



# Apunts

## Bifurcations in 1D systems: saddle-node, transcritical, pitchfork

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Escola Superior d'Enginyeries Industrial,  
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**Nonlinear systems, chaos and  
control in engineering**  
Course 2020-2021  
Bachelors degrees in:  
Aerospace technology engineering  
Aerospace vehicle engineering  
Industrial technology engineering

# Bifurcations in 1D systems

saddle-node, transcritical, pitchfork

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## Summary Part 1

- Flows on the line = first-order ordinary differential equations.

$$dx/dt = f(x)$$

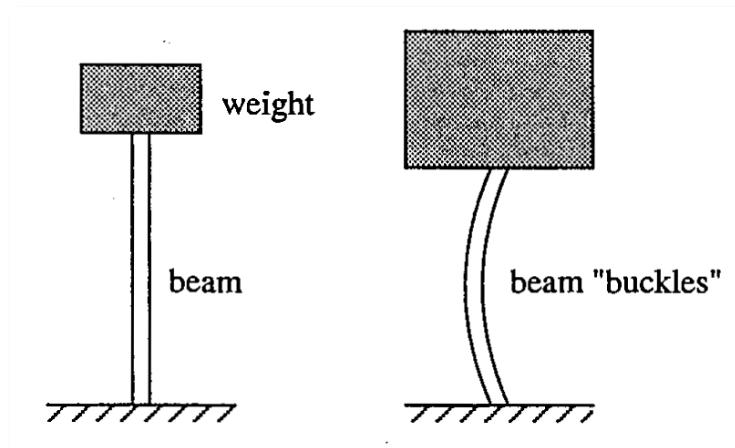
- Fixed point solutions:  $f(x^*) = 0$ 
  - stable if  $f'(x^*) < 0$
  - unstable if  $f'(x^*) > 0$
  - neutral (bifurcation point) if  $f'(x^*) = 0$
- There are no periodic solutions; the approach to fixed point solutions is monotonic (sigmoidal or exponential).

# Outline

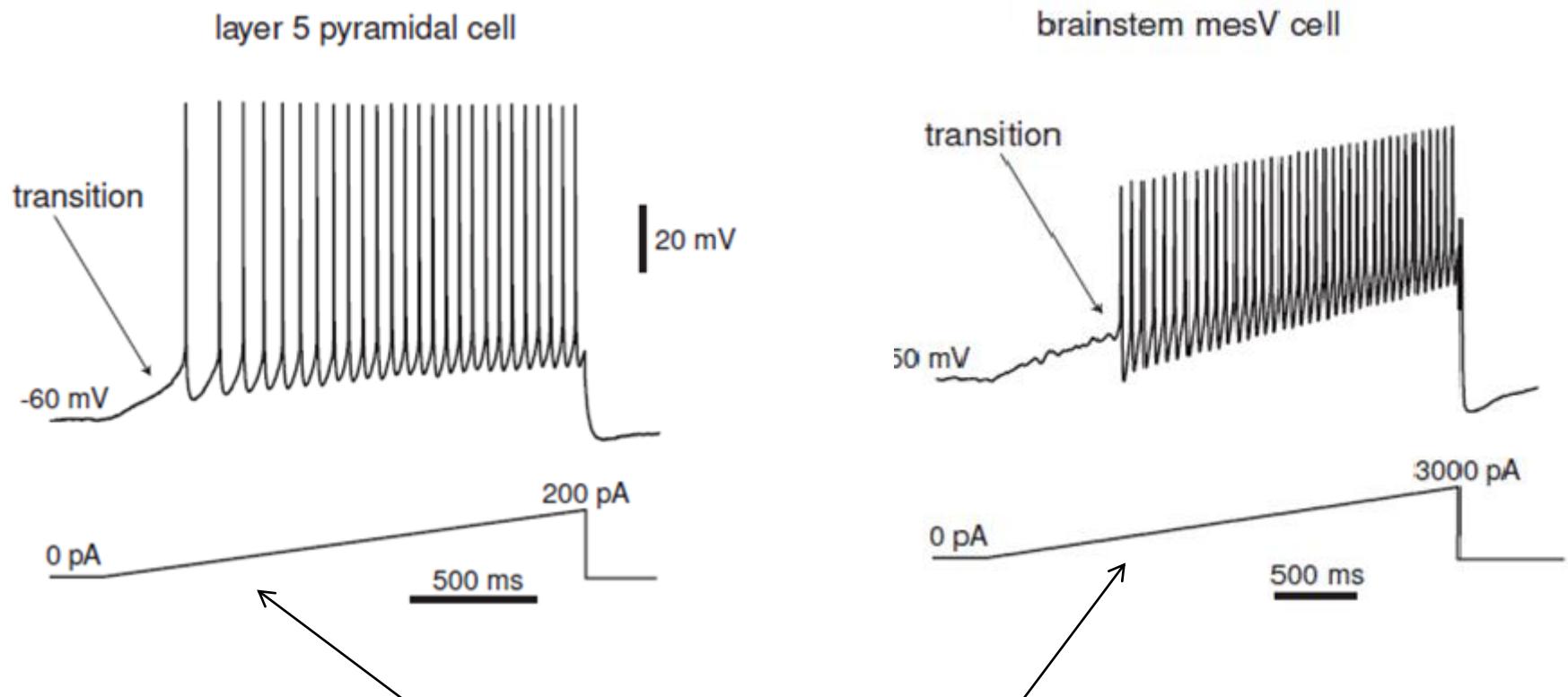
- Introduction to bifurcations
- Saddle-node, transcritical and pitchfork bifurcations
- Examples
- Imperfect bifurcations & catastrophes

# What is a bifurcation?

- A qualitative change (in the structure of the phase space) when a **control parameter is varied**:
  - Fixed points can be created or destroyed
  - The stability of a fixed point can change
- There are many examples in physical systems, biological systems, etc.

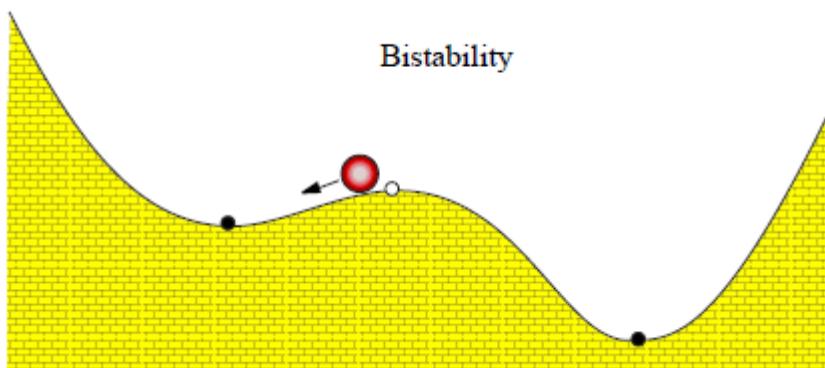
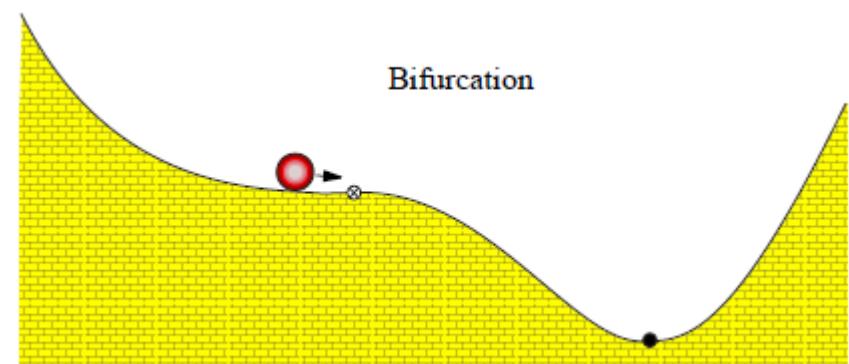
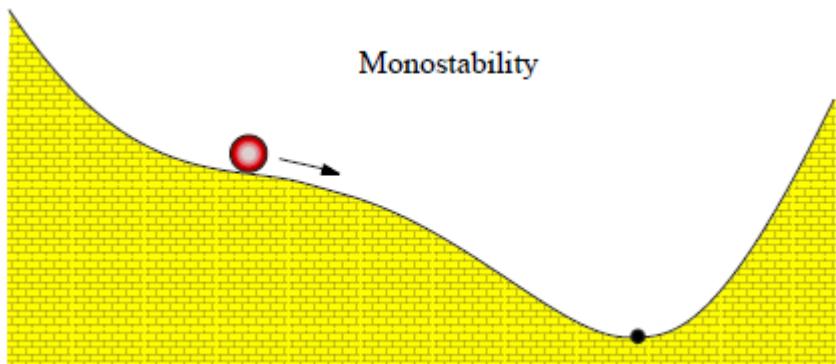


# Example



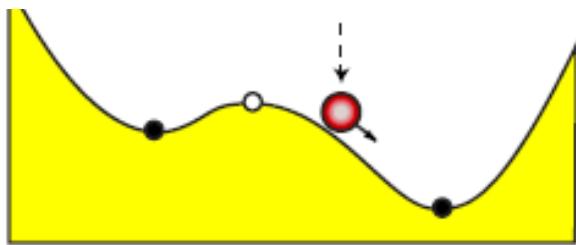
Control parameter increases in time

# Bifurcation and potential

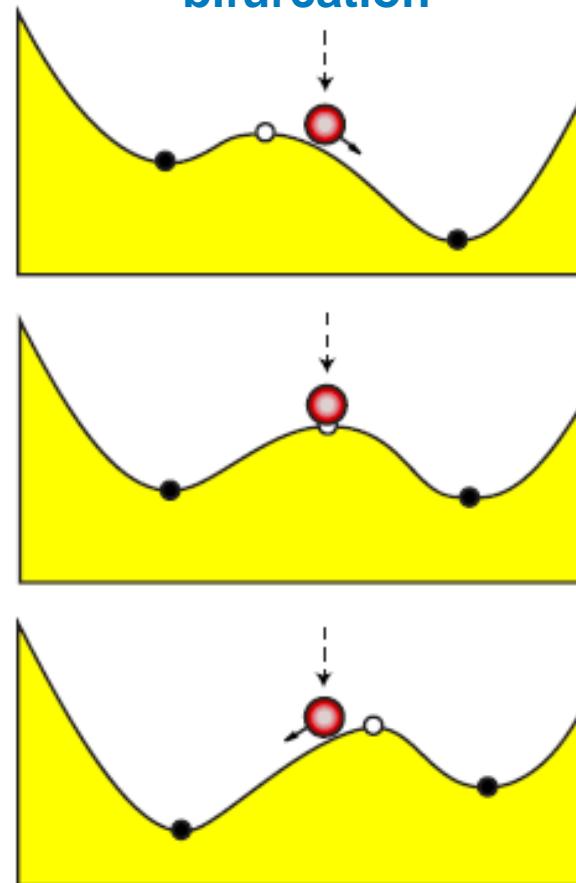


# A bifurcation is not equivalent to a qualitative change of behavior

Bifurcation but no change of behavior



Change of behavior but no bifurcation



# Outline

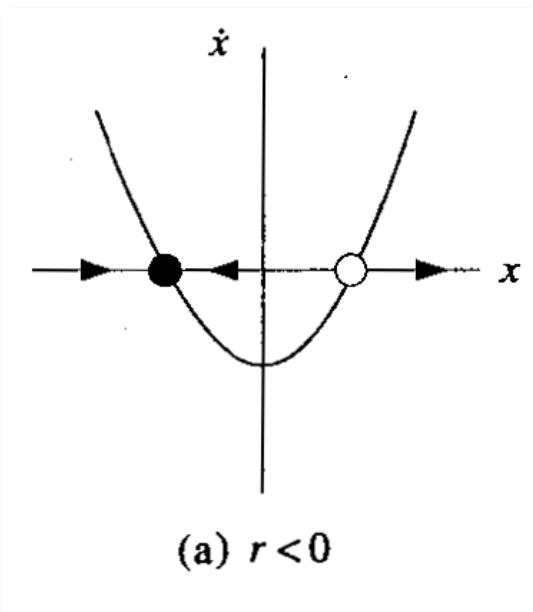
- Introduction to bifurcations
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# Saddle-node bifurcation

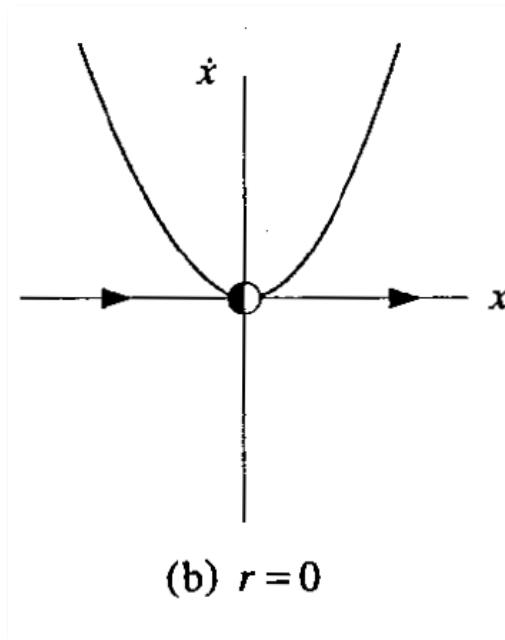
Basic mechanism for the creation or the destruction of fixed points

$$\dot{x} = f(x) = r + x^2$$

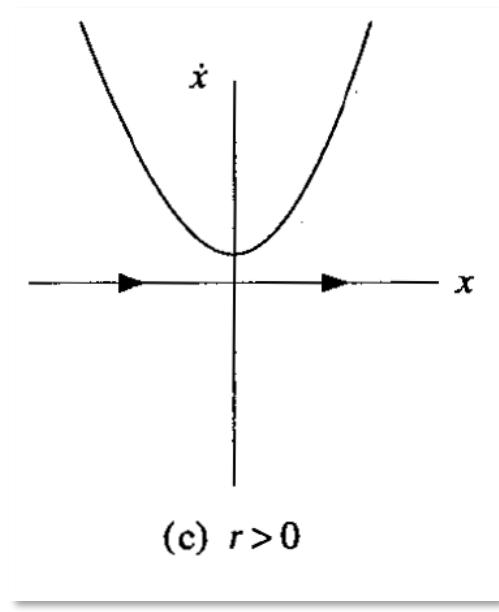
$$x^* = \pm\sqrt{-r}$$



(a)  $r < 0$



(b)  $r = 0$



(c)  $r > 0$

$$f'(-\sqrt{-r}) = -2\sqrt{-r}$$

Stable if  $r < 0$

$$f'(\sqrt{-r}) = 2\sqrt{-r}$$

unstable

At the bifurcation point  $r^* = 0$ :  $f'(x^*) = 0$

## Example

$$\dot{x} = r - x^2$$

- Calculate the fixed points and their stability as a function of the control parameter  $r$

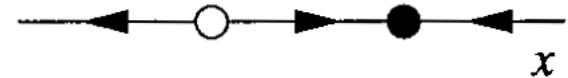
$$x^* = \pm\sqrt{r}$$



$$r < 0$$



$$r = 0$$



$$r > 0$$

## Normal forms

- Are representative of all saddle-node bifurcations.
- Close to the saddle-node bifurcation the dynamics can be approximated by

$$\dot{x} = r - x^2$$

or

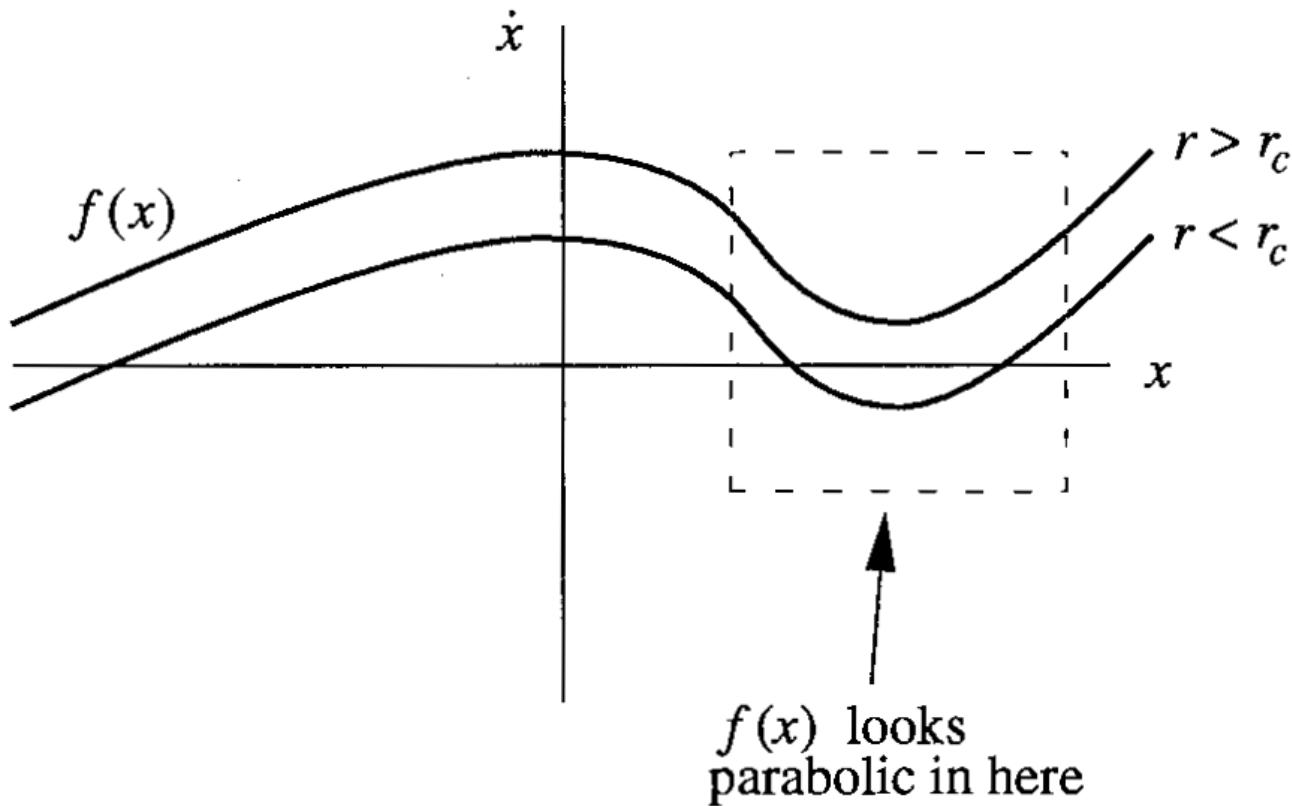
$$\dot{x} = r + x^2$$

Example:  $\dot{x} = r - x - e^{-x}$

$$= r - x - \left[ 1 - x + \frac{x^2}{2!} + \dots \right]$$

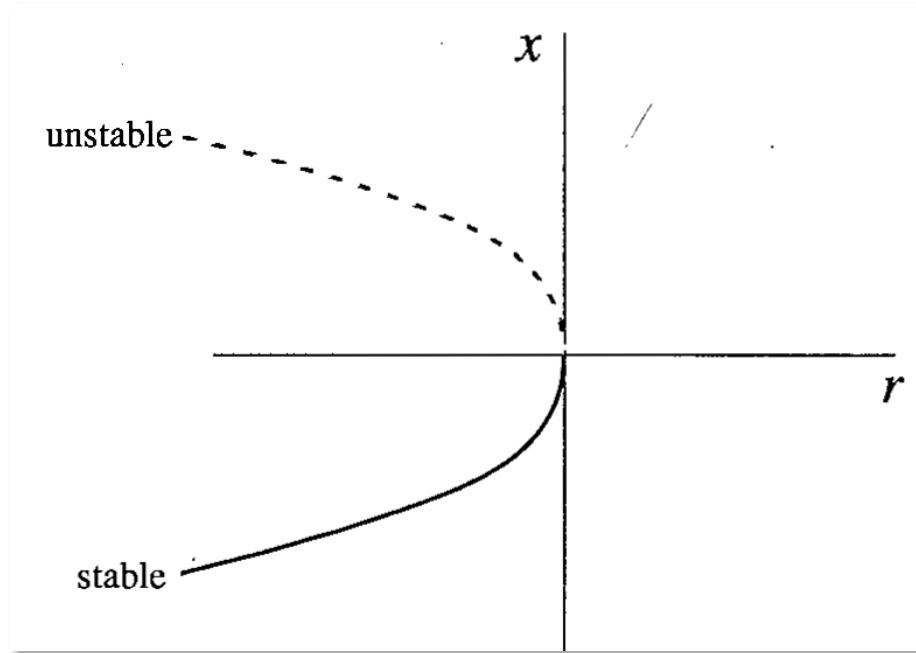
$$= (r - 1) - \frac{x^2}{2} + \dots$$

# Near a saddle-node bifurcation



# Bifurcation diagram

$$\dot{x} = r + x^2$$



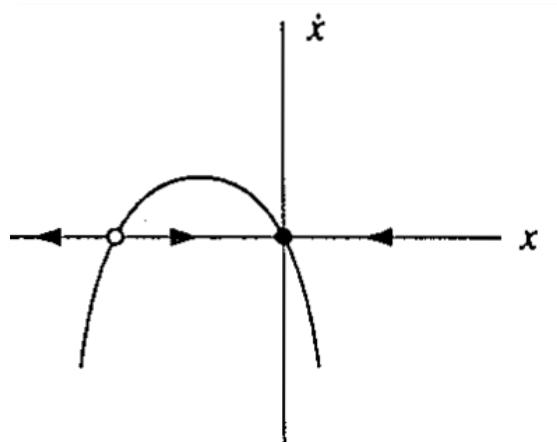
Two fixed points  $\rightarrow$  one fixed point  $\rightarrow$  0 fixed point

A pair of fixed points appear (or disappear) out of the “clear blue sky” (“blue sky” bifurcation, Abraham and Shaw 1988).

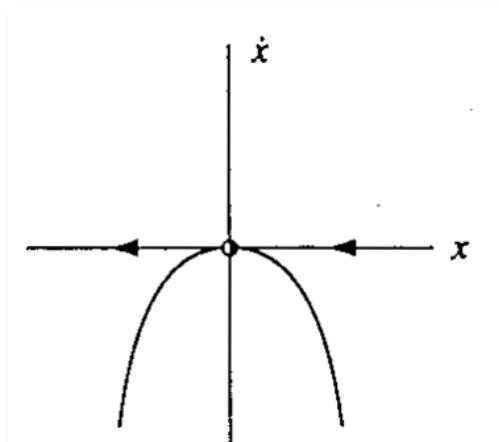
# Transcritical bifurcation

$$\dot{x} = rx - x^2$$

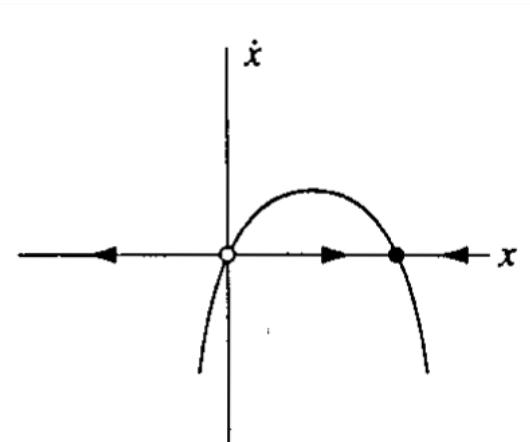
$x^* = 0$   
 $x^* = r$  are the fixed points for all  $r$



(a)  $r < 0$



(b)  $r = 0$



(c)  $r > 0$

Transcritical bifurcation: general mechanism for changing the stability of fixed points.

# Bifurcation diagram

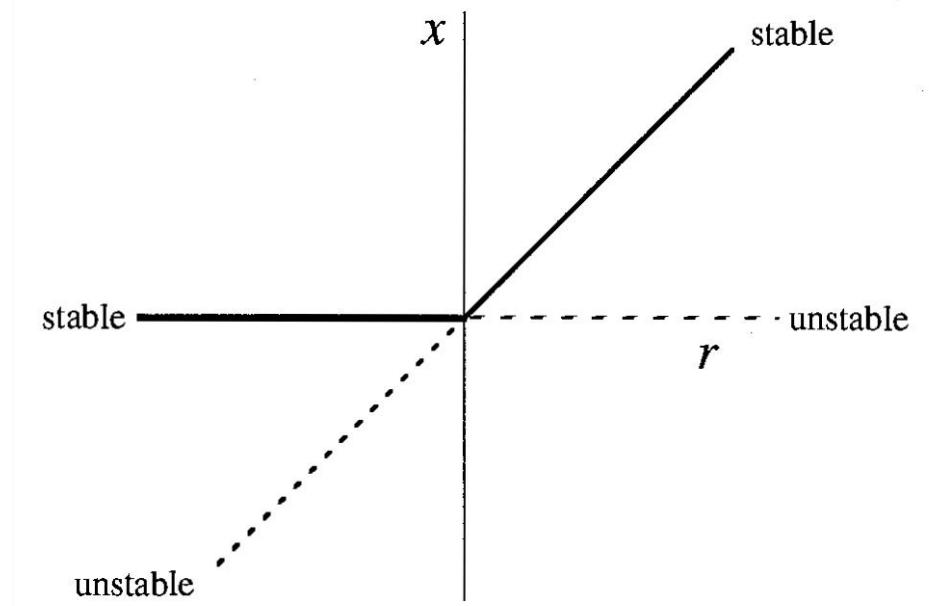
$$\dot{x} = rx - x^2$$

fixed points  $x^* = 0$  and  $x^* = r$

$$f'(x) = r - 2x$$

$$f'(0) = r$$

$$f'(r) = -r$$



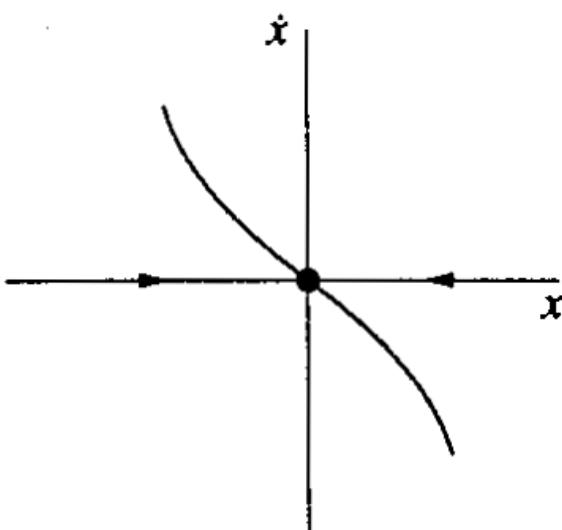
- Exchange of stability at  $r = 0$ .
- **Exercise:**  $\dot{x} = r \ln x + x - 1$

show that a transcritical bifurcation occurs near  $x=1$   
(hint: consider  $u = x-1$  small)

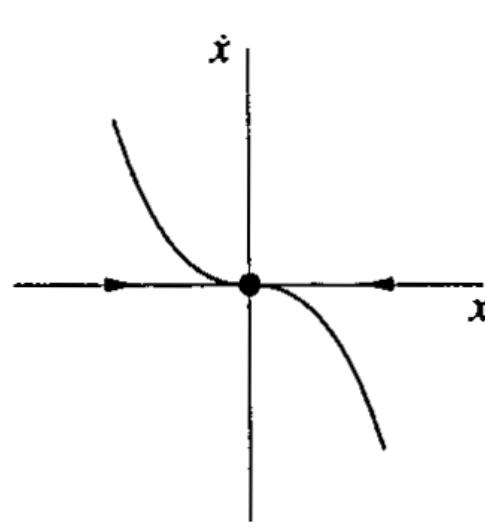
# Pitchfork bifurcation

$$\dot{x} = rx - x^3$$

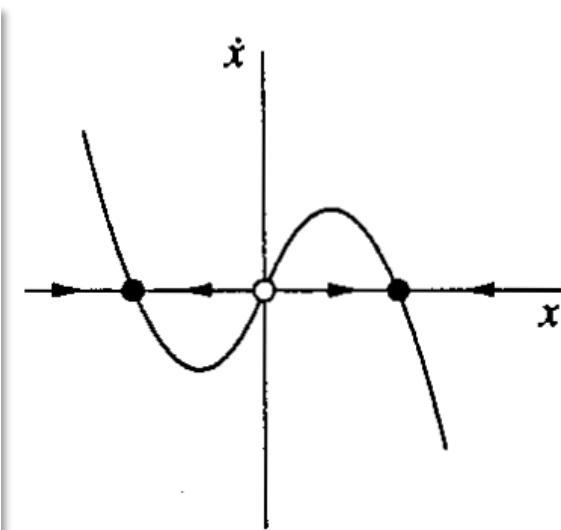
Symmetry  $x \rightarrow -x$



(a)  $r < 0$



(b)  $r = 0$

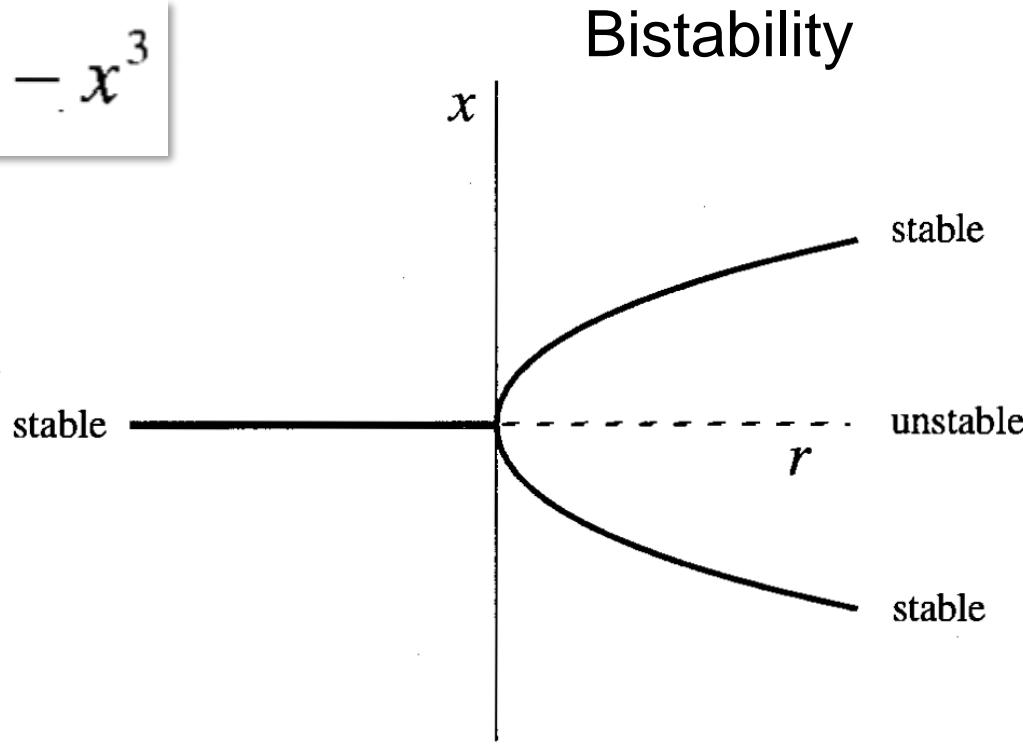


(c)  $r > 0$

One fixed point  $\rightarrow$  3 fixed points

# Bifurcation diagram

$$\dot{x} = rx - x^3$$



The governing equation is symmetric:  $x \rightarrow -x$   
but for  $r > 0$ : symmetry broken solutions.

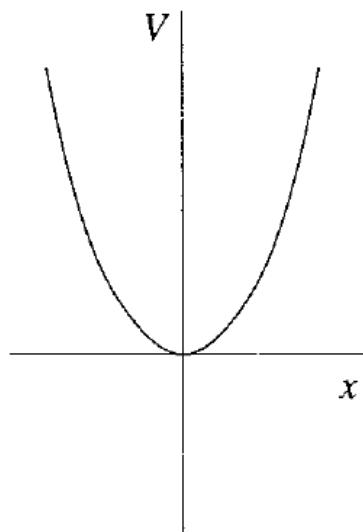
# Potential

$$\dot{x} = rx - x^3$$

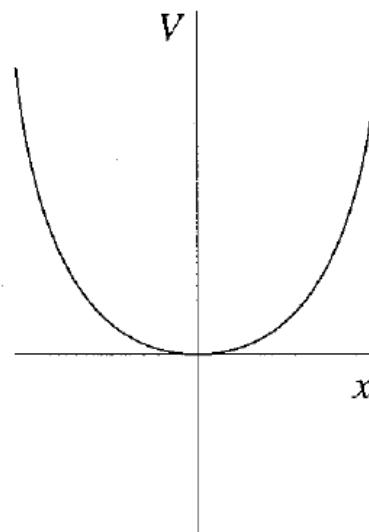
$$\dot{x} = f(x)$$

$$f(x) = -dV/dx$$

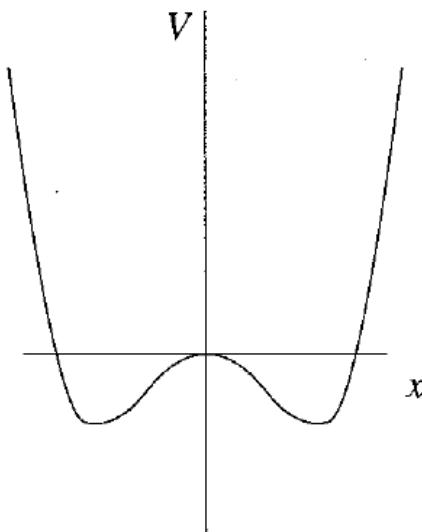
$$V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4$$



$$r < 0$$



$$r = 0$$

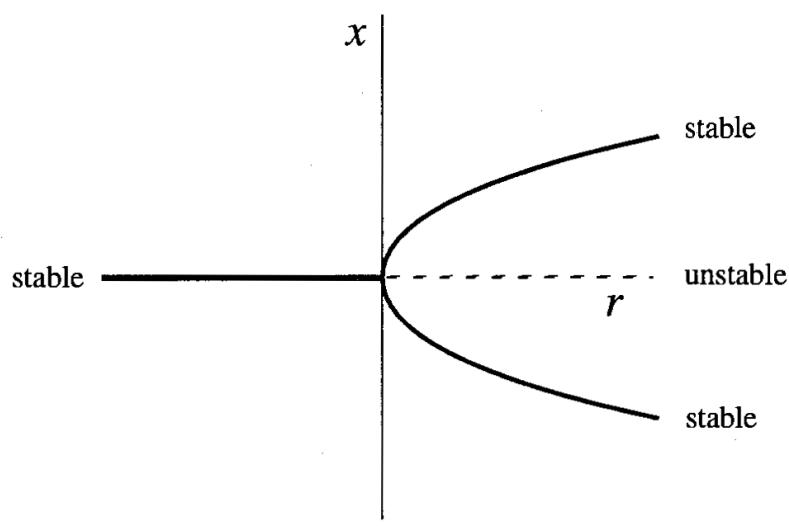


$$r > 0$$

# Pitchfork bifurcations

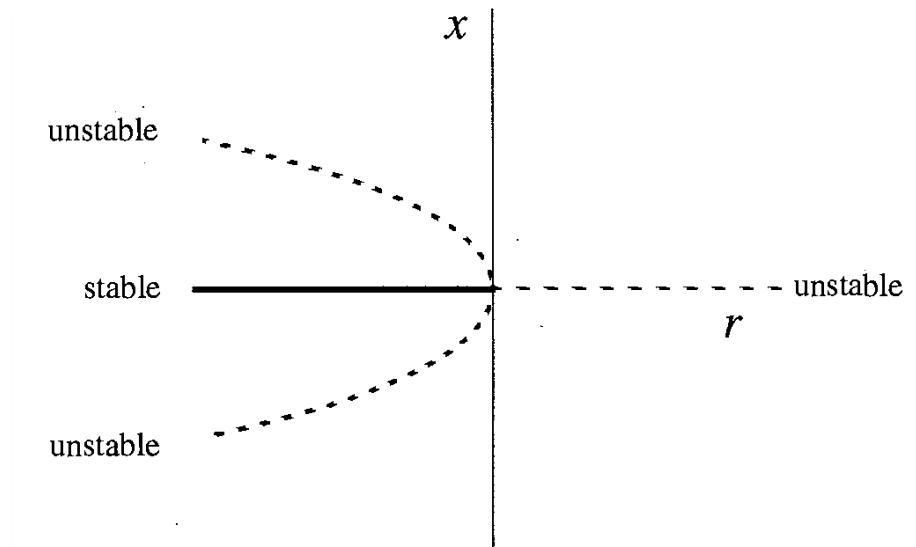
Supercritical:  
 $x^3$  is stabilizing

$$\dot{x} = rx - x^3$$



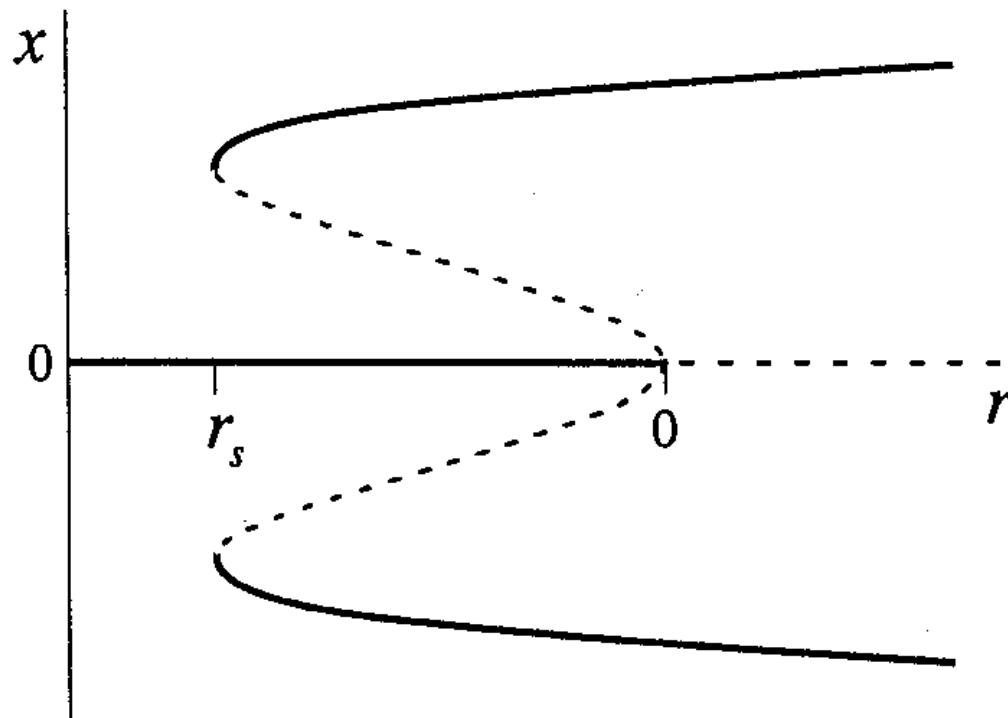
Subcritical:  
 $x^3$  is destabilizing

$$\dot{x} = rx + x^3$$

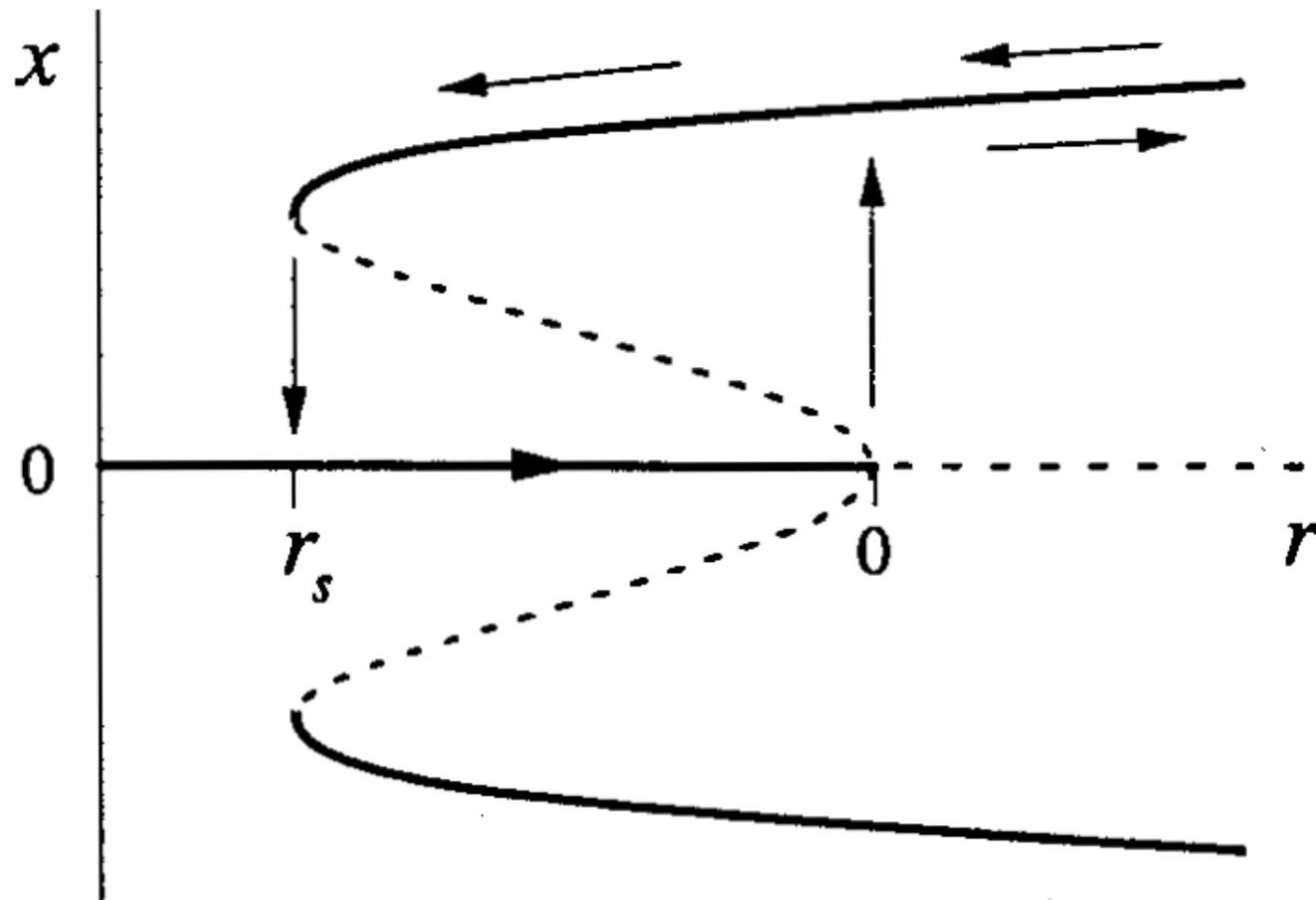


## Exercise 4: find the fixed points and compute their stability

$$\dot{x} = rx + x^3 - x^5$$
$$x(r + x^2 - x^4)$$



## Subcritical bifurcation: Hysteresis



Critical or dangerous transition! A lot of effort in trying to find “early warning signals” (more latter)

# Hysteresis: sudden changes in visual perception



Fischer  
(1967):  
experiment  
with 57  
students.

“When do you  
notice an  
abrupt change  
in perception?”

# Summary

- Bifurcation condition: change in the stability of a fixed point

$$f'(x^*) = 0$$

- In first-order ODEs: three possible bifurcations
  - Saddle node
  - Pitchfork
  - Trans-critical
- The normal form describes the behavior near the bifurcation.

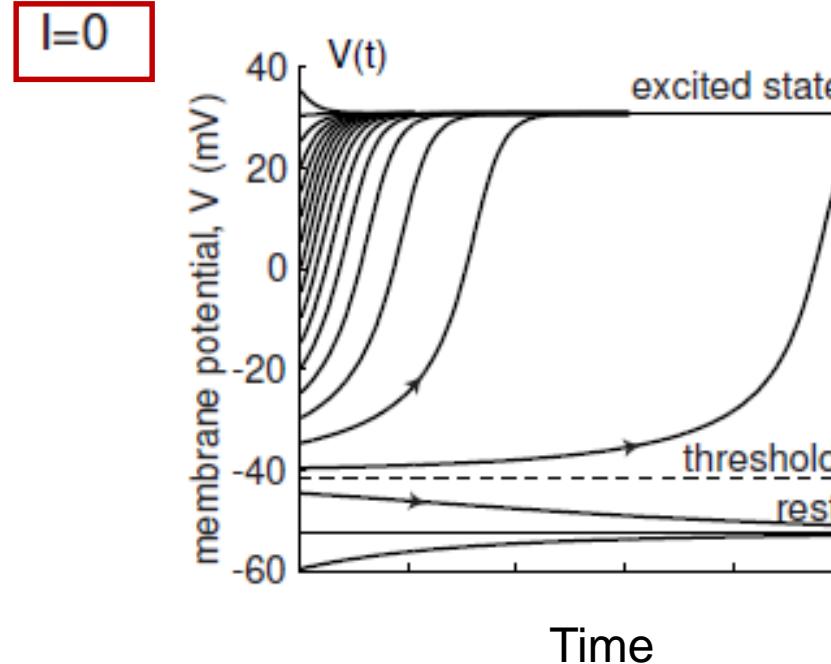
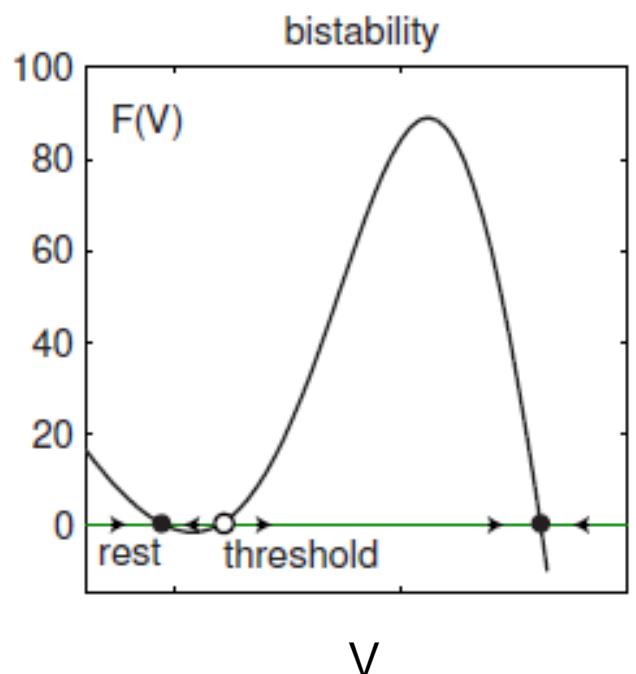
# Outline

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# Example: neuron model

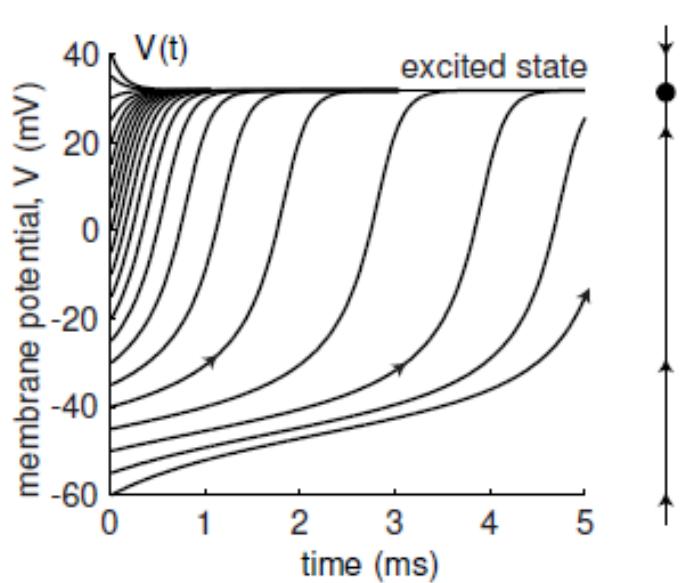
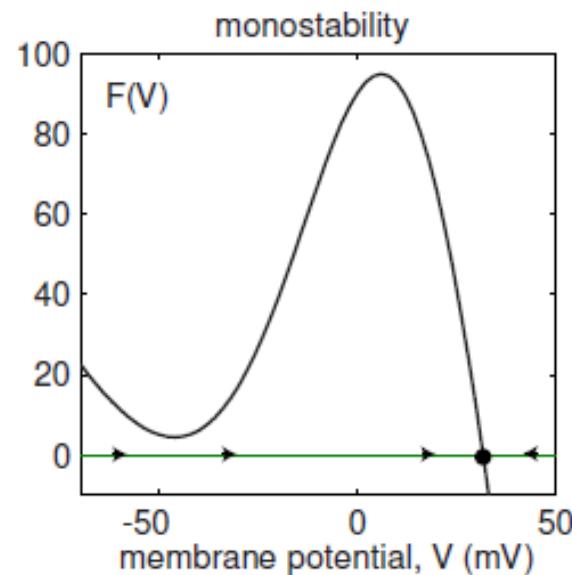
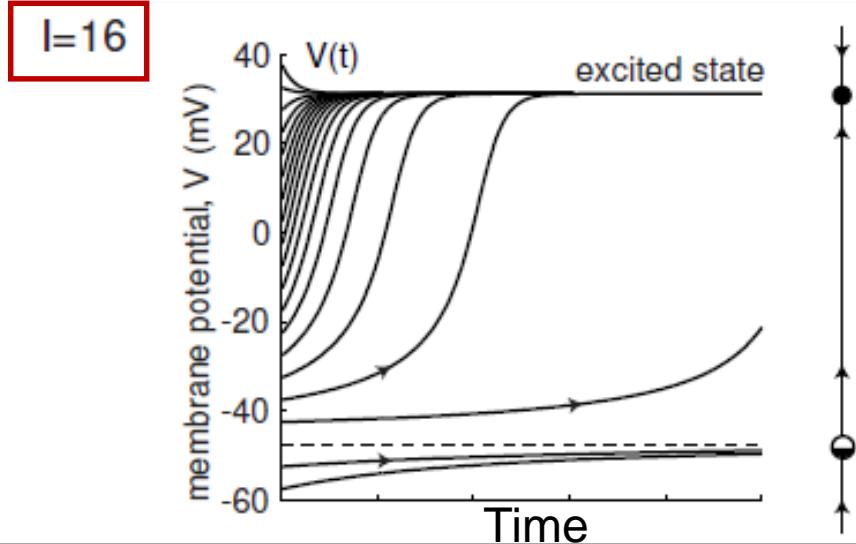
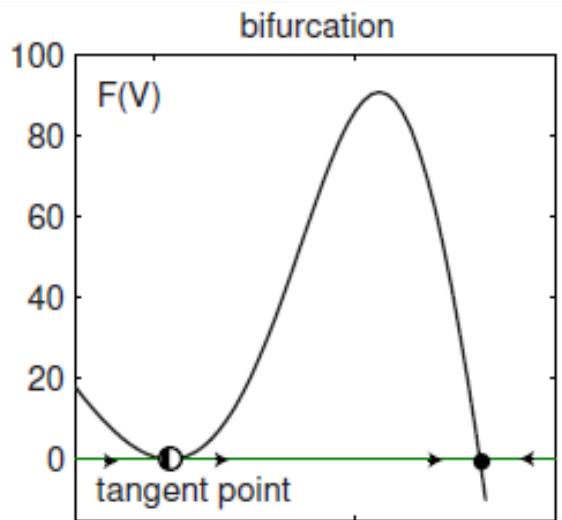
$$C \dot{V} = I - g_L(V - E_L) - \overbrace{g_{\text{Na}} m_\infty(V) (V - E_{\text{Na}})}^{\text{instantaneous } I_{\text{Na,p}}} \\ m_\infty(V) = 1/(1 + \exp \{(V_{1/2} - V)/k\})$$

$$C = 10 \text{ } \mu\text{F}, \quad I = 0 \text{ pA}, \quad g_L = 19 \text{ mS}, \quad E_L = -67 \text{ mV}, \\ g_{\text{Na}} = 74 \text{ mS}, \quad V_{1/2} = 1.5 \text{ mV}, \quad k = 16 \text{ mV}, \quad E_{\text{Na}} = 60 \text{ mV}$$

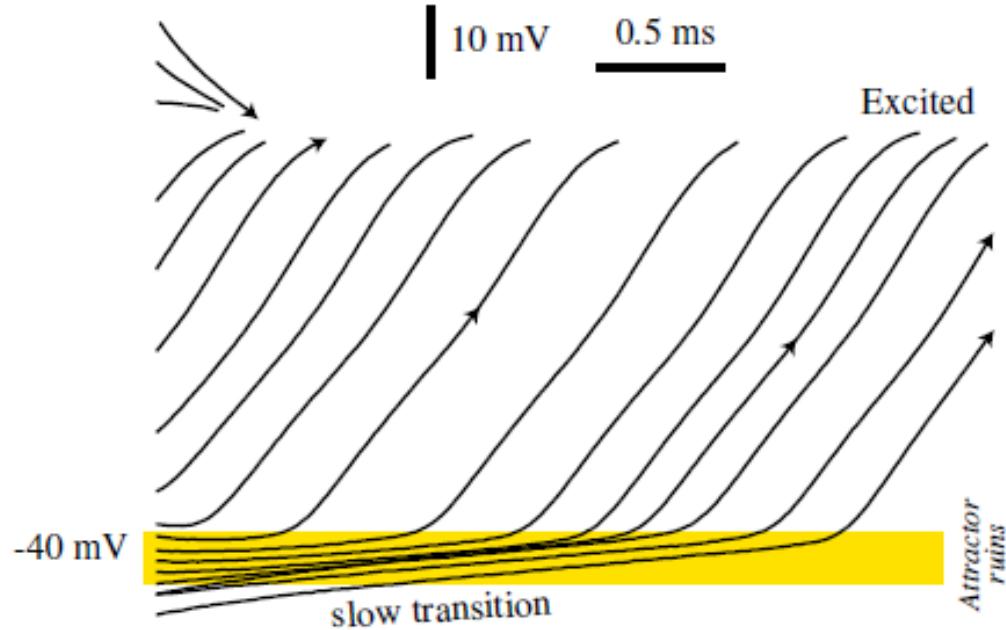
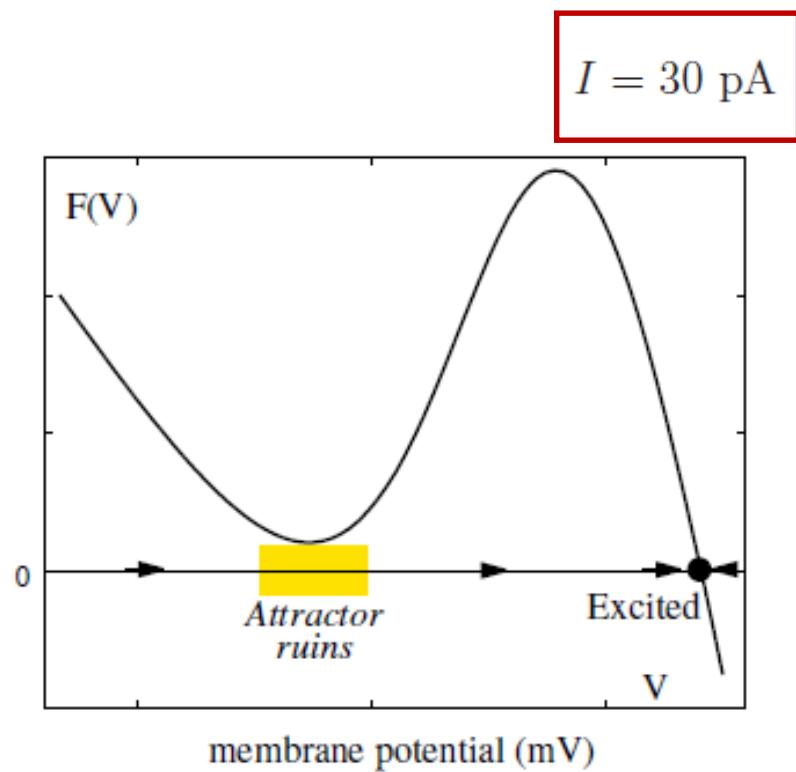


E. M. Izhikevich: *Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting* (MIT Press 2010).

# Saddle-node Bifurcation

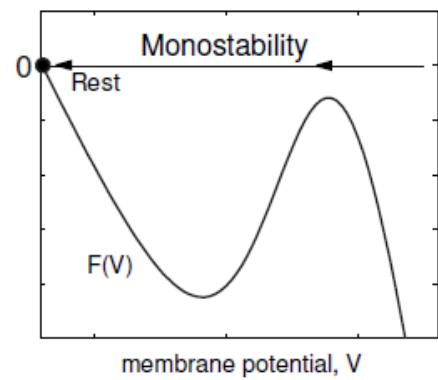
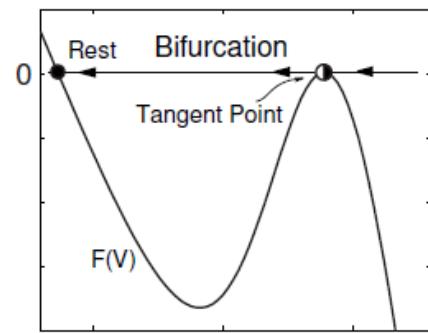
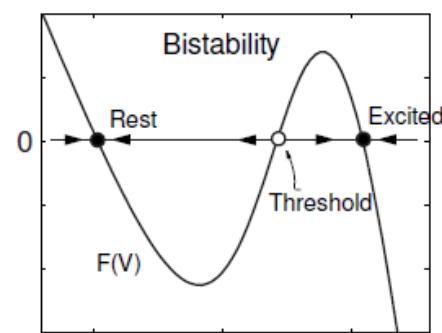


# Near the bifurcation point: slow dynamics

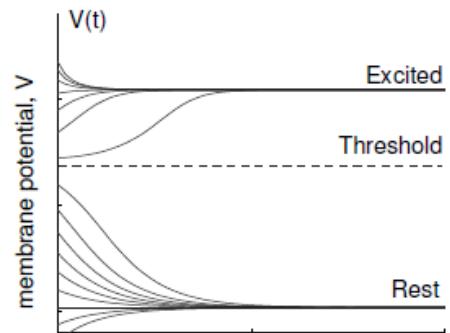


This slow transition is an “early warning signal” of a critical or dangerous transition ahead (more latter)

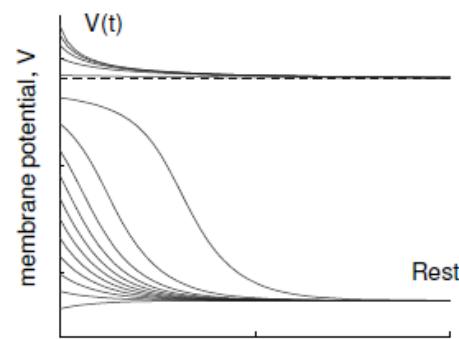
# If the control parameter now decreases



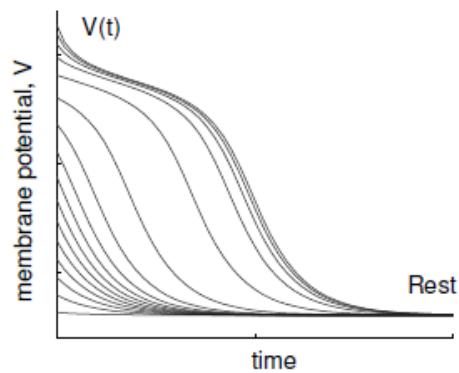
$I = -400$



$I = -890$

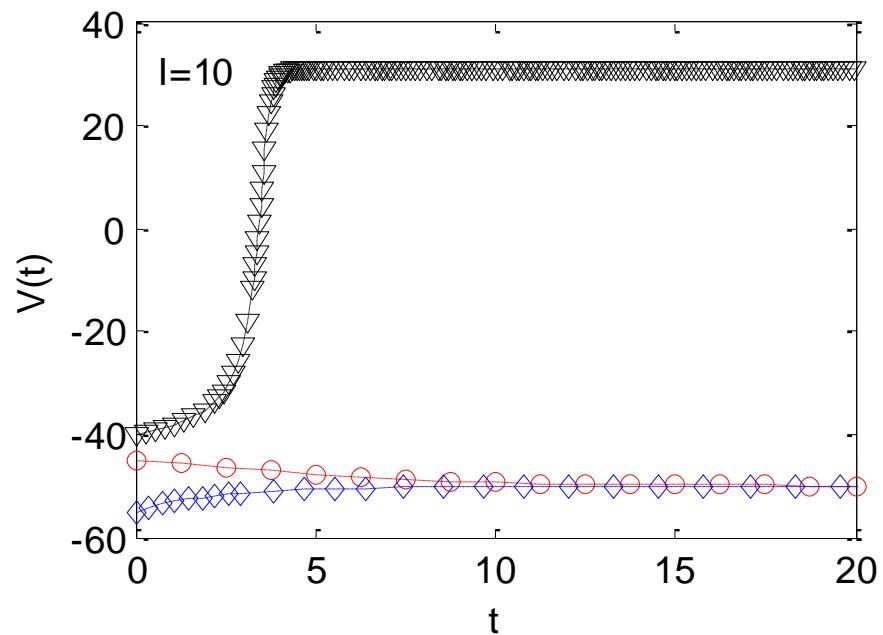
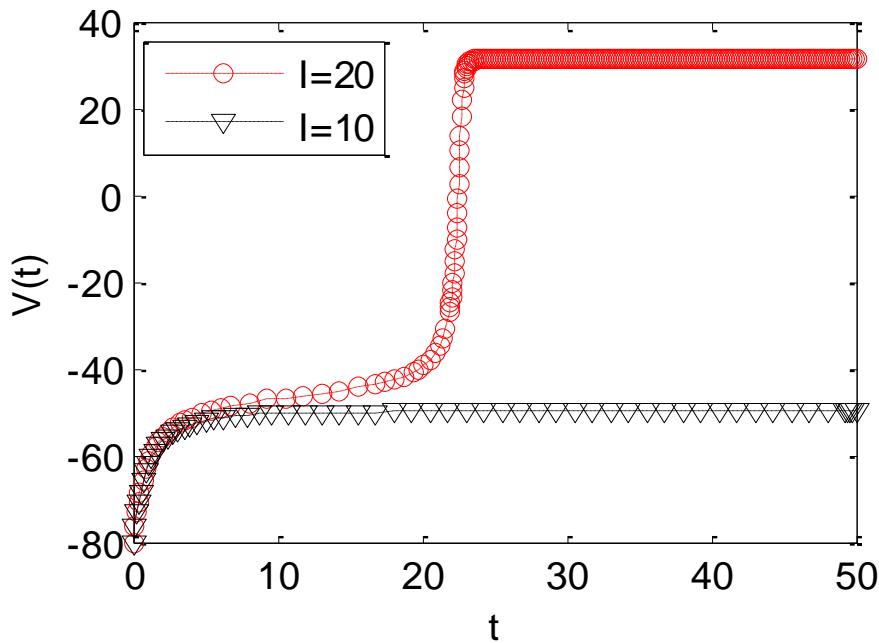


$I = -1000$



## Exercise 5

Simulate the neuron model with different values of the control parameter  $I$  and/or different initial conditions.

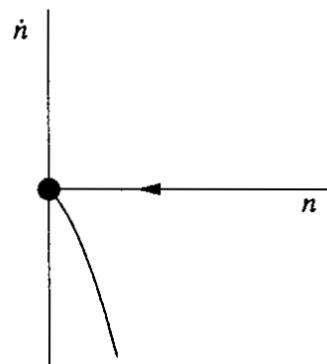
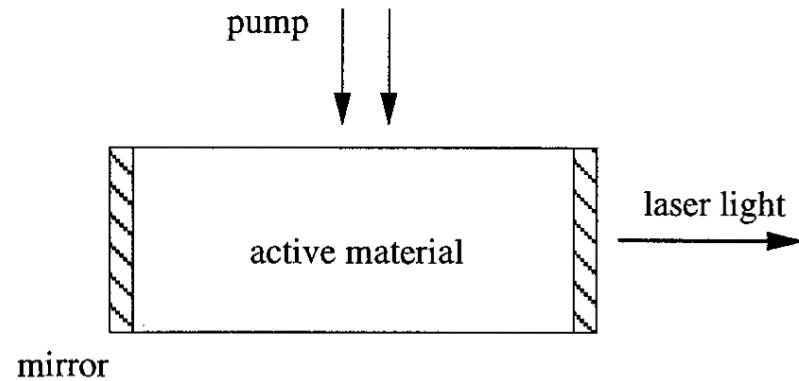


## Example: laser threshold

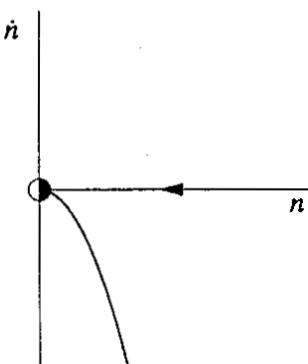
$$\begin{aligned}\dot{n} &= \text{gain} - \text{loss} \\ &= GnN - kn.\end{aligned}$$

$$N(t) = N_0 - \alpha n$$

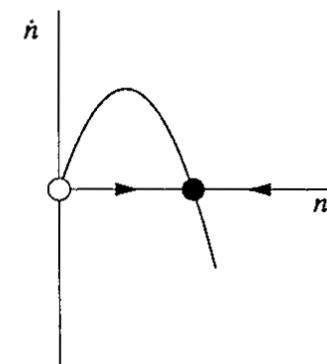
$$\begin{aligned}\dot{n} &= Gn(N_0 - \alpha n) - kn \\ &= (GN_0 - k)n - (\alpha G)n^2 \quad \dot{x} = rx - x^2\end{aligned}$$



$$N_0 < k/G$$

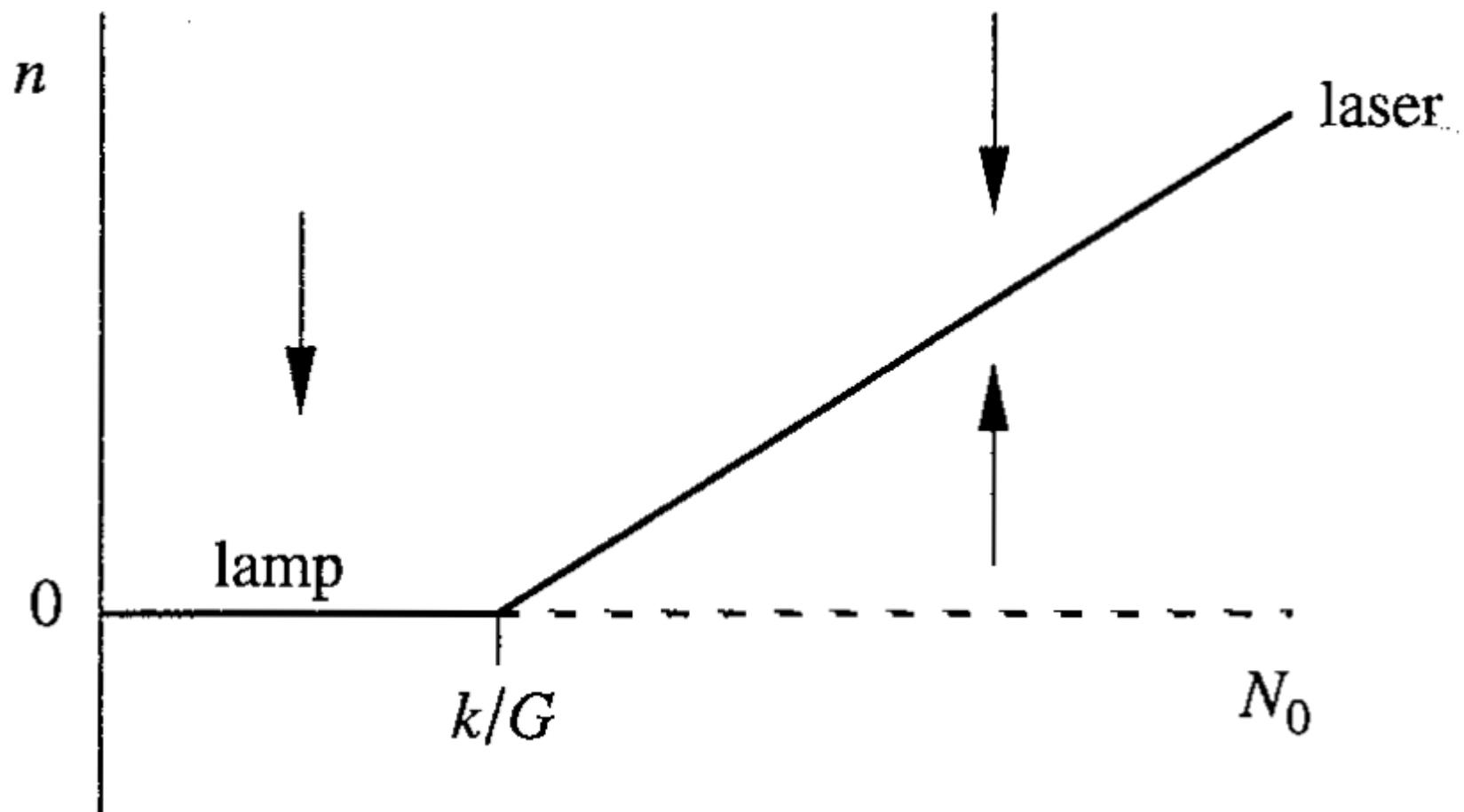


$$N_0 = k/G$$



$$N_0 > k/G$$

# Transcritical Bifurcation



## “Imperfect” bifurcation due to noise

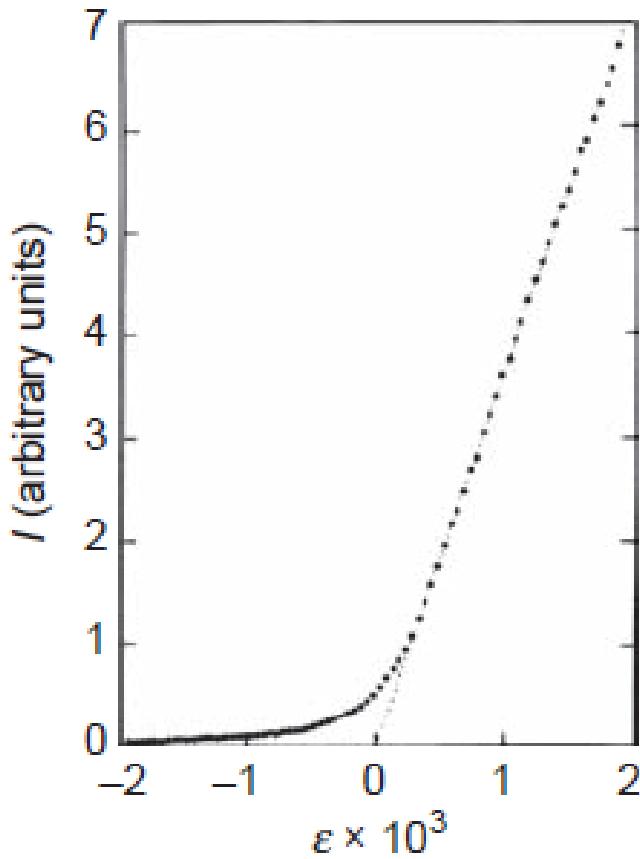


Fig. 1.17 Imperfect bifurcation for a laser in the presence of spontaneous emission, measured for a He-Ne laser. Reprinted Figure 1 with permission from Corti and Degiorgio [42]. Copyright 1976 by the American Physical Society.

## Laser turn on

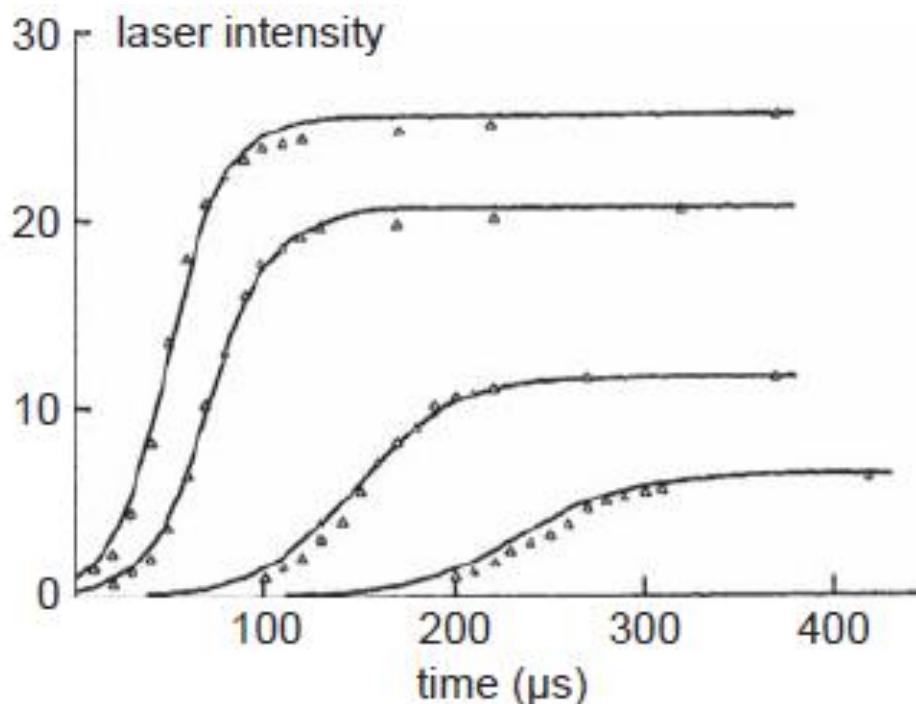


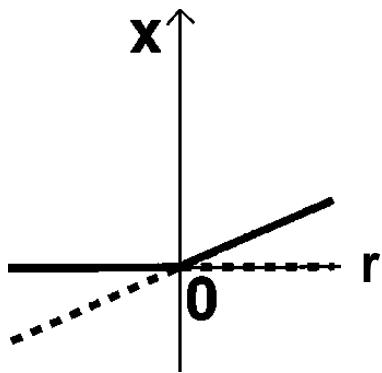
Fig. 1.3 He-Ne gas laser output as a function of time. From the lower to the upper time traces, the pump parameter above threshold is gradually increased. Reprinted Figure 2 with permission from Pariser and Marshall [30]. Copyright 1965 by the American Institute of Physics.

# Laser turn-on delay

$$\dot{x} = rx - x^2$$

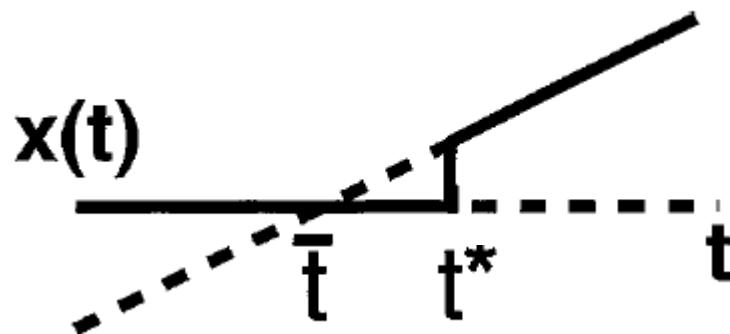
$$r(t) = r_0 + vt$$

$$r_0 < r^* = 0$$



Linear increase of control parameter

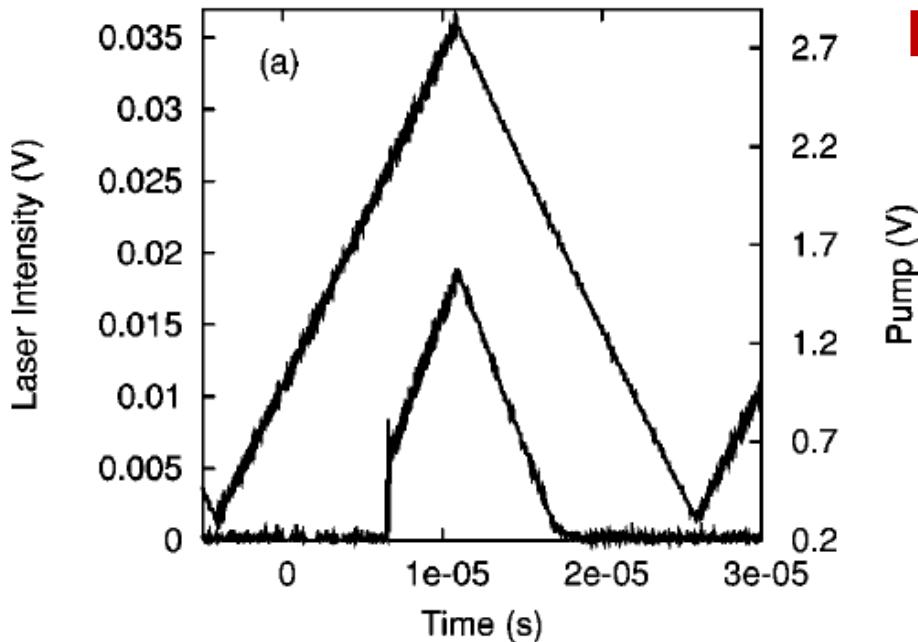
Start before the bifurcation point



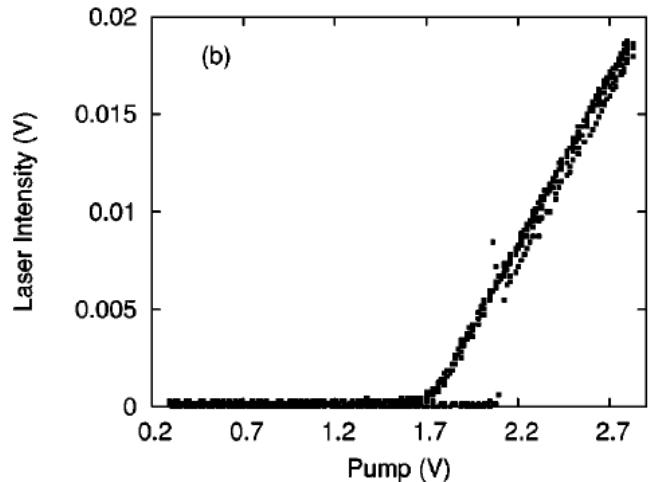
Fixed point solutions  
 $x=0$  stable if  $r<0$ .  
 $x=r$  stable if  $r>0$ .

$t^*$  depends of  $r_0$  and  $v$ .

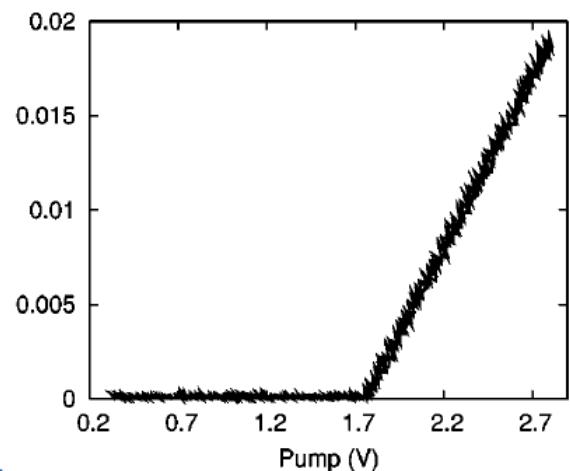
# Comparison with experimental observations



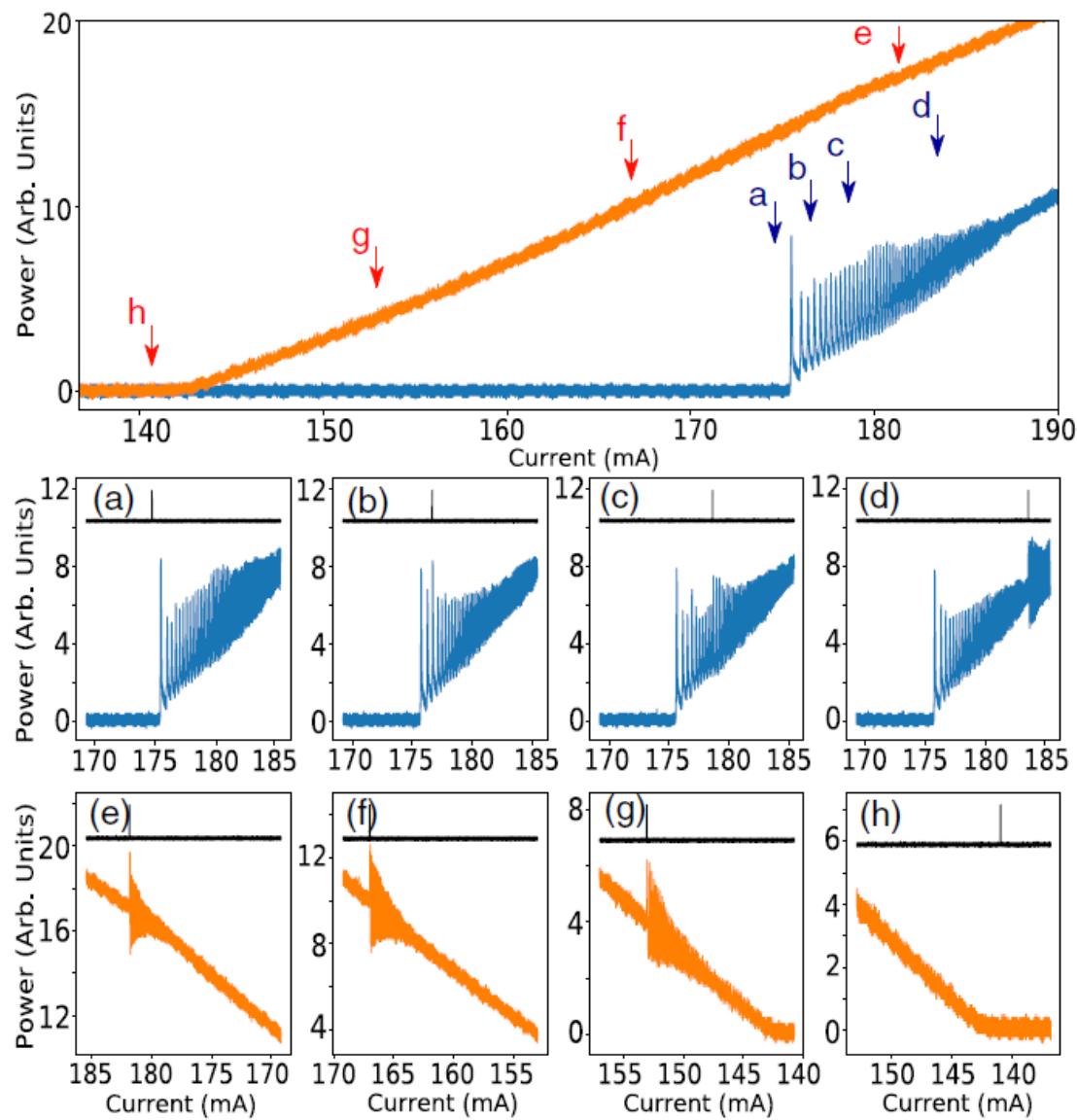
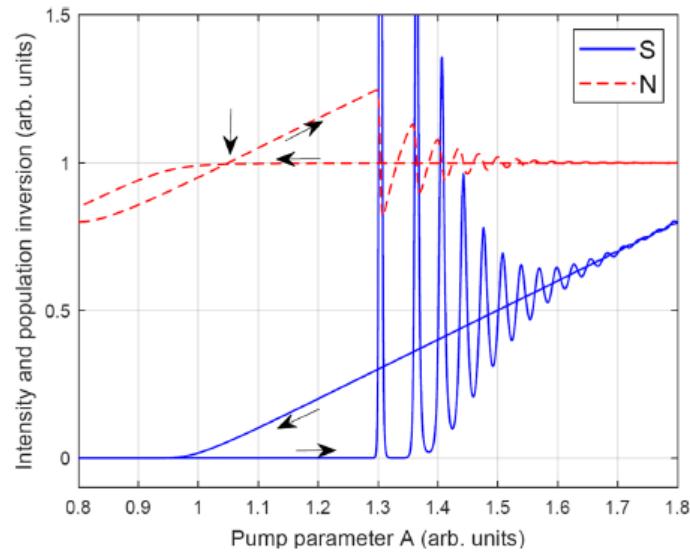
Dynamical hysteresis



Quasi-static very slow  
variation of the control  
parameter



# Recent study on the effect of a perturbation of the time varying parameter



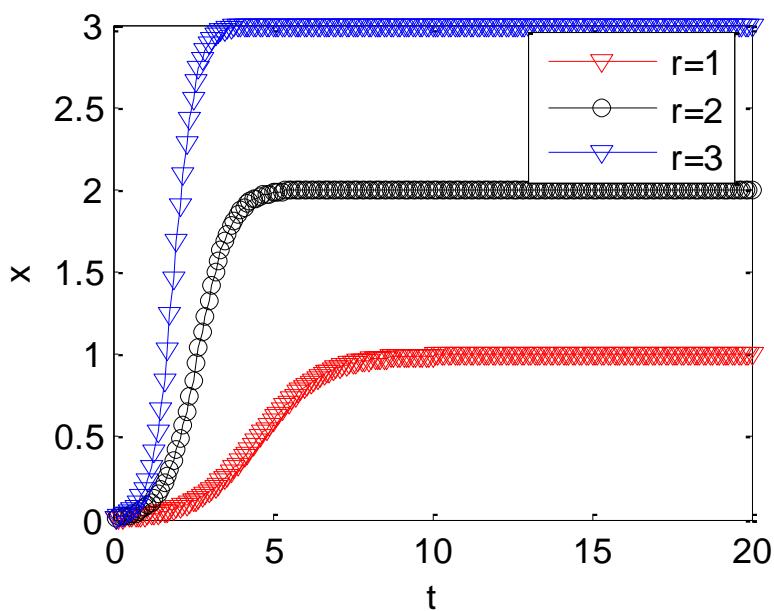
## Exercise 6

$$\dot{x} = rx - x^2$$

6.1 Simulate the “turn on” when  $r$  is constant,  $r > r^* = 0$ .

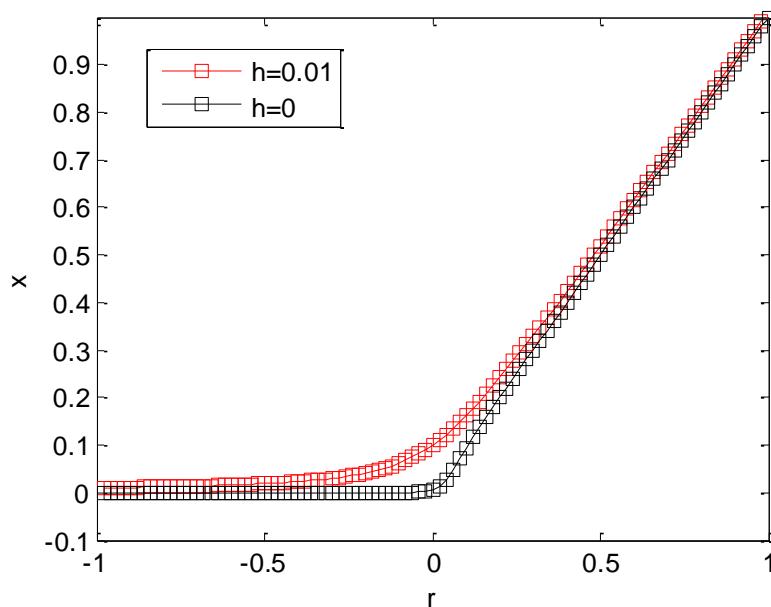
$$r(t) = r$$

$$x_0 = 0.01$$



6.2 Calculate the bifurcation diagram by plotting  $x(t=50)$  vs  $r$ .

$$\dot{x} = rx - x^2 + h$$



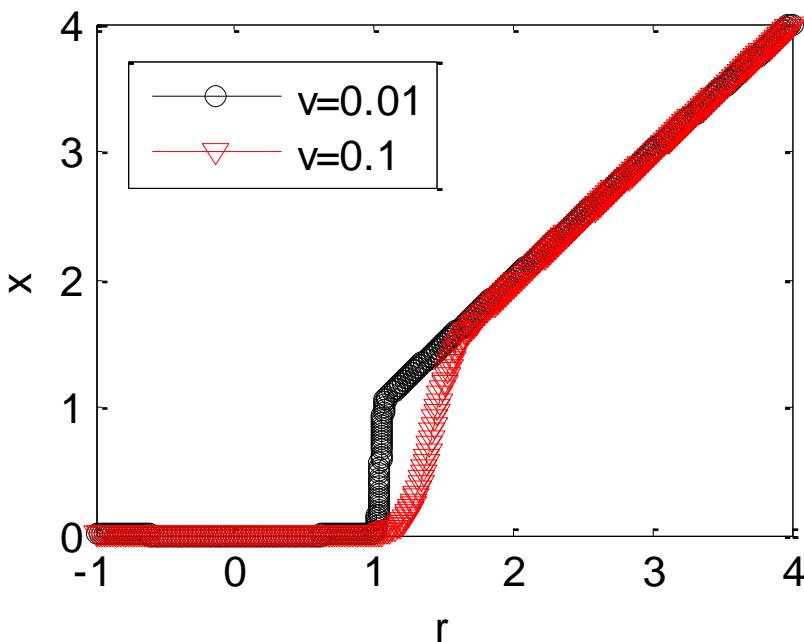
## Exercise 6

$$\dot{x} = rx - x^2$$

6.3 Simulate the equation with  $r$  increasing linearly in time. Consider different variation rate ( $v$ ) and/or different initial value of the parameter ( $r_0$ ).

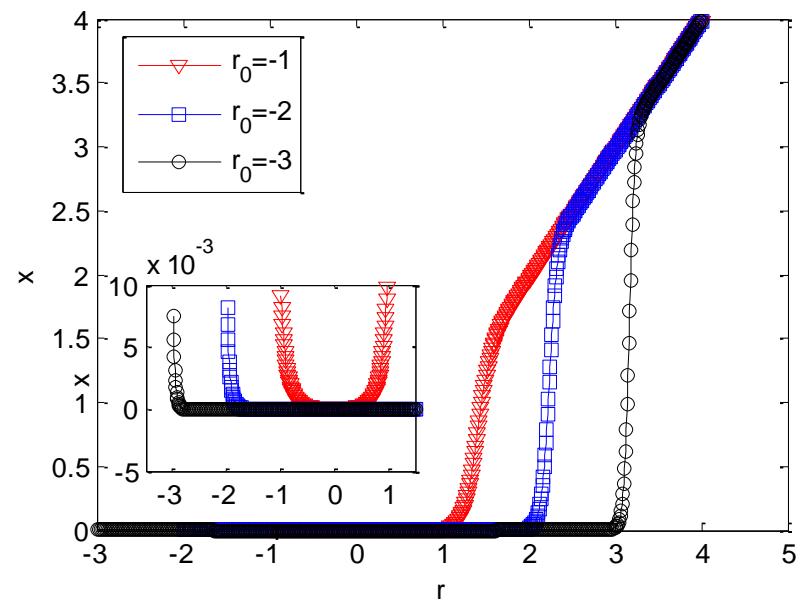
$$r(t) = r_0 + vt$$

$$x_0 = 0.01$$



$$v = 0.1$$

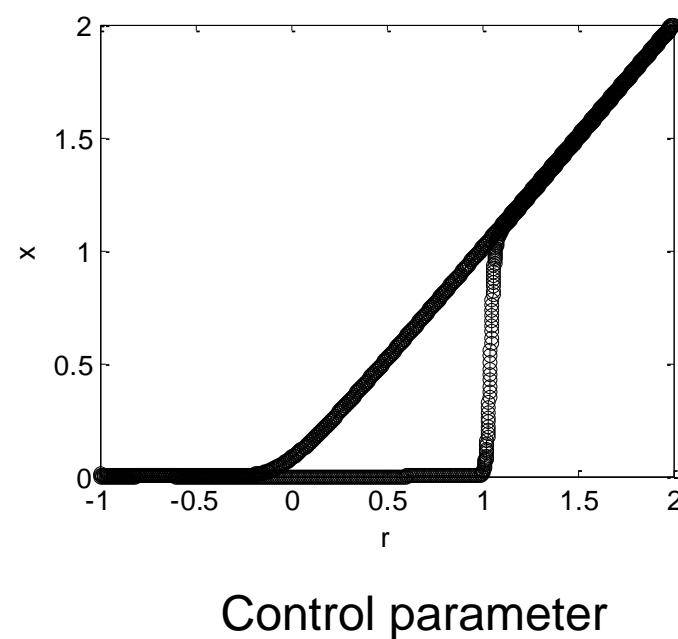
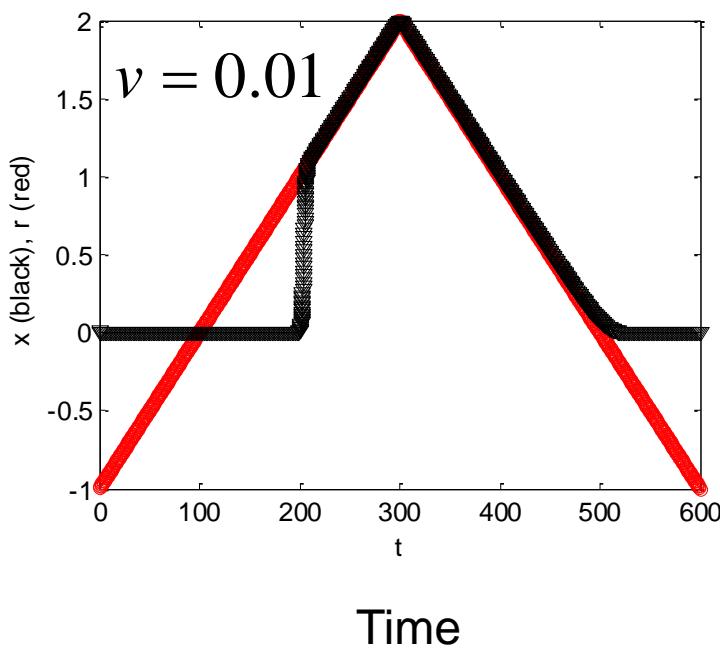
$$x_0 = 0.01$$



## Exercise 6

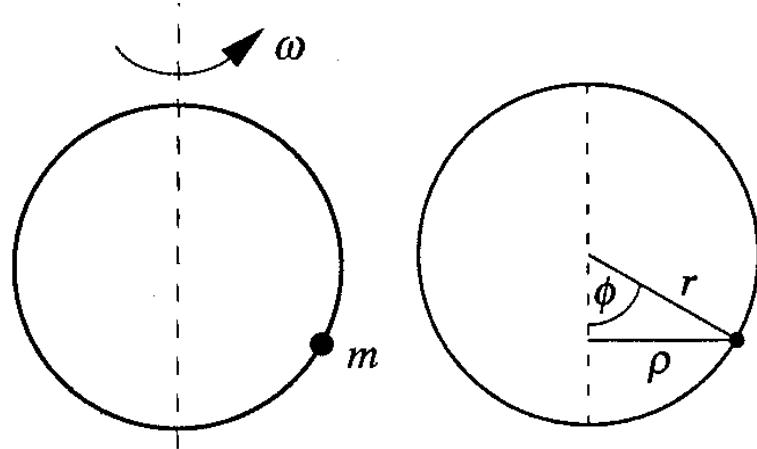
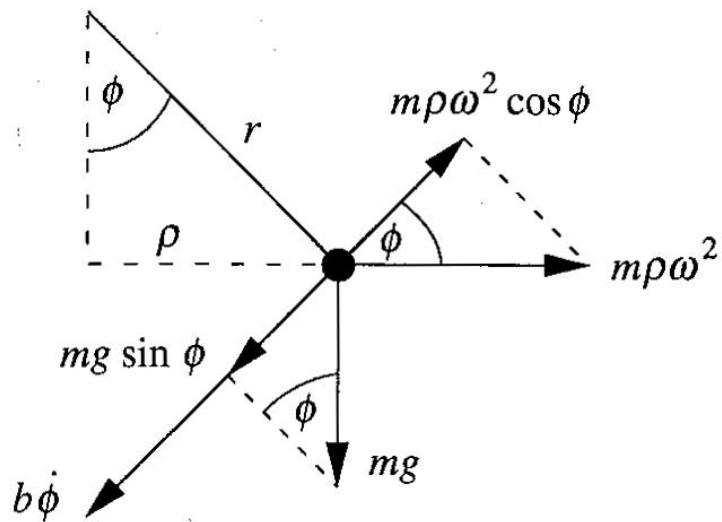
$$\dot{x} = rx - x^2$$

6.4 Consider that the control parameter  $r$  **increases and then decreases** linearly in time: plot  $x$  and  $r$  vs.  $t$  and plot  $x$  vs.  $r$ .



## Example: particle in a rotating wire hoop

- A particle moves along a wire hoop that rotates at constant angular velocity



$$mr\ddot{\phi} = -b\dot{\phi} - mg \sin \phi + mr\omega^2 \sin \phi \cos \phi$$

# Fixed points of a particle particle in a rotating wire hoop

- Neglect the second derivative (more latter)

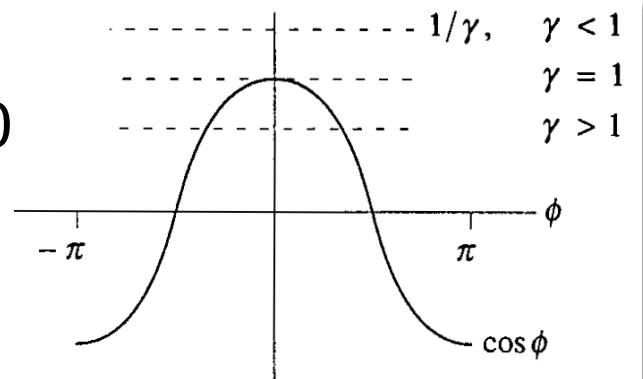
$$\begin{aligned} b\dot{\phi} &= -mg \sin \phi + mr\omega^2 \sin \phi \cos \phi \\ &= mg \sin \phi \left( \frac{r\omega^2}{g} \cos \phi - 1 \right) \end{aligned}$$

- Fixed points from:  $\sin \phi = 0$   
 $\phi^* = 0$  (the bottom of the hoop) and  $\phi^* = \pi$  (the top).  
stable

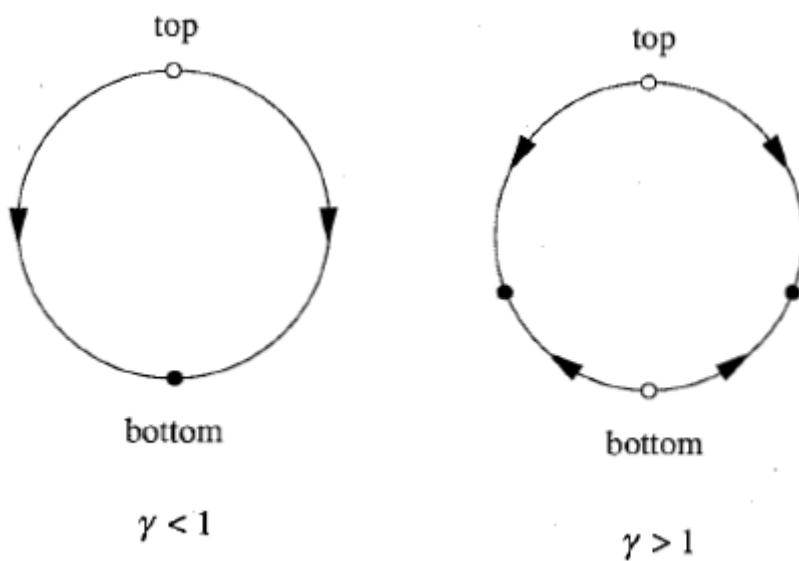
unstable

- Fixed points from:  $\gamma \cos \phi - 1 = 0$

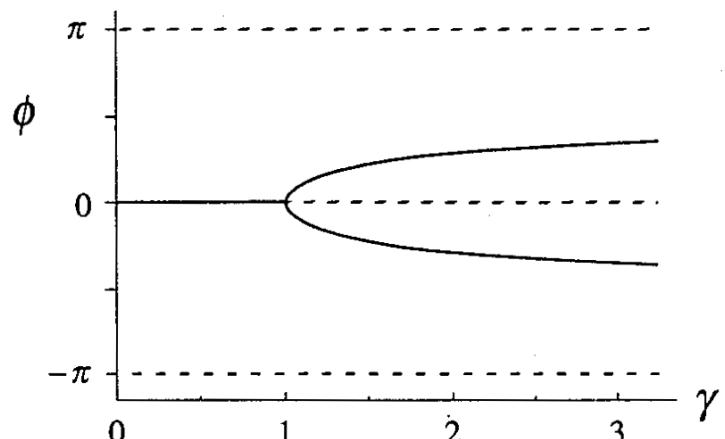
$$\gamma = \frac{r\omega^2}{g}$$



# Stability of the fixed points



Pitchfork Bifurcation:



When is this “first-order” description valid?

When is ok to neglect the second derivative  $d^2x/dt^2$  ?

Dimensional analysis and scaling

# Dimensionless equation

- Dimensionless time

( $T$  = characteristic time-scale)

$$\tau = \frac{t}{T}$$

$$\left( \frac{r}{gT^2} \right) \frac{d^2\phi}{d\tau^2} = - \left( \frac{b}{mgT} \right) \frac{d\phi}{d\tau} - \sin \phi + \left( \frac{r\omega^2}{g} \right) \sin \phi \cos \phi$$

- We want the lhs very small, we define  $T$  such that

$$\frac{r}{gT^2} \ll 1 \quad \text{and} \quad \frac{b}{mgT} \approx O(1) \Rightarrow T = \frac{b}{mg}$$

$$\frac{r}{g} \left( \frac{mg}{b} \right)^2 \ll 1 \Rightarrow b^2 \gg m^2 gr$$

- Define:  $\varepsilon = \frac{m^2 gr}{b^2} \Rightarrow \boxed{\varepsilon \frac{d^2\phi}{d\tau^2} = - \frac{d\phi}{d\tau} - \sin \phi + \gamma \sin \phi \cos \phi}$   $\gamma = \frac{r\omega^2}{g}$

## Over damped limit

$$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin \phi + \gamma \sin \phi \cos \phi$$

- The dimension less equation suggests that the first-order equation is valid in the over damped limit:  $\varepsilon \rightarrow 0$
- Problem: second-order equation has two independent initial conditions:  $\phi(0)$  and  $d\phi/d\tau(0)$
- But the first-order equation has only one initial condition  $\phi(0)$ ,  $d\phi/d\tau(0)$  is calculated from

$$\frac{d\phi}{d\tau} = -\sin \phi + \gamma \sin \phi \cos \phi$$

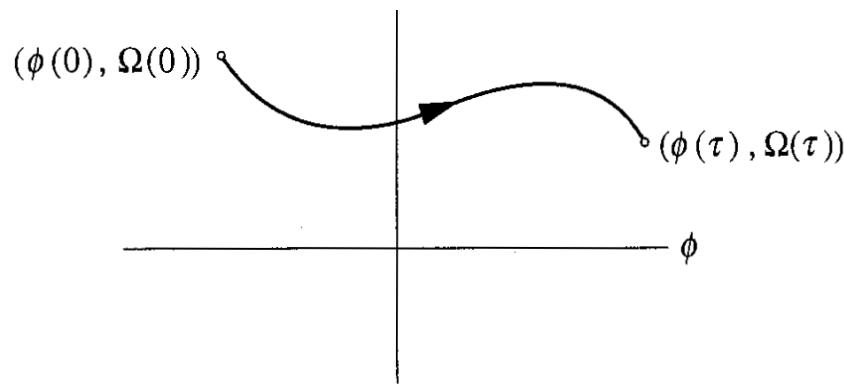
- Paradox: how can the first-order equation represent the second-order equation?

# Trajectories in phase space

- First order system:

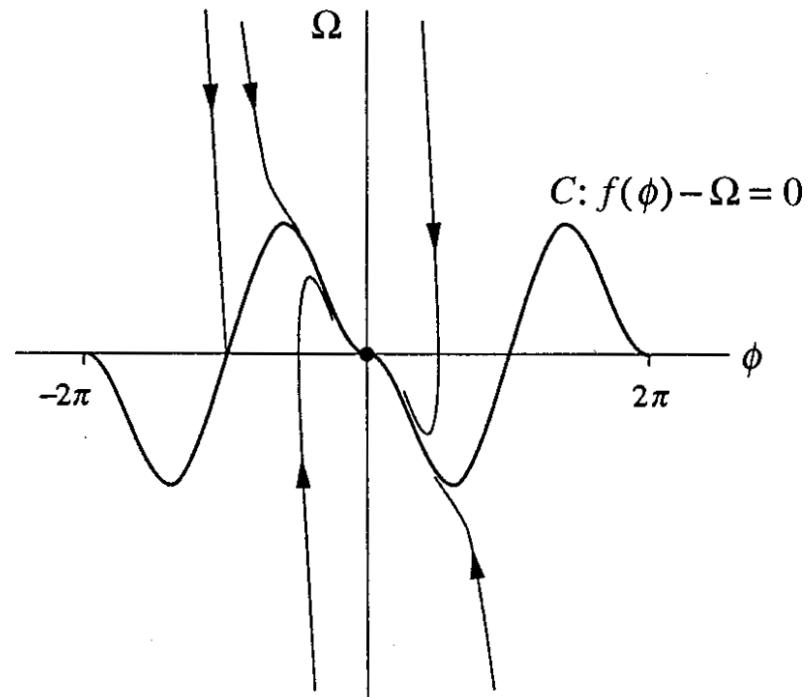
$$\frac{d\phi}{d\tau} = f(\phi) - \sin \phi + \gamma \sin \phi \cos \phi$$

$$\Omega = \phi' \equiv d\phi/d\tau$$



- Second order system:

$$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin \phi + \gamma \sin \phi \cos \phi$$



Second order system:

$$\varepsilon \rightarrow 0$$

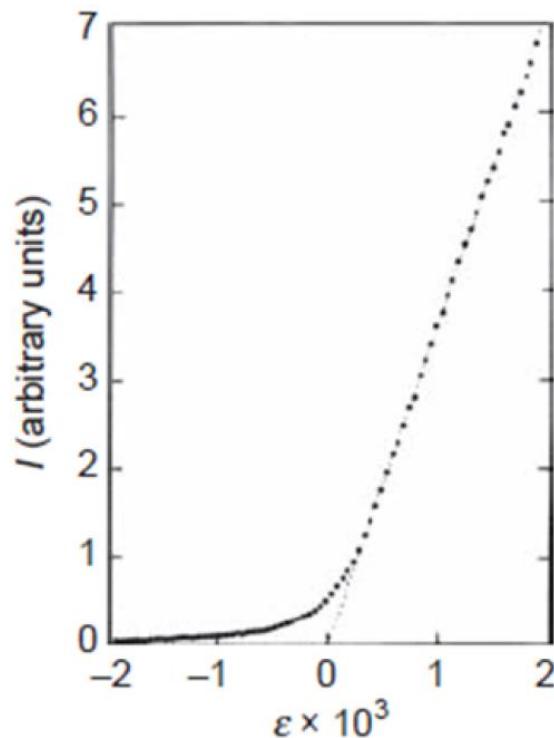
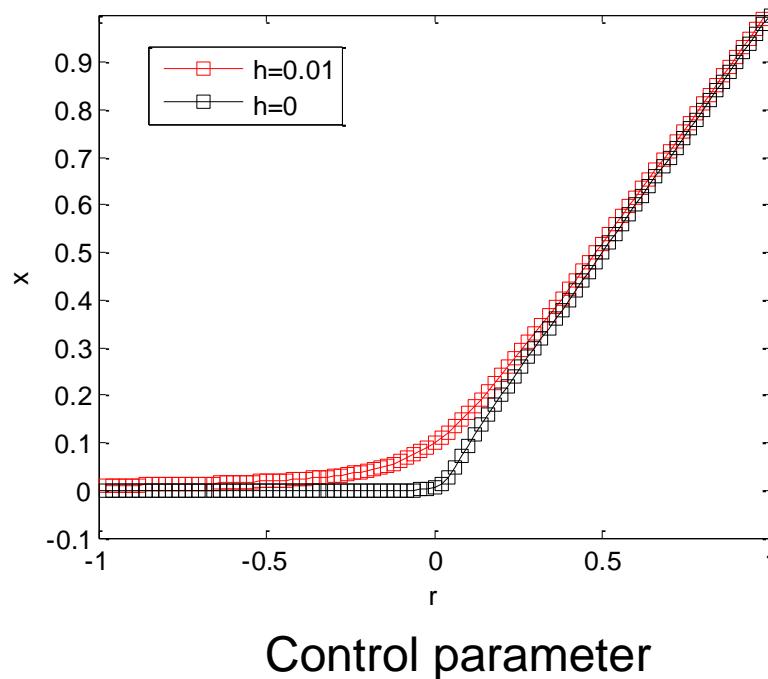
limit, all trajectories slam straight up or down onto the curve  $C$  defined by  $f(\phi) = \Omega$ , and then slowly ooze along this curve until they reach a fixed point

# Outline

- Introduction to bifurcations
- Saddle-node, transcritical and pitchfork bifurcations
- Examples
- Imperfect bifurcations & catastrophes

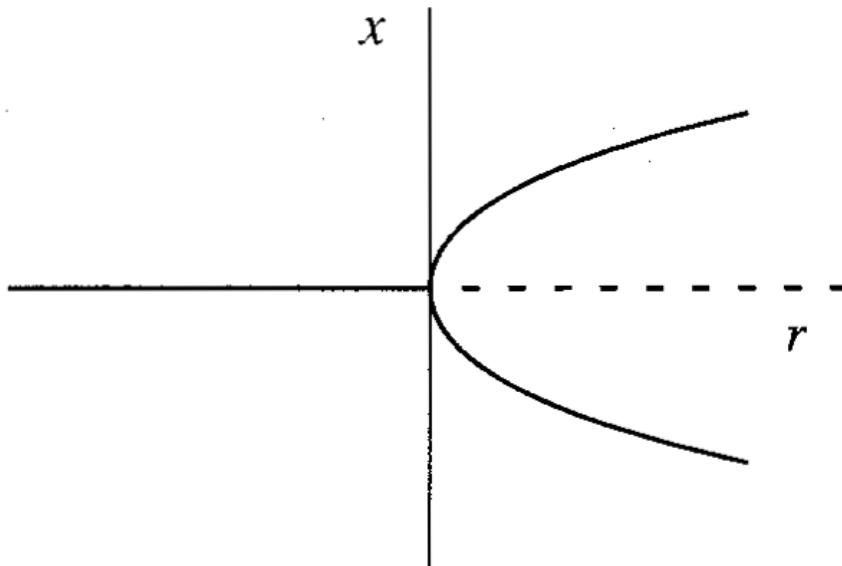
# “Imperfect” bifurcation

$$\dot{x} = rx - x^2 + h$$

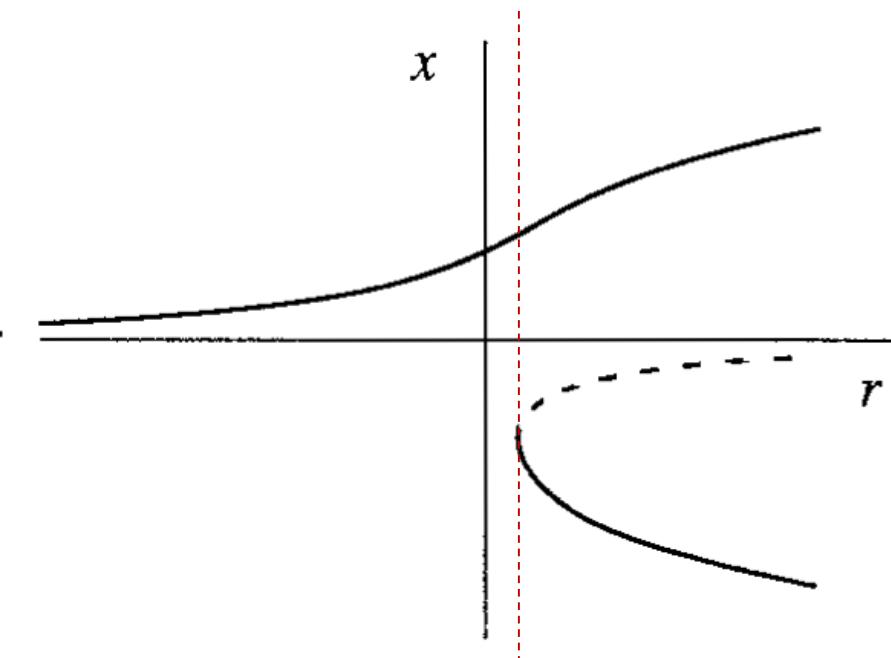


# Imperfect bifurcations

$$\dot{x} = h + rx - x^3$$



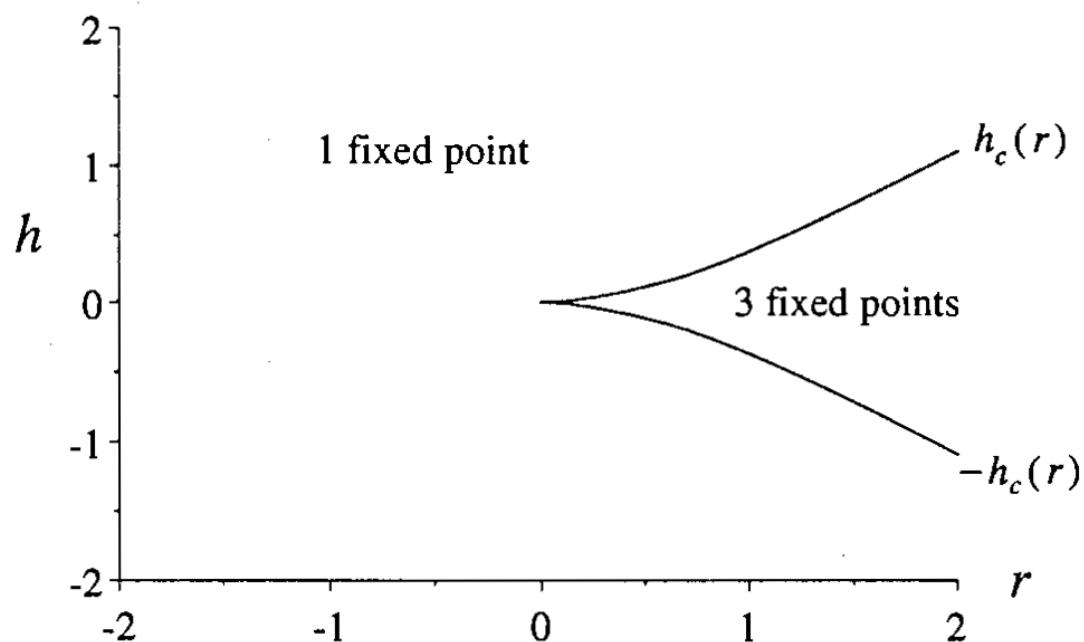
(a)  $h = 0$



(b)  $h \neq 0$

**The number of solutions depends on the parameters ( $h, r$ )**

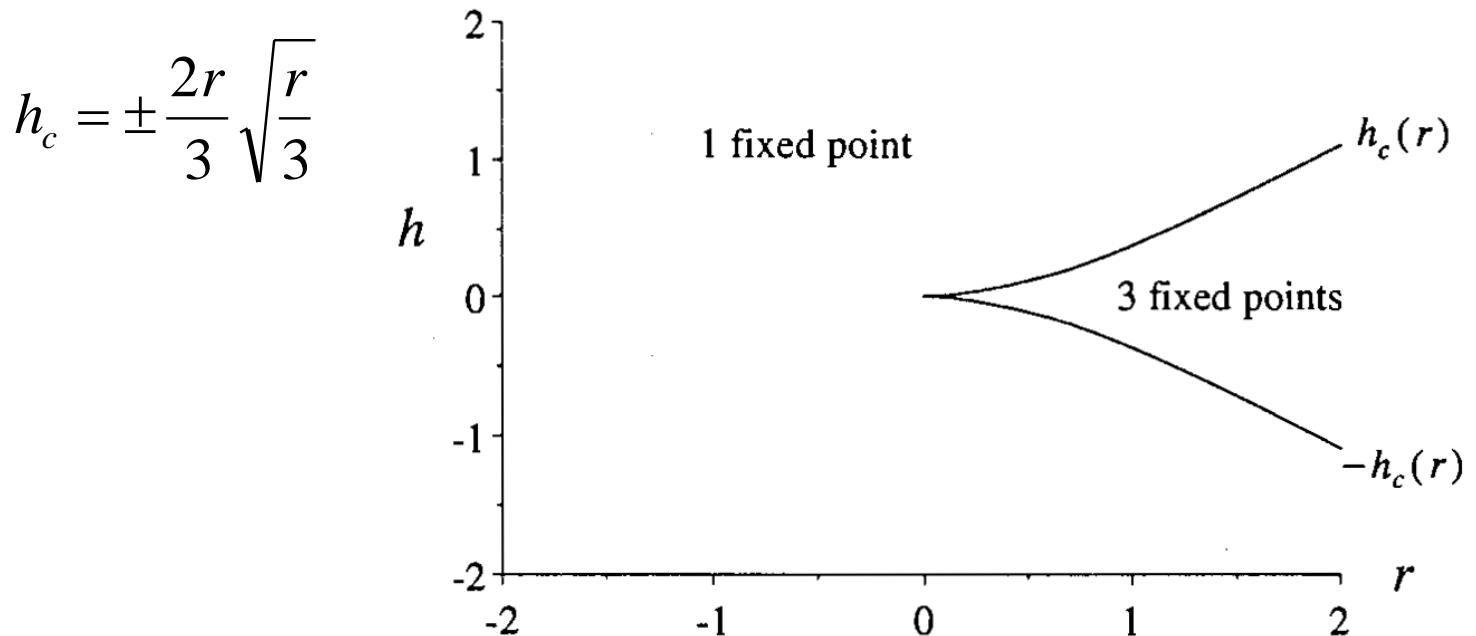
$$\dot{x} = h + rx - x^3$$



## Exercise 7: Calculate the analytical expression of $h_c(r)$

Tip: use the equations of

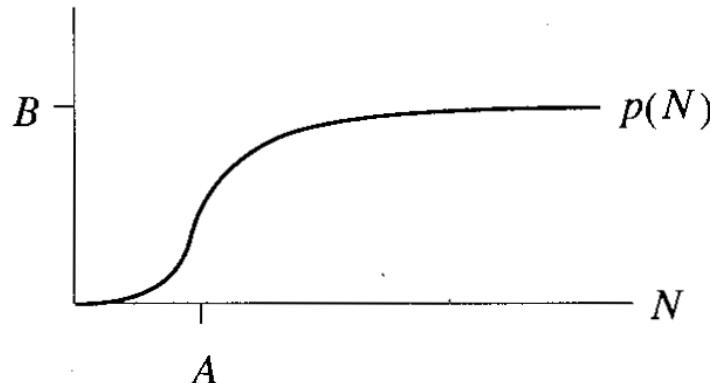
- The fixed points:  $f(x^*) = 0$
- Of the saddle node bifurcation:  $f'(x^*) = 0$



## Example: insect outbreak

$$\dot{N} = RN \left(1 - \frac{N}{K}\right) - p(N)$$

- Budworms population grows logistically ( $R > 0$  grow rate)
- $p(N)$ : dead rate due to predation
- If no budworms ( $N \approx 0$ ): no predation: birds look for food elsewhere
- If  $N$  large,  $p(N)$  saturates: birds eat as much as they can.



$$p(N) = \frac{BN^2}{A^2 + N^2} \quad A, B > 0$$

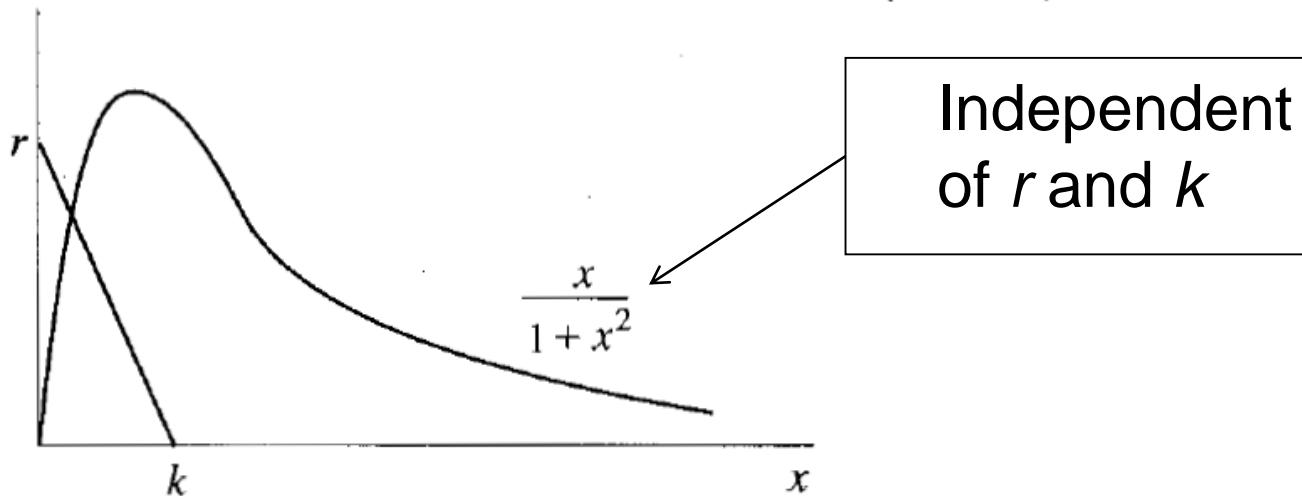
# Dimensionless formulation

$$x = N/A \quad \tau = \frac{Bt}{A}, \quad r = \frac{RA}{B}, \quad k = \frac{K}{A}$$

$$\frac{dx}{d\tau} = rx \left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

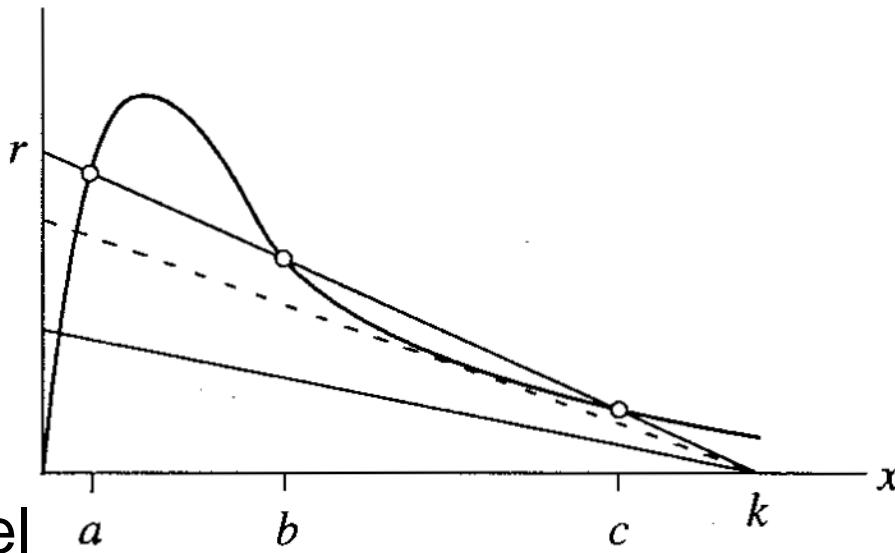
- Fixed point:  $x^*=0$       **Exercise 8:** show that  $x^*=0$  is unstable.
- Other FPs are the solutions of

$$r \left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$$



## Fixed point solutions

When the line intersects the curve tangentially (dashed line): a saddle-node bifurcation occurs (two new FPs appear).

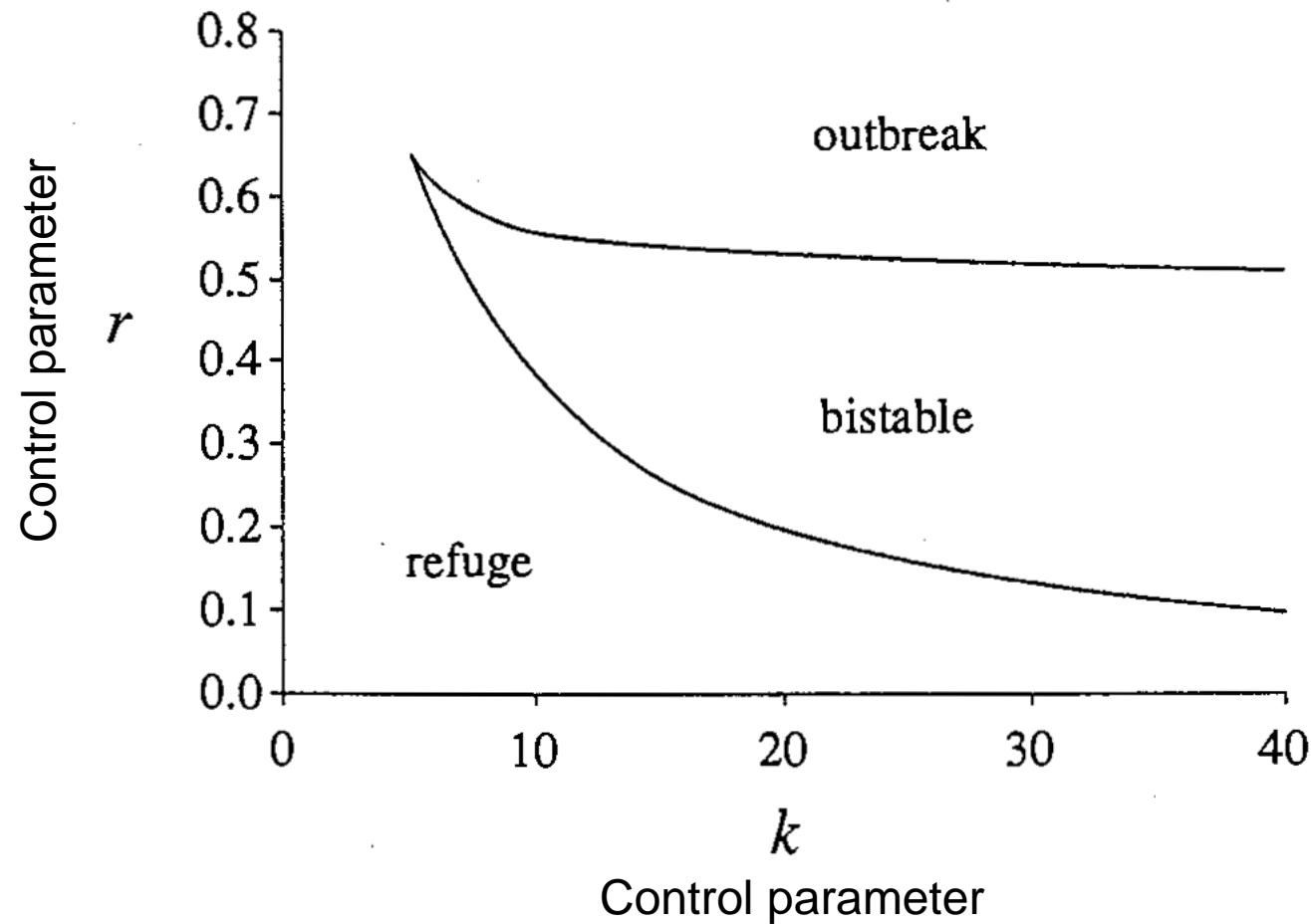


a: Refuge level  
of the budworm  
population

b: threshold

c: Outbreak level (pest)

# Insect outbreak solutions in the parameter space ( $k$ , $r$ )



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