FÍSICA

APRÈN UPC

Apunts

Introduction to dynamical systems and flows on the line

Cristina Masoller Alonso

Assignatura: Nonlinear systems, chaos and control in engineering

Titulacions: Grau en Enginyeria en Tecnologies Aeroespacials (pla 2010)
Grau en Enginyeria en Vehicles Aeroespacials (pla 2010)
Grau en Enginyeria en Tecnologies Industrials (pla 2010)

Curs: 4t Quadrimestre: 2n

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

Idioma: Anglès

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UNIVERSITAT POLITÈCNICA DE CATALUNYA
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Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa

Nonlinear systems, chaos and control in engineering
Course 2020-2021
Bachelors degrees in:
Aerospace technology engineering
Aerospace vehicle engineering
Industrial technology engineering

Introduction to dynamical systems and flows on the line

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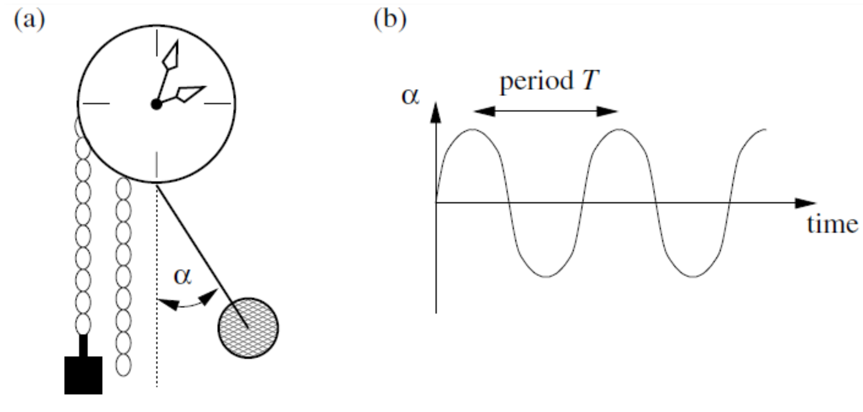
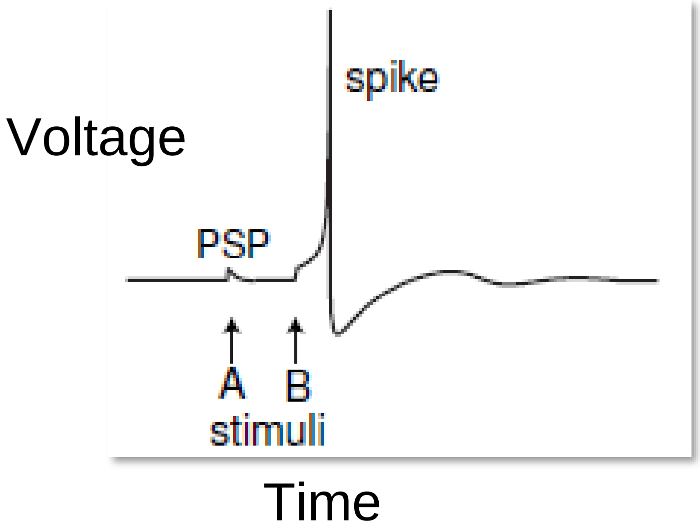
Structure of the first part of the course

- **Introduction to dynamical systems**
- Introduction to flows on the line
- Fixed points and linear stability
- Solving equations with computer

What is a Dynamical System?

- Systems that evolve in time.
- Examples:
  - Pendulum clock
  - Neuron



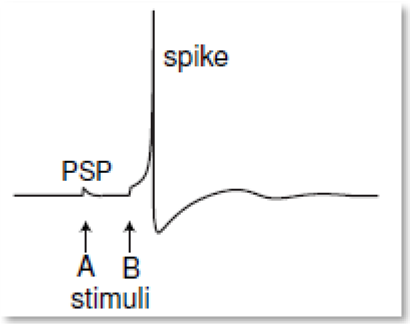


- Dynamical systems can be:
  - *linear or nonlinear (harmonic oscillator – pendulum);*
  - *deterministic or stochastic;*
  - *low or high dimensional;*
  - *continuous time or discrete time.*

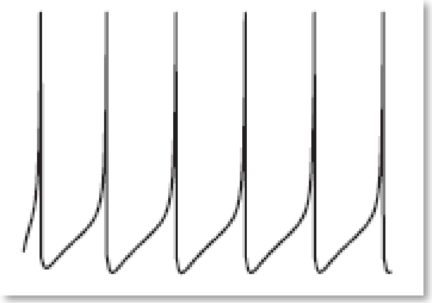
In this course: nonlinear systems (**Nonlinear Dynamics**)

Possible temporal evolution

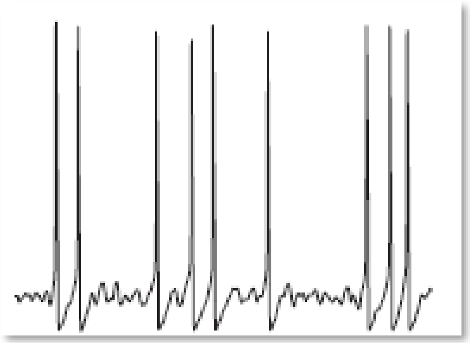
- After a transient the systems settles down to equilibrium (rest state or “fixed point”).



- Keeps spiking in cycles (“limit cycle”).



- More complicated: **aperiodic** evolution (“chaos”).



The beginning of the mathematical modelling of dynamical systems: Newtonian mechanics

- Mid-1600s: Ordinary differential equations (ODEs)

- **Isaac Newton**: studied planetary orbits and solved analytically the “two-body” problem (earth around the sun).

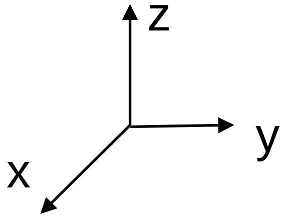
- Since then: a lot of effort for solving the “three-body” problem (earth-sun-moon) – Impossible.

Late 1800s

- **Henri Poincare** (French mathematician).

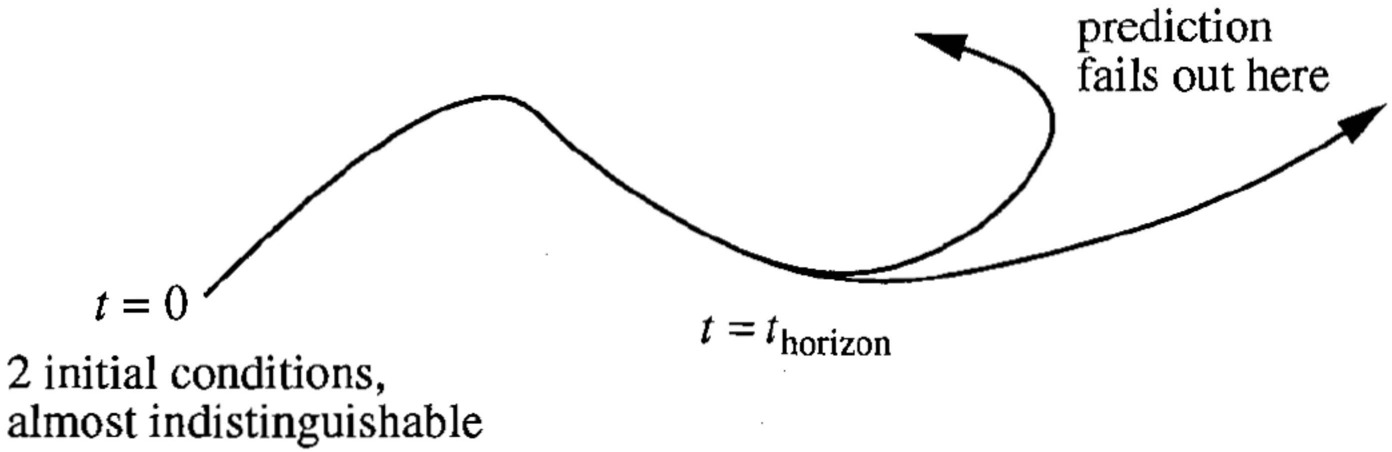
  Instead of asking "*which are the exact positions of planets (trajectories)*?"

  he asked: "*is the solar system* ***stable*** *for ever, or will planets eventually run away*?"
- He developed a **geometrical** approach to solve the problem.
- Introduced the concept of "phase space".



- *Poincaré recurrence theorem*: certain systems will, after a sufficiently long but finite time, return to a state very close to the initial state.
- He also had the intuition of the possibility of chaos**.**

*Poincare: "The evolution of a **deterministic** system can be aperiodic, unpredictable, and strongly depends on the initial conditions".*



Deterministic system: the initial conditions fully determine the future state.
Deterministic **chaotic** system: there is no randomness but the system can be, in the long term, unpredictable.

A problem in time series analysis: How to determine the prediction horizon? With what reliability?

1950s: First computer simulations

- Computes allowed to experiment with equations.
- Huge advance in the field of “*Dynamical Systems*”.

- 1960s: **Eduard Lorenz** (American mathematician and meteorologist at MIT): simple model of convection rolls in the atmosphere.

$$\begin{aligned}
\frac{dx}{dt} &= -\sigma x + \sigma y, \\
\frac{dy}{dt} &= -xz + rx - y, \\
\frac{dz}{dt} &= xy - bz.
\end{aligned}$$

- Famous **chaotic** attractor.

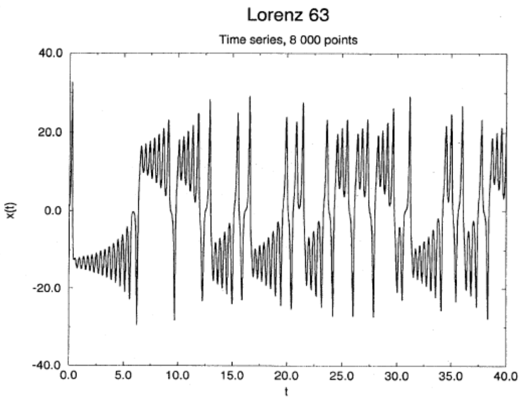
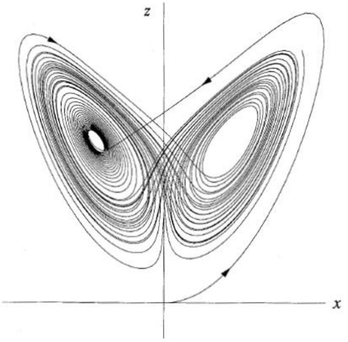
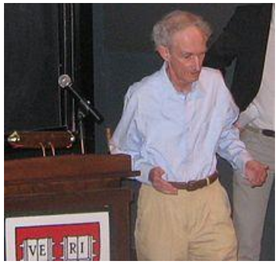


FIG. 1. Chaotic time series $x(t)$ produced by Lorenz (1963) equations (11) with parameter values $r=45.92$, $b=4.0$, $\sigma=16.0$.



The 1970s



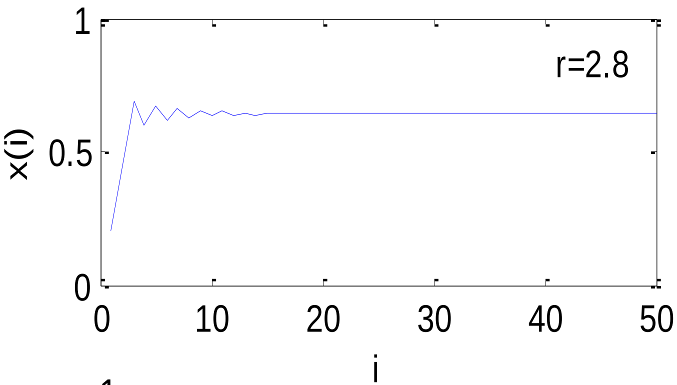
- **Robert May** (Australian, 1936): population biology
- "Simple mathematical models with very complicated dynamics", *Nature* (1976).

$$x_{t+1} = f(x_t)$$

A classical example: The Logistic map $f(x) = r\,x(1-x)$
$x \in (0,1)$, $r \in (0,4)$

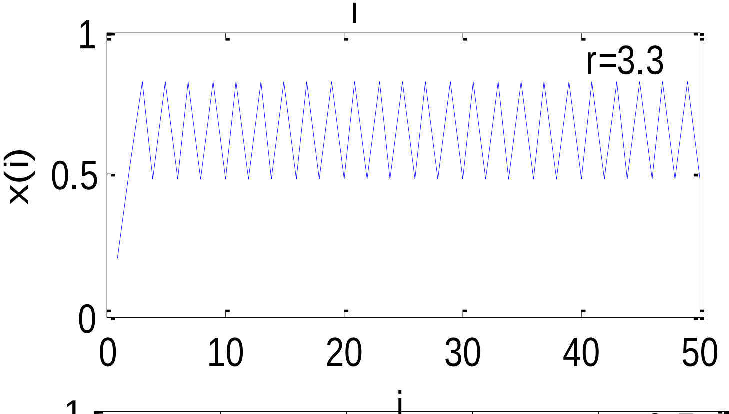
- Difference equations ("iterated maps"), in spite of being simple and deterministic, can exhibit: **stable points**, **stable cycles**, and **apparently random fluctuations**.

The logistic map: $x(i+1) = r\, x(i)[1 - x(i)]$ $\quad x \in (0,1),\ r \in (0,4)$



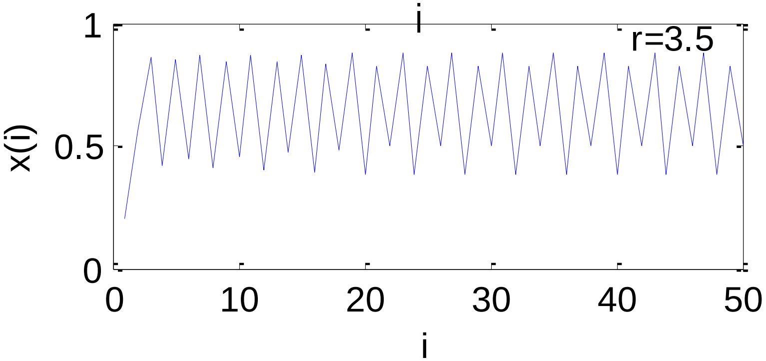
r=2.8, Initial condition: x(1) = 0.2

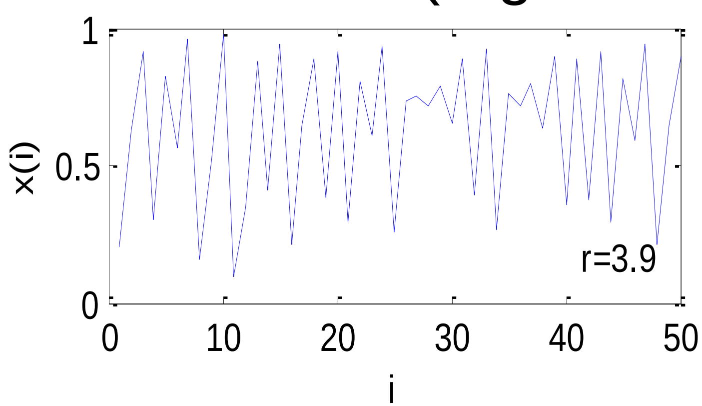
Transient relaxation → long-term stability



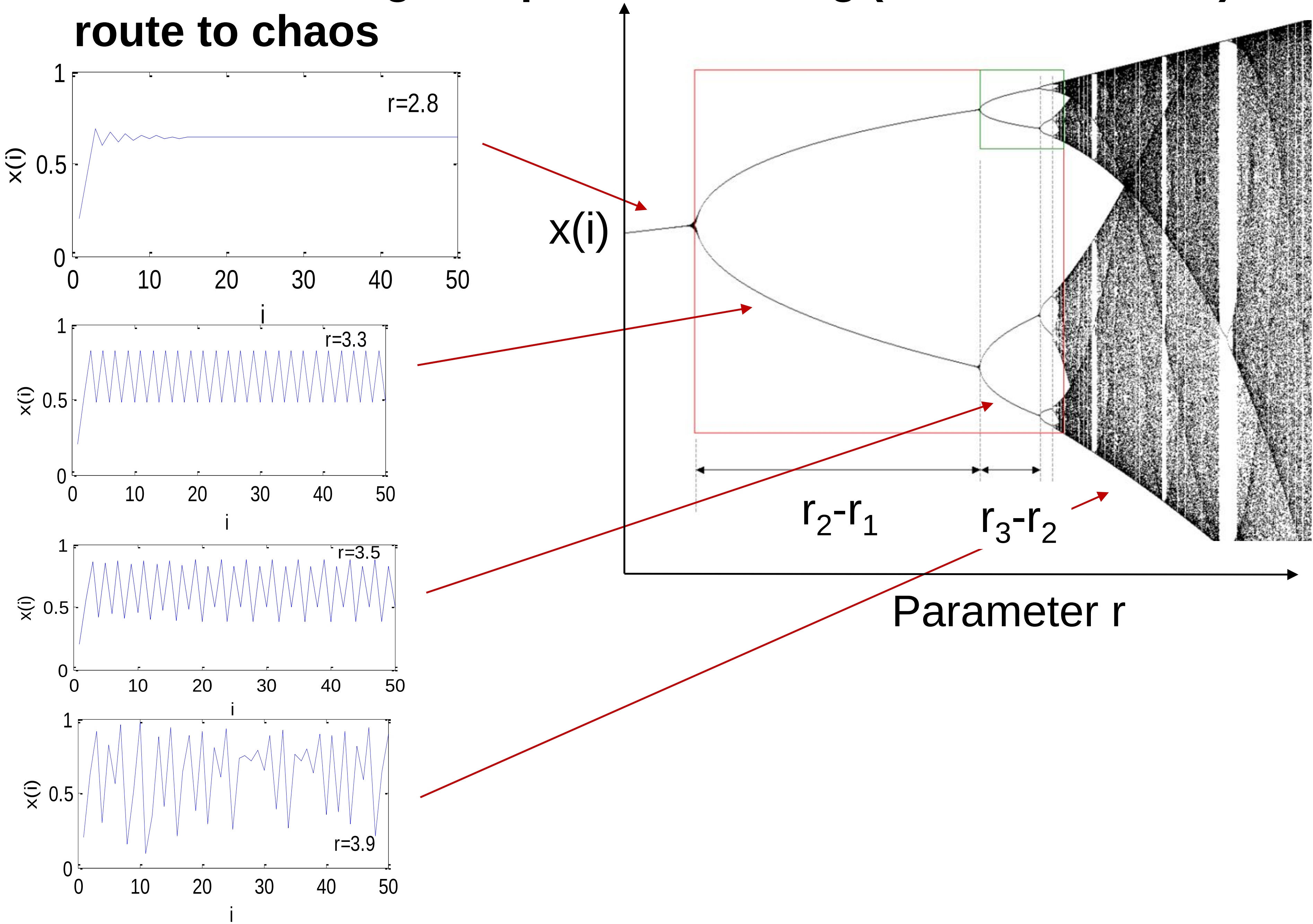
The fixed point is the solution of: $x = r\,x\,(1-x) \Rightarrow x = 1 - 1/r$

Transient dynamics → oscillations (regular or irregular)



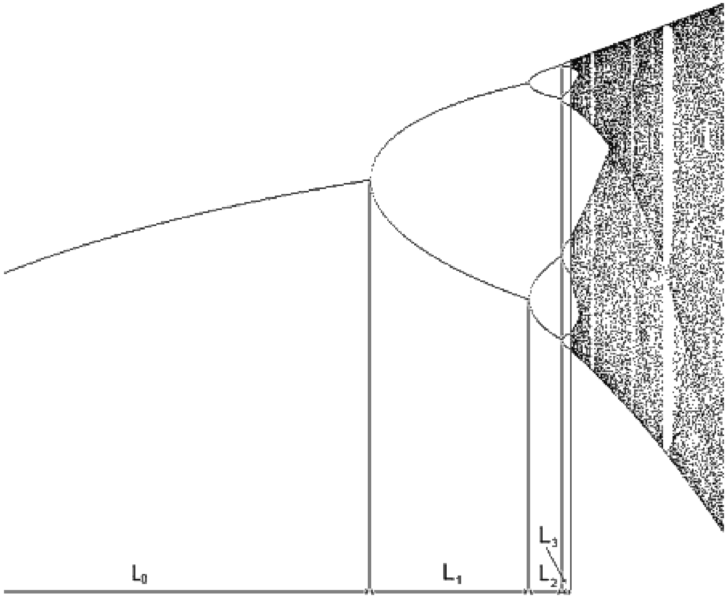


Bifurcation diagram: period-doubling (or subharmonic) route to chaos

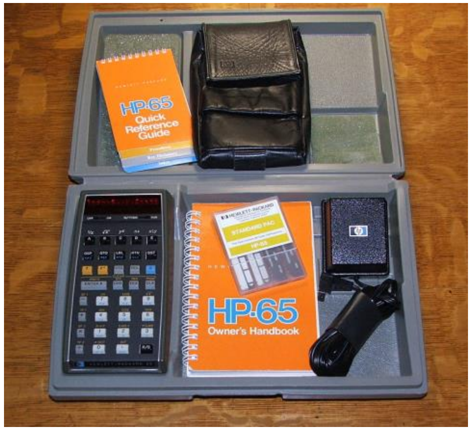


Order within chaos and a “hidden” law in the subharmonic route to chaos

In 1975, **M. Feigenbaum** (American mathematician and physicist 1944-2019), using a small HP-65 calculator, discovered the <u>scaling law of the bifurcation points of the Logistic map.</u>



$$\delta = \lim \frac{L_i}{L_{i+1}} = 4.669201...$$



HP-65 calculator: the first magnetic card-programmable handheld calculator

A universal law

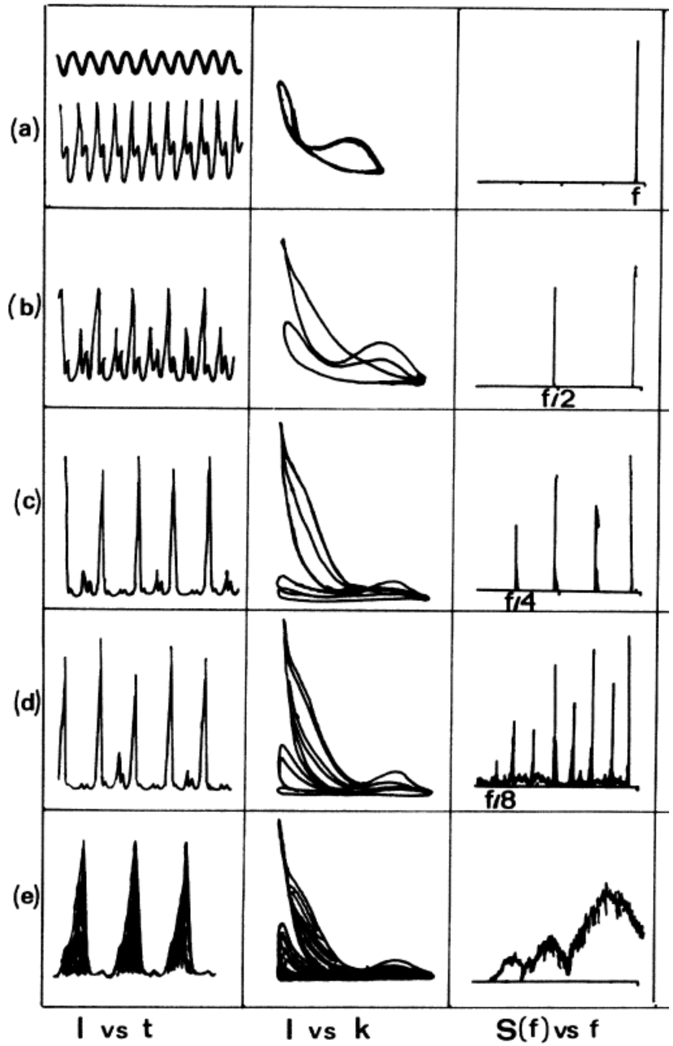
Feigenbaum showed that the same behavior, with the same mathematical constant, occurs for a wide class of functions (functions with a quadratic máximum).

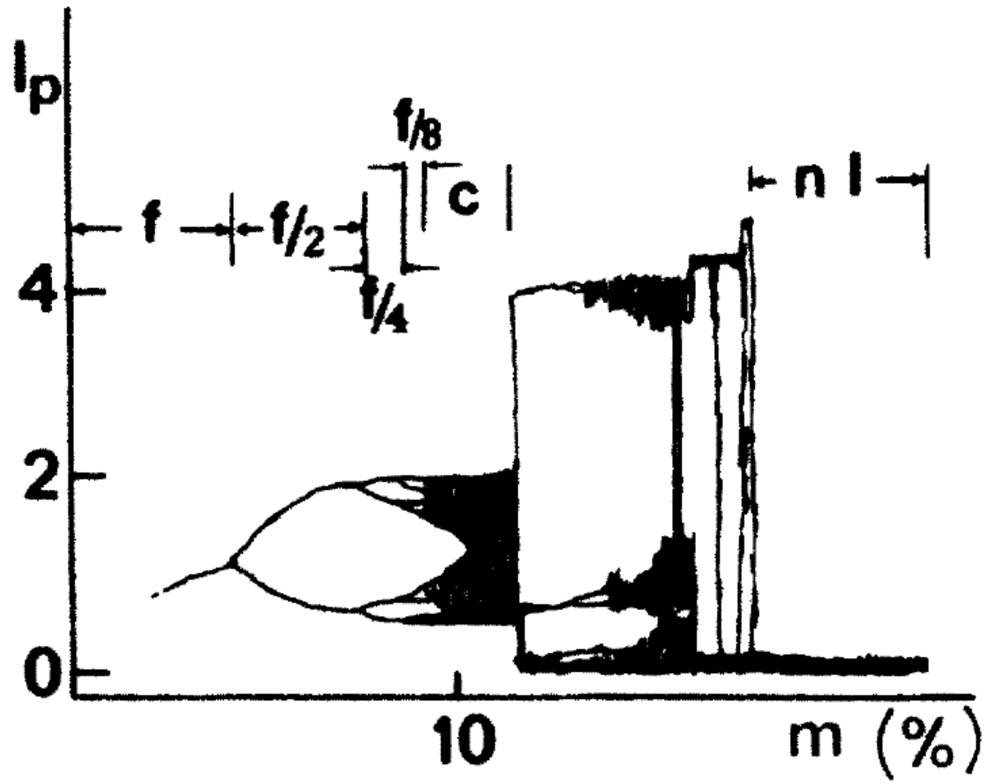
Very different systems (in chemistry, biology, physics, etc.) go to chaos in the same way, quantitatively.

Watch this short video for some very interesting examples: https://www.youtube.com/watch?v=ovJcsL7vyrk

Early experiments: a periodically modulated CO_2 laser

Constant modulation frequency, increasing the modulation amplitude





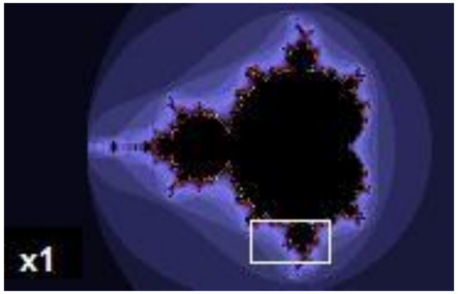
Tredicce et al, Phys. Rev. A 34, 2073 (1986).

The late 1970s

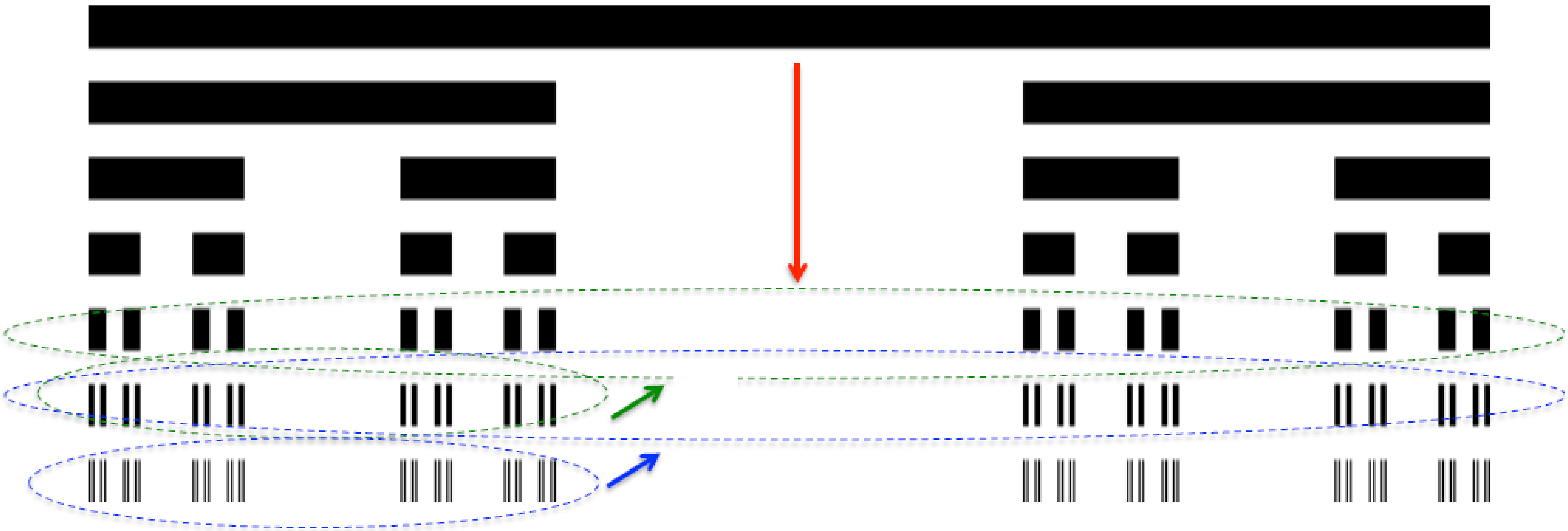
- **Benoit Mandelbrot** (Polish-born, French and American mathematician 1924-2010): “self-similarity” and **fractal objects:**

  each part of the object is like the whole object but smaller.

- Because of his access to IBM's computers, Mandelbrot was one of the first to use **computer graphics** to create and display fractal geometric images.



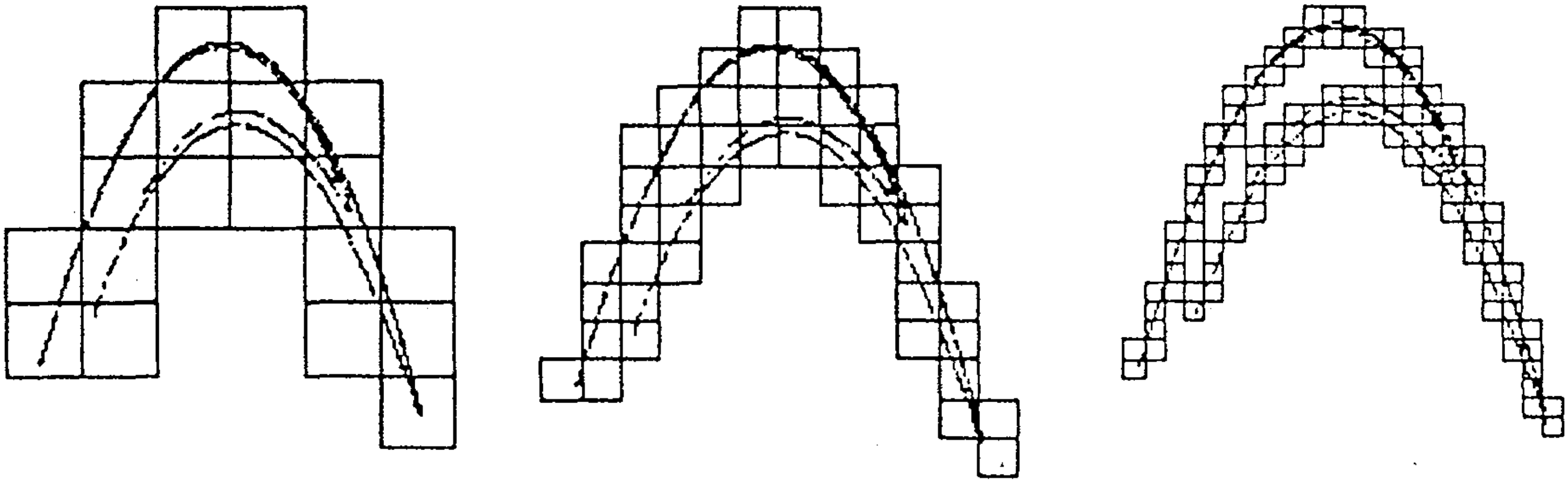
Cantor set (introduced by German mathematician Georg Cantor in 1883): remove the middle third of a line segment and then repeat the process with the remaining shorter segments



Fractal structure: a part of the object resembles the hole object.

D=0.63

How to estimate the dimension of a fractal?

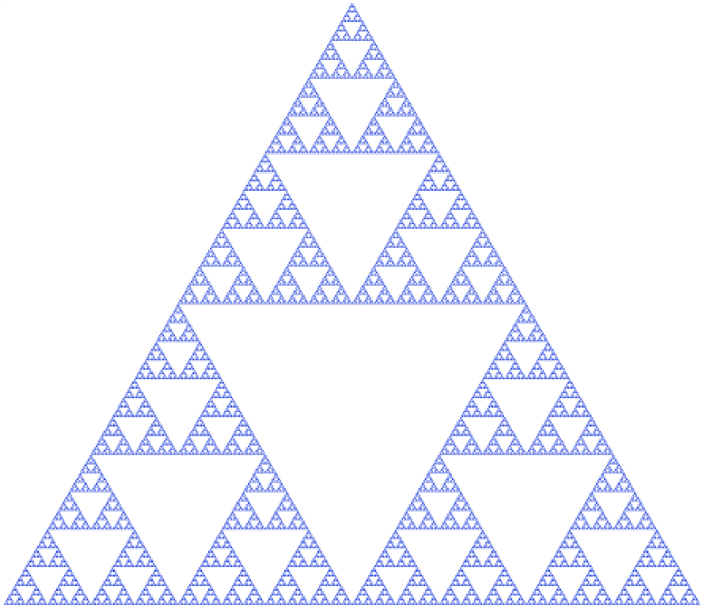


Box counting:
(more latter)

$$\text{bulk} \propto \text{size}^{\text{dimension}}$$

Abarbanel et al, Reviews of Modern Physics 65, 1331 (1993).

Sierpiński triangle



D=1.585

Source: Wikipedia

Fractal objects: characterized by a “fractal” dimension that measures roughness.

Broccoli
D=2.66

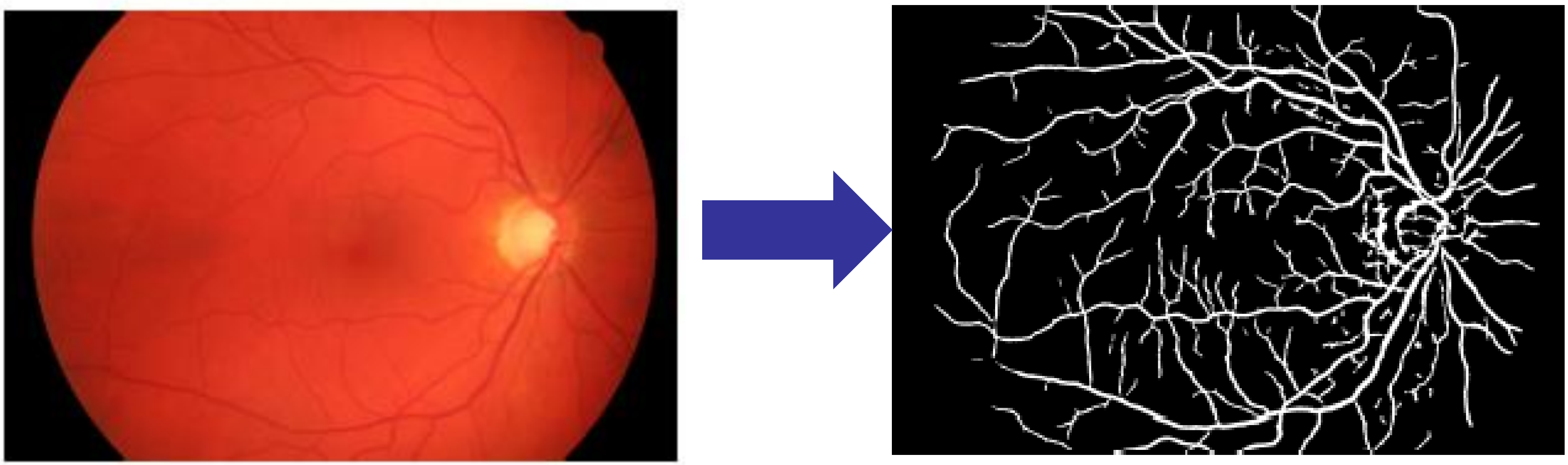
Human lung
D=2.97

Coastline of Ireland
D=1.22

A lot of research is focused on detecting fractal behavior in observed data.

Video: http://www.ted.com/talks/benoit_mandelbrot_fractals_the_art_of_roughness#t-149180

Application of fractal analysis



The fractal dimension of the blood vessels in the normal human retina is about 1.7 while it tends to increase with the level of diabetic retinopathy.

P. Amil et al., PLoS ONE 14, e0220132 (2019).

Spatio-temporal patterns: how “self-organization” emerges?

- **Ilya Prigogine** (Belgium, born in Moscow, Nobel Prize in Chemistry 1977).
- Studied thermodynamic systems far from equilibrium.

- Discovered that, in chemical systems, the interplay of (external) **input of energy** and **dissipation** can lead to “self-organized” patterns.

MARINE ECOLOGY Ruiz-Reynés et al., Sci. Adv. 2017; 3: e1603262 2 August 2017

Fairy circle landscapes under the sea

Daniel Ruiz-Reynés,[1] Damià Gomila,[1]* Tomàs Sintes,[1] Emilio Hernández-García,[1]
Núria Marbà,[2] Carlos M. Duarte[3]

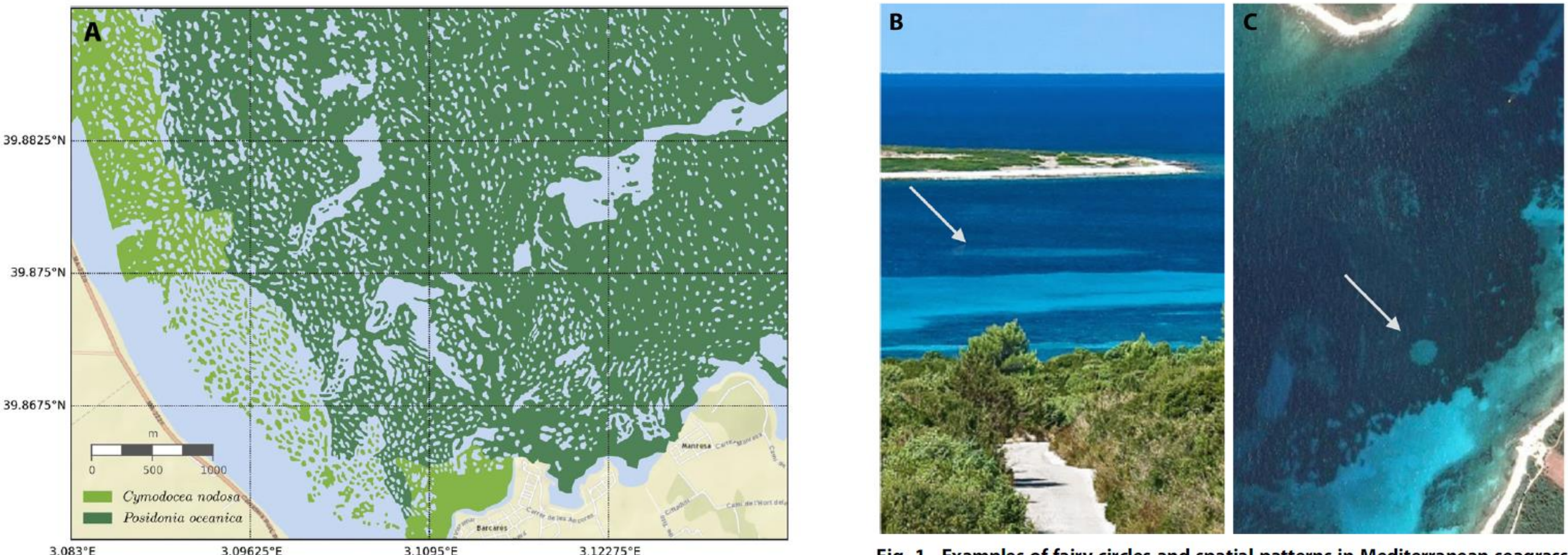
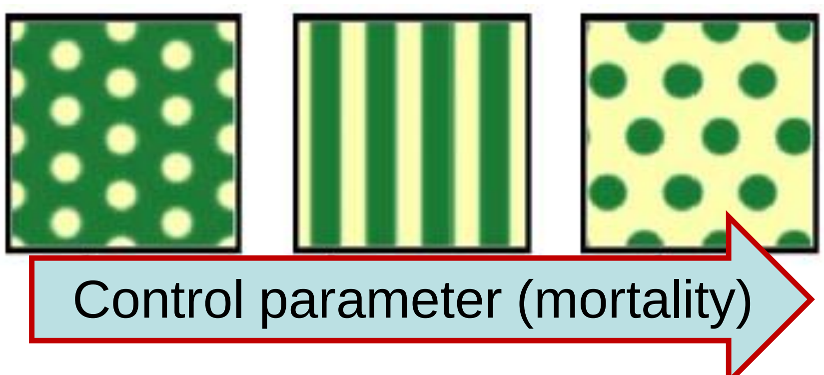


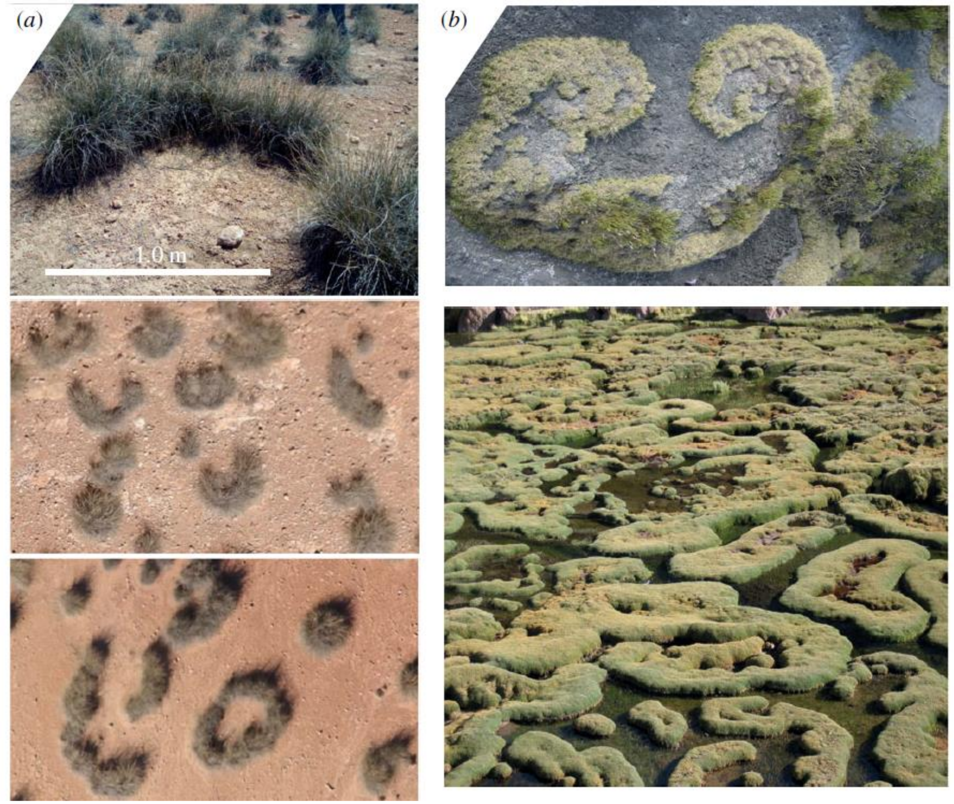
Fig. 1. Examples of fairy circles and spatial patterns in Mediterranean seagrass meadows. (**A**) Side-scan image of a seagrass meadow in Pollença bay (Mallorca



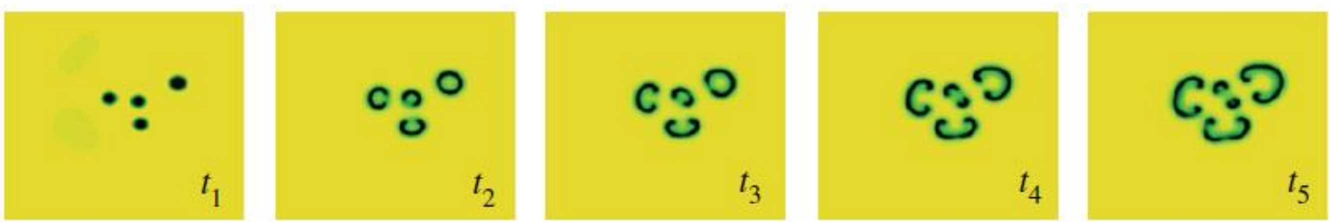
Observation and modelling of vegetation spirals and arcs in isotropic environmental conditions: dissipative structures in arid landscapes

M. Tlidi[1], M. G. Clerc[2], D. Escaff[3], P. Couteron[4],
M. Messaoudi[5], M. Khaffou[5] and A. Makhoute[5]

Phil. Trans. R. Soc. A **376** 20180026 (2018)



Morocco



Model simulation showing the temporal transition from localized patterns to arcs and spirals.

The study of spatio-temporal patterns has uncovered striking similarities in nature

Honey bees do a spire wave to scare away predators

https://www.youtube.com/watch?v=Sp8tLPDMUyg

Rotating waves occur in the heart during ventricular fibrillation

Hurricane Maria (Wikipedia)

https://media.nature.com/original/nature-assets/nature/journal/v555/n7698/extref/nature26001-sv6.mov

In the 80's: can we observe chaos experimentally?

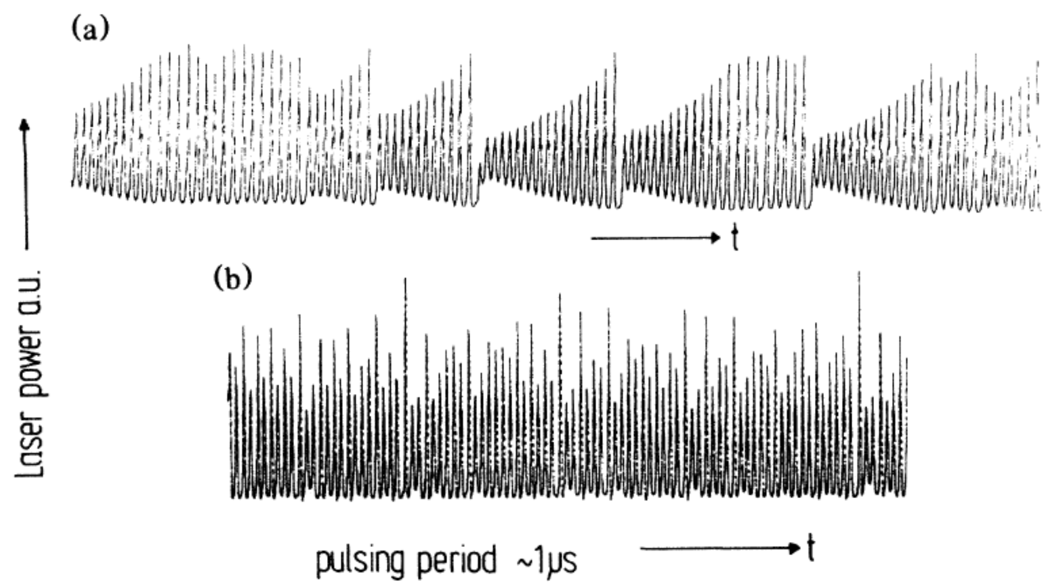
VOLUME 57, NUMBER 22 PHYSICAL REVIEW LETTERS 1 DECEMBER 1986

Evidence for Lorenz-Type Chaos in a Laser

C. O. Weiss and J. Brock[a]

Physikalisch-Technische Bundesanstalt, D-3300 Braunschweig, Federal Republic of Germany

(Received 18 April 1986)



optically pumped NH_3 laser

Can we control a chaotic system?

- **Ott**, **Grebogi** and **Yorke** (OGY) method (1990)

  Unstable periodic orbits can be used for control: wisely chosen **periodic kicks** can maintain the system near the desired orbit.

- **Pyragas** method (1992)

  Control by using a **continuous** self-controlling **feedback** signal, whose intensity is practically zero when the system evolves close to the desired periodic orbit but increases when it drifts away.

E. Ott, C. Grebogi and J. A. Yorke, Phys. Rev. Lett. 64, 1196 (1990).
K. Pyragas, Physics Letters A 170, 421 (1992).

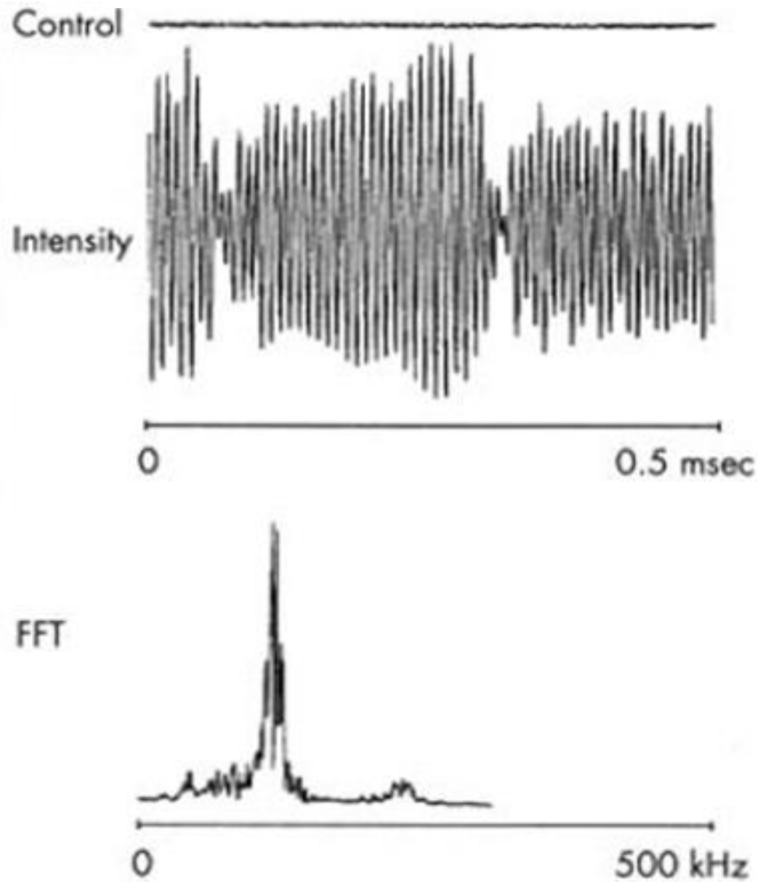
Experimental demonstration of control of optical chaos

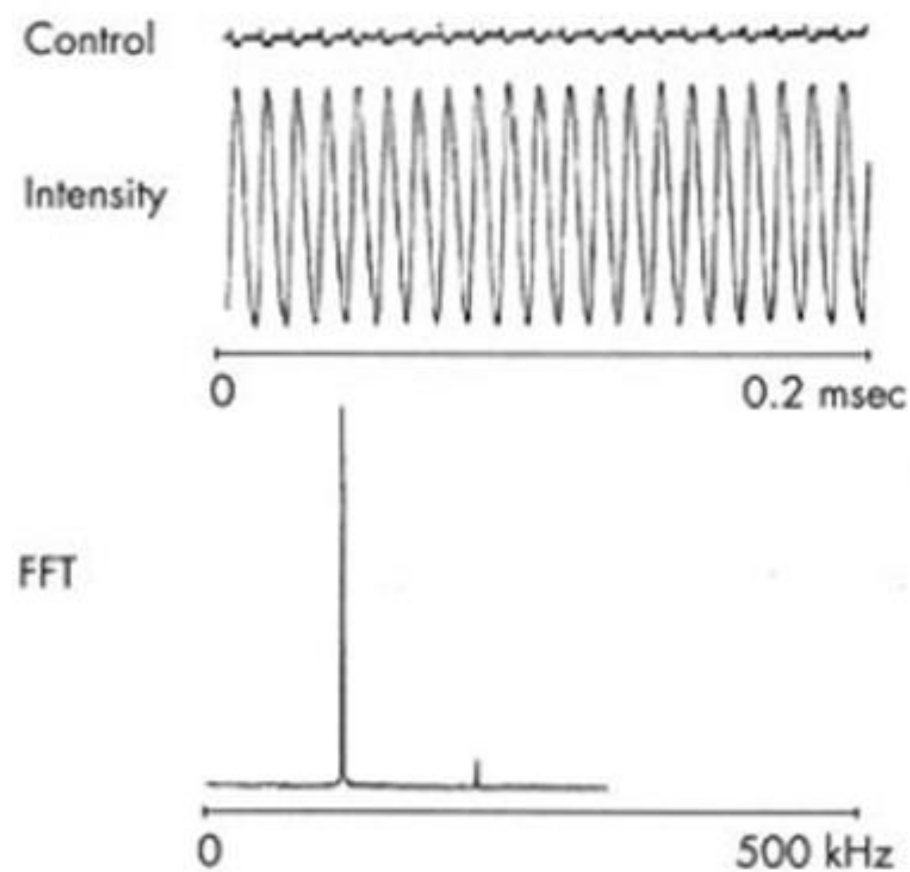
VOLUME 68, NUMBER 9 PHYSICAL REVIEW LETTERS 2 MARCH 1992

Dynamical Control of a Chaotic Laser: Experimental Stabilization of a Globally Coupled System

Rajarshi Roy, T. W. Murphy, Jr., T. D. Maier, and Z. Gills
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332

E. R. Hunt
Department of Physics and Astronomy, Ohio University, Athens, Ohio 45701-2979
(Received 25 November 1991)





Demonstration of delayed feedback control (Pyragas method)

PHYSICAL REVIEW E VOLUME 49, NUMBER 2 FEBRUARY 1994

Controlling unstable periodic orbits by a delayed continuous feedback

Serge Bielawski, Dominique Derozier, and Pierre Glorieux
Laboratoire de Spectroscopie Hertzienne, Université des Sciences et Technologies de Lille, F-59655 Villeneuve d'Ascq Cedex, France

(Received 22 October 1993)

A method is presented for stabilizing unstable periodic orbits of a dynamical system by applying continuous feedback on a control parameter. The feedback signal is proportional to the difference between two values of a dynamical variable, separated by a time equal to the unstable orbit periodicity. This method has been checked numerically and experimentally on the control of the unstable orbits of a CO_2 laser with modulated losses.

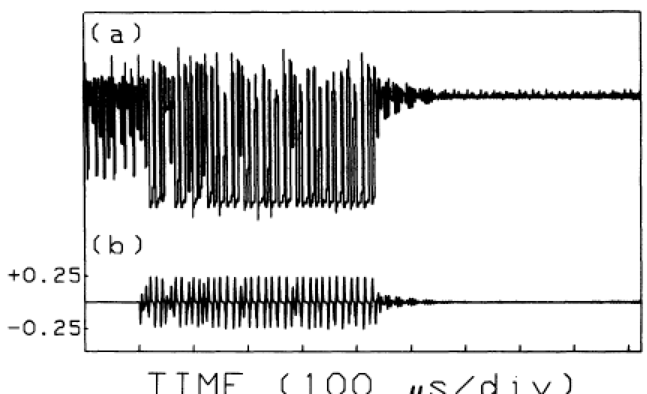


FIG. 3. Stabilization of the unstable T-periodic orbit: (a) periodic sampling of the laser power in arbitrary units. (b) Correction applied to the modulator in units of the modulation index $[p(t)/m]$.

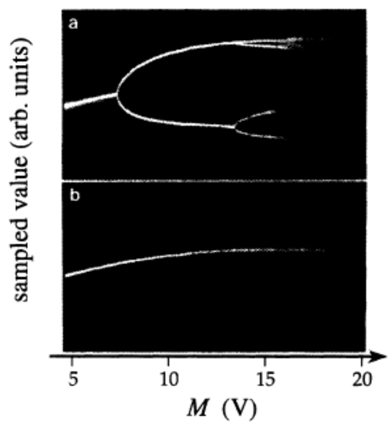


FIG. 4. Bifurcation diagram without (a) and with (b) stabilization of the unstable T-periodic orbit. The stable T-periodic orbit is destabilized for a modulation index of $M = 7.2$ V and the unstable orbit is embedded in the chaotic attractor for $M > 16.8$ V.

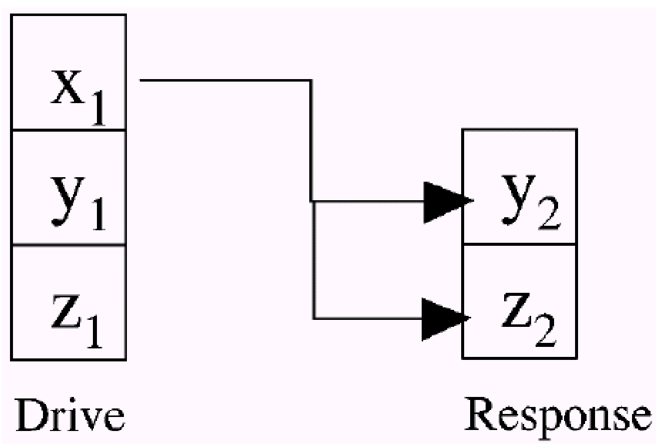
The 1990s: synchronization of two chaotic systems

VOLUME 64, NUMBER 8 PHYSICAL REVIEW LETTERS 19 FEBRUARY 1990

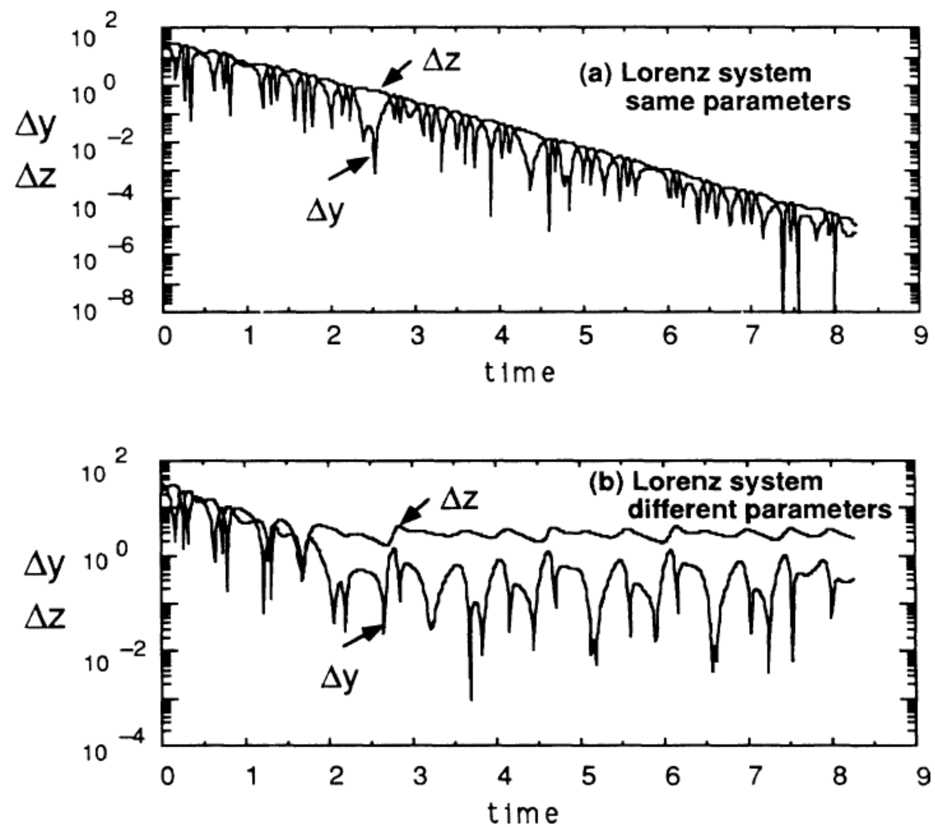
Synchronization in Chaotic Systems

Louis M. Pecora and Thomas L. Carroll
Code 6341, Naval Research Laboratory, Washington, D.C. 20375
(Received 20 December 1989)

Unidirectional coupled
Lorenz systems



$$t \to \infty \quad |y_2 - y_1| \to 0, \ |z_2 - z_1| \to 0$$



Actually, the first observation of synchronization was much earlier (mutual *entrainment* of two pendulum clocks)

In mid-1600s **Christiaan Huygens** (Dutch mathematician) noticed that two pendulum clocks mounted on a common board synchronized and swayed in opposite directions (in-phase also possible).

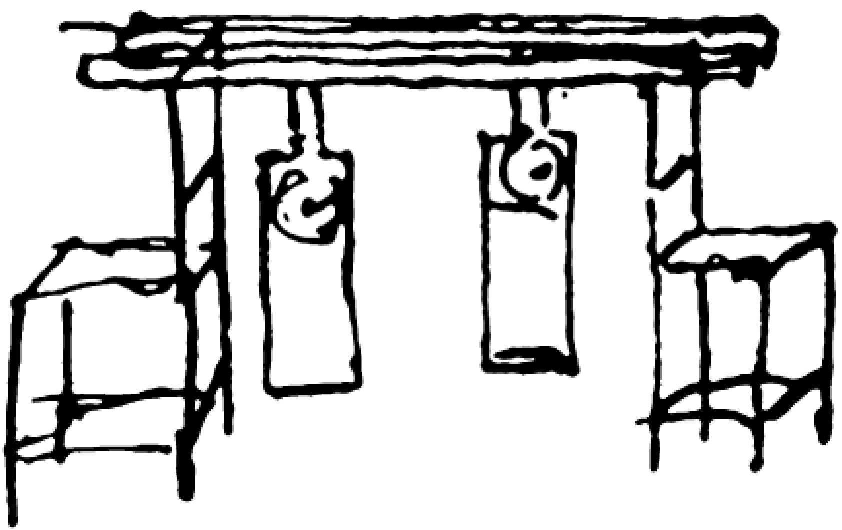
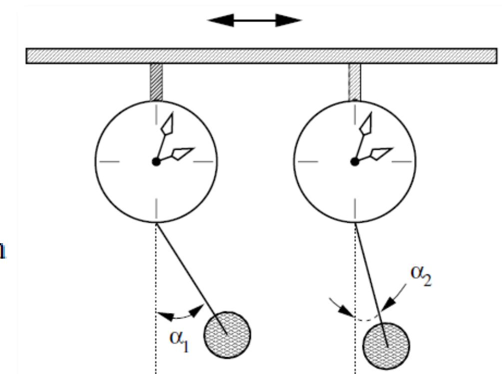


Figure 1.2. Original drawing of Christiaan Huygens illustrating his experiments with two pendulum clocks placed on a common support.



Different types of synchronization

Unidirectional coupling:

$$\frac{dx}{dt} = f(x)$$

$$\frac{dy}{dt} = f(y) + \eta\, g(x - y)$$

Bidirectional (mutual) coupling:

$$\frac{dx}{dt} = f(x) + \rho\, h(y - x)$$

$$\frac{dy}{dt} = f(y) + \eta\, g(x - y)$$

- Complete: ***y*(*t*) = *x*(*t*)** (identical systems)
- Phase: the phases of the oscillations are synchronized, but the amplitudes are not.
- Lag: ***y*(*t*+τ) = *x*(*t*)**
- Generalized: ***y*(*t*) = *F*(x(*t*-τ))** (*F and* τ can depend on the coupling strengths, η and ρ)

Another problem of time series analysis:

How to detect synchronization? How to quantify it?

Experimental observation of synchronization of lasers

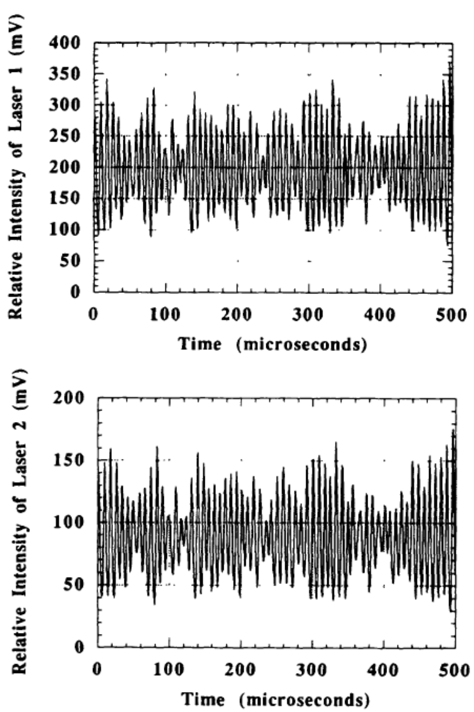
VOLUME 72, NUMBER 13 PHYSICAL REVIEW LETTERS 28 MARCH 1994

Experimental Synchronization of Chaotic Lasers

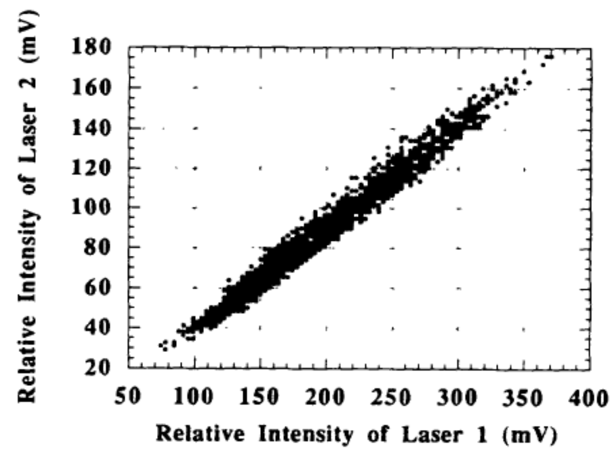
Rajarshi Roy and K. Scott Thornburg, Jr.
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332
(Received 30 August 1993)

We report the observation of synchronization of the chaotic intensity fluctuations of two Nd:YAG lasers when one or both the lasers are driven chaotic by periodic modulation of their pump beams.

(a)

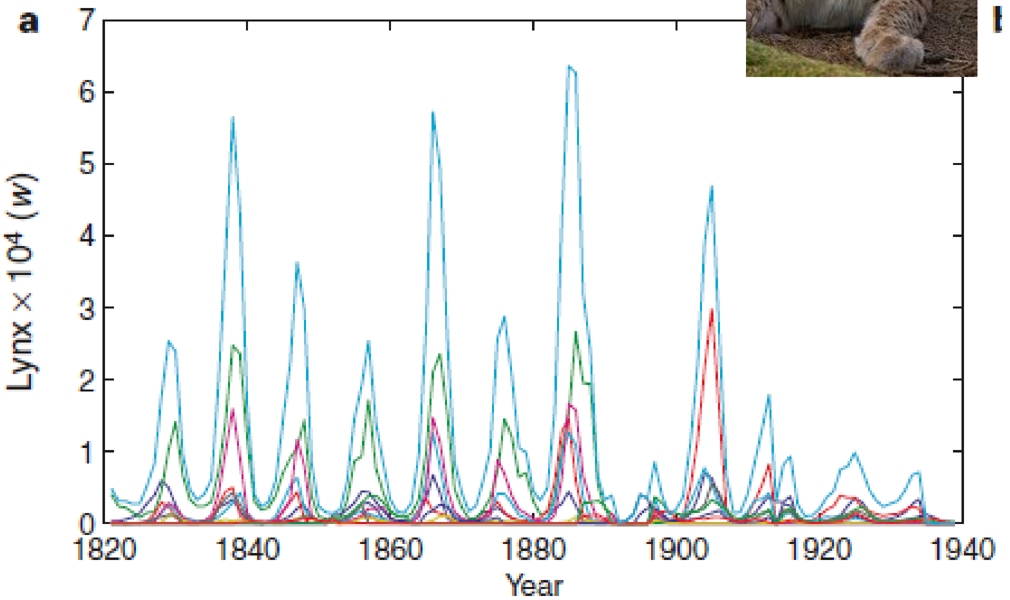


(b)



An example of *phase synchronization*: the abundances of the Lynx populations in six regions in Canada

The size of the populations oscillate regularly and periodically in phase, but with irregular and uncorrelated peaks.



Foodwebs (that represent the interactions of vegetation and populations of herbivores and predators) can display very complex oscillatory behaviors.

B. Blasius et al, Nature 399, 354 (1999).

In the late 90s early 2000s: synchronization of a large number of coupled oscillators

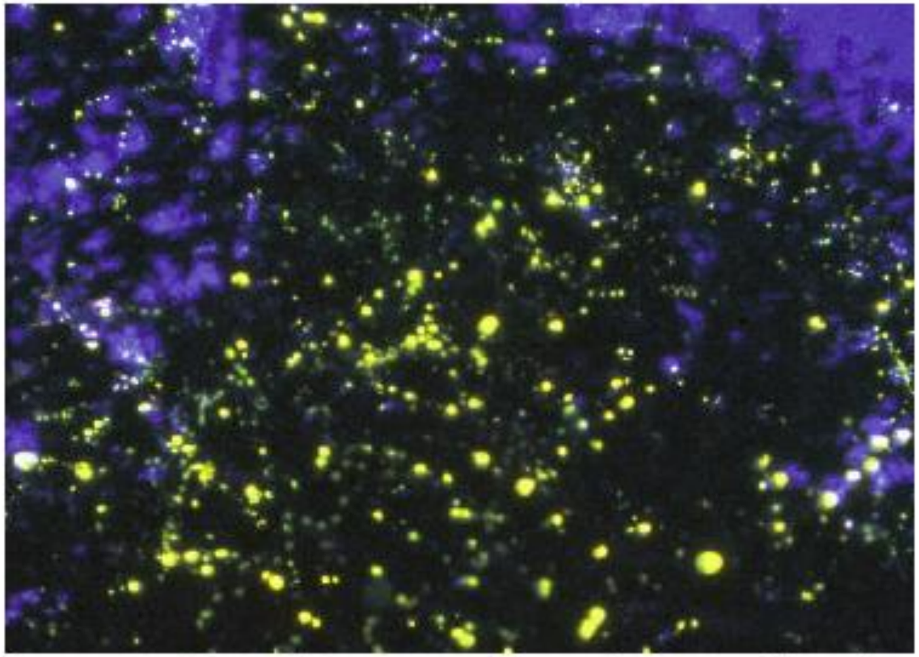


Figure 1 | Fireflies, fireflies burning bright. In the forests of the night, certain species of firefly flash in perfect synchrony — here *Pteroptyx malaccae* in a mangrove apple tree in Malaysia. Kaka *et al.*[2] and Mancoff *et al.*[3] show that the same principle can be applied to oscillators at the nanoscale.

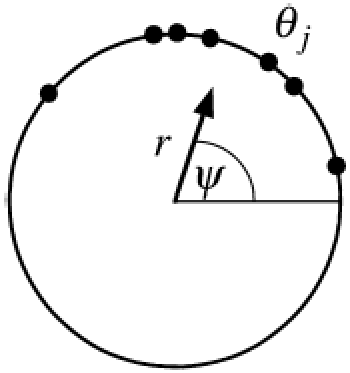
London Millennium Bridge Opening

Kuramoto model

(Japanese physicist, 1975)

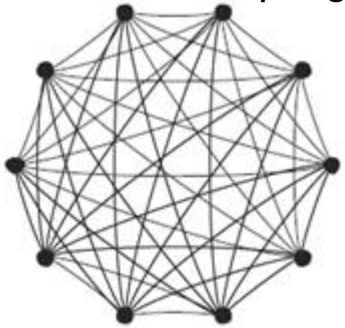
Model of **all-to-all** coupled **phase oscillators**.

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \xi_i, \qquad i = 1 \ldots N$$



K = coupling strength, ξ_i = stochastic term (noise)

All-to-all coupling



Describes the emergence of collective behavior

How to quantify?

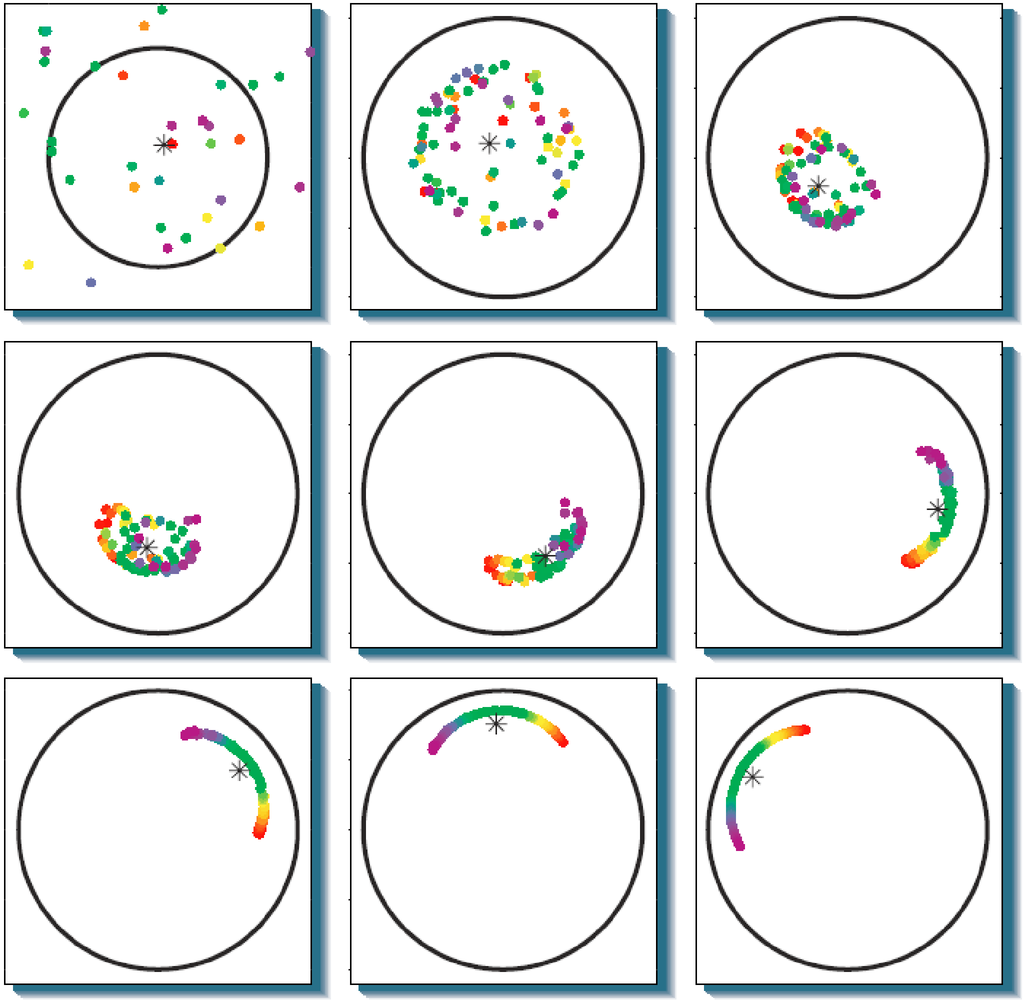
With the **order parameter**:

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

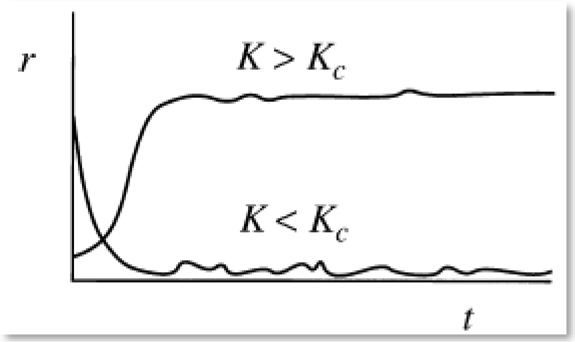
r =0 incoherent state (oscillators scattered in the unit circle)

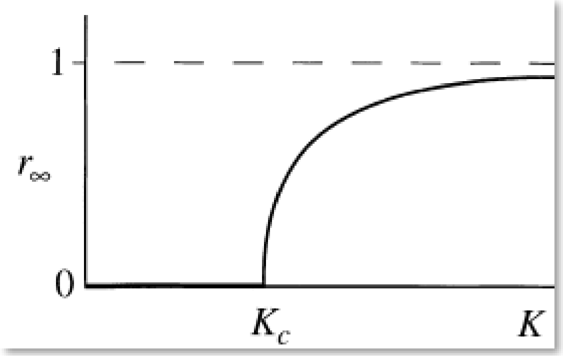
r =1 all oscillators are in phase ($\theta_i=\theta_j \; \forall$ i,j)

Synchronization transition as the coupling strength increases



Strogatz and others, late 90’



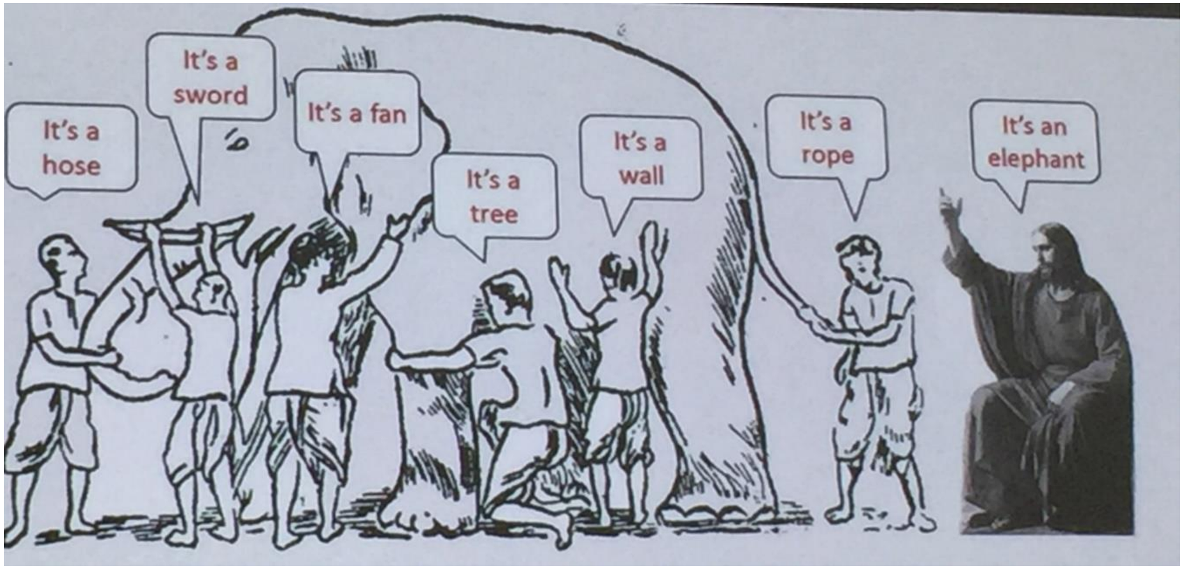


Strogatz, Nature 2001

Video: https://www.ted.com/talks/steven_strogatz_on_sync

2000s to present: from chaotic systems to complex systems

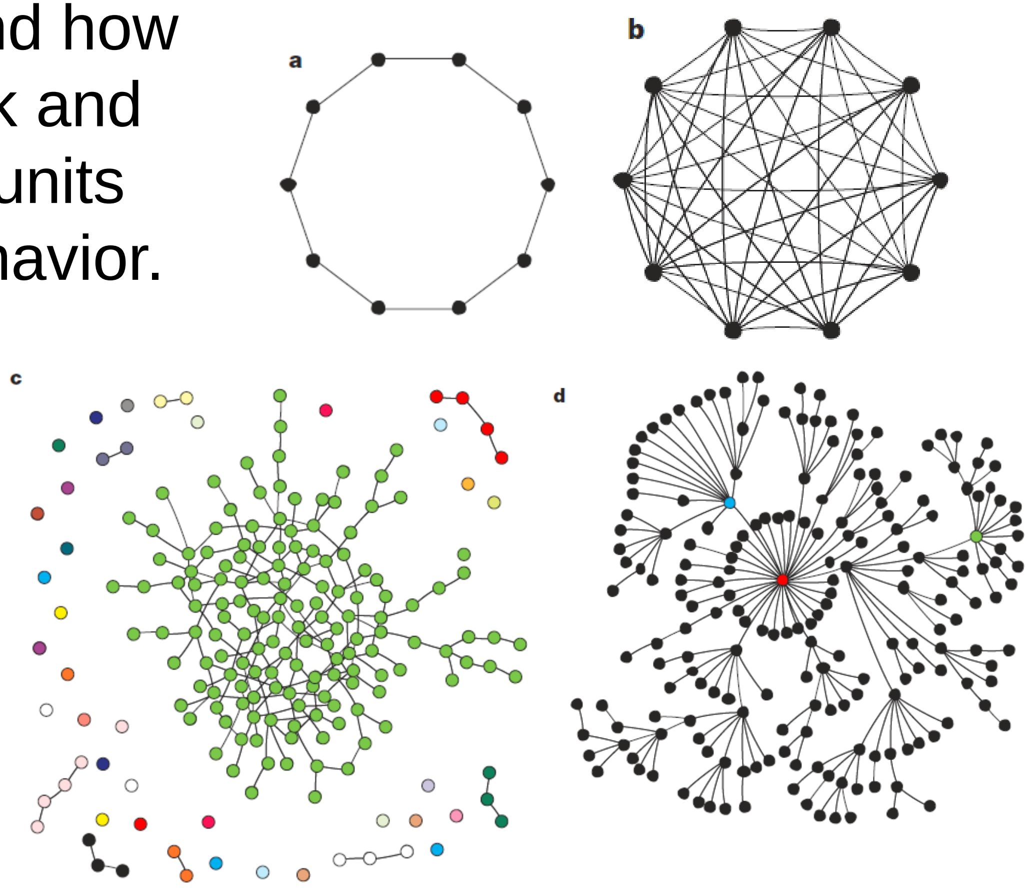
- Complicated systems (large sets of linear elements) are not complex.
- Complex systems: large number of elements, where the elements and/or their interactions are **nonlinear**.
- Main difference: *the whole is not equal to the sum of the parts*.



(a good meal is another example: it is much more than the sum of its ingredients)

Network science

- **Networks** (or **graphs**) are used for mathematical modelling of complex systems.
- **Complexity science**: study of the emergent properties, not present in the individual elements.
- The challenge: to understand how the **structure** of the network and the **dynamics** of individual units determine the collective behavior.
- Applications
  - Epidemics
  - Rumor spreading
  - Transport networks
  - Financial crises
  - Brain, physiology, etc.

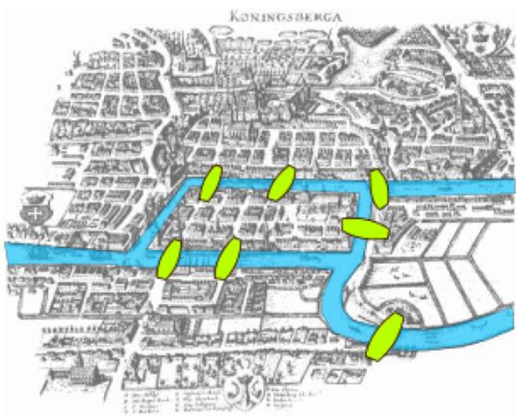
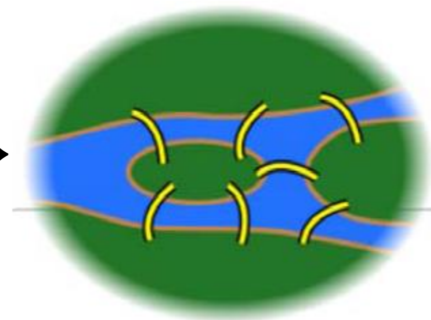
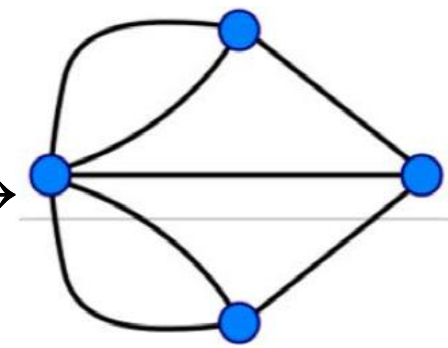


S. Strogatz, Nature 2001

The start of Graph Theory: The Seven Bridges of Königsberg

(Prussia, now Russia)

- The problem was to devise a walk through the city that would cross each of those bridges once and only once.

 →  → 

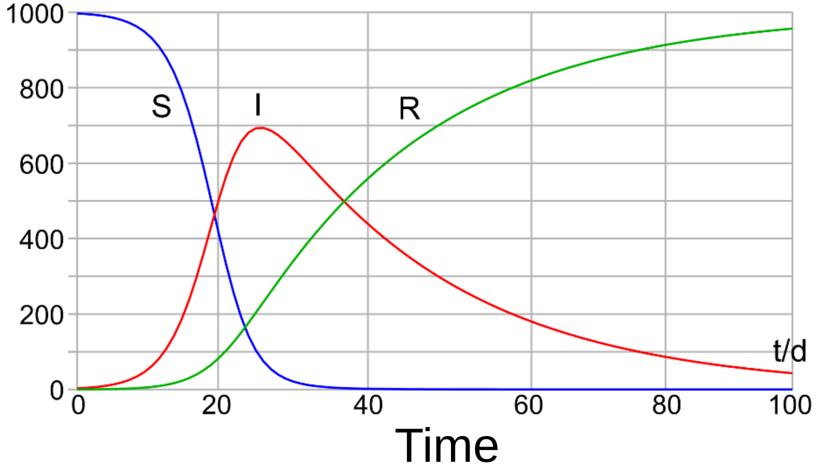
Source: Wikipedia

- By considering the number of odd/even links of each “node”, **Leonhard Euler** (Swiss mathematician) demonstrated in 1736 that is impossible.

The SIR epidemic model (early 1900s)

In its simplest version the SIR model consists of three rate equations for:

- S(t): individuals not yet infected (susceptible).
- I(t): infected individuals that are capable of spreading the disease to those susceptible.
- R(t): individuals that have been infected and can't be re-infected nor transmit the infection to others (either due to immunization or due to death).
- N = S(t) + I(t) + R(t) constant.
- The model predicts the existence of a threshold that separates grow from extinction.



Source: Wikipedia

Many extensions of the SIR model

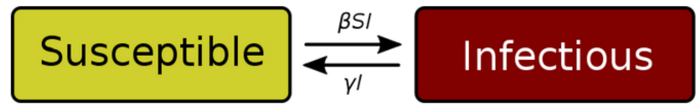
- Immunity that lasts only a certain time interval (after which individuals are back in the susceptible group).
- Additional populations
  - E: exposed people that could have been infected;
  - C: susceptible people that are protected in a confinement compartment;
  - Q: infected people in quarantine;
  - B, D: births and deaths
  - Etc.
- Many extensions of the model to take into account **diffusion in “networks”.**

https://www.investigacionyciencia.es/revistas/investigacion-y-ciencia/una-crisis-csmica-798/cmo-modelizar-una-pandemia-18561?utm_source=Facebook&utm_medium=Social&utm_campaign=fb+web

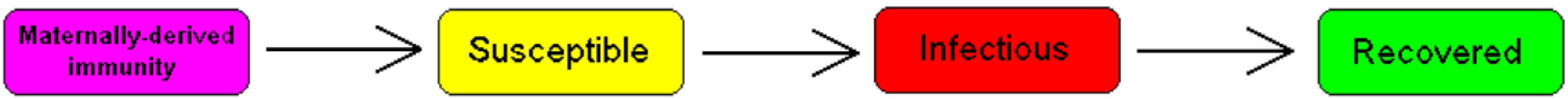
Pastor-Satorras et al, Rev. Mod. Phys. 87, 925 (2015)

A few examples of epidemic models

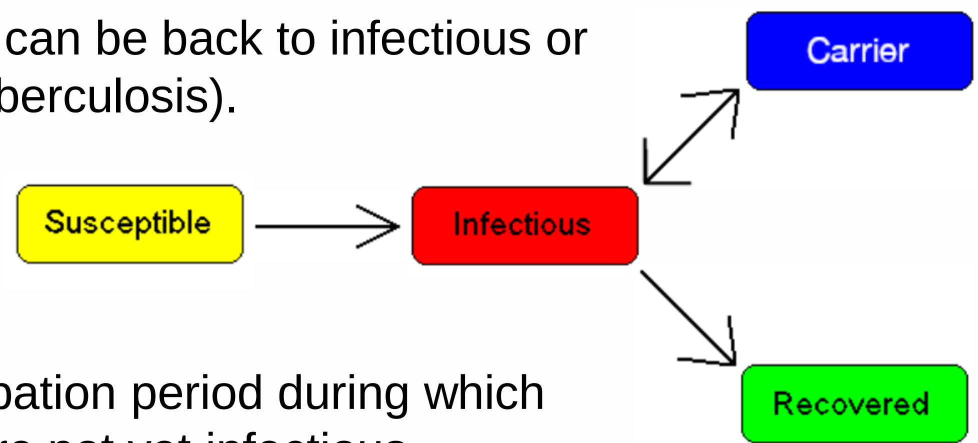
SIS: No long lasting immunity (example: cold).



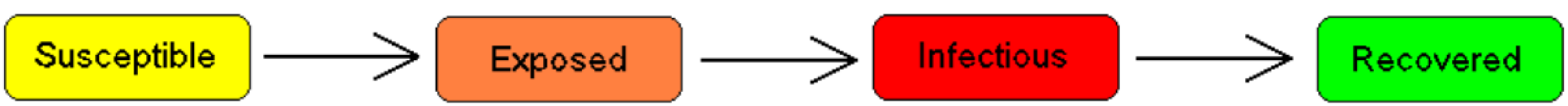
MSIR: Babies have some initial immunity.



Some people might not recover and can be back to infectious or carry disease with symptoms (ex: tuberculosis).

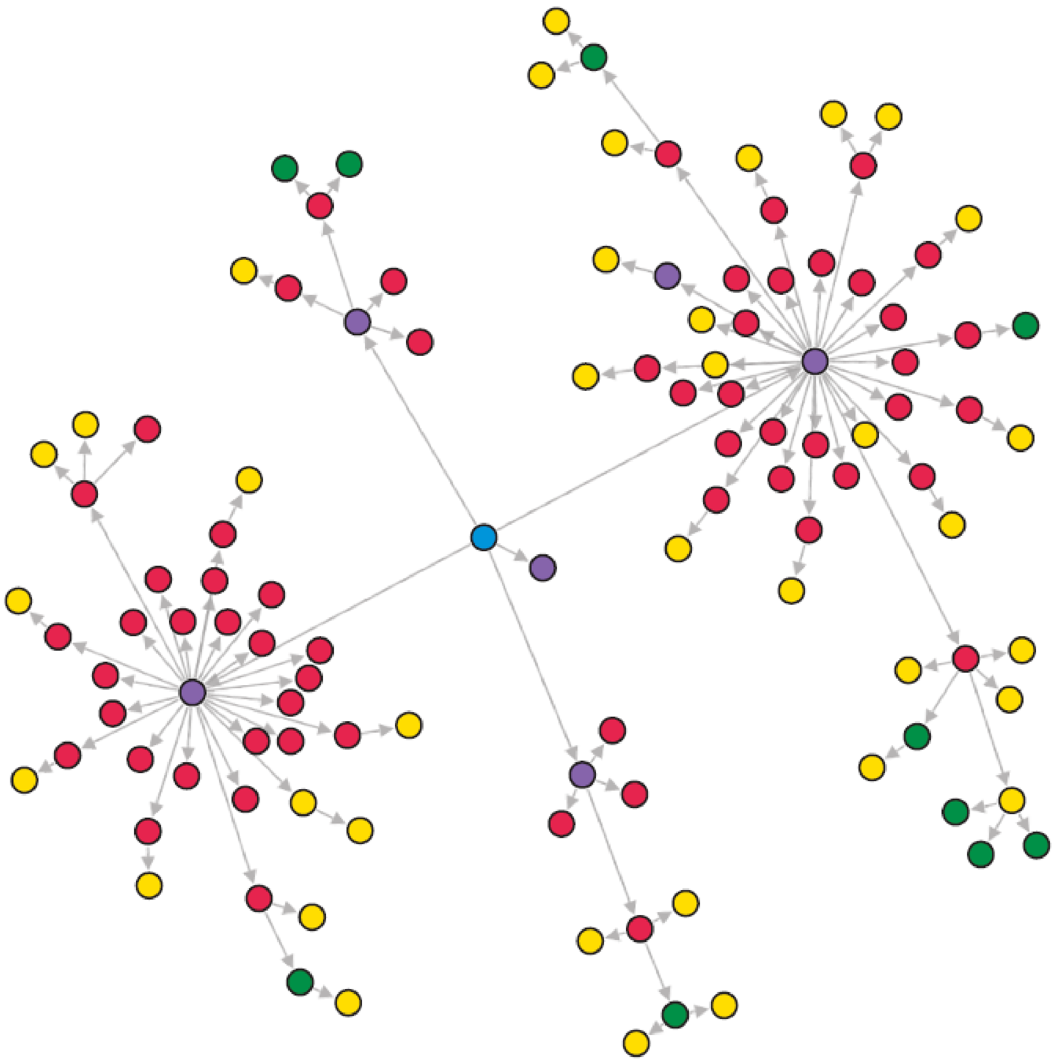


For some infections there is an incubation period during which individuals have been infected but are not yet infectious.



Source: Wikipedia

Example of transmission network of Covid-19



Transmission network seeded by an unknown infected individual (**blue**) who attended a training course with other fitness instructors (**purple**).

The fitness instructors spread the infection to students in their classes (**red**), to family (yellow), and to coworkers (**green**).

Source: Alison Hill, The math behind epidemics, https://physicstoday.scitation.org/doi/10.1063/PT.3.4614

Revisiting the Kuramoto model

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_j \sin\left(\theta_j - \theta_i\right) + \xi_i \quad \Rightarrow \quad \boxed{\frac{d\theta_i}{dt} = \omega_i + \sum_j A_{ij} G\left(\theta_i, \theta_j\right) + \xi_i}$$

Different synchronization regimes can occur, depending on:

- The coupling function (attractive / repulsive).
- The network topology (homogeneous / heterogeneous).
- The number of units ("crowd synchrony")
- The properties of the individual units, in relation to the network:
  - relation between the # of links an element has and the # of links the neighbors have.
  - relation between the # of links that an element has and its properties.
- The synchronization transition can be gradual or explosive.
- Synchronized and unsynchronized oscillations can co-exist ("chimera states").
- Bi-stability: the network can synchronize, depending on the initial conditions.

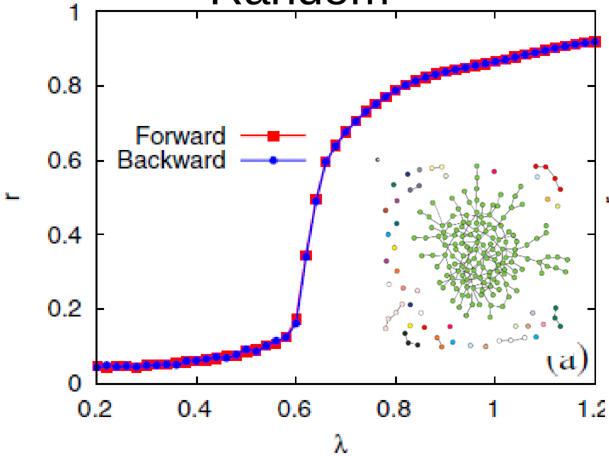
Example: "Explosive" synchronization

$$\dot{\theta}_i = \omega_i + \lambda \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i)$$

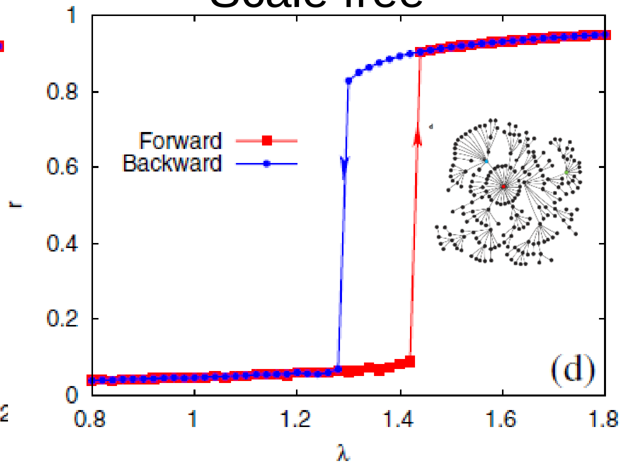
$$\boxed{\omega_i = k_i = \sum_{j=1}^{N} A_{ij}}$$

Fast oscillators have many links; slow oscillators only few links

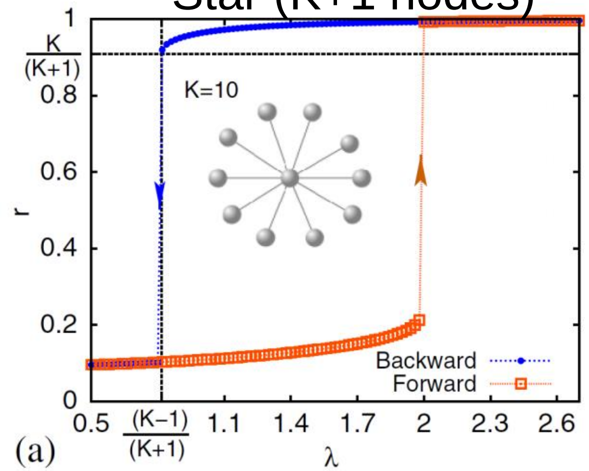
Random



Scale free



Star (K+1 nodes)

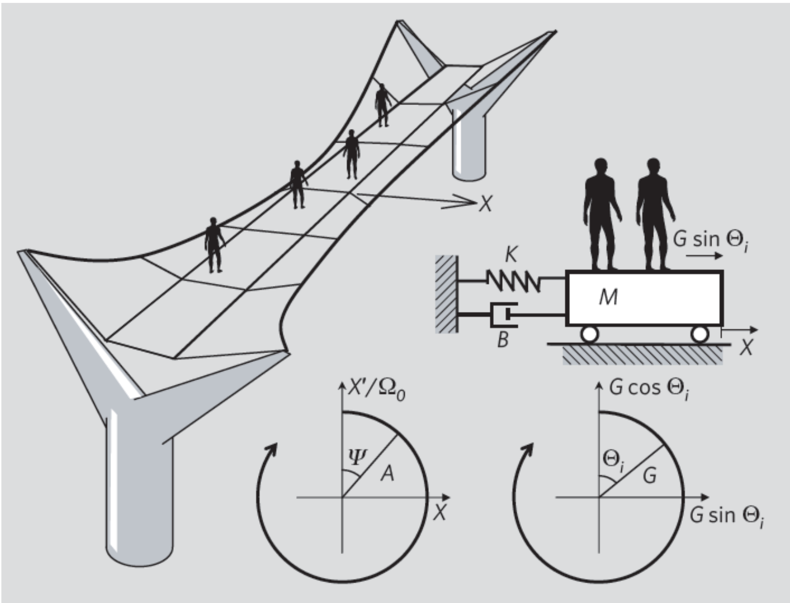


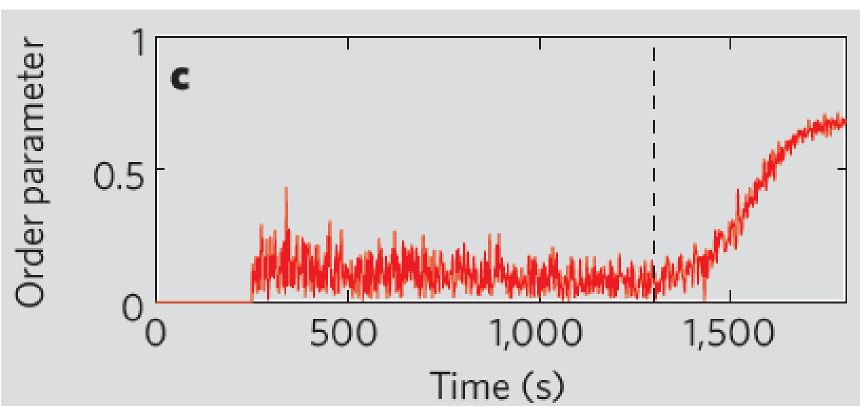
ω_1=K
ω_i=1 i≠1

Explosive sync. has been found in coupled lasers and in electronic circuits.

J. Zamora et al., Phys. Rev. Lett. 105, 264101 (2010).
J. Gomez-Gardeñes et al., Phys. Rev. Lett. 106, 128701 (2011).
I. Leyva et al, Phys. Rev. Lett. 108, 168702 (2012).

"Crowd synchrony": the millennium footbridge starts to sway when packed with pedestrians that synchronize their steps.





Model the bridge as a weakly damped and driven harmonic oscillator:

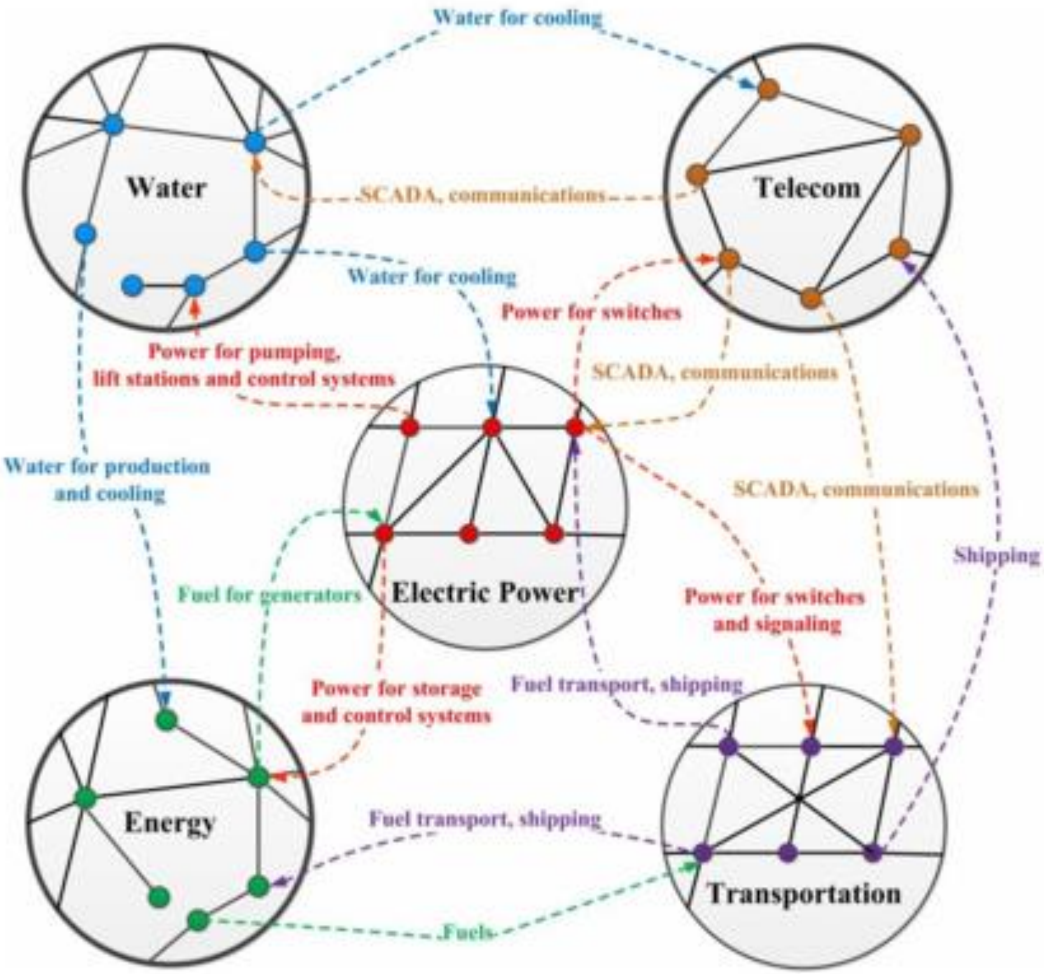
$$M\frac{d^2X}{dt^2} + B\frac{dX}{dt} + KX = G\sum_{i=1}^{N} \sin\Theta_i$$

The bridge's movement alters each pedestrian's gait:

$$\frac{d\Theta_i}{dt} = \Omega_i + CA\sin(\Psi - \Theta_i + \alpha)$$

S. Strogatz, Nature 438, 43 (2005).

Interactions between networks: interdependent networks



Time series analysis problems: how to predict a critical (or extreme) event in one network? (a failure of a link or a node) How will it affect other networks?

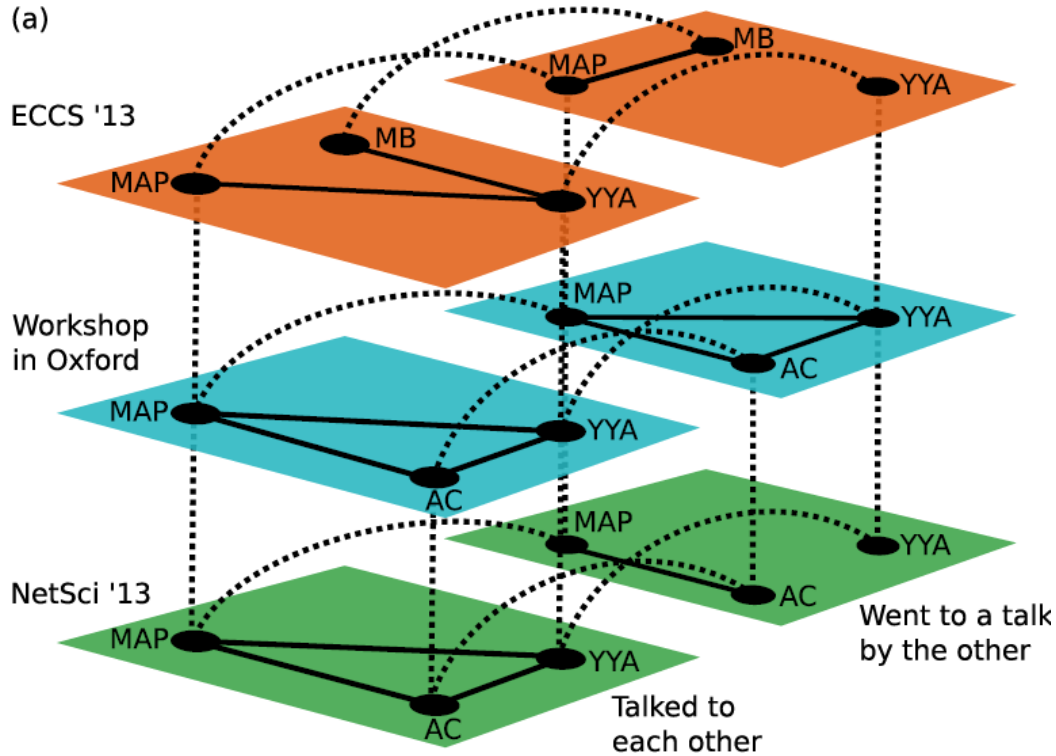
Source: Wikipedia

Multilayer networks

Facebook

Twitter

Linkedin



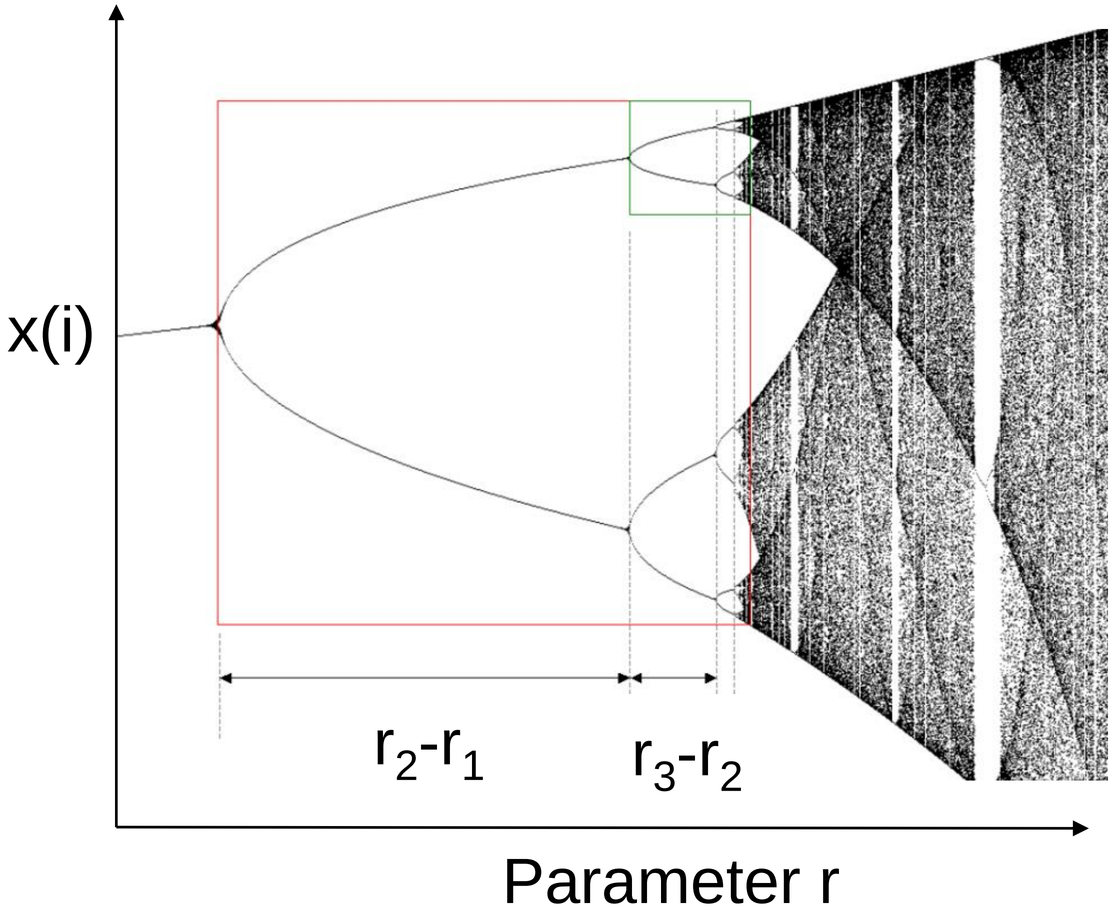
Kivela et al, J. Complex Netw. 2, 203 (2014).

Summary

- Dynamical systems allow to
  - understand low-dimensional systems,
  - uncover “order within chaos”,
  - uncover universal features
  - control chaotic behavior.
- Complexity “network” science: understanding emerging phenomena in large sets of interacting units.
- Dynamical systems and complexity science are interdisciplinary research fields with many applications.

Exercise 1: analyze the logistic map

$$x(i+1) = r\,x(i)[1 - x(i)]$$



- For r=3.5, calculate the 10 values, x(i) with i=1…10, that follow x(0)=0.2. Plot x(i) vs. i
- Plot the bifurcation diagram for r in the interval (3.5,4)
- Estimate $\delta = (r_2 - r_1)/(r_3 - r_2)$

Outline

- Introduction to dynamical systems
- **Introduction to flows on the line**
- Solving equations with computer
- Fixed points and linear stability

Types of dynamical systems

- **Continuous time**: differential equations
  - Ordinary differential equations (ODEs). Example: damped oscillator

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

  - Partial differential equations (PDEs). Example: heat equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

- **Discrete time**: difference equations or “iterated maps”. Example: the logistic map

x(i+1)=**r** x(i)[1-x(i)]

ODEs can be written as first-order differential equations

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0 \quad \Rightarrow \quad \begin{aligned}\dot{x}_1 &= f_1(x_1,\ldots,x_n)\\ &\vdots\\ \dot{x}_n &= f_n(x_1,\ldots,x_n)\end{aligned} \qquad \boxed{\dot{x}=f(x)}$$

- First example: harmonic oscillator $m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$

$$x_1 = x \text{ and } x_2 = \dot{x}$$

$$\dot{x}_2 = \ddot{x} = -\tfrac{b}{m}\dot{x} - \tfrac{k}{m}x = -\tfrac{b}{m}x_2 - \tfrac{k}{m}x_1 \quad \Rightarrow \quad \boxed{\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\tfrac{b}{m}x_2 - \tfrac{k}{m}x_1\end{aligned}}$$

- Second example: pendulum

$$\ddot{x} + \tfrac{g}{L}\sin x = 0 \quad \Rightarrow \quad \boxed{\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\tfrac{g}{L}\sin x_1\end{aligned}}$$

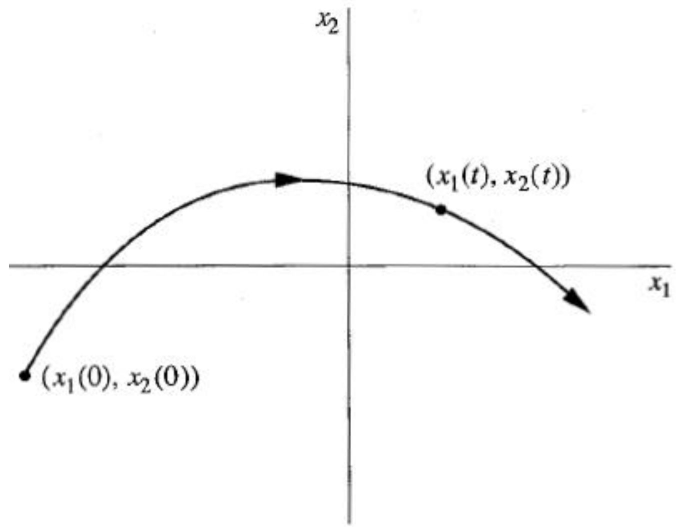
Trajectory in the phase space

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\tfrac{b}{m}x_2 - \tfrac{k}{m}x_1$$

- Given the initial conditions, $x_1(0)$ and $x_2(0)$, we predict the evolution of the system by solving the equations: $x_1(t)$ and $x_2(t)$.
- $x_1(t)$ and $x_2(t)$ are solutions of the equations.

- The evolution of the system can be represented as a <u>trajectory</u> in the <u>phase space</u>.

$\Rightarrow$ two-dimensional (2D) dynamical system.



Key argument (Poincare): find out how the trajectories look like, without solving the equations explicitly.

Classification of dynamical systems described by ODEs (I/II)

$$\boxed{\dot{x} = f(x) + \xi(t)}$$

- *f(x)* linear: in the function *f*, *x* appears to first order only (no x^2, $x_1 x_2$, *sin(x)* etc.). Then, the behavior can be understood from the sum of its parts.
- *f(x)* nonlinear: superposition principle fails!
- Example of linear system: harmonic oscillator

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \quad \Rightarrow \quad \boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{b}{m}x_2 - \tfrac{k}{m}x_1 \end{aligned}}$$

In the right-hand-side x_1 and x_2 appear to first power (no products etc.)

- Example of nonlinear system: pendulum

$$\ddot{x} + \tfrac{g}{L}\sin x = 0 \quad \Rightarrow \quad \boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{L}\sin x_1 \end{aligned}}$$

Classification of dynamical systems described by ODEs (II/II)

$$\boxed{\dot{x} = f(x) + \xi(t)}$$

- $\xi=0$: deterministic.
- $\xi\neq0$: stochastic (real life) –simplest case: additive dynamical noise.
- x: vector with few variables ($n<4$): low dimensional.
- x: vector with many variables: high dimensional.

- f does not depend on time: autonomous system.
- f depends on time: non-autonomous system.

Example of non-autonomous system: a forced oscillator

$$m\ddot{x} + b\dot{x} + kx = F\cos t$$

$$\boxed{\dot{x} = f(x)}$$

- Can also be written as first-order ODE

$$x_1 = x \text{ and } x_2 = \dot{x}$$

$$x_3 = t \qquad \dot{x}_3 = 1$$

$$\Rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tfrac{1}{m}\left(-kx_1 - bx_2 + F\cos x_3\right) \\ \dot{x}_3 &= 1 \end{aligned}$$

- Three-dimensional system: to predict the evolution we need to know the present state (*t, x, dx/dt*).

So…what is a "flow on the line"?

- A one-dimensional autonomous dynamical system described by a first-order ordinary differential equation

$$\dot{x} = f(x)$$

- x $\in \Re$
- *f* does not depend on time

Summarizing

Number of variables

	N=1	N=2	N=3	N>>1	N=∞ (PDEs DDEs)
Linear	RC circuit	Harmonic oscillator	$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = A\mathbf{x}(t) + B\mathbf{u}(t)$ $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))$		• Heat equation, • Maxwell equations • Schrodinger equation
Nonlinear	Logistic population grow	Pendulum	• Forced oscillator • Lorentz model	• Kuramoto phase oscillators	• Navier-Stokes (turbulence)

"flow on the line"

PDEs=partial differential eqs.
DDEs=delay differential eqs.

Outline

- Introduction to dynamical systems
- Introduction to flows on the line
- **Solving equations with computer**
- Fixed points and linear stability

Numerical integration of an ordinary differential equation

$$\dot{x} = f(x) \qquad \frac{dx}{dt} = f(x) \qquad dx = f(x)dt \qquad dx = x_1 - x_0;\; dt = t_1 - t_0$$

Euler first order (*dx*, *dt* small):

$$x_1 - x_0 = f(x_0)(t_1 - t_0)$$

$$x(t_1) \approx x_1 = x_0 + f(x_0)(t_1 - t_0)$$

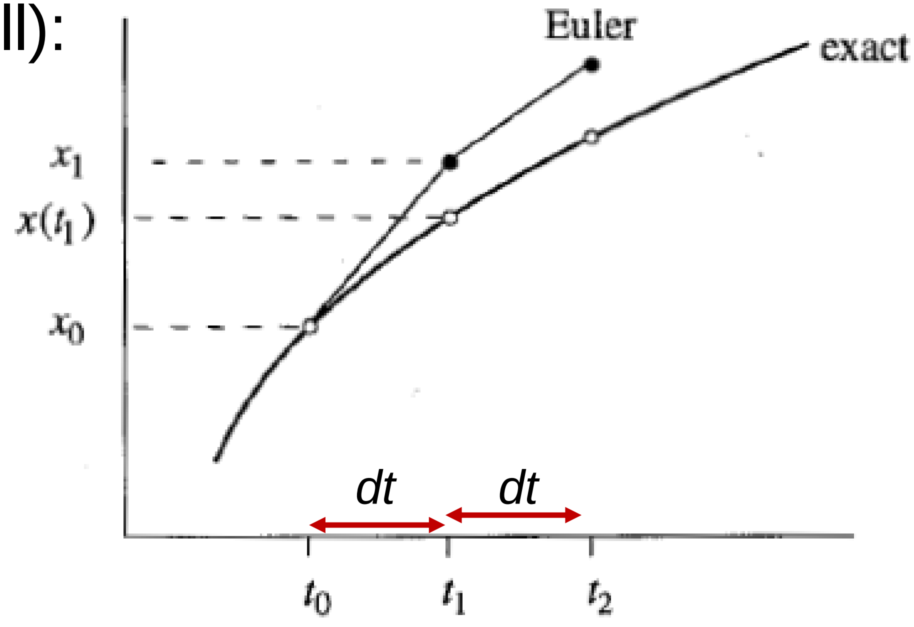
$$x(t_2) \approx x_2 = x_1 + f(x_1)(t_2 - t_1)$$

$$dt = (t_2 - t_1) = (t_1 - t_0)$$

$$t_1 = t_0 + dt$$

$$t_2 = t_1 + dt = t_0 + 2dt \quad \ldots$$

$$t_n = t_0 + ndt$$



$$x_1 = x_0 + f(x_0)dt \qquad x_2 = x_1 + f(x_1)dt \quad \ldots \quad \boxed{x_{n+1} = x_n + f(x_n)dt}$$

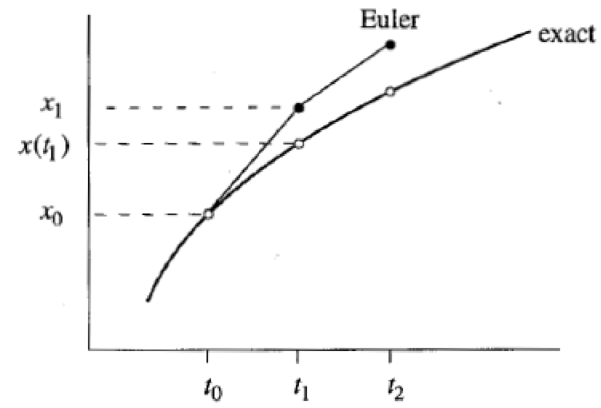
Exercise 2: test the accuracy of the Euler formula

$$\boxed{x_{n+1} = x_n + f(x_n)dt} \qquad t_n = t_0 + ndt$$

- Integrate *analytically* $\dot{x} = -x$ with the initial condition x(0)=1. Which is the value of x(1)?
- Use Euler formula with dt=1 to estimate x(1).
- Repeat for dt=0.1, 0.01, 0.001, 0.0001, complete the table, and plot (in log-log scale) the error vs. dt

Improved (second order) Euler method

$$\dot{x} = f(x) \qquad \boxed{x_{n+1} = x_n + f(x_n)dt}$$



Instead of using the derivative only at the *left end* of the time interval between t_n and t_{n+1}, we use an "*average*" derivative:

$$\tilde{x}_{n+1} = x_n + f(x_n)\Delta t \qquad \text{(the trial step)}$$

$$x_{n+1} = x_n + \tfrac{1}{2}\left[f(x_n) + f(\tilde{x}_{n+1})\right]\Delta t\,. \qquad \text{(the real step)}$$

This is the basis of the "Runge Kutta" method.

“Runge Kutta” method

$$y_{n+1} = y_n + \frac{1}{6} h \left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$t_{n+1} = t_n + h$$

$$\begin{aligned}
k_1 &= f(t_n, y_n), \\
k_2 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right), \\
k_3 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right), \\
k_4 &= f\left(t_n + h, y_n + hk_3\right).
\end{aligned}$$

Source: Wikipedia

Outline

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- Introduction to flows on the line
- Solving equations with computer
- **Fixed points and linear stability**

Example

$$\dot{x} = \sin x$$

Analytical Solution: $t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|$

- Starting from $x_0 = \pi/4$, what is the long-term behavior (what happens when $t \to \infty$?)

- And for any arbitrary condition x_o?

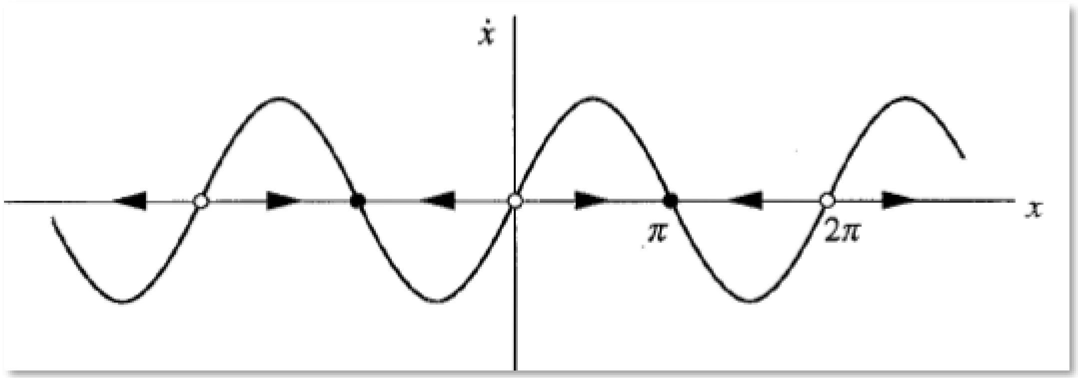
- We look at the “phase portrait”: geometrically, picture of all possible trajectories (without solving the ODE analytically).

- Imagine: x is the position of an imaginary particle restricted to move in the line, and dx/dt is its velocity.

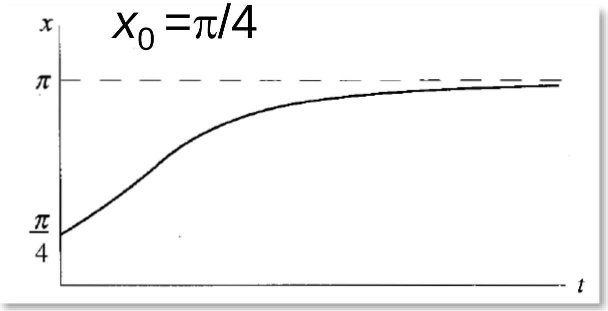
Imaginary particle moving in the horizontal axis

$\dot{x} = \sin x$

Flow to the right when $\dot{x} > 0$
Flow to the left when $\dot{x} < 0$

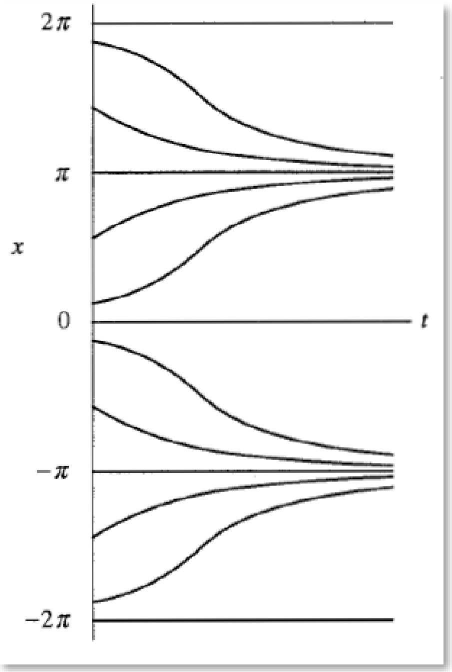


$x_0 = \pi/4$

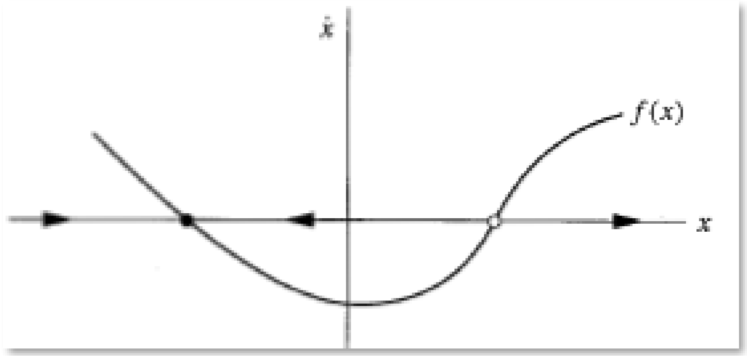


$\dot{x} = 0$ "Fixed points"

x_0 arbitrary



Two types of FPs: stable & unstable



Fixed points

$$\dot{x} = f(x) \qquad f(x^*) = 0$$

$x = x^*$ initially, then $x(t) = x^*$ for all time

Fixed points = equilibrium solutions

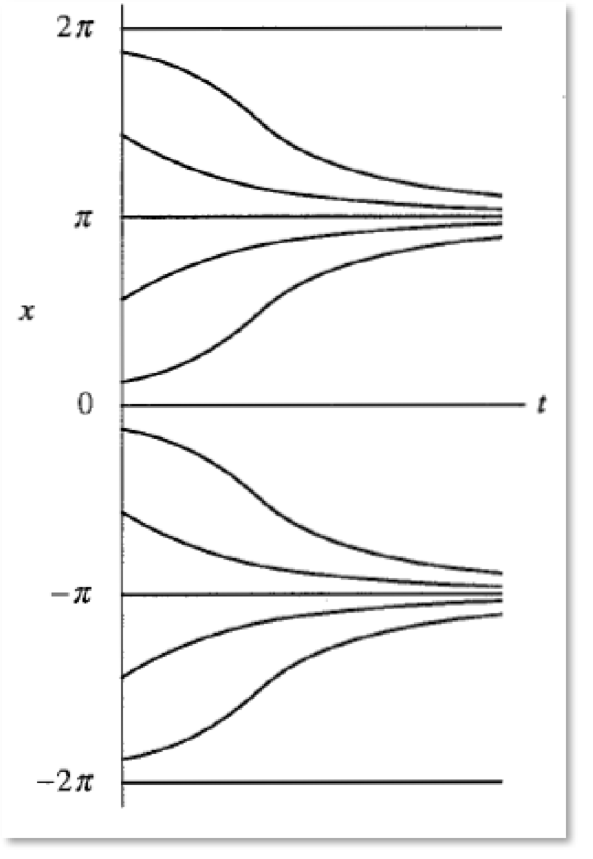
- Stable (attractor or sink): nearby trajectories are attracted

  π and $-\pi$

- Unstable: nearby trajectories are repelled

  0 and $\pm 2\pi$

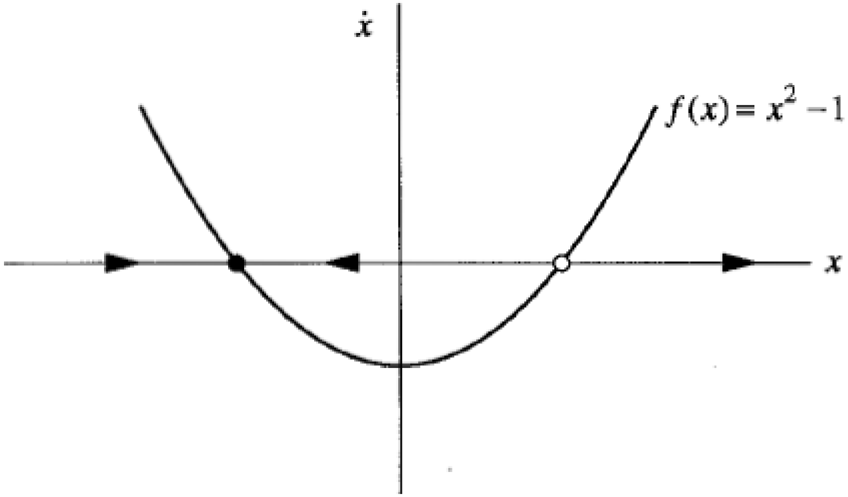
$$\dot{x} = \sin x$$



Example 1

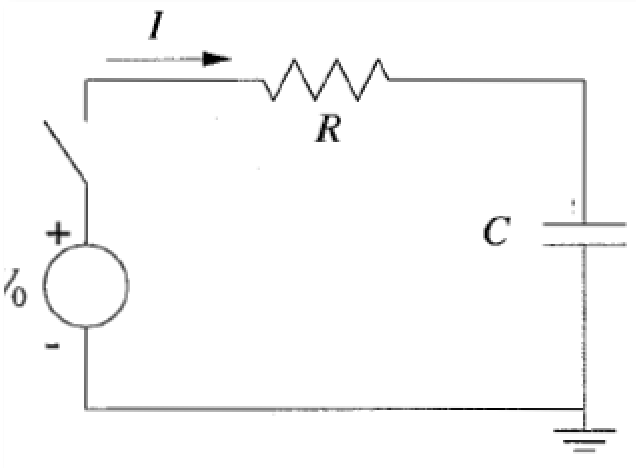
$$\dot{x} = x^2 - 1$$

- Find the fixed points and classify their stability



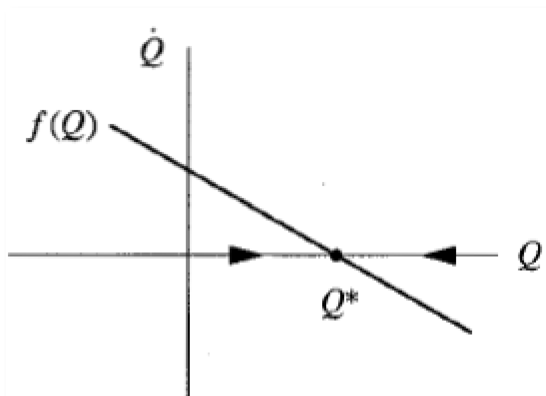
$x^* = -1$ is stable, and $x^* = 1$ is unstable

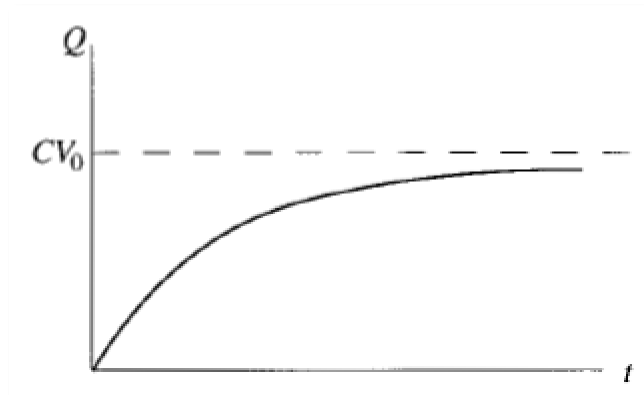
Example 2



$$-V_0 + R\dot{Q} + Q/C = 0$$

$$\dot{Q} = f(Q) = \frac{V_0}{R} - \frac{Q}{RC}$$





Example 3: population model for single species (e.g. bacteria)

- *N*(t): size of the population of the species at time t

$$\frac{dN}{dt} = \text{births} - \text{deaths} + \text{migration}$$

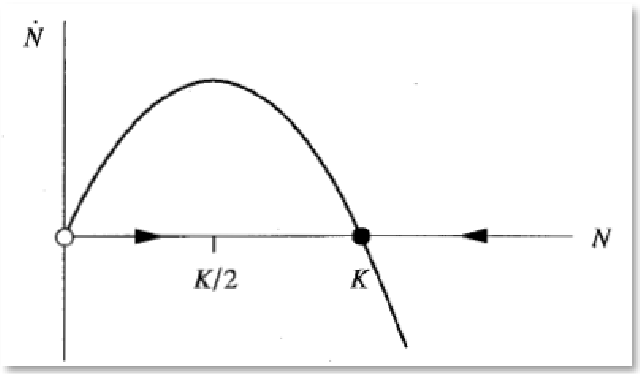
- Simplest model (Thomas Malthus 1798): no migration, births and deaths are proportional to the size of the population

$$\frac{dN}{dt} = bN - dN \quad \Rightarrow \quad N(t) = N_0 e^{(b-d)t}$$

Exponential grow!

More realistic model: the logistic equation

- To account for limited food (Verhulst 1838): $\dot{N} = rN\left(1 - \frac{N}{K}\right)$

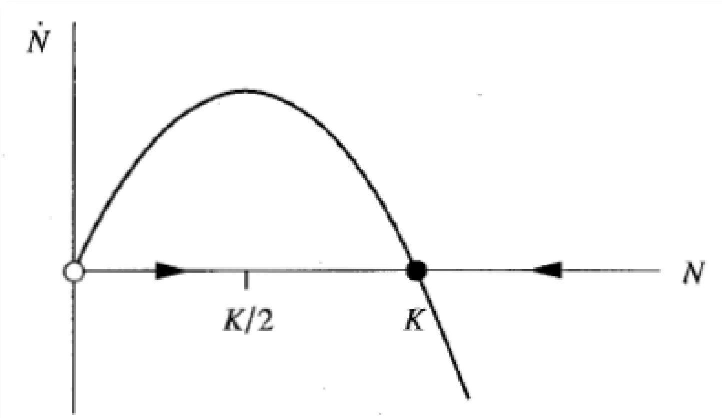


- If *N*>*K* the population decreases
- If *N*<*K* the population increases

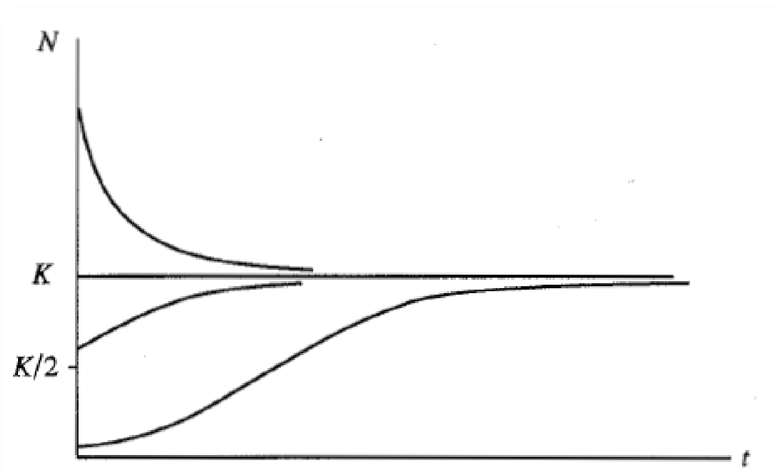
- K = “carrying capacity”

- The carrying capacity of a biological species in an environment is the maximum population size of the species that the environment can sustain indefinitely, given the food, habitat, water, etc.

How does a population approaches the carrying capacity?



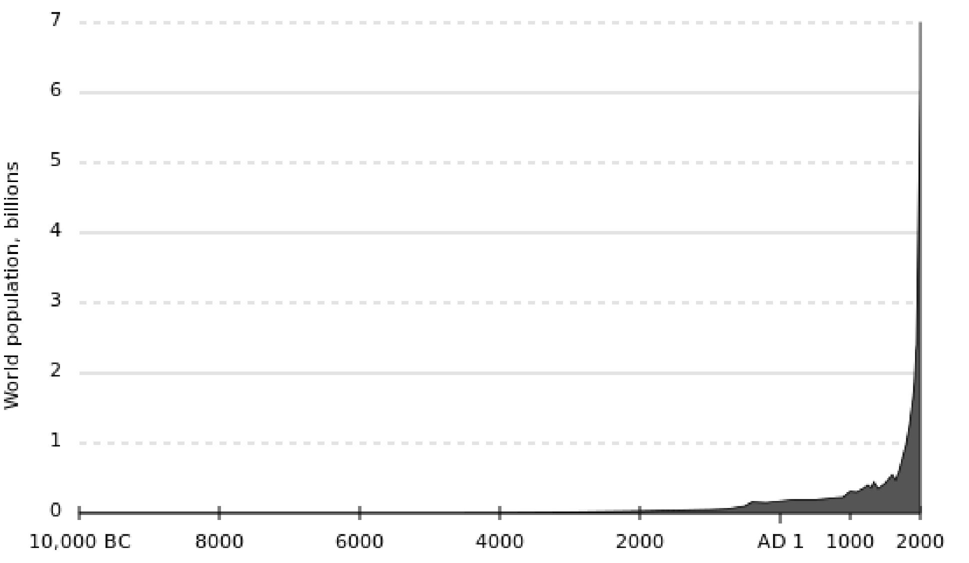
$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

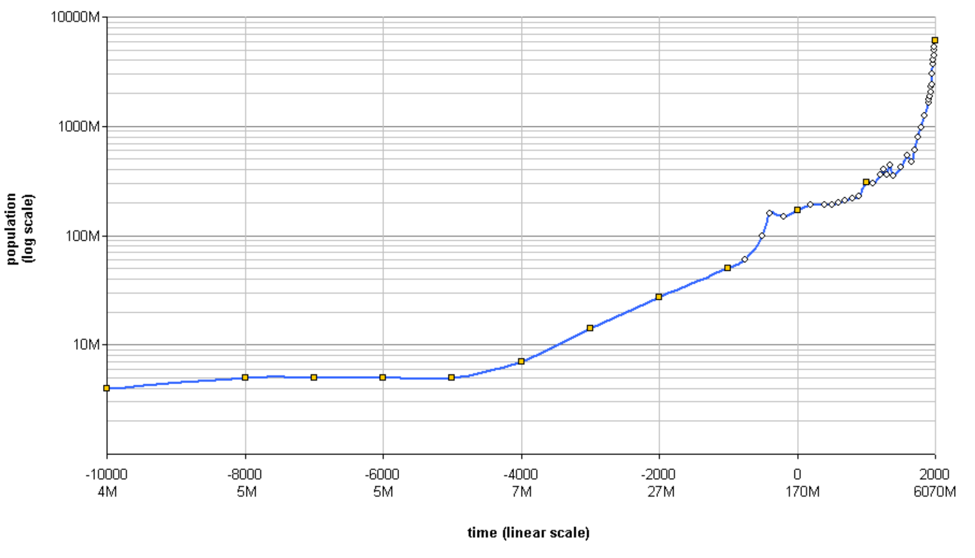


- Exponential or sigmoid approach.
- Good model only for simple organisms that live in constant environments.

How is the evolution of the human population?

Hyperbolic grow !
 Technological advance
 → increase in the carrying capacity of land for people
→ demographic growth
→ more people
→ more potential inventors
→ acceleration of technological advance
→ accelerating growth of the carrying capacity…





Source: wikipedia

Linearization near a fixed point

$\dot{x} = f(x)$ $\quad f(x^*) = 0$ $\quad \eta(t) = x(t) - x^*$ $\quad \eta$ = tiny perturbation

$$\dot{\eta} = \tfrac{d}{dt}(x - x^*) = \dot{x}$$

$$\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$$

Taylor expansion $\quad f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2)$

$f(x^*) = 0$ $\quad \boxed{\dot{\eta} = \eta f'(x^*) + O(\eta^2)}$

The slope *f'(x*)* at the fixed point determines the stability

$f'(x^*) > 0$ the perturbation η grows exponentially

$f'(x^*) < 0$ the perturbation η decays exponentially

$f'(x^*) = 0$ Second-order terms can not be neglected and a nonlinear stability analysis is needed.

Bifurcation (more latter)

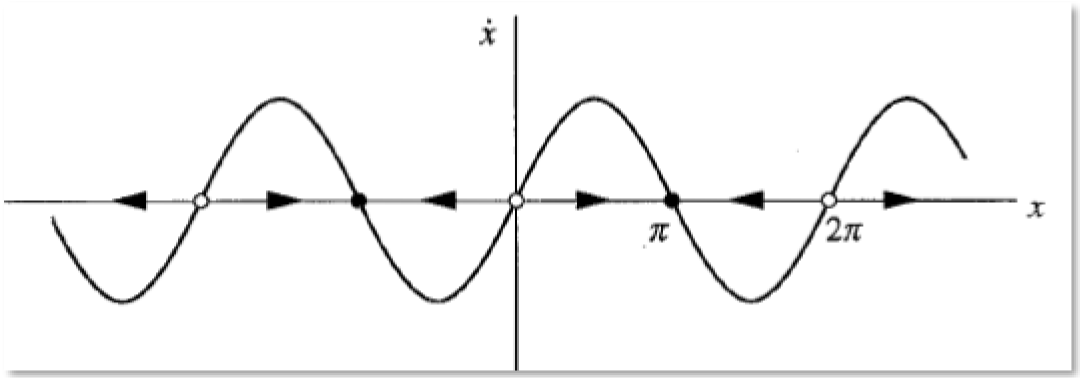
$1/|f'(x^*)|$ Characteristic time-scale

Example 1

- Linear stability of the fixed points of $\dot{x} = \sin x$

$$x^* = k\pi$$

$$f'(x^*) = \cos k\pi = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd}. \end{cases}$$



- Stable: π and $-\pi$
- Unstable: 0, $\pm 2\pi$

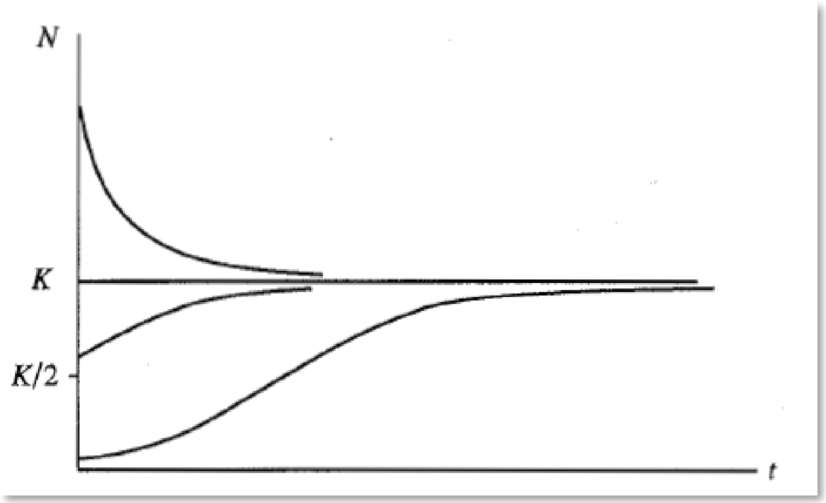
Example 2

- Logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

$$N^* = 0 \text{ and } N^* = K$$

$$f'(0) = r \text{ and } f'(K) = -r \quad \Rightarrow \quad \begin{array}{l} N^* = 0 \text{ is unstable} \\ N^* = K \text{ is stable} \end{array}$$

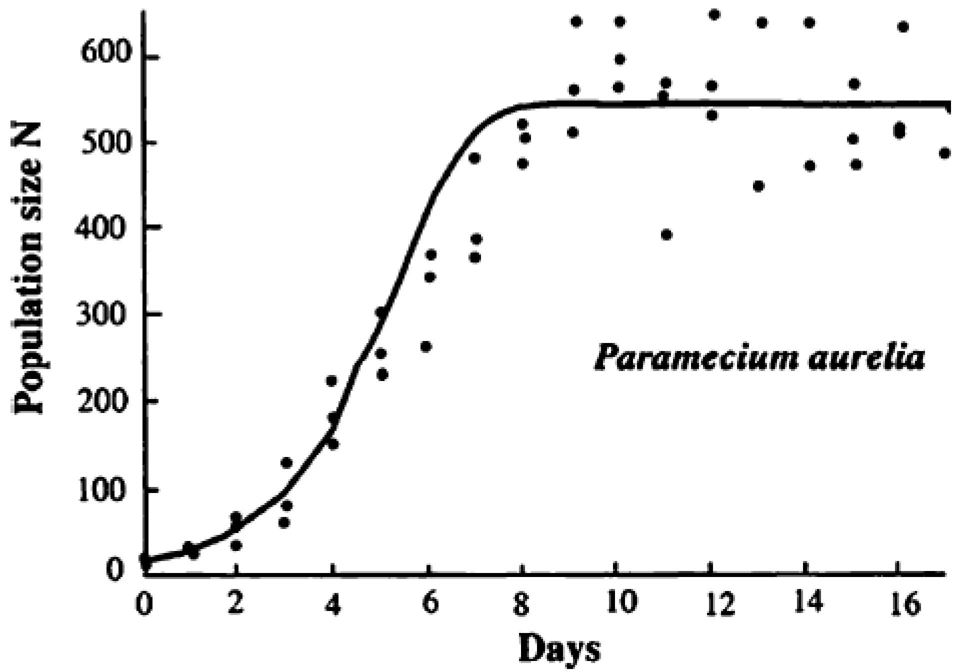


The two fixed points have the same characteristic time-scale:

$$1/|f'(N^*)| = 1/r$$

Good agreement with controlled population experiments

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

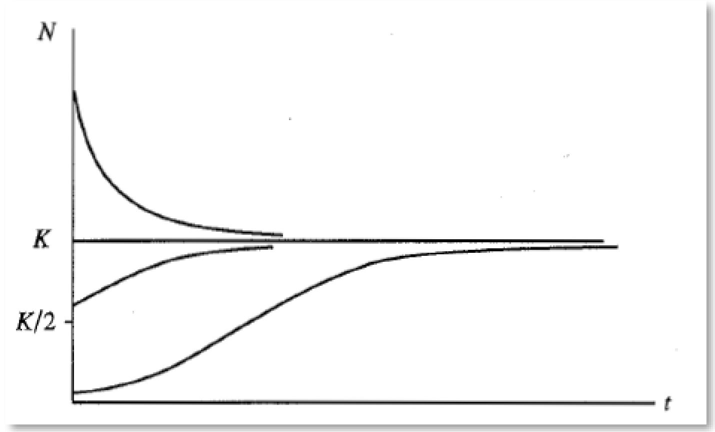


The population growth of the protozoan *Paramecium* in test tubes is a typical example (Figure 1.5). Under the conditions of the experiment, the population stopped growing when there were about 552 individuals per 0.5 ml. The time points show some scatter, which is caused both by the difficulty in accurately measuring population size (only a subsample of the population is counted) and by environmental variations over time and between replicate test tubes. A linear regression of the data N'/N versus N gives $r = 0.99$ and $K = 552$.

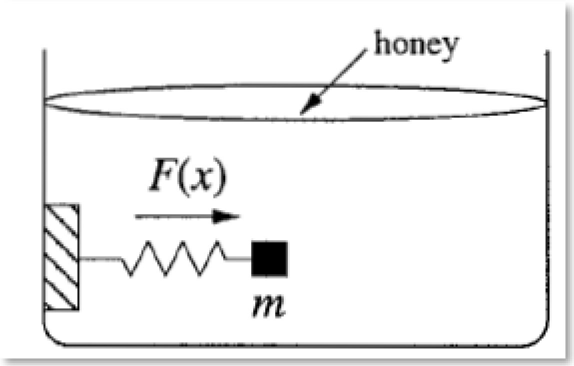
Lack of oscillations

$$\dot{x} = f(x)$$

General observation: only sigmoidal or exponential behavior, the approach is monotonic, **no oscillations**



Analogy:



$$m\ddot{x} + b\dot{x} = F(x)$$

Strong damping (over damped limit) $b\dot{x} >> m\ddot{x}$

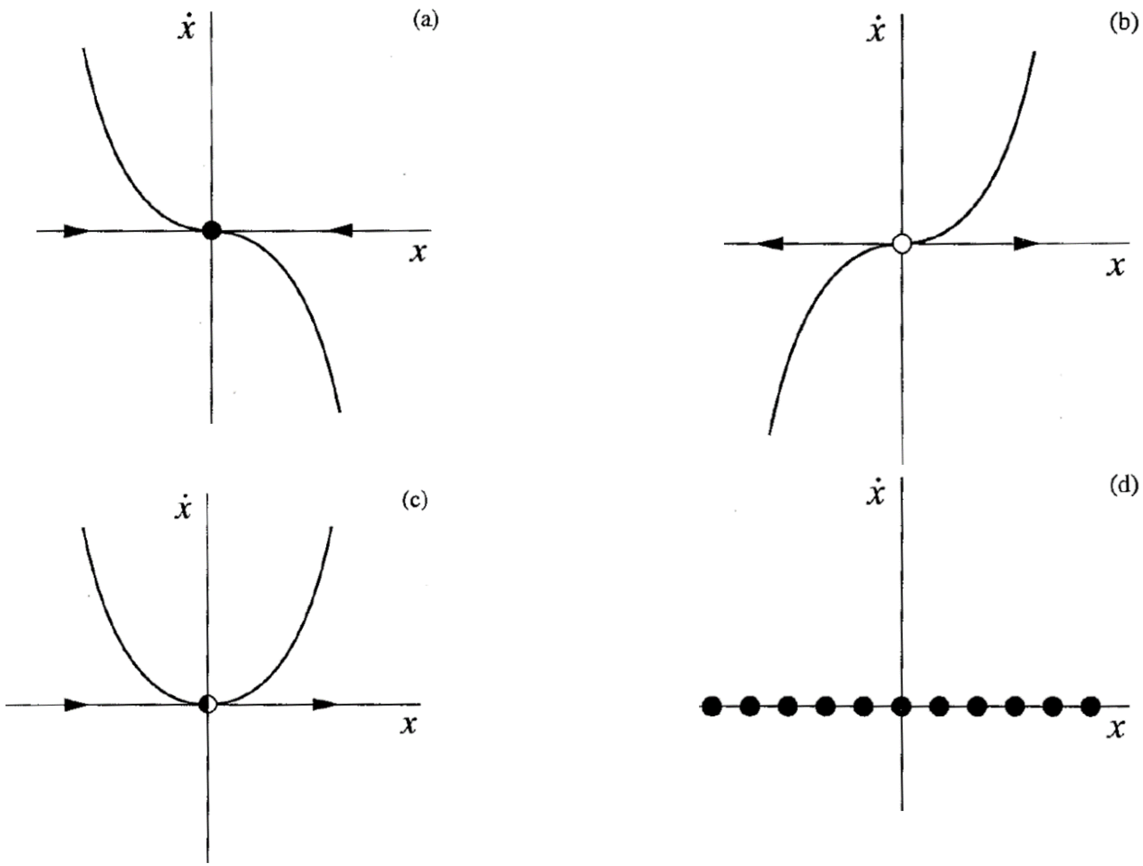
$$b\dot{x} = F(x)$$

To observe oscillations we need to keep the second derivative (weak damping).

Stability of the fixed point x* when f '(x*)=0?

(a) $\dot{x} = -x^3$ (b) $\dot{x} = x^3$ (c) $\dot{x} = x^2$ (d) $\dot{x} = 0$

In all these systems: $x^* = 0$ with $f'(x^*) = 0$

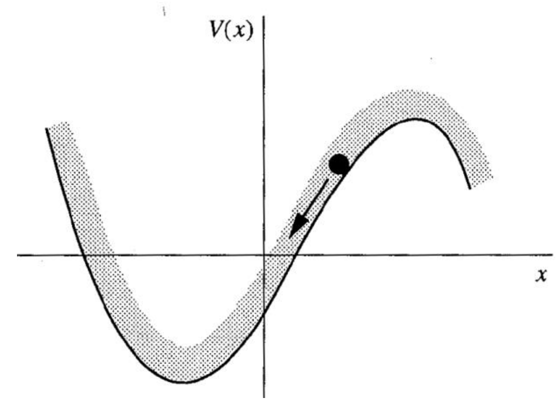


When f'(x*) = 0 nothing can be concluded from the linearization but these plots allow to see what goes on.

Potentials

$$\dot{x} = f(x) \qquad f(x) = -\frac{dV}{dx} \qquad \frac{dx}{dt} = -\frac{dV}{dx}$$

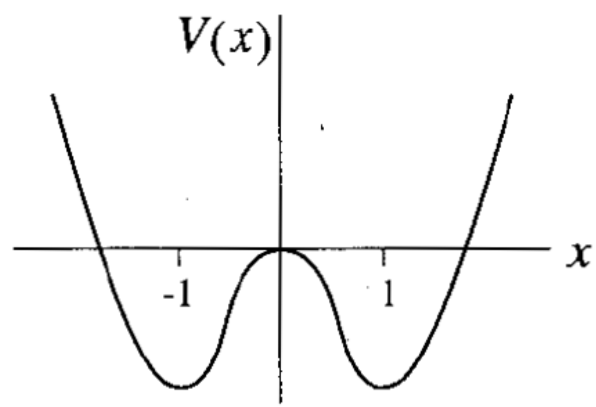
$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} \qquad \frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0$$



V(t) decreases along the trajectory.

- Example: $\dot{x} = x - x^3$

$$V = -\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 + C$$



Two stable fixed points: x=1; x=-1 *(Bistability)*.

Summary

- Flows on the line = first-order ODE: $dx/dt = f(x)$

- Fixed point solutions: $f(x^*) = 0$
  - stable if $f'(x^*) < 0$
  - unstable if $f'(x^*) > 0$
  - neutral (bifurcation point) if $f'(x^*) = 0$

- There are no transient oscillations and there are no periodic solutions; the approach to a fixed point is monotonic (sigmoidal or exponential).

Exercise 3: $$\frac{dN}{dt} = -aN\ln(bN)$$

a) Calculate the fixed points and their linear stability.

b) Demonstrate that when ***a=b*** the analytical solution of $\frac{dN}{dt} = -bN\ln(bN)$ is

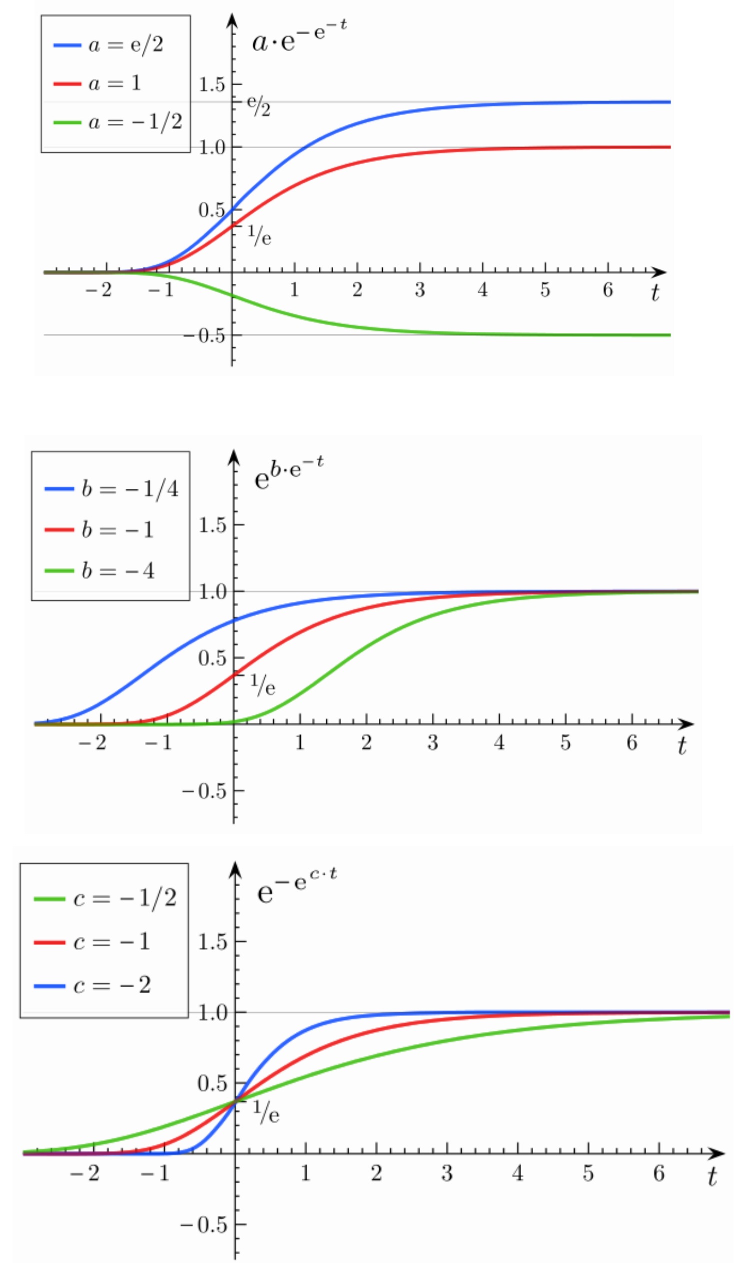
$$N(t) = \frac{e^{ce^{bt}}}{b}$$ where ***c*** is a constant that depends on the initial condition.

c) Integrate the equation numerically and compare the numerical and analytical solutions.

The solution of the previous exercise is a particular case of the Gompertz function:

$$x(t) = ae^{be^{ct}}$$

- Is a mathematical model for a time series: a sigmoid function that describes a process whose growth is slowest at the start (t=-∞) and at the end of the process.
- It describes:
  - How tumors grow.
  - The sales of mobile phones (costs are initially high, then there is a period of rapid growth, followed by saturation).



Source: Wikipedia

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