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Quantum nonlocality certification from Bell inequalities in three-level many-body systems

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Quantum nonlocality certification from Bell inequalities in three-level many-body systems

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Abstract. In this work, we aim at studying the applicability of the recently discovered multipartite permutationally invariant three-outcome Bell inequality involving at most second order correlation functions (3-outcome PIBI) presented in [1] to systems of physical interest. In particular, we display a class of unitary equivalent states for which it is possible to detect nonlocality only with individual spin-1 measurements, and therefore physically realizable. Conversely, we have identified nonlinear interactions, beyond $\mathfrak{su}(2)$, to be key in the nonlocalization mechanism. Finally, we reveal nonlocality for the ground state of the three-level Lipkin-Meshkov-Glick model, not only for infinite-range, but also power-law interactions. This breaks ground to use 3-outcome PIBIs as a certification tool in long-range quantum simulators.

Keywords: Bell inequality, nonlinear, nonlocality, quantum correlations, three-level systems

1. Introduction

The principles of locality and realism are successful assumptions in the description of classical phenomena. In 1935, Einstein, Podolsky and Rosen (EPR) suggested a thought experiment about measuring individually a space-like separated entangled pair. According to them, within the local-realistic frame, a set of local deterministic hidden variables (LHVs) were necessary to explain the correlations between both measurements outcomes [2].

It took nearly thirty years to obtain an operational test to verify whether a set of correlations could be reproduced by a local-realistic model. In particular, the seminal work by J.S. Bell provided bounds on the correlations accessible within a LHV theory in the form of the so-called Bell inequalities (BIs). We shall call classical a deterministic LHV theory on a local-realistic model. Remarkably, some quantum-originated correlations violate BIs [3]. In such cases, the correlations cannot be explained by a LHV theory and we call them conventionally nonlocal 1. Bell's result allows for many experimental implementations, for instance, in photonic platforms by means of polarization [4].

In multipartite scenarios, the derivation of BIs is in general not possible due to the combinatorial complexity of the involved full-body correlations which makes the problem intractable both theoretically and numerically. Remarkably, J. Tura et. al. proposed a set of scalable permutation invariant BIs involving only one-body and two-body correlation functions (PIBIs) [5]. More recently, A. Aloy et. al, extends the previous work from two outcomes to three, tailoring the study of nonlocality in many-body three-level systems [1].

In this project, we focus on a particular family of three-level PIBI, studying the type of correlations necessary for violation. We expect to find correlation mechanisms beyond spin-1 and $\mathfrak{su}(2)$ due to the device independence (DI) [6] of the derivation of the PIBI, not relying upon representations. Furthermore, we aim at testing the inequality with physical particles 2 by considering individual spin-1 measurements along some direction.

Finally, we want to identify models of interest whose ground state (GS) exhibit nonlocality detectable with 3-outcome PIBI. This supposes a first step towards considering its implementation on the novel long-range quantum simulators being developed, offering a tool to explore strongly correlated phenomena. In this context, the certification of non-classicality would impact at the fundamental level, for example to characterize new phase transitions, as well as practically, providing a resource for quantum information tasks [7].

The report is organized as follows. In Section 2, we settle the starting point, define BIs, explain its power to detect nonlocality and describe the particular PIBI addressed throughout the project. In Section 3, we present a set of maximally violating symmetric states whose nonlocality can be detected by the analogue spin-1 $\mathfrak{su}(3)$ local observables. Finally, in Section 4, we reveal nonlocality for the ground state of the three-level Lipkin-Meshkov-Glick model with power-law interactions.

2. Preliminaries

The starting point of this work is a DI Bell scenario, which is described by n parties $K = \{k_1, k_2, .., k_n\}$ with m $\{x_1, x_2, .., x_m\}$ d-outcome $\{a_1, a_2, .., a_d\}$ measurements each on a shared resource ρ (e.g a quantum state). After several rounds of collecting statistics, a set of correlations $p(a_{i_1} a_{i_2} ... \, a_{i_n} | x^{k_1}_{j_1} x^{k_2}_{j_2} ... \, x^{k_n}_{j_n})_{i \in \{1,..,d\} \; j \in \{1,..,m\}}$ can be build. To ease the notation, in the next paragraph we will set $n = 2$. The generalization to the multipartite

1 To avoid confusion this notion of nonlocality in quantum information is different from the usual definition in quantum mechanics/quantum field theory regarding commutation of space-like separated observables.

2 By physical particle we understand a unitary representation of the Lorentz-Poincaré algebra $\mathfrak{so}(3+1)$, i.e an irreducible representation of $\mathfrak{su}(2)$ localized at a point of our 3+1 space-time.

case is straightforward.

If the two outcomes are correlated, the correlations will not factorize $p(a_{i_1}a_{i_2}|x_{j_1}^{k_1}x_{j_2}^{k_2}) \neq p(a_{i_1}|x_{j_1}^{k_1})p(a_{i_2}|x_{j_2}^{k_2})$. If they can be reproduced with a LHV theory $p(a_{i_1}a_{i_2}|x_{j_1}^{k_1}x_{j_2}^{k_2}) = \sum_\lambda p(a_{i_1}a_{i_2}|x_{j_1}^{k_1}x_{j_2}^{k_2},\lambda)$, namely if they factorize as $p(a_{i_1}a_{i_2}|x_{j_1}^{k_1}x_{j_2}^{k_2},\lambda) = p(a_{i_1}|x_{j_1}^{k_1},\lambda)p(a_{i_2}|x_{j_2}^{k_2},\lambda)$ where λ labels the LHVs, then the correlations are within the so-called local polytope L; i.e., L is a convex set with finite number of vertices containing all possible local-realist correlations [8]. The problem that we want to address is, given the set of correlations, do they belong to L?

A natural way to solve it is by characterizing the facets of L, which can be written as a linear combination of correlations,

$$\mathcal{B}(n,m,d) = \sum_{o=0 \text{ s.t. } k\in\mathfrak{S}(o,n)}^{n} \sum_{i\in\mathfrak{C}(o,d)} \sum_{j\in\mathfrak{C}(o,m)} \alpha^{(o)}_{i_1i_2\ldots i_o|j_1j_2\ldots j_o}(k_1,k_2,..,k_o)\ p(a_{i_1}a_{i_2}\ldots\ a_{i_o}|x_{j_1}^{k_1}x_{j_2}^{k_2}\ldots\ x_{j_o}^{k_o})\ , \tag{1}$$

where $\mathfrak{S}(o,n)$ is the set with all subsets of o elements of K ,$\mathfrak{C}(o,b)$ is the set of o-tuples over $\{0,1,..,b-1\}$ and $\alpha\in\mathbb{R}$, such that for a given locally reproducible probability distribution p, $\mathcal{B}(n,m,d)\geq 0$, which is the so-called BI. The independent term $\alpha^{(0)}$ is the classical bound.

In reference [5], a family of BIs valid for any n was derived for the scenario $d=2, m=2$, by considering up to second order correlations $\alpha^{(o>2)}=0$ and permutation invariance (PI), $\alpha\neq\alpha(k)$. The latter, allows us to factorize α and define the collective distributions $P(a_i|x_j) = \sum_{k_1} p(a_i|x_j^{k_1})$, $P(a_{i_1}a_{i_2}|x_{i_1}x_{i_2}) = \sum_{k_1\neq k_2} p(a_{i_1}a_{i_2}|x_{j_1}^{k_1}x_{j_2}^{k_2})$. We shall label the outcomes a_i and measurements x_i with natural numbers $\{1,2,...,d\}$ or the traceless combination $\{-s,-s+1,...,s-1,s\}$ with $s=(d-1)/2$ when we introduce operators. Likewise, in the present thesis, we will label the parties or particles with $i=k_1,\ j=k_2\in\{1,2,...n\}$.

2.1. Three-outcome many-body permutationally invariant Bell inequality (PIBI)

In reference [1], the following 3-outcome PIBI was obtained valid for any n under the above restrictions for two measurements and three outcomes, (i.e $m=2$ and $d=3$),

$$\mathcal{B} = \tilde{P}_0 + \tilde{P}_{00} - 2\tilde{P}_{01} \geq 0\ , \tag{2}$$

where $\tilde{P}_0 = P(0|0)+P(0|1)+P(1|0)+P(1|1)$ is the symmetrized unipartite term and $\tilde{P}_{00} = P(00|00)+P(00|11)+P(11|00)+P(11|11)$, $\tilde{P}_{01} = P(01|01)+P(01|10)$ are the symmetrized bipartite terms.

In order to study which mechanisms lead to nonlocal correlations that violate inequality (2), we determine frustration on the classical theory that describes our 3-outcome PIBI. We start by associating a classical variable $\sigma_{0,1}^{(i)}\in\{-1,0,1\}$ to the outcomes of the local measurements $\{0,1\}$. Then, the ground energy of the two-component nonlinear Ising model

$$\mathcal{K} = 2\sum_i \left(\sigma_0^{2(i)} + \sigma_1^{2(i)}\right) + \sum_{i\neq j}\left((\sigma_0^2 - \sigma_1^2)^{(i)}(\sigma_0^2 - \sigma_1^2)^{(j)} + (\sigma_0 + \sigma_1)^{(i)}(\sigma_0 + \sigma_1)^{(j)}\right) \quad , \quad (3)$$

which follows (2) such that $\overline{\mathcal{K}} = \mathcal{B}$ for any local state with $^{-}$ a thermal average, is the classical bound c.f. [9]. From $\mathcal{K} \geq 0$, if any distribution $\mathcal{B} < 0$, then the system it describes is nonlocal or Bell correlated. Using single update Metropolis algorithm we verified that the GS is in general achieved non-trivially due to the degeneration provided by the infinite range interactions giving raise to antiferromagnetic (AFM) correlations $C_{ab} = \sum_{i\neq j}\sigma_a^{(i)}\sigma_b^{(j)} < 0$, and thus frustrated. Furthermore, we observe $M = \sum_i(\sigma_0 + \sigma_1)^{(i)} = 0 \iff \sum_i \sigma_0^{(i)} = -\sum_i \sigma_1^{(i)} \equiv m$ for the GS because of measurement exchange and global spin flip invariance.

From the quantum side, violations of Bell inequalities are possible due to observable incompatibility. In particular, for spin-1/2 systems, spin squeezing, i.e, the variance shrink below the standard quantum limit in the plane perpendicular to the finite mean spin, has been proposed as a sufficient condition for Bell correlation [10]. In order to find out whether spin-squeezing plays a role in the violation of inequality (2), we consider spin-1 individual measurements in the next section.

3. Class of nonlocal symmetric states detectable with inequality (2)

Firstly, we consider only the distributions that are attainable within the quantum theory with local Hilbert space H of dimension three (hence $\mathbb{C}^3 \sim H$), with the corresponding Von-Neumann measurements $\{\hat{M}_0 = -1\times\hat{P}_{00}+1\times\hat{P}_{10}+0\times\hat{P}_{20},\ \hat{M}_1 = -1\times\hat{P}_{01}+1\times\hat{P}_{11}+0\times\hat{P}_{21}\}$, where $\hat{P}_{ax}$ are projectors.

In the next step, we assume that the local measurements are spin-1 operators $\{\hat{M}_0 = \hat{S}_0, \hat{M}_1 = \hat{S}_1\}$, whose generators are $\hat{\mathbf{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z) \in \mathfrak{su}_3(2)$. The elements of the group are labeled with the Pauli vector $\mathbf{v}_\mu \in \mathbb{R}^3$ as $\mathbf{v}_\mu \cdot \mathbf{S} \equiv \hat{S}_\mu$. We shall see that spin-1 operators are not sufficient to obtain the expectation values and correlations involved in inequality (2). Indeed, by defining $\hat{S}_\pm = (\hat{S}_0 \pm \hat{S}_1)/|\mathbf{v}_0 \pm \mathbf{v}_1|$, where $|\mathbf{v}_0|^2 = |\mathbf{v}_1|^2 = 1$, the 3-outcome PIBI (2) is rewritten as

$$\sum_i(|\mathbf{v}_+|^2\langle\hat{N}_{++}^{(i)}\rangle + |\mathbf{v}_-|^2\langle\hat{N}_{--}^{(i)}\rangle) + \frac{1}{2}\sum_{i\neq j}\left(2|\mathbf{v}_+|^2\langle\hat{S}_+^{(i)}\otimes\hat{S}_+^{(j)}\rangle + |\mathbf{v}_+|^2|\mathbf{v}_-|^2\langle\hat{N}_{+-}^{(i)}\otimes\hat{N}_{+-}^{(j)}\rangle\right) \geq 0 \ , \quad (4)$$

where $\mathbf{v}_\pm = \mathbf{v}_0 \pm \mathbf{v}_1$ and the nematic tensor $\hat{N}_{\mu\nu} = (\hat{S}_\mu\hat{S}_\nu + \hat{S}_\nu\hat{S}_\mu)/2$ appear naturally accounting for the spin nonlinearities beyond $\mathfrak{su}(2)$. Since the expression (4) is not linearly closed within $\mathfrak{su}(2)$, we conclude a full characterization of the measurements as $\mathfrak{su}(3)$ elements is needed.

3.1. Structure of the $SU(3)$ group

Having started with spin-1 operators, it is natural to extend this Lie basis by introducing some linearly independent nematic components to obtain the generators of $SU(3)$ [11],

$$\{\hat{S}_x, \hat{S}_y, \hat{S}_z, \hat{Q}_{xy}, \hat{Q}_{yz}, \hat{Q}_{zx}, \hat{D}_{xy}, \hat{Y}\} \ , \quad (5)$$

where $\hat{Q}_{\mu\nu} = 2\hat{N}_{\mu\nu}$ for $\mu \neq \nu$, $\hat{D}_{xy} = \hat{N}_{xx} - \hat{N}_{yy}$ and $\hat{Y} = (2\hat{N}_{zz} - \hat{N}_{xx} - \hat{N}_{yy})/\sqrt{3}$.

With this extension, not only $\{\hat{S}_x, \hat{S}_y, \hat{S}_z\}$ is an $\mathfrak{su}(2)$ subalgebra, there are more triads $\{\hat{G}_\mu, \hat{G}_\nu, \hat{G}_\eta\} \in \mathfrak{su}(3)$ that follow an analogue algebra $[\hat{G}_\mu, \hat{G}_\nu] = \mathbb{i}c\sum_\eta \epsilon^\eta_{\mu\nu}\hat{G}_\eta$, where ϵ is the Levi-Civita symbol. They can be classified completely in type-1 or type-2 unitary disjoint equivalent classes depending on the value of the structure's constant norm c.

In the one hand, type-1 triads are the ones unitary equivalent to spin-1 $\{\hat{S}_x, \hat{S}_y, \hat{S}_z\}$, for example $\{\hat{Q}_{zx}, \hat{Q}_{yz}, \hat{S}_z\}$. In the other hand type-2 triads are the ones unitary equivalent to two mode operators. For instance, $\{\hat{D}_{xy}, \hat{Q}_{xy}, \hat{S}_z\}$ for modes $\{-1, 1\}$. Similar triads can be found for modes $\{-1, 0\}$ and $\{0, 1\}$.

In each triad, spin squeezing can be defined the same way as in spin-1/2 and for type-2, finite spin polarization is no longer necessary to generate squeezing.

3.2. Class of nonlocal three-level symmetric states

In this subsection, a class of symmetric states that exhibit nonlocality with the 3-outcome PIBI (2) is found by minimizing the Bell functional $\mathcal{B} = \mathcal{B}(\hat{S}_0, \hat{S}_1, \rho)$ to obtain the optimal Bell expectation value in the n-body symmetric subspace $\rho_S \in Sym\{H^{\otimes n}\}$ with fixed measurements. The space $Sym\{H^{\otimes n}\}$ is spanned by qutrit Dicke states $|\{\boldsymbol{n} = (n_0, n_1, n_2) \in \mathfrak{T}^3_n\}\rangle \propto \sum_{\pi\in\mathfrak{P}_n} \pi(|0\rangle^{\otimes n_0}|1\rangle^{\otimes n_1}|2\rangle^{\otimes n_2})$, where $\mathfrak{P}_n$ is the permutation group of $n = n_0 + n_1 + n_2$ elements and $\mathfrak{T}^3_n$ is the partition of n in 3 elements. By definition, the two-body reduced density matrices (2-RDMs) of ρ_S are pairwise equal and we will denote them by $\rho^{(2)}_S$.

We perform the optimization based on a solution of the quantum marginal problem for symmetric states posed as an efficient semidefinite programming (SDP) [12]. In the Dicke basis, the SDP reads

$$\min_{\rho^{(2)}_S} \mathcal{B}(\hat{S}_0, \hat{S}_1, \rho_S) = \mathrm{tr}\left\{\hat{\mathcal{B}}_n(\hat{S}_0, \hat{S}_1)\rho^{(2)}_S\right\} \text{ s.t. } \mathrm{tr}\left\{A^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}\rho_S\right\} = \rho^{(2)\boldsymbol{\alpha}}_{S\ \boldsymbol{\beta}}\ \forall \boldsymbol{\alpha}, \boldsymbol{b} \in \mathfrak{T}^3_n\ , \rho_S \geq 0\ , \quad (6)$$

where $\hat{\mathcal{B}}_n$ is the two-body reduced Bell operator and A encodes the compatibility constrains of $\rho^{(2)}_S$ to have an n-body symmetric extension ρ_S. The minimum found is not necessarily the optimal of inequality (2) since this could be realized in a higher dimensional space with positive-operator valued measures (POVMs), nor we have guarantee that it is the maximal violation of its restriction in H, as it could be accomplished by an irreducible representation of $\mathfrak{P}_n$ with mixed symmetry or a degenerate state.

With the extension to $\mathfrak{su}(3)$, apart from spin-1 operators, we can consider other triads belonging to type-1 or type-2 subspaces as local measurements since they follow the same algebra. Concretely, without loss of generality, we can parametrize the measurements with $\theta \in (0, \pi]$ as $\{\hat{S}_0 = \hat{S}_z\ , \hat{S}_1 = \cos\theta\hat{S}_z + \sin\theta\hat{S}_x\}$ for type-1 and $\{\hat{S}_0 = \hat{S}_z\ , \hat{S}_1 = \cos\theta\hat{S}_z + \sin\theta\hat{D}_{xy}\}$ for type-2. For type-2 measurements, the results are trivial with an optimum state $|\psi\rangle = |0\rangle^{\otimes n}$ and no violation because type-2 measurements are two-mode reducible and consequently nonlinearities vanish, i.e $\hat{N}_{\mu\nu} = \delta_{\mu\nu}\mathbb{I}$, where δ is the Kronecker's delta and $\mathbb{I}$ is the identity 3×3 matrix. Figure (1) shows the symmetric optimized Bell expectation value and an example of an optimum state for type-1 measurements.

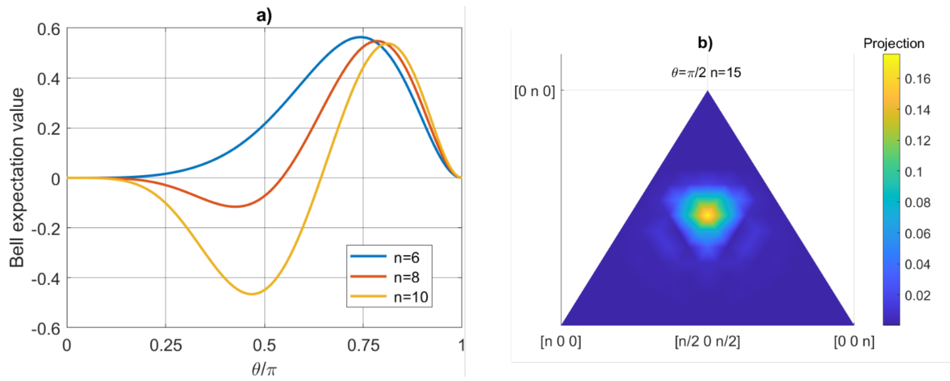

FIGURE 1: *a) Symmetric optimum Bell expectation value of inequality* (4) *for type-1 measurements* $\{\hat{S}_z\ , \cos\theta\hat{S}_z + \sin\theta\hat{S}_x\}$, $\mathcal{B}_S(\theta)$, *for different number of parties obtained by solving* (6). *Nonlocality can be certified for* $n > 7$. *By increasing* n, *the violation becomes larger specially in the vicinity of* $\theta = \pi/2$. *b) example of maximally violating symmetric state* $|\psi\rangle$ *in the Dicke representation,* $|\langle\{\mathbf{n}\}|\psi\rangle|^2$, *for* $\theta = \pi/2$ *and* 15 *parties achieving a violation of* $\mathcal{B}_S = -1.611$.

The state presented in Figure (1) exhibits AFM correlations in accordance with the relevant correlations identified previously with the classical model (3) and are unpolarized, $m = \langle m\rangle = 0$.

From an experimental point of view, given the technological impossibility to measure the particles individually $\hat{E}^{(i)}$, one may consider the so-called Bell correlation certification [13]. In this case, the analogue spin and nematic components (or alternatively spin and second moments) are measured collectively $\mathbb{E}(\hat{E}) \equiv \sum_i \hat{E}^{(i)}/n$ 3. With the previous definition, we define its expected value $\langle\mathbb{E}(\hat{E})\rangle = \sum_i\langle\hat{E}^{(i)}\rangle/n$, the mean individual fluctuations $\mathbb{E}((\Delta\hat{E})^2) = \sum_i(\langle\hat{E}^{2(i)}\rangle - \langle\hat{E}^{(i)}\rangle^2)/n^2 = \langle\mathbb{E}(\hat{E}^2)\rangle - \langle\mathbb{E}(\hat{E})\rangle^2 \geq 0$ and the collective fluctuations $(\Delta\mathbb{E}(\hat{E}))^2 = \sum_{i,j}(\langle\hat{E}^{(i)} \otimes \hat{E}^{(j)}\rangle - \langle\hat{E}^{(i)}\rangle\langle\hat{E}^{(j)}\rangle)/n^2 = \sum_{i\neq j}\langle\hat{E}^{(i)} \otimes \hat{E}^{(j)}\rangle/n^2 + \mathbb{E}((\Delta\hat{E})^2) \geq 0$ [14], the inequality (4) becomes

$$\frac{2|\mathbf{v}_+|^2(\Delta\mathbb{E}(\hat{S}_+))^2 + 2|\mathbf{v}_-|^2\mathbb{E}((\Delta\hat{S}_-)^2) + |\mathbf{v}_+|^2|\mathbf{v}_-|^2(\Delta\mathbb{E}(\hat{N}_{+-}))^2}{2|\mathbf{v}_+|^2\langle\mathbb{E}(\hat{S}_+)\rangle^2 + 2|\mathbf{v}_-|^2\langle\mathbb{E}(\hat{S}_-)\rangle^2 + |\mathbf{v}_+|^2|\mathbf{v}_-|^2\mathbb{E}((\Delta\hat{N}_{+-})^2)} \geq 1\ . \tag{7}$$

The comparison of equation (7) with the form of well-known spin-squeezing inequalities for entanglement detection [15], motivates the conjecture that interplay between spin and nematic components or higher moments are crucial to correlate nonlocaly detectable with (4). The fact that no spin-squeezing was obtained in the analysis of the states of Figure (1) supports this idea.

3 We consider a pairwise permutation symmetric ensemble of particles, consequently the averages are taken on a uniform probability distribution.

4. Three-level Lipkin-Meshkov-Glick model with power-law interactions

In this section, we study the model described by the transverse-interaction quantum Ising Hamiltonian

$$\hat{\mathcal{H}} = B \sum_i \hat{S}_z^{(i)} + \frac{1}{2} \sum_{i \neq j} \frac{J}{|\mathbf{r}_i - \mathbf{r}_j|^\alpha} \left(\hat{S}_x^{(i)} \otimes \hat{S}_x^{(j)} + \hat{D}_{xy}^{(i)} \otimes \hat{D}_{xy}^{(j)} + \hat{Q}_{zx}^{(i)} \otimes \hat{Q}_{zx}^{(j)} \right) , \tag{8}$$

where $\mathbf{r}_i \in \mathbb{R}^3$ is the position of particle i. This Hamiltonian is equivalent to the $\mathfrak{su}(3)$ shell model presented in [16] with power-law interactions α and the three-level Lipkin-Meshkov-Glick (LMG) model, with important applications in nuclear physics [17]. Hereinafter, we will consider AFM interactions, $J = 1$, and $B > 0$.

Firstly, we will solve the Hamiltonian by density matrix renormalization group (DMRG) technique. Then, from the GS, we are going to extract the averaged 1-RDM and 2-RDM which are used to compute the expected value of the Bell operator $\hat{\mathcal{B}}$ with the measurements to be optimized.

4.1. Density matrix renormalization group with long-range interactions and optimization over measurements

Provided the GS has full symmetric support, in the infinite-range interactions limit $\alpha = 0$, the Hamiltonian can be exactly diagonalized efficiently in the symmetric subspace. This is not true for $\alpha \neq 0$, due to separable local state exchange invariance breaking. In this case, we employ DMRG with a unidimensional chain geometry of the model with open boundary conditions.

The first step is to introduce a bond space of dimension χ indexed by γ to arrange the Hamiltonian as a matrix product operator (MPO), $\widehat{MPO} = \sum_{s,s' \in \{0,1,2\}} \sum_{\gamma \setminus \gamma^*} A^{s_1 s_1'}_{\gamma_0^* \gamma_1} A^{s_2 s_2'}_{\gamma_1 \gamma_2} ... A^{s_n s_n'}_{\gamma_{n-1} \gamma_n *} |s_1 s_2 .. s_n\rangle \langle s_1' s_2' ... s_n'|$, where $A^{s_i s_i'}_{\gamma_{i-1}\gamma_i} = \lceil A_i^{s_i s_i'} \rceil_{i \in \{1,2,..,n\}}$ are matrices once (γ_{i-1}, γ_i) are fixed. We fit the polynomial decay of the interaction by exponentials $r^{-\alpha} \approx \sum_k^\kappa a_k b_k^{r-1}$. In this case, it exists a set of 'automata' rules $(l, r) \mapsto \lceil A \rceil_{lr}$ independent of the lattice site i, with only nontrivial action when $1 \equiv \gamma_0^* \leq \gamma_{i-1} \leq \gamma_i$, which can reproduce not only the two-body part but also the single particle term as an MPO [18]. Since $\gamma_i \leq 3\kappa + 2 = \chi \equiv \gamma_n^*$, it can be implemented efficiently. A typical value for $\alpha = 0.5$ and $n = 30$ is $\kappa = 6$.

Next, we initialize the DMRG with a random matrix product state (MPS), $|MPS\rangle = \sum_s \mathrm{tr}\{\lfloor A_1^{s_1} \rfloor \lfloor A_2^{s_2} \rfloor ... \lfloor A_n^{s_n} \rfloor\} |s_1 s_2 .. s_n\rangle$, where $\lfloor A_i^{s_i} \rfloor = A^{s_i}_{\gamma_{i-1}\gamma_i}$ are matrices once fixed s_i and the lattice site i. The MPO Hamiltonian is split into three blocks L,C and R of sizes l, 2 and $n - l - 2$ respectively. The ansatz's corresponding RDM is used to renormalize L and R to a rank-4 tensor and then all the inner indices are contracted within C. The resulting tensor is factorized using single value decomposition (SVD) to the right back to the MPS form. This procedure is repeated $l + 1 \leftarrow l$ for several sweeps $l = 1 \leftarrow l = n - 2$. Normally, 20 sweeps is enough for 10^{-7} precision in the ground state. Once the algorithm has converged, we compute the 1-$RDMs$ and 2-$RDMs$ of the obtained ground state MPS contracting over the corresponding indices.

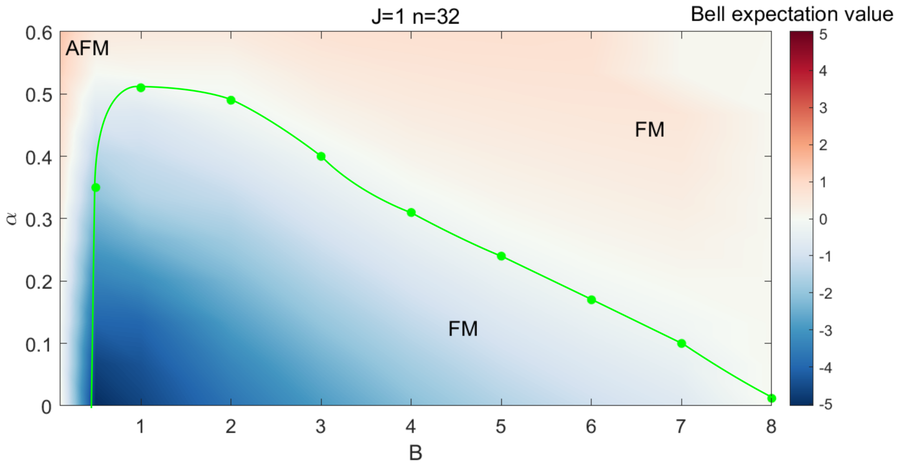

FIGURE 2: *Optimized Bell expectation value of inequality* (2) *on the DMRG resulting ground MPS of the 3-LMG as a function of the interaction polynomial decay* α *and the local energy* B *following Section 4.1 for 32 particles. In green, the data-points with vanishing violation within the measurement subspace generated by* $\{\hat{S}_x, \hat{Q}_{zx}, \hat{D}_{xy}\}$. *The line is intended to ease visualization and delimit violation for smaller* α. *The corresponding dominant correlations* $\mathcal{C}_{zz}$ *are marked, AFM for* $B < 1/2$ *and FM for* $B > 1/2$.

Regarding the optimization over measurements, we use the unitary continuous parametrization $\hat{U}_{k\in\{0,1\}} = \hat{U}(\boldsymbol{\theta}_k)$ elements of the defining representation of the Weyl-Heisenberg group proposed in [1]. Given the roots of unity, $\omega = e^{2\pi \mathrm{i}/3}$, the projectors can be constructed $\{\hat{P}_{0k} = (\hat{U}_k^3 + \hat{U}_k^2 + \hat{U}_k)/3\ ,\hat{P}_{1k} = (\hat{U}_k^3 + \omega\hat{U}_k^2 + \omega^2\hat{U}_k)/3\ ,\hat{P}_{2k} = (\hat{U}_k^3 + \omega^2\hat{U}_k^2 + \omega\hat{U}_k)/3\}\}$. Finally, the parameters $\boldsymbol{\theta}_0, \boldsymbol{\theta}_1 \in \mathbb{R}^{8+1}$ are updated following the conjugated gradient of $\mathcal{B}$ to be minimized 4.

4.2. Nonlocality detection in the ground state of three-level Lipkin-Meshkov-Glick Hamiltonian

In order to delimit which local measurements are able to detect nonlocality with the Bell inequality (4) on the ground state of 3-LMG, we study which phases this system presents with the order parameter magnetization $\mathcal{M}_z = \sum_i \langle \hat{S}_z^{(i)} \rangle / n$ and its fluctuations depending on $\mathcal{C}_{zz} = \sum_{i\neq j} \langle \hat{S}_z^{(i)} \otimes \hat{S}_z^{(j)} \rangle / (n(n-1))$. The results suggest for $B \gtrsim 1/2$, the correlations $\mathcal{C}_{zz}$ are ferromagnetic (FM) and the magnetization is proportional to $-B$, while for $B \lesssim 1/2$, $\mathcal{C}_{zz}$ is weakly AF in the disordered phase 5.

The Figure (2) we display the Bell expectation value obtained applying Section 4.1 as well as with the subspace generated by $\{\hat{S}_x, \hat{Q}_{zx}, \hat{D}_{xy}\}$ transverse to $\hat{S}_z$ and with no spin analogy. The latter restriction was motivated from the negligible imaginary part of the output MPS and the previous result about unpolarized optimal states. The

4 Due to the high non-convexity of the landscape, we can not assert that the minimum found is the absolute.

5 In the sense that when $B \to 0$, $\mathcal{M}_z \to 0$ due to the nonexistent strong correlations.

measurements can be well norm-preserving parametrized with the Gell-Mann matrices ($\hat{\Lambda}_6 = (\hat{S}_x - \hat{Q}_{zx})/\sqrt{2}$, $\hat{\Lambda}_1 = (\hat{S}_x + \hat{Q}_{zx})/\sqrt{2}$, $\hat{\Lambda}_4 = \hat{D}_{xy}$) in the unit sphere given the normalization $\mathrm{tr}\{\hat{\Lambda}_\mu \hat{\Lambda}_\nu\} = 2\delta_{\mu\nu}$.

In Figure (2), in the FM phase we perceive that the violation decays for increasing B as the GS approach the separable $|-1_z\rangle^{\otimes n}$. Similar behaviour is observed for α, as the interaction range is decreased, the antisymmetric content of the GS grows with the contribution of distanced pairs whose 2-RDM tend to $\mathbb{I}/3 \otimes \mathbb{I}/3$ as they decorrelate.

Remarkably, the violation achieved is maximal just before the finite size precursor of a phase transition at $B \approx 1/2$. Conversely, no nonlocality is detected on the AFM phase. In this case, for $\alpha = 0$, the DMRG results seem indicate that the GS belongs to a mixed symmetry representation, in contrast with the FM phase which DMRG and symmetric diagonalization values coincide. This is in agreement with the fact that the DI derived PIBIs are likely to fail in antisymmetric states, as conjectured in the literature and noticed in past experiences [10, 19].

Noteworthy, the violation obtained at first place fits the violation detectable with the transverse observables. Contrary to the state from Figure (1), the present GS exhibits $m \neq 0$ originated mainly from the contribution of $|0_z\rangle$.

5. Conclusions and outlook

To recapitulate, in this project we studied the features of the quantum nonlocality detectable with the 3-outcome PIBI (2) using $\mathfrak{su}(3)$ observables. We expected new types of correlations not present in spin-1/2 systems to be decisive to violate inequality (2) and consequently certify nonlocality.

Indeed, this is the case as we have identified; a) nonlinear correlations at a classical level (3), b) quantum 3-outcome PIBI functional (4) not closed to any analogue $SU(2)$ subspace and c) two-level reducible measurements give zero Bell violation. This results pose nonlinear interactions in many-body spin-1 systems as a key ingredient to achieve Bell correlation of inequality (4), going beyond the celebrated spin squeezing mechanism involving only first moments of collective observables. This phenomena is genuine to 3-level systems without any lower dimension equivalence.

Notwithstanding, a class of balanced $m = 0$ symmetric states with nonlocality detectable only with the information encoded in the local spin degree of freedom is introduced, proving the use of inequality (2) on physical systems. With the extension to $\mathfrak{su}(3)$, the class encompasses analogue spin operators. Furthermore, in the context of the 3-LMG Hamiltonian we show that for $m \neq 0$ it is still possible to certify nonlocality by measuring transversely to the mean spin. In future work, we plan to use this knowledge on other quartic three-level models [20].

Precisely in the last section, we show quantum nonlocality detectability for the 3-LMG GS not only on the infinite range interaction regime, but also with power-law interactions for a decay parameter $\alpha \lesssim 0.5$. The results confirm that permutation invariant BI are less effective in antisymmetric/disordered states and become useful near quantum critical points. In order to elucidate the preference of symmetric states to violate PIBIs, for future work we propose working them in the second quantization where symmetry in operators and indistinguishability in particles is assumed, towards its application on long-range quantum simulators.

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6. References

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