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On the decay of the energy for radial solutions in Moore-Gibson-Thompson thermoelasticity

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Abstract

In this paper we consider the thermoelastic theory of Moore-Gibson-Thompson. We restrict our attention to radially symmetric solutions and we prove the exponential decay with respect the time variable. We show this fact with the help of the energy arguments. Later we give some numerical simulations to see the behaviour.

Keywords

Moore-Gibson-Thompson thermoelasticity, radial symmetry, energy decay, numerical simulations

Introduction

Moore-Gibson-Thompson (MGT) equation has received much interest in the recent ten years from the mathematical point of view Conejero et al. (2015); Dell'Oro et al. (2016, 2017a); Dell'Oro and Pata (2017b); Kaltenbacher et al. (2011); Lasiecka and Wang (2015, 2016); Marchand et al. (2012); Ostoja-Starzewski and Quintanilla (2020); Pellicer and Said-Houari (2017); Pellicer and Sola-Morales (2019). It was proposed in the context of the fluid dynamics, but in a recent work Quintanilla (2019) the author proposed this equation in the context of the heat conduction. The basic idea to see this equation to describe the evolution of the heat is to introduce a relaxation parameter to the type III Green-Naghdi heat equation Green and Naghdi (1992, 1993). That is, as well as the Cattaneo-Maxwell heat conduction equation can be seen after the introduction of the relaxation parameter for the usual Fourier law the Moore-Gibson-Thompson equation can be obtained if we use the same idea in the case of the type III theory. Once time we have the evolution for the temperature we can obtain a thermoelastic theory as it was proposed in Quintanilla (2019). It is worth saying that we can also obtain the system of the MGT-system of thermoelasticity as a particular case of the theory proposed by Gurtin Gurtin (1972), as it can be seen in Bazarra et al. (2020a); Conti et al. (2020c). It is worth saying that this thermoelastic theory has received much attention in the last two years (see Bazarra et al. (2020a,b, 2021a); Conti et al. (2020a,b,c); Fernández and Quintanilla (2021); Jangid and Mukhopadhyay (2020a,b); Kumar and Mukhopadhyay (2020); Ostoja-Starzewski and Quintanilla (2020); Pellicer and Quintanilla (2020); Quintanilla (2020), among others).

It is well known that in the case of the thermoelastic theories one can obtain (in many cases) the exponential decay for the one dimensional theory. However, when the dimension is greater than one we cannot expect this kind of behavior. Nevertheless there exists a subclass of thermoelastic deformations where we can obtain the exponential decay for dimension greater than one. We can see that the decay of solutions can be controlled by an exponential function in the case of the radial solutions. This kind of solutions can be found when the geometry of the body is radially symmetric and examples of these domains are the balls centered at the origin as well as the circular crowns. In this paper we are dealing with radially symmetric thermoelastic solutions. It is worth saying that it has been obtained the exponential decay of radially symmetric solutions for several thermoelastic theories Jiang et al. (1998); Quintanilla and Racke (2003). Our aim in this paper is to see a similar behaviour for the MGT-thermoelastic theory. We use the energy method to prove this fact and we also give several numerical simulations to this behavior.

In the next section we recall the problem we want to study and provide several equations and properties of radially symmetric solutions. In section three we prove the exponential decay of the radially symmetric solutions and in section four we give several numerical simulations.

Basic equations

In this paper we will work in the context of the Moore-Gibson-Thompson thermoelasticity when the material is isotropic and homogeneous. In this case the system of field equations are given by (see Quintanilla (2019) for details)

$$\rho\ddot{u}_i = \mu u_{i,jj} + (\lambda+\mu)u_{j,ji} - \beta(\theta_{,i} + \tau\dot{\theta}_{,i}), \qquad (1)$$

$$c(\tau\ddot{\theta} + \dot{\theta}) = \kappa^*\Delta\alpha + \kappa\Delta\theta - \beta\dot{u}_{r,r}, \qquad (2)$$

In this system we have that u_i is the displacement vector, α is the thermal displacement that satisfies that $\dot{\alpha} = \theta$ is the

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temperature. We know that λ and μ are the usual elasticity constants, ρ is the mass density, κ is the thermal conductivity, κ^* is the rate conductivity, c is the thermal capacity and β is related with the thermal expansion. We will study this system in a bounded domain B with the boundary smooth enough to apply the divergence theorem. As we assume that the material is homogeneous all these parameters are constants.

To propose a well posed problem it is needed to consider initial and boundary conditions. concerning initial conditions we assume that, for all $\boldsymbol{x} \in B$,

$$\begin{aligned} &u_i(\boldsymbol{x},0) = u_i^0(\boldsymbol{x}),\ \dot{u}_i(\boldsymbol{x},0) = v_i^0(\boldsymbol{x}), \\ &\alpha(\boldsymbol{x},0) = \alpha^0(\boldsymbol{x}),\ \dot{\alpha}(\boldsymbol{x},0) = \theta^0(\boldsymbol{x}),\ \ddot{\alpha}(\boldsymbol{x},0) = \zeta^0(\boldsymbol{x}), \end{aligned} \tag{3}$$

and the boundary conditions we impose are

$$u_i(\boldsymbol{x},t) = \frac{\partial \alpha}{\partial \boldsymbol{n}}(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in \partial B, \tag{4}$$

where $\boldsymbol{n}$ is the normal vector to the boundary of B.

To study this system it is usual to assume that

$$\begin{aligned} &\rho > 0,\ \mu > 0,\ \lambda + 2\mu > 0,\ c > 0,\ \kappa^* > 0, \\ &\tau > 0,\ \bar{\kappa} = \kappa - \tau\kappa^* > 0. \end{aligned} \tag{5}$$

It is known that there exist solutions to the problem determined by the system with the boundary and initial conditions proposed previously. To be precise the solutions are generated by a contractive semigroup of linear operators determined in the Hilbert space $\mathbf{W}_0^{1,2} \times \mathbf{L}^2 \times W_*^{1,2} \times W_*^{1,2} \times L_*^2$, where $\mathbf{W}_0^{1,2}$ and $\mathbf{L}^2$ are the usual Sobolev spaces and

$$L_*^2 = \{f \in L^2, \int_B f dv = 0\} \quad \text{and } W_*^{1,2} = W^{1,2} \cap L_*^2$$

However, in this paper we restrict our attention to domains B with a radial symmetry. Furthermore, we assume that the solutions are of the form $u_i(\boldsymbol{x},,t) = x_i U(r,t),\ \ \alpha(\boldsymbol{x},,t) = \alpha(r,t)$ where $r^2 = (x_1^2 + x_2^2)$ for two dimensional domains and $r^2 = (x_1^2 + x_2^2 + x_3^2)$ in the case of three dimensional domains. It is worth noting that in this situation we have that $u_{i,j}(x,t) = u_{j,i}(x,t)$. Therefore, the first equation of our system can be written in an easier way. In fact, it is also satisfied the equality

$$\int_B u_{i,i} u_{j,j} dv = \int_B u_{i,j} u_{i,j} dv,$$

whenever the Dirichlet homogeneous boundary condition is imposed.

Decay of solutions

In this section we prove the exponential decay of the radial solutions to the problem involving the Moore-Gibson-Thompson thermoelasticity theory in the case that the constitutive parameters satisfy the previous conditions proposed. Moreover, to prove the exponential decay we also need to impose that $\beta \neq 0$. In the analysis proposed in this section we assume that $\beta > 0$, but at the end of this section we explain how to adapt the analysis to the case $\beta < 0$.

To this end, we first define the functions:

$$\begin{aligned} E_1(t) = \frac{1}{2}\int_B \Big(&\rho \dot{u}_i \dot{u}_i + (\lambda + 2\mu) u_{i,j} u_{i,j} + c(\tau\dot{\theta} + \theta)^2 \\ &+ \kappa^* |\nabla(\alpha + \tau\theta)|^2 + \tau\overline{\kappa}|\nabla\theta|^2\Big)\, dv, \\ E_2(t) = \frac{1}{2}\int_B \Big(&\rho \ddot{u}_i \ddot{u}_i + (\lambda + 2\mu) \dot{u}_{i,j} \dot{u}_{i,j} + c(\tau\ddot{\theta} + \dot{\theta})^2 \\ &+ \kappa^* |\nabla(\theta + \tau\dot{\theta})|^2 + \tau\overline{\kappa}|\nabla\dot{\theta}|^2\Big)\, dv, \\ E_3(t) = \frac{1}{2}\int_B \Big(&\rho \dot{u}_{i,j} \dot{u}_{i,j} + (\lambda + 2\mu) u_{i,jj} u_{i,kk} \\ &+ c|\nabla(\tau\dot{\theta} + \theta)|^2 + \kappa^* |\Delta(\alpha + \tau\theta)|^2 + \tau\overline{\kappa}|\Delta\theta|^2\Big)\, dv. \end{aligned}$$

We have that

$$\begin{aligned} E_1'(t) &= -\overline{\kappa}\int_B |\nabla\theta|^2\, dv, \\ E_2'(t) &= -\overline{\kappa}\int_B |\nabla\dot{\theta}|^2\, dv, \\ E_3'(t) &= -\overline{\kappa}\int_B |\Delta\theta|^2\, dv. \end{aligned}$$

Now, we consider several auxiliary functions $F_1(t)$ and $F_2(t)$ defined as

$$\begin{aligned} F_1(t) &= \int_B \rho u_i \dot{u}_i\, dv, \\ F_2(t) &= \int_B \rho u_{i,i} \dot{u}_{j,j}\, dv, \end{aligned}$$

and so, we have

$$\begin{aligned} F_1'(t) = &-\int_B \Big[(\lambda + 2\mu) u_{i,j} u_{i,j} + \beta(\theta_{,i} + \tau\dot{\theta}_{,i}) u_i\Big]\, dv \\ &+ \int_B \rho \dot{u}_i \dot{u}_i\, dv, \\ F_2'(t) = &-\int_B \Big[(\lambda + 2\mu) u_{i,jj} u_{i,kk} + \beta(\theta_{,i} + \tau\dot{\theta}_{,i}) u_{i,jj}\Big]\, dv \\ &+ \int_B \rho \dot{u}_{i,i} \dot{u}_{j,j}\, dv. \end{aligned}$$

We also consider the function

$$G(t) = -\int_B c(\tau\ddot{\theta} + \dot{\theta})\dot{\theta}\, dv.$$

It follows that

$$\begin{aligned} G'(t) \quad &= -\int_B c(\tau\dddot{\theta} + \ddot{\theta})\dot{\theta}\, dv - c\tau\int_B |\ddot{\theta}|^2\, dv - c\int_B \dot{\theta}\ddot{\theta}\, dv \\ &= -\int_B (\kappa^*\Delta\theta + \kappa\Delta\dot{\theta} - \beta\ddot{u}_{r,r})\dot{\theta}\, dv - c\tau\int_B |\ddot{\theta}|^2\, dv \\ &\quad -c\int_B \dot{\theta}\ddot{\theta}\, dv \\ &= \int_B \Big[\kappa^*\nabla\theta\,\nabla\dot{\theta} + \kappa|\nabla\dot{\theta}|^2 - \beta\ddot{u}_r\dot{\theta}_{,r}\Big]\, dv - c\tau\int_B |\ddot{\theta}|^2\, dv \\ &\quad -c\int_B \dot{\theta}\ddot{\theta}\, dv \\ &= \int_B \Big[\kappa^*\nabla\theta\,\nabla\dot{\theta} + \kappa|\nabla\dot{\theta}|^2\Big]\, dv - c\tau\int_B |\ddot{\theta}|^2\, dv \\ &\quad -c\int_B \dot{\theta}\ddot{\theta}\, dv \\ &\quad -\frac{\beta}{\rho}\int_B \Big[(\lambda + 2\mu)u_{i,jj} - (\beta\theta_{,i} + \beta\tau\dot{\theta}_{,i})\Big]\dot{\theta}_{,i}\, dv. \end{aligned}$$

Now, we define the functions

$$\begin{aligned} H(t) &= \int_B c(\tau\dot{\theta} + \theta)\dot{u}_{r,r}\, dv, \\ J(t) &= -\int_B \kappa^*\Delta\alpha u_{j,j}\, dv, \end{aligned}$$

and we obtain

$$H'(t) = \int_B c(\tau\ddot{\theta} + \dot{\theta})\dot{u}_{r,r}\, dv + \int_B c(\tau\dot{\theta} + \theta)\ddot{u}_{r,r}\, dv$$
$$= \int_B (\kappa^*\Delta\alpha + \kappa\Delta\theta - \beta\dot{u}_{i,i})\dot{u}_{r,r}\, dv$$
$$- \int_B c(\tau\dot{\theta}_{,r} + \theta_{,r})\ddot{u}_r\, dv$$
$$= -\beta \int_B \dot{u}_{i,j}\dot{u}_{i,j}\, dv + \int_B (\kappa^*\Delta\alpha + \kappa\Delta\theta)\dot{u}_{j,j}\, dv$$
$$-\frac{c}{\rho}\int_B (\tau\dot{\theta}_{,r} + \theta_{,r})\Big[(\lambda + 2\mu)u_{r,jj} - \beta(\theta_{,r} + \tau\dot{\theta}_{,r})\Big]\, dv,$$
$$J'(t) = -\int_B \kappa^*\Delta\theta\, u_{j,j}\, dv - \int_B \kappa^*\Delta\alpha\, \dot{u}_{j,j}\, dv.$$

Therefore, it follows that

$$H'(t) + J'(t) = -\beta \int_B \dot{u}_{i,j}\dot{u}_{i,j}\, dv + \kappa \int_B \Delta\theta\, \dot{u}_{j,j}\, dv$$
$$-\kappa^* \int_B \Delta\theta\, u_{j,j}\, dv + \frac{c\beta}{\rho}\int_B |\nabla(\theta + \tau\dot{\theta})|^2\, dv$$
$$-\frac{c(\lambda + 2\mu)}{\rho}\int_B (\tau\dot{\theta}_{,r} + \theta_{,r})u_{r,jj}\, dv.$$

If we consider λ^* large enough we see that

$$F_1'(t) + F_2'(t) + G'(t) + \lambda^*(H'(t) + J'(t))$$
$$= -(\lambda + 2\mu)\int_B [u_{i,j}u_{i,j} + u_{i,jj}u_{i,kk}]\, dv$$
$$-\int_B c\tau|\ddot{\theta}|^2\, dv - \frac{\lambda^*\beta}{\rho}\int_B \dot{u}_{i,j}\dot{u}_{i,j}\, dv$$
$$+\int_B (\rho\dot{u}_i\dot{u}_i + \rho\dot{u}_{i,j}\dot{u}_{i,j})\, dv + \kappa^*\int_B \nabla\theta\,\nabla\dot{\theta}\, dv$$
$$+\beta\int_B (\theta_{,i} + \tau\dot{\theta}_{,i})(u_i + u_{i,jj})\, dv$$
$$+\kappa\int_B |\nabla\dot{\theta}|^2\, dv - c\int_B \dot{\theta}\ddot{\theta}\, dv$$
$$-\frac{\beta}{\rho}\int_B (\lambda + 2\mu)u_{i,jj}\dot{\theta}_{,i}\, dv$$
$$+\frac{\beta^2}{\rho}\int_B \theta_{,i}\dot{\theta}_{,i}\, dv + \frac{\beta^2\tau}{\rho}\int_B |\nabla\dot{\theta}|^2\, dv$$
$$+\frac{\lambda^*\kappa}{\rho}\int_B \Delta\theta\, \dot{u}_{j,j}\, dv - \frac{\lambda^*\kappa^*}{\rho}\int_B \Delta\theta\, u_{j,j}\, dv$$
$$-\frac{\lambda^* c(\lambda + 2\mu)}{\rho}\int_B (\tau\dot{\theta}_{,r} + \theta_{,r})u_{r,jj}\, dv$$
$$+\frac{\lambda^* c\beta}{\rho}\int_B |\nabla(\theta + \tau\dot{\theta})|^2\, dv.$$

Now, we consider

$$\mathcal{E}(t) = E_1(t) + E_2(t) + E_3(t)$$
$$+\varepsilon(F_1(t) + F_2(t) + G(t) + \lambda^*(H(t) + J(t))),$$

where $\varepsilon > 0$ is assumed sufficiently small, but positive and λ^* large enough. Thus, we have

$$\mathcal{E}'(t) = -C(\varepsilon)\int_B \Big(|\nabla\theta|^2 + |\nabla\dot{\theta}|^2 + |\Delta\theta|^2 + u_{i,j}u_{i,j}$$
$$+u_{i,jj}u_{i,kk} + |\ddot{\theta}|^2 + \dot{u}_{i,j}\dot{u}_{i,j}\Big)\, dv,$$

where $C(\varepsilon)$ is a positive constant.

We also note that

$$\int_B \ddot{u}_i\ddot{u}_i\, dv \le K_1\int_B (u_{i,jj}u_{i,kk} + |\nabla\theta|^2 + |\nabla\dot{\theta}|^2)\, dv$$

and

$$\int_B |\Delta\alpha|^2\, dv \le K_2\int_B \Big(|\dot{\theta}|^2 + |\ddot{\theta}|^2 + |\dot{u}_{i,i}\dot{u}_{j,j}| + |\Delta\theta|^2\Big)\, dv,$$

where K_1 and K_2 are positive constants.

Therefore, we also obtain

$$\mathcal{E}'(t) = -C^*(\varepsilon)\int_B \Big(|\nabla\theta|^2 + |\nabla\dot{\theta}|^2 + |\Delta\alpha|^2 + |\Delta\theta|^2$$
$$+|\ddot{\theta}|^2 + u_{i,j}u_{i,j} + \dot{u}_{i,j}\dot{u}_{i,j} + u_{i,jj}u_{i,kk}\Big)\, dv,$$

where $C^*(\varepsilon)$ is a positive constant.

It is clear that $\mathcal{E}(t)$ is equivalent to $E(t) = E_1(t) + E_2(t) + E_3(t)$ whenever ϵ is small enough, but positive. We can obtain a constant η such that

$$\mathcal{E}'(t) + \eta E(t) \le 0 \quad \text{for } t \ge 0.$$

Then, it follows the existence of two positive constants η_1 and η_2 such that

$$E(t) \le \eta_1 E(0)e^{-\eta_2 t},$$

which is the exponential decay estimate we wanted to obtain.

The analysis proposed in this section uses that β is positive. In the case that β is negative we can follow a similar argument, but the main difference is that the parameter λ^* should be selected negative, but with absolute value sufficiently large.

Numerical results: discrete energy decay

In this final section, we will shown some numerical simulations for the solutions of the previously studied thermomechanical problem in the case that we have a two-dimensional ball with radius 1. We also assume that the time interval is fixed as $[0, T]$, where T denotes the final time. Moreover, in order to simplify the writing, we do not indicate the dependence on the spatial variable $\boldsymbol{x}$.

By using boundary conditions (4), defining the velocity $\boldsymbol{v} = (v_i)_{i=1}^2 = \dot{\boldsymbol{u}} = (\dot{u}_i)_{i=1}^2$ and the thermal velocity $\zeta = \dot{\theta}$ system (1)-(2) leads to the following variational formulation, for a.e. $t \in [0, T]$ and $\boldsymbol{w} \in V$ $\xi \in E$,

$$\rho(\dot{\boldsymbol{v}}(t), \boldsymbol{w})_H + (\lambda + \mu)(\operatorname{div}\boldsymbol{u}(t), \operatorname{div}\boldsymbol{w})_Y$$
$$+\mu(\nabla\boldsymbol{u}(t), \nabla\boldsymbol{w})_Q = -\beta(\nabla(\theta(t) + \tau\zeta(t)), \boldsymbol{w})_H,$$
$$c(\tau\dot{\zeta}(t) + \zeta(t), \xi)_Y + \kappa(\nabla\theta(t), \nabla\xi)_H$$
$$= -\kappa^*(\nabla\alpha(t), \nabla\xi)_H - \beta(\operatorname{div}\boldsymbol{v}(t), \xi)_Y,$$

where $\boldsymbol{v}(0) = \boldsymbol{v}^0$, $\boldsymbol{u}(0) = \boldsymbol{u}^0$, $\alpha(0) = \alpha^0$, $\theta(0) = \theta^0$ and $\zeta(0) = \zeta^0$, the variational spaces E and V are given by

$$E = \{\xi \in H^1(B) \quad ; \quad \frac{\partial\xi}{\partial\boldsymbol{n}} = 0 \quad \text{on} \quad \partial B\},$$
$$V = \{(w_i) \in [H^1(B)]^2 \,;\, w_i = 0 \text{ on } \partial B \text{ for } i = 1, 2\},$$

and we used the notations $Y = L^2(B)$, $H = [L^2(B)]^2$ and $Q = [L^2(B)]^{2\times 2}$. Here, the displacements, the thermal displacement and the temperature are defined as

$$u_i(t) = \int_0^t v_i(s)\, ds + u_i^0, \quad \theta(t) = \int_0^t \zeta(s)\, ds + \theta^0,$$
$$\alpha(t) = \int_0^t \theta(s)\, ds + \alpha^0.$$

For the spatial approximation of the above problem, we assume the domain $\overline{B}$ is polyhedral and we denote by $\mathcal{T}^h$ a regular triangulation in the sense of Ciarlet (1991). Thus, we construct the finite dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by

$$\begin{aligned} E^h &= \{\xi^h \in C(\overline{B})\ ;\ \xi^h_{|Tr} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h\},\\ V^h &= \{w_i^h \in C(\overline{B})\ ;\ w^h_{i_{|Tr}} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h,\\ &\qquad w_i^h = 0 \text{ on } \partial B,\ \text{for } i = 1, 2\}, \end{aligned}$$

where $P_1(Tr)$ represents the space of polynomials of degree less or equal to one in the element Tr, i.e. the finite element spaces V^h and E^h are composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by u_i^{0h}, v_i^{0h}, α^{0h}, θ^{0h} and ζ^{0h}, are given by

$$\begin{aligned} &u_i^{0h} = \mathcal{P}^h_{V^h} u_i^0, \quad v_i^{0h} = \mathcal{P}^h_{V^h} v_i^0, \quad \alpha^{0h} = \mathcal{P}^h_{E^h}\alpha^0,\\ &\theta^{0h} = \mathcal{P}^h_{E^h}\theta^0, \quad \zeta^{0h} = \mathcal{P}^h_{E^h}\zeta^0, \end{aligned}$$

where $\mathcal{P}^h_{V^h}$ and $\mathcal{P}^h_{E^h}$ are the classical finite element interpolation operators over V^h and E^h, respectively (see, e.g., Ciarlet (1991)).

In order to provide the time discretization of the previously given variational problem, we consider a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$, with constant step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \ldots, N$.

Therefore, using a combination of forward and backward Euler schemes in time, the fully discrete approximation is the following.

Find the discrete velocity field $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete thermal velocity $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset E^h$ such that $\boldsymbol{v}_0^{hk} = \boldsymbol{v}^{0h}$, $\zeta_0^{hk} = \zeta^{0h}$, and, for $n = 1, \ldots, N$ and for all $\boldsymbol{w}^h \in V^h$ and $\xi^h \in E^h$,

$$\begin{aligned} &\rho(\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + (\lambda + \mu)k^2(\operatorname{div} \boldsymbol{v}_n^{hk}, \operatorname{div} \boldsymbol{w}^h)_Y\\ &\quad + \mu k^2(\nabla \boldsymbol{v}_n^{hk}, \nabla \boldsymbol{w}^h)_Q\\ &\quad = \rho(\boldsymbol{v}_{n-1}^{hk}, \boldsymbol{w}^h)_H - \beta k(\nabla(\theta_{n-1}^{hk} + \tau\zeta_{n-1}^{hk}), \boldsymbol{w}^h)_H\\ &\quad - (\lambda + \mu)k(\operatorname{div} \boldsymbol{u}_{n-1}^{hk}, \operatorname{div} \boldsymbol{w}^h)_Y\\ &\quad - \mu k(\nabla \boldsymbol{u}_{n-1}^{hk}, \nabla \boldsymbol{w}^h)_Q,\\ &c(\tau\zeta_n^{hk} + k\zeta_n^{hk}, \xi^h)_Y + \kappa k^2(\nabla\zeta_n^{hk}, \nabla\xi^h)_H\\ &\quad = c(\tau\zeta_{n-1}^{hk}, \xi^h)_Y - \kappa^* k(\nabla\alpha_{n-1}^{hk}, \nabla\xi^h)_H\\ &\quad - \kappa k(\nabla\theta_{n-1}^{hk}, \nabla\xi^h)_H - \beta k(\operatorname{div} \boldsymbol{v}_{n-1}^{hk}, \xi^h)_Y, \end{aligned}$$

where the discrete displacement field $\boldsymbol{u}_n^{hk}$, the discrete thermal displacement α_n^{hk} and the discrete temperature θ_n^{hk} are updated from the following equations:

$$\begin{aligned} &\boldsymbol{u}_n^{hk} = k\sum_{j=1}^n \boldsymbol{v}_j^{hk} + \boldsymbol{u}^{0h}, \quad \alpha_n^{hk} = k\sum_{j=1}^n \theta_j^{hk} + \alpha^{0h},\\ &\theta_n^{hk} = k\sum_{j=1}^n \zeta_j^{hk} + \theta^{0h}. \end{aligned}$$

We note that the above fully discrete problem leads to a uncoupled linear system which is solved by using Cholesky's method. Moreover, thanks to conditions (5), the existence of a unique discrete solution is guaranteed.

Our aim in this section is to show numerically the asymptotic discrete energy behavior. Therefore, we use the following data:

$$\begin{aligned} &T = 50, \quad \rho = 0.1, \quad \mu = 1, \quad \lambda = 5, \quad \beta = 1,\\ &c = 1, \quad \kappa = 1, \quad \kappa^* = 1, \end{aligned}$$

and the initial conditions, for all $(x, y) \in \partial B$,

$$\begin{aligned} &\alpha^0(x, y) = \theta^0(x, y) = \zeta^0(x, y) = 0,\\ &\boldsymbol{u}^0(x, y) = \boldsymbol{v}^0(x, y) = (1 - x^2 - y^2, 1 - x^2 - y^2). \end{aligned}$$

Taking the discretization parameter $k = 0.001$ and using the finite element mesh shown in Figure 1, for some values of the relaxation parameter τ we plot in Figure 2 the evolution in time, in both natural and semi-log scales, of the discrete energy E_n^{hk} given by

$$\begin{aligned} E_n^{hk} = \frac{1}{2}\int_B \Big(&\rho(\boldsymbol{v}_n^{hk})_i(\boldsymbol{v}_n^{hk})_i + (\lambda + 2\mu)(\boldsymbol{u}_n^{hk})_{i,j}(\boldsymbol{u}_n^{hk})_{i,j}\\ &+ c(\tau\zeta_n^{hk} + \theta_n^{hk})^2 + \kappa^*|\nabla(\alpha_n^{hk} + \tau\theta_n^{hk})|^2\\ &+ \tau\overline{\kappa}|\nabla\theta_n^{hk}|^2\Big)\,dv. \end{aligned}$$

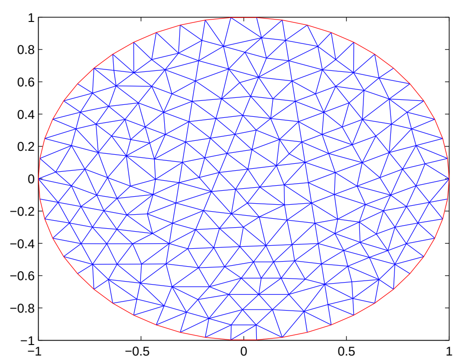

Figure 1. Finite element mesh.

As can be seen, in all cases the discrete energy tends to zero and the theoretical exponential asymptotic behavior seems to be achieved.

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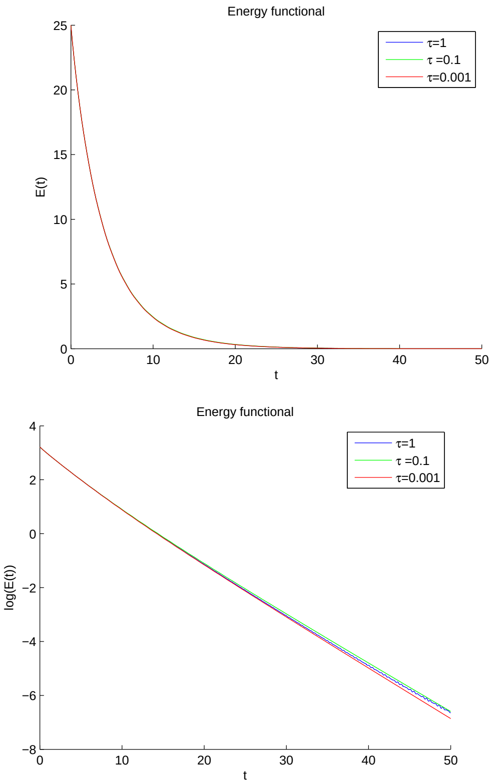

Figure 2. Evolution in time of the discrete energy (natural and semi-log scales).

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