on the critical set: on the critical set, the distribution is spanned by $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, i = 1, \ldots n\}$.*

Using the two last examples and a generalization of Moebius transformations, we can construct b-contact structures on the unit ball with critical set given by the unit sphere.

Example 3.4. *Let us denote the unit ball of dimension n by D^n and the half-space, that is $\mathbb{R}^n$ where the first coordinate is positive, by $\mathbb{R}^n_+$. The Moebius transformation maps the open half-space diffeomorphically to the closed 2-ball minus a point by the following map:*

$$\Phi : \{z \in \mathbb{C} | \Re(z) > 0\} \to D^2 \setminus \{(1, 0)\}$$
$$z \mapsto \frac{z-1}{z+1}.$$

This map can easily be generalized to all dimension and the inverse is given by

$$\Psi : D^n \setminus \{(1, 0, \ldots, 0)\} \to \mathbb{R}^n_+$$
$$(x_1, \ldots, x_n) \mapsto \frac{1}{(x_1 - 1)^2 + \sum_{i=2}^n x_i^2}\big(1 - \sum_{i=1}^n x_i^2, 2x_2, \ldots, 2x_n\big).$$

We now provide $\mathbb{R}^{2n+1}_+$ with the b-contact structures described in Example 3.2 (respectively 3.3) and pull-back the b-contact form. We obtain hence two different b-contact structures on the unit ball minus a point and the critical set is given by the unit sphere S^{2n-2} minus the point $(1, 0, \ldots, 0)$.

It is not possible to compactify this example by adding the point. This can be seen when computing the hyperplane distribution of the pushforward under Φ. Alternatively, this follows as we will see in one of the two cases from Theorem 4.9. However, we will see that the 3-sphere does admit a b-contact structure, induced by a b-symplectic structure, see Example 7.3.

Example 3.5. *A compact example admitting a b-contact structure is given by $S^2 \times S^1$. Let us consider the 2-sphere S^2, with coordinates (θ, h) where $\theta \in [0, 2\pi]$ is the angle and $h \in [0, 1]$ is the height, and the 1-sphere S^1 with coordinate $\varphi \in [0, 2\pi]$. Then $(S^2 \times S^1, \alpha = \sin\varphi d\theta + \cos\varphi \frac{dh}{h})$ is a b-contact manifold. Once more, the rank on the critical set changes when $\cos\varphi = 0$, where instead of a plane-distribution, we are dealing with a line distribution.*

Example 3.6 (Non-orientable example)**.** *Coorientable contact manifolds are always orientable as $\alpha \wedge (d\alpha)^n$ is a volume form. There are b-contact forms on non-orientable manifolds. Consider the example of the b-contact form on the 3-torus given by $(\mathbb{T}^2 \times S^1, \alpha = \cos\theta \frac{dx}{\sin 2\pi x} + \sin\theta dy)$. Consider the group action $\mathbb{Z}/2\mathbb{Z}$ that acts on $(x, y) \in \mathbb{T}^2$ by $\mathrm{Id} \cdot (x, y) = (x, y)$ and $-\mathrm{Id} \cdot (x, y) = (1 - x, y)$. The orbit space under the quotient of this action is the Klein bottle. The b-contact form is invariant under the action of the group and therefore descends to $\mathbb{K} \times S^1$ where $\mathbb{K}$ is the Klein bottle. The manifold $\mathbb{K} \times S^1$ is of course non-orientable.*

Example 3.7 (Product examples)**.** *Let (N^{2n+1}, α) be a b-contact manifold and let $(M^{2m}, d\lambda)$ be an exact symplectic manifold, then $(N \times M, \alpha + \lambda)$ is a b-contact manifold. It is easy to check that $\tilde{\alpha} = \alpha + \lambda$ satisfies $\tilde{\alpha} \wedge (d\tilde{\alpha})^{n+m} \neq 0$.*

In the same way if (N^{2n+1}, α) is a contact manifold and $(M^{2m}, d\lambda)$ be an exact b-symplectic manifold (where exactness is understood in the b-complex), then $(N \times M, \alpha + \lambda)$ is a b-contact manifold. These product examples can even be endowed with additional structures such as group actions or integrable systems. For instance we can produce examples of toric b-contact manifolds combining the product of toric contact manifolds in [Le] *with (exact) toric b-symplectic manifolds (see* [GMPS]*). We can also combine the techniques in* [KM] *for b-symplectic manifolds and* [B] *(among others) for contact manifolds to produce examples of integrable systems on these manifolds.*

4. THE b-CONTACT DARBOUX THEOREM

In usual contact geometry, the Reeb vector field R_α of a contact form α is given by the equations

$$\begin{cases} \iota_{R_\alpha} d\alpha = 0 \\ \alpha(R_\alpha) = 1. \end{cases}$$

In the case where we change the tangent bundle by ${}^bTM$, the existence is given by the same reasoning: $d\alpha$ is a bilinear, skewsymmetric 2-form on the space of b-vector fields ${}^bTM$, hence the rank is an even number. As $\alpha \wedge (d\alpha)^n$ is non-vanishing and of maximum degree, the rank of $d\alpha$ must be $2n$, its kernel is 1-dimensional and α is non-trivial on that line field. So a global vector field is defined by the normalization condition.

By the same reasoning, we can define the b-contact vector fields: for every function $H \in C^\infty(M)$, there exist a unique b-vector field X_H defined by the equations

$$\begin{cases} \iota_{X_H} \alpha = H \\ \iota_{X_H} d\alpha = -dH + R_\alpha(H)\alpha. \end{cases}$$

A direct computation yields that in Example 3.2, the Reeb vector field is given by $\frac{\partial}{\partial t}$. In Example 3.3, the Reeb vector field is given by $z\frac{\partial}{\partial z}$ and hence singular. We will see that, roughly speaking, the Reeb vector field locally classifies b-contact structures.

We now prove a Darboux theorem for b-contact manifolds. The proof follows the one of usual contact geometry as in [Ge]. More precisely, it makes use of Moser's path method. There are two differences from the standard Darboux theorem: the first one is that there exist two local models, depending on whether or not the Reeb vector field is vanishing on the critical set Z. The second one is that in the case where the Reeb vector field is singular, the local expression of the contact form only holds pointwise, see for instance Example 4.7. Furthermore, in the case where the Reeb vector field is singular, this linearization is done up to multiplication of a non-vanishing function. The proof is not following Moser's path method in this case as the flow of the Reeb vector field is stationary.

Theorem 4.1. *Let α be a b-contact form inducing a b-contact structure ξ on a b-manifold (M, Z) of dimension $(2n+1)$ and $p \in Z$. We can find a local chart $(\mathcal{U}, z, x_1, y_1, \dots, x_n, y_n)$ centered at p such that on $\mathcal{U}$ the hypersurface Z is locally defined by $z = 0$ and*

(1) *if $R_p \neq 0$*
   (a) *ξ_p is singular, then*

$$\alpha|_{\mathcal{U}} = dx_1 + y_1 \frac{dz}{z} + \sum_{i=2}^n x_i dy_i,$$

   (b) *ξ_p is regular, then*

$$\alpha|_{\mathcal{U}} = dx_1 + y_1 \frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^n x_i dy_i,$$

(2) *if $R_p = 0$, then $\tilde{\alpha} = f\alpha$ for $f(p) \neq 0$, where*

$$\tilde{\alpha}_p = \frac{dz}{z} + \sum_{i=1}^n x_i dy_i.$$

We call the b-contact form *regular* at $p \in M$ when the Reeb vector field is not vanishing at p and *singular* on the contrary.

Proof. We may assume without loss of generality that $M = \mathbb{R}^{2n+1}$ and that p is the origin of $\mathbb{R}^{2n+1}$. Let us choose linear coordinates on $T_p\mathbb{R}^{2n+1}$. By the non-integrability condition, $d\alpha$ has rank $2n$ and α is non-trivial on the kernel of $d\alpha$. We first choose the vector belonging to the kernel of $d\alpha$ and then complete a symplectic basis of $d\alpha$.

Let us first treat the case where $\ker d\alpha \subset T_pZ$: We choose x_1 such that $\frac{\partial}{\partial x_1} \in \ker d\alpha$ and $\alpha(\frac{\partial}{\partial x_1}) = 1$. Now let us take $V \in \ker\alpha$, but $V \notin T_pZ$ such that $\iota_V d\alpha \neq 0$. As $V \notin T_pZ$, V belongs to the kernel of the a vector bundle morphism

$$ {}^bTM|_Z \to TZ $$

as explained in [GMP]. We take the coordinate z such that $V = z\frac{\partial}{\partial z}$. We then choose a coordinate y_1 such that $\frac{\partial}{\partial y_1} \in \ker\alpha$ and $d\alpha(z\frac{\partial}{\partial z}, \frac{\partial}{\partial y_1}) = 1$.

We complete a symplectic basis of $d\alpha$ and we can choose the remaining $2n-2$ coordinates x_i and y_i in both cases so that for all $i = 2, \dots, n$ that $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \in T_pZ$.

We now set

$$ \alpha_0 = dx_1 + y_1\frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i \text{ when } \ker d\alpha \subset Z \tag{4.2} $$

when ξ_p is singular and when ξ_p is regular we set

$$ \tilde{\alpha}_0 = dx_1 + y_1\frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i \text{ when } \ker d\alpha \subset Z. $$

By the choice of the basis, it is clear that at the origin,

$$ \begin{cases} \alpha = \alpha_0 \\ d\alpha = d\alpha_0 \end{cases} $$

when ξ_p is singular. We only work out the details in this case, as the case ξ_p regular works analogously.

Note that, until this stage, we only used linear algebra arguments, which are more involved due to the structure of the vector bundle ${}^bTM$. Let us now apply Moser's path method. In a neighbourhood of p, we consider the family of b-forms of degree 1

$$ \alpha_t = (1-t)\alpha_0 + t\alpha \text{ for } t \in [0,1]. $$

By the choice of basis, it is clear that at the origin,

$$ \begin{cases} \alpha_t = \alpha \\ d\alpha_t = d\alpha \end{cases} $$

and so α_t is a path of b-contact forms in a neighbourhood of the origin. We want to show that there exist an isotopy $\psi_t : \mathcal{U} \mapsto \mathbb{R}^{2n+1}$ satisfying

$$ \begin{cases} \psi_t^*\alpha_t = \alpha_0 \\ \psi_t(p) = p \\ \psi_t|_Z \subset Z. \end{cases} \tag{4.3} $$

Differentiating the first equation, we obtain $\mathcal{L}_{X_t}\alpha_t + \dot{\alpha}_t = 0$, where $X_t(p) = \frac{d\psi_s}{ds}\big(\psi_t^{-1}(p)\big)\big|_{s=t}$. Inserting the splitting $X_t = H_tR_{\alpha_t} + Y_t$, where $H_t \in C^\infty(M)$ and $Y_t \in \ker\alpha_t$ and applying Cartan's formula, we obtain

$$ \iota_{Y_t} d\alpha_t + dH_t + \dot{\alpha}_t = 0. \tag{4.4} $$

Evaluating this differential equation in the Reeb vector field R_{α_t}, we obtain

$$dH_t(R_{\alpha_t}) + \dot{\alpha}_t(R_{\alpha_t}) = 0. \tag{4.5}$$

This equation can be solved locally around the point p, as we can assume without loss of generality that R_{α_t} does not have closed orbits around that point. This is due to the fact that $R_{\alpha_t} \neq 0$. In fact, by the construction of the coordinate system $R_\alpha = \frac{\partial}{\partial x_1}$. Furthermore, as $\dot{\alpha}_t(p) = 0$, $dH_t(p) = 0$, and we can choose the constant of integration such that $H_t(p) = 0$. Once H_t is chosen, let us take a look at Equation (4.4), given by

$$\iota_{Y_t} d\alpha_t = -(dH_t + \dot{\alpha}_t).$$

We want to solve this equation for Y_t. By the previous observation and the fact that $d\alpha_t$ is a b-symplectic form, we obtain that $Y_t(p) = 0$, so $X_t(p) = 0$. Furthermore, it is clear that Y_t is a b-vector field because $d\alpha$ is a b-form. Integrating the vector field X_t gives us the isotopy ψ_t, satisfying the conditions of (4.3). This proves the first part of the theorem.

Let us now consider the case where $\ker d\alpha \nsubseteq T_pZ$, which corresponds to the case where $R_p = 0$ and $d\alpha$ is a smooth de Rham form. A b-form decomposes as $f\frac{dz}{z} + \beta$, where z is a defining function. As $d\alpha$ is smooth, the function f can only depend on z on Z and hence, $f(p) \neq 0$ as we would be in the smooth case otherwise. We choose a neighbourhood $\mathcal{U}$ around the origin such that f is non-vanishing on that neighbourhood. By dividing by f, the b-form $\tilde{\alpha} = \frac{dz}{z} + \tilde{\beta}$ defines the same distribution. Now take a contractible $2n$-dimensional disk $D^{2n} \ni p$ in $\mathcal{U}$. As $(D, d\alpha)$ is symplectic, we know by applying Darboux theorem for symplectic forms (we assume the disk D small enough), that there exist $2n$ functions x_i, y_i such that locally $d\alpha = \sum_{i=1}^n dx_i \wedge dy_i$. Now consider the b-form $\alpha - \sum_{i=1}^n x_i dy_i - \frac{dz}{z}$. This form is closed and smooth. Hence by Poincaré lemma for smooth forms, there exists a smooth function g such that

$$\tilde{\alpha} = \frac{dz}{z} + dg + \sum_{i=1}^n x_i dy_i.$$

We can change the defining function by $\tilde{z} = e^{-g}z$, so that $\frac{d\tilde{z}}{\tilde{z}} = \frac{dz}{z} + dg$. Now

$$\tilde{\alpha} = \frac{d\tilde{z}}{\tilde{z}} + \sum_{i=1}^n x_i dy_i.$$

As $\tilde{\alpha} \wedge (d\tilde{\alpha})^n = n\frac{d\tilde{z}}{\tilde{z}} \wedge \sum_{i=1}^n dx_i \wedge dy_i \neq 0$, the functions $\tilde{z}, x_i, y_i$ form a basis. □

Remark 4.6. It follows from the b-Darboux theorem that if $(M, \ker\alpha)$ be a b-contact manifold and $\ker\alpha_p$ is regular for $p \in Z$, then there is an open neighbourhood around p where $\ker\alpha$ is regular.

The following example shows that it is possible to have both local models appearing on one connected component of the critical set. Furthermore, it shows in the case where the Reeb vector field is singular, we can only prove the normal form pointwise and does not hold in a local neighbourhood as when the Reeb vector field is regular.

Example 4.7. *$(S^2 \times S^1, \alpha = \sin\varphi d\theta + \cos\varphi\frac{dh}{h})$ where (θ, h) are the polar coordinates on S^2 and φ the coordinate on S^1. The Reeb vector field is given by $R = \sin\varphi\frac{\partial}{\partial\theta} + \cos\varphi h\frac{\partial}{\partial h}$.*

We will prove that there are at least two points where the Reeb vector field is singular in the compact, 3-dimensional case. This will be a corollary of the following. By definition of the b^m-tangent bundle, the Reeb vector field is tangent to the critical set. We can prove that in dimension 3, the Reeb vector field is in fact Hamiltonian with respect to the induced area form from the contact condition. We will prove the following theorem:

Theorem 4.8. *Let $(M, \alpha = u\frac{dz}{z} + \beta)$ be a b-contact manifold of dimension 3. Then the restriction on Z of the 2-form $\Theta = ud\beta + \beta \wedge du$ is symplectic and the Reeb vector field is Hamiltonian with respect to Θ with Hamiltonian function u, i.e. $\iota_R\Theta = du$.*

Proof. In the decomposition, α is given by $\alpha = u\frac{dz}{z} + \beta$. The contact condition implies that $\Theta := ud\beta + \beta \wedge du$ is an area form, so it is symplectic. In the same decomposition, let us write the Reeb vector field as $R_\alpha = g \cdot z\frac{\partial}{\partial z} + X$, where $g \in C^\infty(M)$ and $X \in \mathfrak{X}(Z)$. As R_α is the Reeb vector field, we obtain the following equations:

$$\begin{aligned} g \cdot u + \beta(X) &= 1, \\ -gdu + \iota_X d\beta &= 0, \\ \iota_X du &= 0. \end{aligned}$$

A straightforward computation using those equations yield that $\iota_X \Theta = du$, hence the restriction of R_α to Z is the Hamiltonian vector field for the function $-u$. □

In the compact case, we obtain:

Corollary 4.9. *Let (M, α) be a 3-dimensional compact b-contact manifold. Then there are at least two points where the local normal form of α is described by the singular model of the Darboux theorem.*

Remark 4.10. As shown in Example 3.4, there is a b-contact structure on the unit disk under the pull-back under the Moebius transformation of the regular local model. It follows from the last corollary, that this example can not be compactified.

Example 4.11. *As before, consider $(S^2 \times S^1, \alpha = \sin\varphi d\theta + \cos\varphi\frac{dh}{h})$. The Reeb vector field on the critical set is given by Hamiltonian vector field of the function $-\cos\varphi$ with respect to the area form $d\varphi \wedge d\theta$. Hence, on the critical set, the Reeb vector field vanishes when $\cos\varphi = 0$ and there are no periodic orbits of the Reeb vector field on the critical set.*

A well known result in contact geometry is Gray's stability theorem, asserting that on a closed manifold, smooth families of contact structures are isotopic. The proof uses Moser's path method that works well in b-geometry. One proves in the same line the following stability result for b-contact manifolds.

Theorem 4.12. *Let (M, Z) compact b-manifold and let (ξ_t), $t \in [0, 1]$ be a smooth path of b-contact structures. Then there exists an isotopy ϕ_t preserving the critical set Z such that $(\phi_t)_*\xi_0 = \xi_t$, or equivalently, $\phi_t^*\alpha_t = \lambda_t\alpha_0$ for a non-vanishing function λ_t.*

Proof. Assume that ϕ_t is the flow of a time dependent vector field X_t. Deriving the equation, we obtain

$$d\iota_{X_t}\alpha_t + \iota_{X_t}d\alpha_t + \dot{\alpha} = \mu_t\alpha_t$$

where $\mu_t = \frac{\dot{\lambda}_t}{\lambda_t} \circ \phi_t^{-1}$. If X_t belongs to ξ_t, the first term of the last equation vanishes and applying then the Reeb vector field yields

$$\dot{\alpha}_t(R_{\alpha_t}) = \mu_t.$$

The equation given by

$$\iota_{X_t}d\alpha_t = \mu_t\alpha_t - \dot{\alpha}_t$$

then defines X_t because $(\mu_t\alpha_t - \dot{\alpha}_t)(R_{\alpha_t})$. We integrate the vector field X_t to find ϕ_t and as X_t is a vector field, tangent to the critical set, the flow preserves it. □

The compactness condition is necessary as it is shown in the next example.

Example 4.13. *Consider the path of b-contact structures on $\mathbb{R}^3$ given by $\ker\alpha_t$ where $\alpha_t = (\cos\frac{\pi}{2}t - y\sin\frac{\pi}{2}t)\frac{dz}{z} + (\sin\frac{\pi}{2}t + y\cos\frac{\pi}{2}t)dx$. As $\alpha_0 = \frac{dz}{z} + ydx$ and $\alpha_1 = dx - y\frac{dz}{z}$, the two b-contact structures cannot be isotopic.*

In the same lines, we prove the following semi-local result.

Theorem 4.14. *Let (M, Z) be a b-manifold and assume Z compact. Let $\xi_0 = \ker \alpha_0$ and $\xi_1 = \ker \alpha_1$ be two b-contact structures such that $\alpha_0|_Z = \alpha_1|_Z$. Then there exists a local isotopy ψ_t, $t \in [0, 1]$ in an open neighbourhood $\mathcal{U}$ around Z such that $\psi_t^* \alpha_t = \lambda_t \alpha_0$ and $\psi_t|_Z = \mathrm{Id}$ where λ_t is a family of non-vanishing smooth functions.*

Proof. The proof is done following Moser's path method. Put $\xi_t = (1-t)\xi_0 + t\xi_1$, $t \in [0, 1]$. Because the non-integrability condition is an open condition and $\xi_t|_Z = \xi_0|_Z = \xi_1|_Z$, there exists an open neighbourhood $\mathcal{U}$ containing Z such that ξ_t is a family of b-contact structures. We will prove that there exists an isotopy $\psi_t : \mathcal{U} \mapsto M$ such that $\psi_t^* \alpha_t = \lambda_t \alpha_0$, where λ_t is a non-vanishing smooth function and $\lambda_t|_Z = \mathrm{Id}$. Assume that ψ_t is the flow of a vector field X_t and differentiating, we obtain the following equation:

$$d\iota_{X_t}\alpha_t + \iota_{X_t} d\alpha_t + \dot{\alpha}_t = \mu_t \alpha_t,$$

where $\mu_t = \frac{d}{dt}(\log|\lambda_t|) \circ \psi_t^{-1}$. Taking $X_t \in \xi_t$, this equation writes down

$$\dot{\alpha}_t + \iota_{X_t} d\alpha_t = \mu_t \alpha_t. \tag{4.15}$$

Applying the Reeb vector field to both sides, we obtain the equation that defines μ_t:

$$\mu_t = \dot{\alpha}_t(R_{\alpha_t}).$$

As $\dot{\alpha}_t|_Z = 0$, $\mu_t|_Z = 0$ and hence X_t is zero on Z. By non-degeneracy of $d\alpha_t$ on ξ_t there exists a unique $X_t \in \xi_t$ solving Equation 4.15. Integrating X_t yields the desired result. □

Note that this proof fails if one wants to prove stability of b-contact forms, that is we cannot assume that $\lambda_t = \mathrm{Id}$ in a neighbourhood of Z.

5. b-JACOBI MANIFOLDS

In the symplectic case, it is often helpful to look at b-symplectic manifolds as being the dual of a particular case of Poisson manifold. In contact geometry, Jacobi manifolds play this role.

Recall that a Jacobi structure on a manifold M is a triplet (M, Λ, R) where Λ is a smooth bi-vector field and R a vector field satisfying the following compatibility conditions:

$$[\Lambda, \Lambda] = 2R \wedge \Lambda, \qquad [\Lambda, R] = 0, \tag{5.1}$$

where the bracket is the Schouten–Nijenhuis bracket. We refer the reader [V] and references therein for further information on Jacobi manifolds.

Definition 5.2. Let (M, Λ, R) be a Jacobi manifold of dimension $2n+1$. We say that M is a b-Jacobi manifold if $\Lambda^n \wedge R$ cuts the zero section of $\Lambda^{2n+1}(TM)$ transversally.

Note that this definition is similar to the one of b-Poisson manifolds, in the sense that it also asks the top wedge power to be transverse to the zero section. We denote the hypersurface given by the zero section of $\Lambda^{2n+1}(TM)$ by Z and we call it the *critical set.*

It is well-known that contact manifolds are a particular case of odd-dimensional Jacobi manifolds. A particular case of even-dimensional Jacobi manifolds are given by *locally conformally symplectic manifolds.*

Definition 5.3. A locally conformally symplectic manifold is a manifold M of dimension $2n$ equipped with a non-degenerate two-form $\omega \in \Omega^2(M)$ that is locally closed, which is equivalent to the existence of a closed 1-form $\alpha \in \Omega^1(M)$ such that $d\omega = \alpha \wedge \omega$.

Locally conformally symplectic manifold regained recent attention, notably in the work [CM].

We will prove that b-contact manifolds and b-Jacobi manifolds are dual in some sense, as will be explained in the next two propositions. Before doing so, let us note that in the case where the dimension of the Jacobi manifold is $\dim M = 2n$, we can given an similar definition to the one of Definition 5.2 by asking that Λ^{2n} cuts the zero-section of $\Lambda^{2n}(TM)$ transversally. It should be possible to prove in the same lines that this case corresponds to locally conformally b-symplectic manifold.

Proposition 5.4. *Let $(M, \ker\alpha)$ be a b-contact manifold. Let Λ be the bi-vector field computed as in Equation A.1 in Appendix A and let R be the Reeb vector field. Then (M, Λ, R) is a b-Jacobi manifold.*

Proof. As being b-Jacobi is a local condition, we can work in a local coordinate chart. Outside of the critical set, α is a contact form. Hence we can compute Λ as in Equation A.1 in both local models of the Darboux theorem and Λ can smoothly be extended to the critical set Z. A straightforward computation now yields that for both local models $\Lambda^n \wedge R \pitchfork 0$. □

Recall that to every Jacobi manifold (M, Λ, R), one can associate a homogeneous Poisson manifold. Indeed, $(M \times \mathbb{R}, \Pi := e^{-\tau}(\Lambda + \frac{\partial}{\partial\tau} \wedge R))$ is a Poisson manifold because

$$\begin{aligned}[\Pi, \Pi] &= [e^{-\tau}\Lambda, e^{-\tau}\Lambda] + 2[e^{-\tau}\Lambda, e^{-\tau}\frac{\partial}{\partial\tau} \wedge R] + [e^{-\tau}\frac{\partial}{\partial\tau} \wedge R, e^{-\tau}\frac{\partial}{\partial\tau} \wedge R] \\ &= 2e^{-2\tau}[\Lambda, \Lambda] + 2(-e^{-\tau}\Lambda \wedge R) = 0.\end{aligned}$$

Furthermore, the later is said to be *homogeneous* because the vector field $T = \frac{\partial}{\partial\tau}$ satisfies

$$L_T P = -P.$$

This construction is called Poissonization. The same stays true in the b-scenario, although we need to assume that the b-Jacobi manifold is of odd dimension, as b-Poisson manifold are defined only for even dimensions.

Lemma 5.5. *The Poissonization of a b-Jacobi manifold of odd dimension is a homogeneous b-Poisson manifold.*

Proof. The proof is a straightforward computation:

$$\Pi^{n+1} = -e^{-(n+1)\tau}\frac{\partial}{\partial\tau} \wedge \Lambda^n \wedge R.$$

It follows from the definition of b-Jacobi that Π is transverse to the zero-section. □

Proposition 5.6. *Let (M^{2n+1}, Λ, R) be a b-Jacobi manifold. Then M is a b-contact manifold.*

Proof. The proposition is based on the local normal form of Jacobi structures, which are proved in [DLM]. The main result is recalled in Appendix B. Let (M, Λ, R) be the b-Jacobi structure, so that $\Lambda^n \wedge R \pitchfork 0$. As usual, denote the critical hypersurface by $Z = (\Lambda^n \wedge R)^{-1}(0)$. First note that outside of Z, the leaf of the characteristic foliation is maximal dimensional. This is saying that outside of Z, the Jacobi structure is equivalent to a contact structure.

Consider a point $p \in Z$ and denote the leaf of the characteristic foliation by L. By the transversality condition, the dimension of the leaf needs to be of dimension $2n$ or $2n-1$. Indeed, as $(M \times \mathbb{R}, e^{-\tau}(\frac{\partial}{\partial\tau} \wedge R + \Lambda))$ is b-Poisson, the critical set of $M \times \mathbb{R}$ is foliated by symplectic manifolds of codimension 2, that is of dimension $2n$. Hence the critical set restricted to the hypersurface $\{\tau = 0\}$, which is identified to be the critical set Z of the initial manifold M, is foliated by codimension 1 and codimension 2 leaves.

Let us first consider the case where at the point $x \in Z$, the leaf is of dimension $2n$. We will prove that this case corresponds to the case where the R is singular, vanishing linearly. Let us apply Theorem 5.9 of [DLM]. Hence the Jacobi manifold (N, Λ_N, E_N) (see Theorem 5.9) is of dimension 1, hence Λ_N is zero. Hence Λ is given by

$$\Lambda = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_{i+n}} - \sum_{i=1}^{n} x_{i+n}\frac{\partial}{\partial x_{i+n}} \wedge E_N.$$

We now use the transversality condition on $\Lambda^n \wedge E_N$ to conclude that $E_N = z\frac{\partial}{\partial z}$. which is the same expression for the b-Jacobi structure associated to the b-contact form $\alpha = \frac{dz}{z} + \sum_{i=1}^n x_i dx_{i+n}$.

Let us consider the case where the leaf is of dimension $2n-1$. We will see that this corresponds to the case where the Reeb vector field is regular. According to Theorem 5.11 in [DLM], the bivector field is given by

$$\Lambda = \Lambda_{2n-1} + \Lambda_N + E \wedge Z_N$$

where (N, Λ_N, Z_N) is a homogeneous 2-dimensional Poisson manifold and $\Lambda_{2n-1} = \sum_{i=1}^{n-1}(x_{i+n-1}\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_i}) \wedge \frac{\partial}{\partial x_{i+q}}$. The transversality condition implies that $\Lambda_{2n-1}^{n-1} \wedge \Lambda_N \wedge \frac{\partial}{\partial x_0} \pitchfork 0$, hence Λ_N is a b-Poisson manifold. By [GMP], $\Lambda_N = z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial y}$. The homogeneous vector field Z_N is determined by equation $\mathcal{L}_{Z_N}\Lambda_N = -\Lambda_N$. Hence $Z_N = y\frac{\partial}{\partial y}$. Hence the Jacobi structure is given by $E = \frac{\partial}{\partial x_0}$ and

$$\Lambda = \sum_{i=1}^{n-1}(x_{i+n-1}\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_i}) \wedge \frac{\partial}{\partial x_{i+q}} + z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x_0} \wedge y\frac{\partial}{\partial y},$$

which is the Jacobi structure associated to the contact form $\alpha = dx_0 + y\frac{dz}{z} + \sum_{i=1}^{n-1} x_i dx_{i+q}$. □

6. Geometric structure on the critical set

To determine the induced structure of the b-contact structure on the critical set, we compute the associated Jacobi structure. Let us briefly review some results on Jacobi manifolds, which can all be found in [V]. The Hamiltonian vector fields of a Jacobi manifold (M, Λ, R) are defined by $X_f = \Lambda^\sharp(df) + fR$. It can be shown that the distribution $\mathcal{C}(M) = \{X_f | f \in C^\infty(M)\}$ is involutive and invariant under the Hamiltonian flow. Stefan–Sussmann theorem asserts that $\mathcal{C}(M)$ integrates to a singular foliation, denoted by $\mathcal{F}$. As $\mathcal{C}(M) = \text{Im}\Lambda^\sharp + \langle R \rangle$, the leaves of $\mathcal{F}$ are even-dimensional when $R \in \text{Im}\Lambda^\sharp$ and odd dimensional in the other case. The induced structure on odd-dimensional leaves of $\mathcal{F}$ turns out to be a contact structure. For even dimensional leaves, one obtains locally conformally symplectic leaves. The definition of locally conformally symplectic manifolds is recalled in Definition 5.3.

The computation of a Jacobi structure associated to a contact structure is explained in Appendix A. As we have proved a local norm form theorem, we can use the two local models to compute the associate Jacobi structure and check in both cases if $R \in \Lambda^\sharp$. We will prove

Theorem 6.1. *Let $(M^{2n+1}, \xi = \ker\alpha)$ be a b-contact manifold and $p \in Z$. We denote $\mathcal{F}_p$ the leaf of the singular foliation $\mathcal{F}$ going through p. Then*

(1) *if ξ_p is regular, that is $\mathcal{F}_p$ of dimension $2n$, then the induced structure on $\mathcal{F}_p$ is locally conformally symplectic;*
(2) *if ξ_p is singular, that is $\mathcal{F}_p$ of dimension $2n-1$, then the induced structure on $\mathcal{F}_p$ is contact.*

Proof. By Theorem 4.1, if ξ_p is singular, the Reeb vector field is not singular and the contact form can be written locally as $\alpha = dx_1 + y_1\frac{dz}{z} + \sum_{i=2}^n x_i dy_i)$. The Reeb vector field is given by $R = \frac{\partial}{\partial x_1}$, the dual of $d\alpha$ by $\Pi = z\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial z} + \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$. As Liouville vector field with respect to $d\alpha$, we take $X = \sum_{i=1}^n y_i\frac{\partial}{\partial y_i}$. The Jacobi structure associated to this b-contact structure is given by $\Lambda = \Pi + R \wedge X$.

On the critical set, we have

$$\Lambda\big|_Z = \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i=1}^n y_i\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_i}.$$

Let us check if we can find a one form α such that $\Lambda\big|_Z^\#(\alpha) = \frac{\partial}{\partial x_1}$. For $y_1 = 0$, this cannot be solved, hence the set $\{z = 0, y_1 = 0\}$ is a leaf with an induced contact structure.

If ξ_p is not singular and the Reeb vector is regular, the contact form can be written locally as $\alpha = dx_1 + y_1\frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^n x_i dy_i)$. A direct computation implies that the Reeb vector field lies in the distribution spanned by the bi-vector field Λ, hence the b-contact structure induces a locally conformally symplectic structure on the set $\{z = 0, y_1 \neq 0\}$.

Last, if ξ_p is not singular and the Reeb vector is singular, Theorem 4.1 yields that the Reeb vector field can be written as $z\frac{\partial}{\partial z}$. As the Reeb vector field is vanishing, the critical set equals the $2n$-dimensional leaf spanned by $\mathrm{Im}\Lambda^\sharp$. The induced structure on $\mathcal{F}_p$ is locally conformally symplectic. □

Remark 6.2. Let us consider the case where $\dim M = 3$ and the distribution ξ is singular. Then the induced structure on the critical set is given by $\Lambda|_Z = y_1\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial x_1}$. As the critical set is a surface, it is clear that this is a Poisson structure and furthermore, that it is transverse to the zero section. Hence we obtain an induced b-symplectic structure on the critical set. Note that this is not true for higher dimensions.

7. Symplectization and contactization

Symplectic and contact manifolds are related to each other as follows. It is well-known that a contact manifold can be transformed into a symplectic one by *symplectization*: if (M, α) is a contact manifold, then $(M \times \mathbb{R}, d(e^t\alpha))$ (where t is the coordinate on $\mathbb{R}$) is a symplectic manifold. On the other hand, hypersurfaces of a symplectic manifold (M, ω) are contact, provided that there exist a vector field satisfying $\mathcal{L}_X\omega = \omega$ that is transverse to the hypersurface. Such a vector field is called Liouville vector field. The contact form on the hypersurface is given by the contraction of the symplectic form with the Liouville vector field, i.e. $\alpha = \iota_X\omega$.

We will show that the same holds in the b-category.

Example 7.1. *Let $(\mathbb{R}^4, \omega = \frac{1}{z}dz \wedge dt + dx \wedge dy)$ be a b-symplectic manifold. A Liouville vector field is given by $X = \frac{1}{2}(z\frac{\partial}{\partial z} + 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$. Note that Liouville vector fields are defined up to addition of symplectic vector fields, that is a vector field Y satisfying $\mathcal{L}_Y\omega = 0$. Another Liouville vector field is for example given by $t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$.*

Let us take a b-symplectic manifold (W, ω) of dimension $(2n + 2)$ and a Liouville vector field X on W that is transverse to a hypersurface H of W. Then $(H, \iota_X\omega)$ is a b-contact manifold of dimension $(2n+1)$ as $\iota_X\omega \wedge (d\iota_X\omega)^n = \frac{1}{n+1}\iota_X(\omega^{n+1})$ is a volume form provided that X is transverse to H. If H does not intersect the critical set, one obtains of course a smooth contact form. Due to the b-Darboux theorem, there are two local models for b-contact manifolds and we will see that we can obtain both structures, depending on the relative position of the hypersurface with the Reeb vector field on it.

Example 7.2. *Let us take $(W = \mathbb{R}^4, \omega = \frac{1}{z}dz \wedge dt + dx \wedge dy)$ and the Liouville vector field $X = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$. The contraction of X with the b-symplectic form yields $\iota_X\omega = -\frac{t}{z}dz + xdy$. Let us take different hypersurfaces transverse to X and compute the induced b-contact form.*

- *If we take as hypersurface the hyperplane $M_1 = \{(1, y, -t, z), y, t, z \in \mathbb{R}\}$, which is transverse to X, we obtain $\alpha = dy + t\frac{dz}{z}$, which is the regular local model.*
- *If we take as hypersurface the hyperplane $M_2 = \{(x, y, -1, z), x, y, z \in \mathbb{R}\}$, which is transverse to X, we obtain $\alpha = \frac{dz}{z} + xdy$, which is the singular local model.*

Example 7.3. *The three dimensional sphere admits a b-contact structure. Consider the $\mathbb{R}^4$ with the standard b-symplectic structure $\omega = \frac{dx_1}{x_1} \wedge dy_1 + dx_2 \wedge dy_2$ and denote by S^3 the unit sphere in $\mathbb{R}^4$. The Liouville vector field $X = \frac{1}{2}x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} + \frac{1}{2}(x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2})$ is transverse to the sphere and hence $\iota_X\omega$ defines a b-contact form on S^3. The critical set is a 2 dimensional sphere, S^2, given by the intersection of the sphere with the hyperplane $z = 0$.*

Example 7.4. *The unit cotangent bundle of a b-manifold has a natural b-contact structure. Let (M, Z) be a b-manifold of dimension n with coordinates $z, y_i, i = 2, \dots, n$ as in Example 3.2. It is shown in* [GMP] *that the cotangent bundle has a natural b-symplectic structure defined by the b-form given by the exterior derivative $d\lambda = d\big(x_1\frac{dz}{z} + \sum_{i=2}^n x_i dy_i\big)$. The unit b-cotangent bundle is given by ${}^bT_1^*M = \{(z, y_2, \dots, y_n, x_1, \dots, x_n) \in {}^bT^*M | \sum_{i=1}^n x_i^2 = 1\}$. The vector field $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ defined on the b-cotangent bundle ${}^bT^*M$ is a Liouville vector field, and is transverse to the unit b-cotangent bundle, and hence induces a b-contact structure on it.*

From a Riemannian point of view, the Reeb vector field describes the geodesic flow associated to a b-metric, that is a metric a bundle metric on ${}^bTM$. In the coordinate chart (z, y_i), a b-metric is give by $g = \frac{1}{z^2}dz \otimes dz + \sum_{i=2}^n dy_i \otimes dy_i$ and induces a bundle metric g^* on ${}^bT^*M$. The unit cotangent bundle is alternatively described by ${}^bT_1^*M = \{X \in {}^bT^*M | g^*(X, X) = 1\}$ and the associated Reeb vector field to the associated contact form as described above, is the push-forward under the bundle isomorphism of the geodesic vector field on ${}^bTM$.

We will compute a particular case of the unit cotangent bundle of a b-manifold.

Example 7.5. *Consider the torus $\mathbb{T}^2$ as a b-manifold where the boundary component if given by two disjoint copies of S^1. The unit cotangent bundle $S^*\mathbb{T}^2$, diffeomorphic to the 3-torus $\mathbb{T}^3$ is a b-contact manifold with b-contact form given by $\alpha = \sin\phi \frac{dx}{\sin(x)} + \cos\phi dy$, where ϕ is the coordinate on the fiber and (x, y) the coordinates on $\mathbb{T}^2$.*

We saw that hypersurfaces of b-symplectic manifolds that are transverse to a Liouville vector field have an induced b-contact structure. The next lemma describes which model describes locally the b-contact structure.

Lemma 7.6. *Let (W, ω) be a b-symplectic manifold and X a Liouville vector field transverse to a hypersurface H. Let R be the Reeb vector field defined on H for the b-contact form $\alpha = i_X\omega$. Then $R \in H^\perp$, where $H^\perp$ is the symplectic orthogonal of H.*

Proof. The Reeb vector field defined on H satisfies $i_R(d\alpha)|_H = i_R(di_X\omega)|_H = i_R\omega|_H = 0$. □

Hence if $H^\perp$ is generated by a singular vector field, the contact manifold (H, α) is locally of the second type as in the b-Darboux theorem. In the other case, the local model is given by the first type.

We now come back to the contactization of a b-symplectic manifold.

Theorem 7.7. *Let (M, α) be a b-contact manifold. Then $(M \times \mathbb{R}, \omega = d(e^t\alpha))$ is a b-symplectic manifold.*

Proof. It is clear that ω is a closed b-form. Furthermore, a direct computation yields

$$\big((e^t d\alpha)\big)^{n+1} = e^{t(n+1)} dt \wedge \alpha \wedge (d\alpha)^n,$$

which is non-zero as a b-form by the non-integrability condition. □

It is easy to see that $\frac{\partial}{\partial t}$ is a Liouville vector field of the symplectization $(M \times \mathbb{R}, d(e^t\alpha))$, which is clearly transverse to the submanifold $M \times \{0\}$. Hence, we obtain the initial contact manifold (M, α). This gives us the following proposition.

Proposition 7.8. *Every b-contact manifold can be obtained as a hypersurface of a b-symplectic manifold.*

Remark 7.9. Another close relation between the symplectic and the contact world is the contactization: take an exact symplectic manifold, i.e. $(M, d\beta)$, then $(M \times \mathbb{R}, \beta + dt)$, where t is the coordinate on $\mathbb{R}$, is contact. This remains true in the b-case. Furthermore, it is clear that by this construction, we obtain b-contact forms of the first type, as the Reeb vector field is given by $\frac{\partial}{\partial t}$.

8. Other singularities

In what follows, we consider contact structures with higher order singularities. Let (M^n, Z) be a manifold with a distinguished hypersurface and let us assume that Z is the zero level-set of a function z. The b^m-tangent bundle, which we denote by ${}^{b^m}TM$, can be defined to be the vector bundle whose sections are generated by

$$\Big\{z^m \frac{\partial}{\partial z}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\Big\}.$$

The de Rham differential can be extended to this setting. The notion of b-symplectic manifolds then naturally extends and we talk about b^m-symplectic manifolds, see [Sc, GMW]. In the same fashion, we can extend the notion of b-tangent and b-cotangent bundle, and thus also b-contact manifolds to the b^m-setting[3] we say that a b^m-form $\alpha \in {}^{b^m}\Omega^n(M)$ is b^m-contact if $\alpha \wedge (d\alpha)^n \neq 0$ where the dimension of M is $2n+1$. The proofs of the theorems of the previous sections, in particular Theorem 4.1 and Proposition 5.6 and the construction carried out in Section 7, generalize directly to this setting. For the sake of a clear notation, we do not write down the statements of the generalization, but only assert informally, that b can be replaced by b^m in the statements.

In the next section, we are going to relate the topology of b^m-contact structures to smooth contact structures, but also certain type confoliations that are given by the following definition.

Definition 8.1. A smooth 1-form $\alpha \in \Omega^1(M^{2n+1})$ is a *folded contact form* if $\alpha \wedge (d\alpha)^n$ intersect the zero-section of the bundle $\bigwedge^{2n+1} T^*M$ transversally.

As mentioned before, this is a special kind of confoliation, see [ET] for the definition. By the transversality, the set where the contact condition fails is given by a hypersurface, that we call the *folding hypersurface*.

We do not enter in a more detailed study of folded contact forms but only remark that folded contact structures appear in the work of [M2, JZ].

9. Desingularization of b^m-contact structures

In this section, we desingularize singular contact structures and consequently explain the relation to smooth contact structures. The proof is based on the idea of [GMW]. However, in contrast to the symplectic case, we need an additional assumption in order to desingularize the b^m-contact form.

Recall that from Lemma 2.1, it follows that a b^m-form $\alpha \in {}^{b^m}\Omega^1(M)$ decomposes $\alpha = u\frac{dz}{z^m} + \beta$ where $u \in C^\infty(M)$ and $\beta \in \Omega^1(M)$. In order to desingularize the b^m-contact forms, we will assume that β is the pull-back under the projection of a one-form defined on Z.

Definition 9.1. We say that a b^m-contact structure $(M, \ker\alpha)$ is almost convex if $\beta = \pi^*\tilde{\beta}$, where $\pi : \mathcal{N}(Z) \to Z$ is the projection from a tubular neighbourhood of Z to the critical set and $\tilde{\beta} \in \Omega^1(Z)$. We will abuse notation and write $\beta \in \Omega^1(Z)$. We say that a b^m-contact structure is convex if $\beta \in \Omega^1(Z)$ and $u \in C^\infty(Z)$.

Note that the this notion is to be compared to the one of convex hypersurfaces, which we will recall in the next section. As we will see in the next lemma, almost convex b^m-contact structures are semi-locally isotopic to convex ones.

Lemma 9.2. *Let $(M, \ker\alpha)$ be an almost convex b^m-contact manifold and let the critical hypersurface Z be compact. Then there exists a neighbourhood around the critical set $\mathcal{U} \supset Z$, such that α is isotopic to a convex b^m-contact form relative to Z on $\mathcal{U}$.*

[3] We remark here that further generalizations as contact structures over different Lie algebroids are possible. The symplectic analogue is being treated in [MS].

Proof. Let $\alpha = u\frac{dz}{z^m} + \beta$ where $u \in C^\infty(M)$ and $\beta \in \Omega^1(Z)$. Put $\tilde{\alpha} = u_0\frac{dz}{z^m} + \beta$, where $u_0 = u|_Z \in C^\infty(Z)$, which is convex. Take the linear path between the two b^m-contact structures, which is a path of b^m-contact structures because ξ and $\tilde{\xi}$ equal on Z. Applying Theorem 4.14, we obtain that there exist a local diffeomorphism f preserving Z and a non-vanishing function λ such that on a neighbourhood of Z, $f^*\alpha = \lambda\tilde{\alpha}$. □

The next lemma gives intuition on this definition and gives a geometric characterization of the almost-convexity in terms of the f_ϵ-desingularized symplectization.

Lemma 9.3. *A b^m-contact manifold $(M, \ker\alpha)$ is almost-convex if and only if the vector field $\frac{\partial}{\partial t}$ is a Liouville vector field in the desingularization of the b^m-symplectic manifold obtained by the symplectization of $(M, \ker\alpha)$. Here t denotes the coordinate of the symplectization.*

Proof. Let $(M, \ker\alpha)$ be a almost-convex b^m-contact manifold. The symplectization is given by $(M \times \mathbb{R}, \omega = d(e^t\alpha))$. The desingularization technique of Theorem 2.5 produces a family of symplectic forms $\omega_\epsilon = ue^t dt \wedge df_\epsilon + e^t dt \wedge \beta + e^t du \wedge df_\epsilon + e^t d\beta$. From almost-convexity it follows that $\frac{\partial}{\partial t}$ preserves ω_ϵ, so $\frac{\partial}{\partial t}$ is a Liouville vector field.

To prove the converse, assume that $\frac{\partial}{\partial t}$ is a Liouville vector field in (M, ω_ϵ). It follows from the fact that $\mathcal{L}_{\frac{\partial}{\partial t}}\omega_\epsilon = \omega_\epsilon$ that $\beta \in \Omega^1(Z)$. □

We will see that under almost-convexity, the b^{2k}-contact forms can be desingularized.

Theorem 9.4. *Let $(M^{2n+1}, \ker\alpha)$ a b^{2k}-contact structure with critical hypersurface Z. Assume that α is almost convex. Then there exists a family of contact forms α_ϵ which coincides with the b^{2k}-contact form α outside of an ϵ-neighbourhood of Z. The family of bi-vector fields $\Lambda_{\alpha_\epsilon}$ and the family of vector fields R_{α_ϵ} associated to the Jacobi structure of the contact form α_ϵ converges to the bivector field Λ_α and to the vector field R_α in the C^{2k-1}-topology as $\epsilon \to 0$.*

We call α_ϵ the f_ϵ-desingularization of α.

A corollary of this is that almost-convex b^m-contact forms admit a family of contact structures if m is even, and a family of folded-type contact structures is m is odd.

The proof of this theorem follows from the definition of convexity and makes use of the family of functions introduced in [GMW].

Proof. By the decomposition lemma, $\alpha = u\frac{dz}{z^m} + \beta$. As α is almost convex, the contact condition writes down as follows:

$$\alpha \wedge (d\alpha)^n = \frac{dz}{z^m} \wedge (u(d\beta)^n + n\beta \wedge du \wedge (d\beta)^{n-1}) \neq 0.$$

In an ϵ-neighbourhood, we replace $\frac{dz}{z^m}$ by a smooth form. The expression depends on the parity of m.

Following [GMW] we consider an odd smooth function $f \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $f'(x) > 0$ for all $x \in [-1, 1]$ and satisfying outside that

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for} \quad x < -1, \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for} \quad x > 1. \end{cases} \tag{9.5}$$

Let $f_\epsilon(x)$ be defined as $\epsilon^{-(2k-1)}f(x/\epsilon)$.

We obtain the family of globally defined 1-forms given by $\alpha_\epsilon = udf_\epsilon + \beta$ that agrees with α outside of the ϵ-neighbourhood. Let us check that α_ϵ is contact inside this neighbourhood. Using the almost-convexity condition, the non-integrability condition on the b^m-form α writes down as follows:

$$\alpha_\epsilon \wedge (d\alpha_\epsilon)^n = dz \wedge (f'_\epsilon(z)ud\beta + f'_\epsilon(z)\beta \wedge du - \beta \wedge \frac{\partial\beta}{\partial z}).$$

We see that $\alpha_\epsilon \wedge d\alpha_\epsilon = f'_\epsilon(z) z^m \alpha \wedge d\alpha$ and hence α_ϵ is contact.

We denote by Λ_α and R_α the bi-vector field and vector field of the b-contact form α. Now let us check that the bi-vector field $\Lambda_{\alpha_\epsilon}$ and the vector field of R_{α_ϵ} corresponding to the Jacobi structure of the desingularization converge to Λ_α and R_α respectively.

Let us write R_α and Λ_α in a neighbourhood of a point $p \in Z$.

$$R_\alpha = g z^{2k} \frac{\partial}{\partial z} + X, \quad \Lambda_\alpha = z^{2k} \frac{\partial}{\partial z} \wedge Y_1 + Y_2 \wedge Y_3$$

where $g \in C^\infty(M)$ and $X, Y_i \in \mathfrak{X}(Z)$ for $i = 1, 2, 3$. The Jacobi structure associated to the desingularization is given by

$$R_{\alpha_\epsilon} = g \frac{1}{f'_\epsilon(z)} \frac{\partial}{\partial z} + X, \quad \Lambda_{\alpha_\epsilon} = \frac{1}{f'_\epsilon(z)} \frac{\partial}{\partial z} \wedge Y_1 + Y_2 \wedge Y_3.$$

The C^{2k-1}-convergence follows from this formulas. □

Similarily, we can prove a desingularization theorem for b^{2k+1}-contact forms.

Theorem 9.6. *Let $(M^{2n+1}, \ker \alpha)$ a b^{2k+1}-contact structure with critical hypersurface Z. Assume that α is almost convex. Then there exists a family of folded contact forms α_ϵ which coincides with the b^{2k+1}-contact form α outside of an ϵ-neighbourhood of Z.*

Proof. By the decomposition lemma, $\alpha = u \frac{dz}{z^{2k+1}} + \beta$. As α is almost convex, the contact condition writes down as follows:

$$\alpha \wedge (d\alpha)^n = \frac{dz}{z^{2k+1}} \wedge (u (d\beta)^n + n\beta \wedge du \wedge (d\beta)^{n-1}) \neq 0.$$

In an ϵ-neighbourhood, we replace $\frac{dz}{z^{2k+1}}$ by a smooth form.

Following [GMW] we consider an even smooth function given by $f_\epsilon(x) := \frac{1}{\epsilon^{2k}} f(\frac{x}{\epsilon})$ where $f \in C^\infty(\mathbb{R})$ satisfies

- $f > 0$ and $f(x) = f(-x)$,
- $f'(x) > 0$ if $x < 0$,
- $f(x) = -x^2 + 2$ if $x \in [-1, 1]$,
- $f(x) = \log(|x|)$ if $k = 0, x \in \mathbb{R} \setminus [-2, 2]$.
- $f(x) = \frac{-1}{(2k+2)x^{2k+2}}$ if $k > 0$, $x \in \mathbb{R} \setminus [-2, 2]$.

We define $\alpha_\epsilon = u df_\epsilon + \beta$. We see that $\alpha_\epsilon \wedge d\alpha_\epsilon = f'_\epsilon(z) dz \wedge (u(d\beta)^n + n\beta \wedge du \wedge (d\beta)^{n-1})$ and hence α_ϵ is folded contact: indeed f'_ϵ vanishes transversally at zero, and away from zero, this last expression is non-zero.

□

An alternative proof of this theorem would be to use the symplectization as explained in Section 7 and to use immediately Theorem 2.5 in the symplectization. The almost convex condition makes sure that the vector field in the direction of the symplectization is Liouville in the desingularization, see Lemma 9.3. Hence the induced structure is contact. Without the almost-convexity, the induced structure of the desingularized symplectic form on the initial manifold is not necessarily contact. This is saying that almost-convexity is a sufficient condition, but not a necessary condition to apply the desingularization method.

In the next section of the article, we will see that in presence of convex hypersurface in contact manifolds, a construction that transforms the convex hypersurface into the critical set of a b^m-contact manifold, holds.

10. Existence of singular contact structures on a prescribed submanifold

Existence of contact structures on odd dimensional manifolds has been one of the leading questions in the field. The first result in this direction was proved for open odd-dimensional manifolds by Gromov [Gr]. The case for closed manifolds turned out to be much more subtle. The 3-dimensional case was proved by Martinet–Lutz [Lu, M1]. In dimension 5, the existence problem was solved in [CPP], whereas the higher dimensional case was only solved in the celebrated article by Borman–Eliashberg–Murphy [BEM].

Theorem 10.1. *Any almost contact closed manifold M^{2n+1} admits a contact structure.*

We give in this section an answer to the question whether or not closed manifolds also admit b^m-contact structures. The result relies on convex hypersurface theory, which was introduced by Giroux [Gir1].

Definition 10.2. Let $(M, \ker \alpha)$ be a contact manifold. A vector field X is contact if it preserves ξ, that is $\mathcal{L}_X \alpha = g\alpha$ for $g \in C^\infty(M)$. A hypersurface Z in M is convex if there exists a contact vector field X that is transverse to Z.

It follows from this definition that the contact structure can be written under vertically invariant form in a neighbourhood of Z, that is $\alpha = g \cdot (udt + \beta)$, where the contact vector field X is given by $\frac{\partial}{\partial t}$, $u \in C^\infty(Z)$, $\beta \in \Omega^1(Z)$ and g is a function that is not necessarily vertically invariant. Note that the definition of convex b^m-contact forms, that is Definition 9.1, is the analogue of this definition in the b-setting. As was proved by Giroux [Gir1], in dimension 3, there generically all closed surfaces are convex.

Theorem 10.3 ([Gir1])**.** *Any generic closed surface in a 3-dimensional contact manifold is a convex surface.*

In higher dimension, this result does not hold for generic hypersurfaces, see [M]. However, even though genericity does not hold, examples are given by boundaries of tubular neighbourhoods of Legendrian submanifolds.

In the theory of convex hypersurfaces, a fundamental role is played by the set Σ given by the points of the convex hypersurface where the transverse contact vector field belongs to the contact distribution. It is a consequence of the non-integrability condition that Σ is a codimension 1 submanifold in Z. When M is of dimension 3, a connected component of Σ is called the dividing curve. Loosely speaking, the dividing curves determine the germ of the contact structure on a neighbourhood of the convex surface. For a precise statement, see [Gir1, Gir2].

We will prove that convex hypersurfaces can be realized as the critical set of b^{2k}-contact structures. A similar result holds for b^{2k+1}-contact structures. However, in this case the critical set has two connected components, which correspond to two convex hypersurfaces arbitrarily close to a connected component of the given convex hypersurface.

Theorem 10.4. *Let (M, ξ) be a contact manifold and let Z be a convex hypersurface in M. Then M admits a b^{2k}-contact structure for all k that has Z as critical set. The codimension 2 submanifold Σ corresponds to the set where the rank of the distribution drops and the induced structure is contact.*

Using Giroux's genericity result, we obtain the following corollary in dimension 3:

Corollary 10.5. *Let M be a 3-dimensional manifold. Then for a generic surface Z, there exists a b^{2k}-contact structure on M realising Z as the critical set.*

Proof of the Corollary. Using Gromov's result in the open case and Lutz–Martinet for M closed, we can equip M with a contact form. As is proved in [Gir2], a generic surface Z is convex and the conclusion follows from Theorem 10.4. □

Proof of Theorem 10.4. Using the transverse contact vector field, we find a tubular neighbourhood of Z diffeomorphic to $Z \times \mathbb{R}$ such that the contact structure is defined by the contact form $\alpha = udt + \beta$, where t is the coordinate on $\mathbb{R}$, $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$. Note that in general, α is multiplied by a non-vanishing function g that is not vertically invariant. As g is non-vanishing, the form divided by the function gives a contact form defining the same contact structure. The non-integrability condition then is equivalent of saying that $u(d\beta)^n + n\beta \wedge du \wedge d\beta$ is a volume form on Z. We will change the contact form to a b^{2k}-contact form.

Take $\epsilon > 0$. Let us take a function s_ϵ (that is smooth outside of $x = 0$) such that

(1) $s_\epsilon(x) = x$ for $x \in \mathbb{R} \setminus [-2\epsilon, 2\epsilon]$,
(2) $s_\epsilon(x) = -\frac{1}{x^{2k-1}}$ for $x \in [-\epsilon, 0[\cup]0, \epsilon]$,
(3) $s'_\epsilon(x) > 0$ for all $x \in \mathbb{R}$.

Now consider $\alpha_\epsilon = uds_\epsilon + \beta$. By construction, α_ϵ is a b^{2k}-form that coincides with α outside of $Z\times(\mathbb{R}\setminus[-2\epsilon, 2\epsilon])$. Furthermore, α_ϵ satisfies the non-integrability condition on $Z\times]-2\epsilon, 2\epsilon[$ because $s'_\epsilon > 0$.

The rest of the statement follows from the discussion of Section 6. □

Remark 10.6. Note that there are many different choices for the function s_ϵ yielding the same result: the function s_ϵ only needs to allow singularities of the right order and have positive derivative. We call (M, α_ϵ) the s_ϵ-singularization of the contact manifold (M, α).

This proof only works for b^m-contact forms where m is even because it is essential that $s'_\epsilon > 0$. In the case where the complimentary set of the convex hypersurface is connected, the contact condition obstructs the existence of b^{2k+1}-contact structures on M having Z as critical set. This is because the contact condition induces an orientation on the manifold, whereas in the b^{2k+1}-contact case, the orientation changes when crossing the critical set. The same holds for symplectic surfaces: see for example [MP] where this orientability issues were formulated using colorable graphs.

Lemma 10.7. *Let M be an orientable manifold with Z a hypersurface such that $M \setminus Z$ is connected. Then there exist no b^{2k+1}-contact form with critical set Z.*

Proof. Assume by contradiction that there is a b^{2k+1}-contact form. Let z be a defining function for the critical set. The contact condition writes down as $\frac{dz}{z^{2k+1}}\nu$, where ν is volume form on M. This expression has opposite signs on either side of Z. As $M \setminus Z$ is connected, $\alpha \wedge (d\alpha)^n$ must vanish in $M \setminus Z$, which is in contradiction with the contact condition. □

To overcome this orientability issue, we prove existence of b^{2k+1}-contact structures with two disjoint critical sets contained in a tubular neighbourhood of a given convex hypersurface.

Theorem 10.8. *Let (M, ξ) be a contact manifold and let Z be a convex hypersurface in M. Then M admits a b^{2k+1}-contact structure for all k that has two diffeomorphic connected components Z_1 and Z_2 as critical set. The codimension 2 submanifold Σ corresponds to the set where the rank of the distribution drops and the induced structures is contact. Additionally, one of the hypersurfaces can be chosen to be Z.*

Proof. The proof follows from the same considerations as before, but replacing the vertically invariant contact form α defining the contact ξ by $\alpha_\epsilon = uds_\epsilon + \beta$, where $s_\epsilon :] - \epsilon, \epsilon[\to \mathbb{R}$ is given by

- $s_\epsilon(t) = |t|$ for $|t| \in [3\epsilon/4, \epsilon]$,
- $s_\epsilon(t) = \log |t - 3\epsilon/8|$ for $|t| \in [\epsilon/4, \epsilon/2]$ if $m = 1$,
- $s_\epsilon(t) = \frac{1}{2k(x-3\epsilon/8)^{2k}}$ for $|t| \in [\epsilon/4, \epsilon/2]$ if $m = 2k + 1 \neq 1$,
- s_ϵ is odd, i.e. $s_\epsilon(-t) = -s_\epsilon(t)$,
- $s'_\epsilon(t) \neq 0$.

As before, $s'_\epsilon \neq 0$ assures that α_ϵ is a b^{2k+1}-contact form. As any other function with non-vanishing derivative and the right order of singularities gives rise to a b^{2k+1}-contact form, one of the two hypersurfaces can be chosen to be the initial convex hypersurface. □

Remark 10.9. Given a contact manifold with a convex hypersurface such that the complementary set of the hypersurface is not connected, it may, in some particular cases, also be possible to construct a b^{2k+1}-contact form admitting a unique connected component as critical set. This is related to extending a given contact form in a neighbourhood of a contact manifold with boundary to a globally defined contact form. More precisely, let α be the contact form. In a tubular neighbourhood around the convex hypersurface, we replace as before $\alpha = udt + \beta$ by $\alpha_\epsilon = uds_\epsilon + \beta$ where s_ϵ is given by

- $s_\epsilon(t) = t$ for $t > 2\epsilon$,
- $s_\epsilon(t) = \log t$ for $0 < t < \epsilon$,
- $s'_\epsilon(t) > 0$ for $t > 0$,
- s_ϵ is even, i.e. $s_\epsilon(-t) = s_\epsilon(t)$.

The form α_ϵ is a b^{2k+1}-contact form that agrees with α for $t > 2\epsilon$. However, it does not agree with α for $t < -2\epsilon$ and in fact, it may not always be possible to extend α_ϵ.

Remark 10.10. Recall that a b^m-contact form $\alpha = u\frac{dz}{z^m} + \beta$ is convex if $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$, see Definition 9.1. Note that the s_ϵ-singularization of is by construction a convex b^m-contact manifold.

As a corollary of Theorem 10.8, we will prove that any contact manifold is folded contact.

Corollary 10.11. *Let Z be a convex hypersurface in a contact manifold $(M, \ker\alpha)$. Then M admits a folded-contact form that has two connected components Z_1 and Z_2 as folded hypersurface, diffeomorphic to Z. In particular in dimension 3, a generic surface $Z \subset M$ can be realized as a connected component of the folded hypersurface of a folded contact form.*

Proof. The proof is a direct application of the existence theorem for $k = 1$ and the desingularization theorem. First, by Theorem 10.8, the hypersurface Z can be realized as one of two connected components of the critical set of a b-contact structure. As the obtained b-contact form is convex, see Remark 10.10, we then use the desingularization theorem (Theorem 9.6) to obtain a folded contact structure.

The genericity statement in dimension 3 follows as in Corollary 10.5. □

Remark 10.12. It follows from [BEM] that a necessary and sufficient condition for a manifold to admit a contact structure is that it is almost contact. It would be interesting to ask whether the almost contact condition can be relaxed to prove the existence of b^m-contact structures on closed manifolds. For an example it is well known that $SU(3)/SO(3)$ does not admit a contact structure, see [Ge]. Another indication for this is given by examples of cooriented b^m-contact structures on non-orientable manifolds (see Example 3.6).

11. Open problems

We finish this article by open problems concerning b^m-contact manifolds that will be the content of an upcoming paper. In this article, we neglected the study of the dynamical properties of the Reeb vector field associated to a b^m-contact form. A natural question that arises is whether or not its flow always admits periodic orbits. In the case of a positive answer, do those orbit always exist away from the critical set or on the critical set? Those are generalizations of the well-known Weinstein conjecture in contact geometry.

This is particularly interesting bearing in mind applications to celestial mechanics. It has been proved that b^m-symplectic structures appear naturally in the study of celestial mechanics as for example the restricted three body problem, see [BDMOP, DKM, KMS]. As b^m-contact manifolds appear as certain kind of level-sets of Hamiltonians and the Reeb flow is a reparametrization of the Hamiltonian flow on this level-set, results in the direction of the before-mentioned generalization of Weinstein conjecture are of great importance in the study of celestial mechanics. These lines of research may put some new light in studying the contact geometry of the restricted three body problem, in a similar vein as in [AFKP].

APPENDIX A. CONTACT MANIFOLDS AS JACOBI MANIFOLDS

It is well-known that every contact manifold is a particular case of Jacobi manifold, see [V]. Indeed, if (M,α) is a contact manifold, then (M,Λ,R) is a Jacobi structure, where R is the Reeb vector field and the bi-vector field Λ is defined by

$$\Lambda(df,dg)=d\alpha(X_f,X_g),$$

where X_f,X_g are the contact Hamiltonian vector fields of f and g. We give an alternative way to compute the Jacobi structure associated to the contact structure.

Let us denote the bi-vector field, dual to $d\alpha$, by Π. Furthermore, we denote by X a Liouville vector field relatively to $d\alpha$, i.e. $\mathcal{L}_X d\alpha=d\alpha$. Eventually, we define the bi-vector field

$$\Lambda=\Pi+R\wedge X. \tag{A.1}$$

We have the following identities:

- $\mathcal{L}_X\Pi=\Pi$,
- $\mathcal{L}_R\Pi=0$,
- $[\Pi,\Pi]=0$.

The following lemma characterizes the Jacobi structure.

Lemma A.2. *The Jacobi structure associated to (M,α) is given by Λ and R if and only if $R\wedge[X,R]\wedge X=0$.*

Proof. Let us check the two conditions of a Jacobi manifold, which are $[\Lambda,\Lambda]=2R\wedge\Lambda$ and $[\Lambda,R]=0$. The second equation writes

$$[\Lambda,R]=[\Pi+R\wedge X,R]=[\Pi,R]+[\Pi,R\wedge X]=0+[\Pi,R]\wedge X-R\wedge[\Pi,X]=R\wedge\Pi=0.$$

As for the first one, we do the following computation:

$$[\Lambda,\Lambda]=[\Pi,\Pi]+2[\Pi,R\wedge X]+[R\wedge X,R\wedge X].$$

Here, the first term is zero. The second term, using a well-known identity of the Schouten-bracket, gives us

$$2[\Pi,R\wedge X]=2[\Pi,R]\wedge X-2R\wedge[\Pi,X]=0+2R\wedge\Pi=0.$$

For the third term, using the same identity, we obtain

$$[R\wedge X,R\wedge X]=R\wedge[X,R]\wedge X+[R,R]\wedge X\wedge X-R\wedge R\wedge[X,X]-R\wedge[R,X]\wedge X$$

$$=2R\wedge[X,R]\wedge X.$$

□

APPENDIX B. LOCAL MODEL OF JACOBI MANIFOLDS

We recall local structure theorems of Jacobi manifolds, proved in [DLM]. Let us first introduce some notation.

- $\Lambda_{2q} = \sum_{i=1}^{q} \frac{\partial}{\partial x_{i+q}} \wedge \frac{\partial}{\partial x_i}$
- $Z_{2q} = \sum_{i=1}^{q} x_{i+q} \frac{\partial}{\partial x_{i+q}}$
- $R_{2q+1} = \frac{\partial}{\partial x_0}$
- $\Lambda_{2q+1} = \sum_{i=1}^{q} (x_{i+q} \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_i}) \wedge \frac{\partial}{\partial x_{i+q}}$

Theorem B.1 ([DLM]). *Let (M^m, Λ, R) be a Jacobi manifold, x_0 a point of M and S be the leaf of the characteristic foliation going through x_0.*

If S is of dimension $2q$, then there exist a neighbourhood of x_0 that is diffeomorphic to $U_{2q} \times N$ where U_{2q} is an open neighbourhood containing the origin of $\mathbb{R}^{2q}$ and (N, Λ_N, R_N) is a Jacobi manifold of dimension $m - 2q$. The diffeomorphism preserves the Jacobi structure, where the Jacobi structure on $U_{2q} \times N$ is given by

$$R_{U_{2q} \times N} = \Lambda_N, \qquad R_{U_{2q} \times N} = \Lambda_{2q} + \Lambda_N - Z_{2q} \wedge R_N.$$

If S is of dimension $2q+1$, then there exist a neighbourhood of x_0 that is diffeomorphic to $U_{2q+1} \times N$ where U_{2q+1} is an open neighbourhood containing the origin of $\mathbb{R}^{2q+1}$ and (N, Λ_N, R_N) is a homogeneous Poisson manifold of dimension $m - 2q - 1$. The diffeomorphism preserves the Jacobi structure, where the Jacobi structure on $U_{2q} \times N$ is given by

$$R_{U_{2q+1} \times N} = R_{2q+1}, \qquad \Lambda_{U_{2q+1} \times N} = \Lambda_{2q+1} + \Lambda_N + E_{2q+1} \wedge Z_N.$$

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