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Eigenvalues with respect to a weight for general boundary value problems on networks

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Abstract In this work we analyze self-adjoint boundary value problems on networks for Schrödinger operators, in which a part of the boundary with a Neumann condition is always considered. We first characterize when the energy is positive semi-definite on the space of functions satisfying the null boundary conditions. To do this, the fundamental tools are the Doob transform and the discrete version of the trace function. Then, we raise eigenvalue problems with respect to a weight for general boundary value problems and we prove the discrete version of the Mercer Theorem. Finally, we apply the obtained results to a Dirichlet-Robin boundary value problem on a star-like network.

Keywords Schrödinger operators · Eigenvalues · Green operators · Positive semi-definitess · Discrete trace · Mercer Theorem

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1 Introduction

The spectral theory for the Poisson equation, for the Dirichlet problem or even for the Neuman problem on networks, has been widely analized in the literature for the combinatorial Laplacian or for the normalized Laplacian, see [13–15, 17, 20, 23, 24]. In addition, the study of eigenvalues of the Poisson equation for

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Schrödinger operators has been treated in [9]. In most of the above mentioned works no weights are considered, except for the case of the Neumann eigenvalue where the weight is the degree of the vertices, [13]. Moreover, recently B. Hua *et. alt.* ([21,22]) have considered eigenvalue Robin type problems for the probabilistic Laplacian in the context of Cheeger estimates.

In this work we consider general self-adjoint boundary value problems in which there always exists a part of the boundary with Neumann or Robin condition, since the other boundary conditions have already been considered, see for instance [1]. Another novelty of the present work is the consideration of Schrödinger operators with arbitrary potential. Moreover, we raise the eigenvalue problems with respect to general weights on the vertex set. Therefore, our work includes the usual case where the weight is given by the degree. This kind of problems represents the discrete version of self-adjoint boundary value problems for second order elliptic differential operators in divergence form on a compact Riemannian manifold with boundary and can be formulated as

$$\mathcal{L}_q(u) = f \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = g \quad \text{on} \quad F_{N} \qquad \text{and} \qquad u = h \quad \text{on} \quad F_{D}.$$

In the discrete setting, the above mixed Dirichlet-Robin boundary value problem can be interpreted as the linear system whose coefficient matrix is irreducible and has the following block structure

$$\mathsf{A} = \begin{pmatrix} \mathsf{L} & -\mathsf{C} \\ -\mathsf{C}^T & \mathsf{D} \end{pmatrix}$$

where D is a diagonal matrix with positive diagonal entries, $\mathsf{C} \geq 0$ and L is a symmetric Z-matrix . Therefore, the results here obtained could be applied in this matrix context. From the probabilistic point of view the presence of a mixed boundary condition corresponds to a random walk with both absorbent states and reflecting barriers.

This manuscript is organized as follows. We first define the discrete version of the trace function between Sobolev spaces, which allows us to consider null Robin boundary conditions and to reduce the dimension of the problem by incorporating the boundary conditions to the Schrödinger operator. This technique generalizes the methods used in [16,21,22], where the Neumann problems on subgraphs were reduced to problems of Laplacians on graphs without boundary; that is, Poisson equations.

Following the guidelines provided by the differential case, we characterize when the Energy is positive semi-definite on the subspace of functions that vanish the boundary conditions. For that, the description of admissible potentials throughout Doob potentials that also verify a discrete Poincaré type inequality is essential. As a consequence we get the discrete version of the variational characterization of the solution; that is, the Dirichlet Principle.

In Section 4 we define the different resolvent kernels associated with the boundary value problem and we show their relationship through the addition of suitable projectors. Among the kernels, we emphasize the so-called Green kernel whose associated matrix is the group inverse of the matrix associated with the boundary value problem. As $\mathcal{L}_q$ is a Hilbert-Schmith operator, we prove through the spectral theorem the existence of an orthonormal basis of eigenfunctions with respect to an arbitrary weight and hence, we manage to prove the corresponding Mercer Theorem.

We end the paper with the application of the obtained results to the case of a star-like network. In particular, we compute the eigenvalues and eigenfunctions for a Dirichlet-Robin boundary value problem on a star-like network and we give explicit expressions of the Green operators and of the inverse operator in the regular case.

2 Preliminaries

In this section we briefly present the main definitions and results about linear difference operators on networks that will be useful in the rest of the paper, see [10] for a more detailed description.

Given a nonempty finite set V, the set of real valued functions on V, respectively the set of non-negative functions, is denoted by $\mathcal{C}(V)$, respectively by $\mathcal{C}^+(V)$, and for any $x \in V$, $\varepsilon_x \in \mathcal{C}(V)$ stands for the Dirac function at x. If $u \in \mathcal{C}(V)$, its *support* is given by $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$. Moreover, if F is a nonempty subset of V, $|F|$ denotes its cardinal, $\chi_{_F}$ its characteristic function and we consider the sets $\mathcal{C}(F) = \{u \in \mathcal{C}(V) : \mathsf{supp}(u) \subset F\}$ and $\mathcal{C}^+(F) = \mathcal{C}(F) \cap \mathcal{C}^+(V)$. We call *weight on* F any function $\sigma \in \mathcal{C}^+(F)$ such that $\mathsf{supp}(\sigma) = F$ and denote by $\Omega(F)$ the set of weights on F. Alternatively, when $u \in \mathcal{C}(V)$ the notation $u \geq 0$ on F means that $u\chi_{_F} \in \mathcal{C}^+(F)$, whereas the notation $u > 0$ on F means that $u\chi_{_F} \in \Omega(F)$.

For any $u \in \mathcal{C}(F)$, we define the *integral of* u as

$$\int_F u = \int_F u(z)dz = \sum_{z \in V} u(z).$$

Given $\sigma \in \Omega(F)$, the map $\langle \cdot, \cdot \rangle_\sigma : \mathcal{C}(V) \times \mathcal{C}(V) \longrightarrow \mathbb{R}$ defined as

$$\langle u, v \rangle_\sigma = \int_F uv\sigma$$

is bilinear and positive semi-definite. The associated seminorm is denoted by $||\cdot||_\sigma$ and clearly $||u||_\sigma = 0$ iff $u = 0$ on F. Therefore, $\langle \cdot, \cdot \rangle_\sigma$ determines an inner product on $\mathcal{C}(F)$. When $\sigma = \chi_{_F}$, then $\langle \cdot, \cdot \rangle_\sigma$ and $||\cdot||_\sigma$ are denoted by $\langle \cdot, \cdot \rangle_{_F}$ and $||\cdot||_{_F}$, respectively. In the sequel we consider $\Omega_{_F}$ the set of normalized weights defined as $\Omega_{_F} = \left\{\sigma \in \Omega(F) : \displaystyle\int_F \sigma = |F|\right\}$. Therefore, $\chi_{_F}$ is the unique constant weight belonging to $\Omega_{_F}$.

Any function $K : V \times V \longrightarrow \mathbb{R}$ is called *a kernel on* V and we denote by $\mathcal{C}(V \times V)$ the space of kernels. Moreover, given two nonempty subsets $F, H \subseteq V$, $\mathcal{C}(F \times H)$ denotes the set of kernels $K \in \mathcal{C}(V \times V)$ such that

$K(x,y) = 0$ if $(x,y) \notin F \times H$. If we label the elements of F and H, then each function $u \in \mathcal{C}(H)$ can be identified with a vector order $|H|$ given by $\mathsf{u} = \big(u(x)\big)_{x \in H}$ and each kernel on $F \times H$ can be identify with the matrix of order $|F| \cdot |H|$ given by $\mathsf{K} = \big(K(x,y)\big)_{\substack{x \in F \\ y \in H}}$. Throughout this paper we strongly use these identifications.

If $K \in \mathcal{C}(F \times H)$ is a kernel, for each $x \in F$ and any $y \in H$ we denote by K^x and K_y the functions of $\mathcal{C}(F)$ and $\mathcal{C}(H)$ defined respectively by $K^x(y) = K_y(x) = K(x,y)$. Moreover, we define *the homomorphism associated with* K as $\mathcal{K} \colon \mathcal{C}(H) \longrightarrow \mathcal{C}(F)$ that assigns to each $f \in \mathcal{C}(H)$, the function $\mathcal{K}(f)(x) = \displaystyle\int_H K(x,y)\, f(y)\, dy$ for all $x \in V$. Notice that $K_y = \mathcal{K}(\varepsilon_y)$ for any $y \in H$.

An important example of kernels in $\mathcal{C}(F \times H)$ are those associated with *projectors*. Specifically, given $u \in \mathcal{C}(F)$, $v \in \mathcal{C}(H)$, we call *projector on* u *along* v the kernel $u \otimes v$, defined as $(u \otimes v)(x,y) = u(x)v(y)$ for any $x, y \in V$. Notice that $K \in \mathcal{C}(F \times H)$ is a projector iff its associated matrix has rank 1 and then its associated homomorphism is $\mathcal{K}(f) = \langle v, f \rangle_H u$.

The relationship between kernels, integral operators and endomorphisms is given by the following result. Its first part can be seen as a discrete version of *Schwartz's Kernel Theorem*, because of the natural identification between $\mathcal{C}(F)$ and its dual space, see [3, Proposition 5.1].

Proposition 1 (Kernel Theorem) *Each endomorphism of* $\mathcal{C}(F)$ *is an integral operator associated with a kernel on* F *which is uniquely determined. Moreover, if* $\mathcal{K}$ *is an integral operator on* F *and* K *is its associated kernel, then* $\mathcal{K}$ *is self-adjoint on* $\mathcal{C}(F)$*; that is,* $\langle \mathcal{K}(u), v \rangle_F = \langle \mathcal{K}(v), u \rangle_F$ *for any* $u, v \in \mathcal{C}(F)$ *iff* K *is symmetric, that is* $K(x,y) = K(y,x)$*, for all* $x, y \in F$*. Moreover, if* $\emptyset \neq A \subseteq F$*, then* $\mathsf{Img}\, \mathcal{K} \subset \mathcal{C}(A)$ *iff* $K \in \mathcal{C}(A \times F)$ *and* $\mathcal{C}(F \setminus A) \subset \mathsf{ker}\, \mathcal{K}$ *iff* $K \in \mathcal{C}(F \times A)$.

The symmetric kernels $K \in \mathcal{C}(F \times F)$ such that $K(x,y) \leq 0$ for $x \neq y$ play a fundamental role in many areas of Applied Mathematics. After labeling F, they can be identified with symmetric Z-matrices of order $|F|$, see [8]. Moreover if for $t \in \mathbb{R}$ we consider $K_t = K + tI$, where $I(x,y) = 0$ for $x \neq y$ and $I(x,x) = 1$, then the matrix identified with K_t is positive definite for t large enough and hence it is an M-matrix. Taking into account the properties of irreducible M-matrices, we have the following key result, see [8, Th. 4.16] for its proof.

Lemma 1 *Consider* $K \in \mathcal{C}(F \times F)$ *a symmetric kernel such that* $K(x,y) \leq 0$ *for* $x \neq y$ *and such that the matrix of order* $|F|$ *identified with it is irreducible. If* $\mathcal{K} \colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F)$ *is the endomorphism determined by* K*, then there exists* $\omega \in \Omega(F)$ *and* $\alpha \in \mathbb{R}$ *such that* $\mathcal{K}(\omega) = \alpha\omega$.

In the framework of Discrete Mathematics it is usual to introduce the kernels with the properties mentioned in the above lemma as the main operators on a network. More specifically, if we consider a symmetric function $c \colon V \times V \longrightarrow [0, +\infty)$ satisfying that $c(x,x) = 0$, then the pair $\Gamma = (V, c)$ is called network, where c is named the *conductance* of the network and we say that vertex x is *adjacent* to vertex y iff $c(x,y)>0$. We always assume that Γ is *connected.*

The function $\kappa \in \mathcal{C}^+(V)$ defined as $\kappa(x) = \sum\limits_{y\in V} c(x,y)$ for any $x\in V$ is called the *degree of* Γ. The connectivity assumption on V implies that $\kappa \in \Omega(V)$.

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u\in\mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y\in V} c(x,y)\,\big(u(x)-u(y)\big), \quad x\in V.$$

Given $q\in\mathcal{C}(V)$, the *Schrödinger operator* on Γ with *potential* q is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u\in\mathcal{C}(V)$ the function $\mathcal{L}_q(u)=\mathcal{L}(u)+qu$. The kernel associated with $\mathcal{L}_q$ is

$$L_q(x,y) = \begin{cases} -c(x,y), & y\neq x,\\ \kappa(x)+q(x), & y=x\end{cases}$$

and hence, if L_q is the matrix identified with it, then L_q is an irreducible symmetric Z-matrix. Conversely, each irreducible symmetric Z-matrix of order $|V|$ can be identified with a Schrödinger operator on a connected network whose vertex set is V. This identification has been strongly used by the authors, see [5].

Given $F\subseteq V$, the *boundary of* F is the set $\delta(F)=\big\{z\in V\setminus F : c(z,y)>0 \text{ for some } y\in F\big\}$ and the *closure of* F is the set $\bar{F}=F\cup\delta(F)$. Clearly, F is a proper subset; that is, $\emptyset\neq F\subset V$, iff $\delta(F)\neq\emptyset$ and then, the *boundary degree of* F is $\kappa_{_F}\in\Omega\big(\delta(F)\big)$ defined as $\kappa_{_F}(x)=\sum\limits_{y\in F} c(x,y)$ for any $x\in\delta(F)$.

It is easy to prove that when F is connected if $F\subseteq H\subseteq\bar{F}$, then H is also a connected set.

If F is a proper subset of V, for each $u\in\mathcal{C}(\bar{F})$ we define the *normal derivative of* u *on* F as the function in $\mathcal{C}(\delta(F))$ given by

$$\frac{\partial u}{\partial \mathsf{n}_{_F}}(x) = \sum_{y\in F} c(x,y)\big(u(x)-u(y)\big) = \kappa_{_F}(x)u(x) - \sum_{y\in F} c(x,y)u(y),$$

for any $x\in\delta(F)$.

The *normal derivative on* F is the operator $\dfrac{\partial}{\partial\mathsf{n}_{_F}} : \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(\delta(F))$ that assigns to any $u\in\mathcal{C}(\bar{F})$, its normal derivative on F.

Given $F\subset V$ a proper subset we consider $c_{_F}=c\cdot\chi_{_{(\bar{F}\times\bar{F})\setminus(\delta(F)\times\delta(F))}}$. Clearly, if $u\in\mathcal{C}(\bar{F})$, then

$$\mathcal{L}(u)(x) = \sum_{y\in\bar{F}} c_{_F}(x,y)\big(u(x)-u(y)\big),\ x\in F,$$
$$\frac{\partial u}{\partial\mathsf{n}_{_F}}(x) = \sum_{y\in\bar{F}} c_{_F}(x,y)\big(u(x)-u(y)\big),\ x\in\delta(F).$$

Therefore, if we define $\Gamma(F)$ as the network whose vertex set is $\bar{F}$ and whose conductance is c_F and we consider $\mathcal{L}^F$ its combinatorial Laplacian, then for any $u \in \mathcal{C}(\bar{F})$ we have that $\mathcal{L}^F(u) = \mathcal{L}(u)$ on F, whereas $\mathcal{L}^F(u) = \dfrac{\partial u}{\partial \mathsf{n_F}}$ on $\delta(F)$. In particular, $\kappa_F(x)$ is the degree of $x \in \delta(F)$ in this new network.

The above identities allow us to show that the relation between the values of the Schrödinger operators on F and the values of the normal derivative at $\delta(F)$ is given by the *First Green Identity*, see for instance [3,4]:

$$\begin{aligned}\int_F v\,\mathcal{L}_q(u) =& \frac{1}{2}\int_{\bar{F}}\int_{\bar{F}} c_F(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,dxdy\\ &+\int_F quv - \int_{\delta(F)} v\,\frac{\partial u}{\partial \mathsf{n_F}},\end{aligned}$$

for any $u,v \in \mathcal{C}(\bar{F})$. A direct consequence of the above identity is the so-called *Second Green Identity*

$$\int_F \Big(v\,\mathcal{L}_q(u) - u\,\mathcal{L}_q(v)\Big) = \int_{\delta(F)}\left(u\,\frac{\partial v}{\partial \mathsf{n_F}} - v\,\frac{\partial u}{\partial \mathsf{n_F}}\right), \quad \text{for any } u,v \in \mathcal{C}(\bar{F}).$$

We define the *energy associated with F and q* as the symmetric bilinear form $\mathcal{E}_q^F : \mathcal{C}(\bar{F}) \times \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ given for any $u,v \in \mathcal{C}(\bar{F})$ by

$$\mathcal{E}_q^F(u,v) = \frac{1}{2}\int_{\bar{F}}\int_{\bar{F}} c_F(x,y)\,\big(u(x)-u(y)\big)\,\big(v(x)-v(y)\big)\,dx\,dy + \int_{\bar{F}} q\,u\,v.$$

Clearly to evaluate the energy on the subspace $\mathcal{C}(\bar{F})$, the values of the potential q on $V \setminus \bar{F}$ are irrelevant and hence we can suppose, without loss of generality, that $q \in \mathcal{C}(\bar{F})$. From the First Green Identity, for any $u,v \in \mathcal{C}(\bar{F})$ we get that

$$\mathcal{E}_q^F(u,v) = \int_F v\mathcal{L}_q(u) + \int_{\delta(F)} v\,\left[\frac{\partial u}{\partial \mathsf{n_F}} + qu\right]. \tag{1}$$

3 Discrete trace function and admissible potentials

In this section we study mixed boundary value problems for Schrödinger operators. Specifically, we are interested in obtaining necessary and sufficient conditions for the existence and uniqueness of solution of such a problems.

Throughout the paper, we consider $F \subset V$ a proper subset, $\emptyset \neq F_N \subset \delta(F)$ and $F_D = \delta(F) \setminus F_N$; that is, $\delta(F) = F_D \cup F_N$ is a partition of $\delta(F)$, where F_D can be the empty set. In addition, we always assume that $F \cup F_N$ is a connected subset on the network $\Gamma(F)$; that is, it is connected with respect to the conductance c_F. Then, we define the *outer degree of F, with respect to F_D*, as the function $p_F \in \mathcal{C}(F)$ given by

$$p_F(x) = \sum_{y \in F_D} c(x,y) = \sum_{y \in F_D} c_F(x,y) \quad \text{for any } x \in F. \tag{2}$$

Therefore, $p_{_F} \in \mathcal{C}^+(F)$ and moreover $p_{_F} = 0$ when $F_{_D} = \emptyset$, whereas $\emptyset \neq \mathsf{supp}(p_{_F}) \subseteq \delta(V \setminus F)$ when $F_{_D} \neq \emptyset$.

Our aim is to study self-adjoint boundary value problems associated with the Schrödinger operator with potential $q \in \mathcal{C}(F \cup F_{_N})$. Specifically, for any $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(F_{_N})$ and $h \in \mathcal{C}(F_{_D})$ the *boundary value problem on F with data f, g, h*, BVP in the sequel, consists on finding $u \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(u) = f \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = g \quad \text{on} \quad F_{_N} \qquad \text{and} \qquad u = h \quad \text{on} \quad F_{_D}. \tag{3}$$

Any $u \in \mathcal{C}(\bar{F})$ satisfying the above identities is called *solution of the BVP*. When $g = 0$ and $h = 0$, then Problem (3) becomes

$$\mathcal{L}_q(u) = f \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = 0 \quad \text{on} \quad F_{_N} \qquad \text{and} \qquad u = 0 \quad \text{on} \quad F_{_D} \tag{4}$$

and it is called *semi-homogeneous boundary value problem with data f*. The *associated homogeneous boundary value problem* consists in finding $u \in \mathcal{C}(\bar{F})$ such that $\mathcal{L}_q(u) = 0$ on F, $\dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = 0$ on $F_{_N}$ and $u = 0$ on $F_{_D}$. The space of solutions of the homogeneous boundary value problem is denoted by $\mathcal{V}_q^H$, whereas the subspace of $\mathcal{C}(\bar{F})$ formed by the functions vanishing the boundary conditions is denoted by $\mathcal{V}_q$. Therefore,

$$\mathcal{V}_q = \left\{ u \in \mathcal{C}(\bar{F}) : \frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = 0 \text{ on } F_{_N} \text{ and } u = 0 \text{ on } F_{_D} \right\}$$

and clearly, we have $\mathcal{V}_q^H \subset \mathcal{V}_q \subset \mathcal{C}(F \cup F_{_N})$. The boundary value problem is called *regular* when $\mathcal{V}_q^H = \{0\}$. Observe that for any $u \in \mathcal{C}(F \cup F_{_N})$, we have that

$$\mathcal{E}_q^F(u,u) = \frac{1}{2} \int_{F \cup F_N} \int_{F \cup F_N} c_{_F}(x,y) \big(u(x) - u(y)\big)^2 dx\, dy + \int_{F \cup F_N} (q + p_{_F})\, u^2. \tag{5}$$

Moreover, if $P_{_F} = \displaystyle\min_{x \in \mathsf{supp}(p_F)} \{p_{_F}(x)\}$, then for any $q \geq -P_{_F} \chi_{\mathsf{supp}(p_F)}$, it is satisfied that

$$\mathcal{E}_q^F(u,u) \geq 0$$

and hence the energy is positive semi-definite. We can interpret the value $P_{_F}$ as a discrete version of the Poincaré constant.

The following result is a by-product of the Green Identities, summarized in Identity (1).

Lemma 2 (Self-adjointness) *The boundary value problem* (3) *is self-adjoint; that is,*

$$\int_F v\, \mathcal{L}_q(u) = \int_F u\, \mathcal{L}_q(v) = \mathcal{E}_q^F(u,v), \qquad \textit{for all } u, v \in \mathcal{V}_q.$$

Problem (3) is generically known as a *Mixed Dirichlet-Robin problem* and, in particular, includes the following boundary value problems:

(i) *Neumann problem*: $q = 0$ on F_N, $F_N = \delta(F)$ and hence $F_D = \emptyset$.

(ii) *Robin problem*: $q \neq 0$ on F_N, $F_N = \delta(F)$ and hence $F_D = \emptyset$.

(iii) *Mixed Dirichlet-Neumann problem*: $F_N, F_D \neq \emptyset$ and $q = 0$ on F_N.

The boundary value problem (3) when $q \in \mathcal{C}^+(\bar{F})$ has been extensively treated in the literature, see for instance [2,4,7,13,18] where the existence and uniqueness of solutions was established. It corresponds to linear systems with *d.d.* M-matrices as coefficient matrix, see [5]. The analysis for Dirichlet Problem ($F_N = \emptyset$) and Poisson equation ($F = V$) for more general potentials has been analyzed in [3,4] and correspond to linear systems with a general symmetric M-matrix as coefficient matrix. As we are assuming that $F_N \neq \emptyset$, we are not considering here neither Dirichlet nor Poisson problems.

3.1 The trace map

The most common way to solve self-adjoint boundary value problems in PDE is to raise the weak formulation and to apply the so-called Dirichlet Principle on the associated affine subspace. The discrete version of this methodology will be considered in the following section. If the boundary value problem is of Dirichlet type, another common methodology is to built a smooth enough function such that the value on the boundary coincides with the boundary data and hence apply again the Dirichlet principle to a vector subspace. However, for mixed boundary value problems this technique is unusual because it would requiere a very precise knowledge of the trace function to obtain a smooth enough function satisfying the boundary conditions, see [19, Vol. 2]. In the discrete case, the difficulties due to the regularity of functions and domains disappear and in very general conditions (that are in force when the Dirichlet principle is applicable), the trace map can be explicitly described.

Therefore, we first establish that under a simple condition, for any data the BVP (3) can be transformed into a semi-homogeneous one; that is, a BVP in which the data is supported by F, or equivalently in which the boundary conditions are null. Recall that we are assuming $F_N \neq \emptyset$ and $F \cup F_N$ is connected in $\Gamma(F)$.

Lemma 3 *Suppose that $q(x) + \kappa_F(x) \neq 0$ for any $x \in F_N$. If given $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(F_N)$ and $h \in \mathcal{C}(F_D)$, we consider $u_g \in \mathcal{C}\big(F_N\big)$ and $f_{g,h} \in \mathcal{C}(F)$ defined respectively as*

$$u_g = \frac{g}{q + \kappa_F} \qquad \text{and} \qquad f_{g,h} = \mathcal{L}(u_g + h) \cdot \chi_F,$$

then u is a solution of BVP (3) with data f, g, h iff $u = v + u_g + h$ where v is a solution of the semi-homogeneous BVP (4) with data $f - f_{g,h}$.

Observe that for any $x \in F$ we have

$$\mathcal{L}(u_g)(x) = -\sum_{y \in F_N} \frac{c(x,y)g(y)}{q(y)+\kappa_F(y)} \quad \text{and} \quad \mathcal{L}(h)(x) = -\sum_{y \in F_D} c(x,y)h(y),$$

and hence $f_{g,h}$ depends only on the values of the data g and h.

Next we prove that under the same hypothesis than before, the values on F_N of any solution of BVP (3), depend only on its values on F.

Proposition 2 *Suppose that $q(x) + \kappa_F(x) \neq 0$ for any $x \in F_N$. Then, for any $u \in \mathcal{C}(F)$ there exists a unique extension of u to $F \cup F_N$, $\gamma(u)$, such that $\gamma(u) \in \mathcal{V}_q$. Moreover, $\gamma \colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F \cup F_N)$ is given by*

$$\gamma(u)(x) = \frac{1}{q(x)+\kappa_F(x)} \sum_{y \in F} c(x,y)u(y), \quad x \in F_N,$$

and establishes an isomorphism onto $\mathcal{V}_q$. Therefore, $\dim \mathcal{V}_q = |F|$ and when $\kappa_F + q > 0$ on F_N, then $\omega \in \Omega(F)$ iff $\gamma(\omega) \in \Omega(F \cup F_N)$.

Proof Given $u \in \mathcal{C}(F \cup F_N)$, then $u \in \mathcal{V}_q$ iff for any $x \in F_N$ we have

$$\big(q(x) + \kappa_F(x)\big)u(x) = \sum_{y \in F} c(x,y)u(y),$$

which, in particular, implies that the values of u on F_N are uniquely determined by the values of u on F; in fact on $\delta(V \setminus F)$. Therefore, if for any $u \in \mathcal{C}(F)$ we define $\gamma(u) \in \mathcal{C}(F \cup F_N)$ as $\gamma(u) = u$ on F and as

$$\gamma(u)(x) = \frac{1}{q(x)+\kappa_F(x)} \sum_{y \in F} c(x,y)u(y), \quad x \in F_N,$$

then $\gamma(u)$ extends u to $F \cup F_N$ and moreover, $\gamma(u) \in \mathcal{V}_q$.

Clearly $\gamma \colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F \cup F_N)$ is linear and moreover $\operatorname{Ker}\gamma = \{0\}$, since $\gamma(u) = 0$ on $F \cup F_N$ implies that $u = 0 = \gamma(u)$ on F. In addition, given $u \in \mathcal{V}_q$ it is obvious that $u = \gamma(u \cdot \chi_F)$, and hence $\operatorname{Img}\gamma = \mathcal{V}_q$; that is, γ is an isomorphism onto $\mathcal{V}_q$.

Finally, when $q+\kappa_F > 0$ on F_N, the last property follows from the expression of $\gamma(\omega)$. $\square$

According with its continuous analogue, the map γ defined in the above proposition, will be named *trace map for $F \cup F_N$*.

Notice that if $q(x) \leq -\kappa_F(x)$ for some $x \in F_N$, then $\Omega(F \cup F_N) \cap \mathcal{V}_q = \emptyset$ and hence, no weight on F can be extend to a weight on $F \cup F_N$ satisfying the boundary conditions of the BVP (3).

On the other hand, for any $g \in \mathcal{C}(F_N)$ and for any $u \in \mathcal{C}(F)$, the function $v = \gamma(u) + u_g$ is the unique extension of u to $F \cup F_N$ such that $\dfrac{\partial v}{\partial \mathsf{n}_F} + qv = g$ on F_N.

Next, we use the trace function to transform the mixed BVP into a Poisson Equation on a network without boundary. Thus, we reduce the dimension of the problem and moreover, this new formulation will allow us to tackle the spectral analysis in the next section.

Suppose that $q+\kappa_F>0$ on F_N and consider the network $\widehat{\Gamma}(F)=(F,\widehat{c})$ and the potential $\widehat{q}\in\mathcal{C}(F)$, where

$$\widehat{c}(x,y)=c(x,y)+\sum_{z\in F_N}\frac{c(x,z)c(z,y)}{q(z)+\kappa_F(z)},\qquad \widehat{q}(x)=q(x)+\sum_{z\in F_N}\frac{c(x,z)q(z)}{q(z)+\kappa_F(z)},$$

for any $x,y\in F$, see Figure 1.

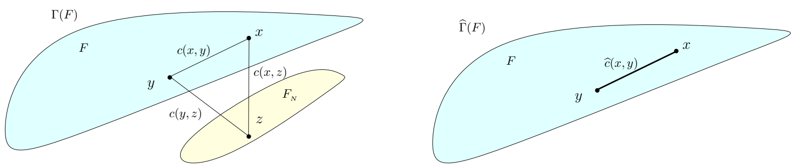

Fig. 1 Example of a network $\Gamma(F)$ and its associated $\widehat{\Gamma}(F)$.

Compare the conductance of the above network with the expressions considered in [16, pg. 581] and [22, pg. 2169].

Proposition 3 *The network $\widehat{\Gamma}(F)$ is connected and for any $u\in\mathcal{C}(F)$*

$$\widehat{\mathcal{L}}_{\widehat{q}}(u)=\mathcal{L}_q\big(\gamma(u)\big)\quad\text{on } F.$$

Moreover, $\mathcal{L}_q$ is positive semi-definite (definite) on $\mathcal{V}_q$ iff $\widehat{\mathcal{L}}_{\widehat{q}}$ is positive semi-definite (definite) on $\mathcal{C}(F)$ and given $v\in\mathcal{V}_q$, it is satisfied $\mathcal{L}_q(v)=0$ on F iff $\widehat{\mathcal{L}}_{\widehat{q}}(v\cdot\chi_F)=0$ on F.

Proof Given $x,y\in F$ there exist $x_0,x_1,\dots,x_\ell\in F\cup F_N$ such that $x=x_0$, $x_\ell=y$ and $c_F(x_i,x_{i+1})>0$ for $i=0,\dots,\ell-1$, since $F\cup F_N$ is connected with the conductance c_F.

If $x_i,x_{i+1}\in F$, then $\widehat{c}(x_i,x_{i+1})\ge c(x_i,x_{i+1})=c_F(x_i,x_{i+1})>0$.

If $x_i\in F$ and $x_{i+1}\in F_N$, then $x_{i+2}\in F$ and hence $c(x_i,x_{i+1})c(x_{i+1},x_{i+2})>0$, which implies that $\widehat{c}(x_i,x_{i+2})>0$. Therefore, there exists a path form x to y in $\widehat{\Gamma}(F)$.

On the other hand, for any $u\in\mathcal{C}(F)$ it is satisfied that for any $x\in F$

$$\begin{aligned}\mathcal{L}_q\big(\gamma(u)\big)(x)&=\big(q(x)+\kappa(x)\big)u(x)-\sum_{y\in F}c(x,y)u(y)-\sum_{z\in F_N}c(x,z)\gamma(u)(z)\\&=\big(q(x)+\kappa(x)\big)u(x)-\sum_{y\in F}\left[c(x,y)+\sum_{z\in F_N}\frac{c(x,z)c(z,y)}{q(z)+\kappa_F(z)}\right]u(y).\end{aligned}$$

Moreover, since $\widehat{q}+\widehat{\kappa}=q+\kappa$ on F, it is satisfied that for any $x\in F$

$$\widehat{\mathcal{L}}_{\widehat{q}}(u)(x)=\mathcal{L}_q\big(\gamma(u)\big)(x).$$

Finally, the above identity implies that for any $u\in\mathcal{C}(F)$,

$$\widehat{\mathcal{E}}_{\widehat{q}}(u,u)=\mathcal{E}_q\big(\gamma(u),\gamma(u)\big)$$

and the last claims follow. □

We remark that the above proposition transforms a mixed boundary value problem on $F\cup F_N$ into a Poisson equation on F with respect to a new Schrödinger operator. This is due to the fact that the values of a function verifying the boundary condition on F_N are uniquely determined by the values of the function on F. A particular version of this technique was use in [16, 22] in the context of Neumann boundary value problems for the combinatorial Laplacian. Therefore, given $f\in\mathcal{C}(F)$, then $\widehat{\mathcal{L}}_{\widehat{q}}(u)=f$ iff $\gamma(u)$ is a solution of the semihomogeneus BVP (4) with data f. Moreover, for any semihomogeneous mixed boundary value problem the Fredholm Alternative is in force: Given $f\in\mathcal{C}(F)$, Problem (4) has a solution iff $\langle v,f\rangle_F=0$, for any $v\in\mathcal{V}_q^H$. In particular, Problem (4) has a unique solution iff $\mathcal{V}_q^H$ is the trivial subspace.

Observe that the procedure we have just described is the operational version of the Schur complement method to solve linear systems. In this case,

$$\widehat{\mathsf{L}}_{\widehat{q}}=\mathsf{L}_q/\mathsf{D}=\mathsf{L}_q(F;F)-\mathsf{C}(F;F_N)\mathsf{D}^{-1}\mathsf{C}(F_N;F),$$

where D is the diagonal matrix whose diagonal elements are $k_F(x)+q(x)$, $x\in F_N$. Moreover, the matrix associated with γ is

$$\begin{bmatrix}\mathsf{I}\\ -\mathsf{D}^{-1}\mathsf{C}(F_N;F)\end{bmatrix}.$$

3.2 Positive semi-definiteness of the energy

We now study the existence and uniqueness of solution for the BVP (3), see [4] for a similar analysis when $q\in\mathcal{C}(\bar{F})$. Clearly, a sufficient condition for $\mathcal{V}_q^H=\{0\}$ is that the energy is positive definite on $\mathcal{V}_q$ and, in turns, this property holds when the energy is positive definite on $\mathcal{C}(F\cup F_N)$. Motivated by this, a potential $q\in\mathcal{C}(F\cup F_N)$ is called *admissible* when the Schrödinger operator $\mathcal{L}_q$; that is, the energy $\mathcal{E}_q$, is positive semi-definite on $\mathcal{C}(F\cup F_N)$.

Clearly, from Identity (5), the potential $q\in\mathcal{C}(F\cup F_N)$ is admissible when $q\ge -p_F$ on $F\cup F_N$. Moreover, $\mathcal{E}_q$ is positive definite except when $q=-p_F$ on $F\cup F_N$ in which case $\mathcal{E}_q^F(u,u)=0$ for $u\in\mathcal{C}(F\cup F_N)$ iff $u=a\chi_{F\cup F_N}$, $a\in\mathbb{R}$. Notice that the inequality $q\ge -p_F$ allows q to take negative values on $\mathsf{supp}(p_F)\subset\delta(F^c)\subset F$. Besides, the above inequality implies that $q\ge 0$ on F_N and hence $q+\kappa_F>0$ on F_N.

We can improve the above result in the sense that we can accurately determine the positive semi-definiteness of $\mathcal{E}_q^F$ on $\mathcal{C}(F\cup F_N)$ and hence characterize when q is admissible. To do this, it is useful to introduce a special class of potentials. Given $\omega\in\Omega(F\cup F_N)$, the *Doob potential associated with* ω is the function $q_\omega\in\mathcal{C}(\bar F)$ defined as

$$q_\omega=-\frac{1}{\omega}\,\mathcal{L}(\omega)\ \text{on}\ F,\quad q_\omega=-\frac{1}{\omega}\,\frac{\partial\omega}{\partial \mathsf{n}_F}\ \text{on}\ F_N\quad\text{and}\ q_\omega=0\ \text{on}\ F_D,\qquad(6)$$

that clearly satisfies $q_\omega\in\mathcal{C}(F\cup F_N)$, $q_\omega>-\kappa\chi_F-\kappa_F\chi_{F_N}$, and hence $q_\omega+\kappa_F>0$ on F_N. Notice that if $\omega\in\Omega(F\cup F_N)$, then $q_\omega=q_\mu$ for any $\mu=a\omega$, $a>0$, see also Lemma 5 below. This motivates us to consider for any $\sigma\in\Omega_F$ the set

$$\Omega_\sigma(F\cup F_N)=\Big\{\omega\in\Omega(F\cup F_N):||\omega||_\sigma=1\Big\}.$$

In particular, when $\sigma=\chi_F$ the above set is denoted by $\Omega_1(F\cup F_N)$. Notice that for any $\sigma\in\Omega_F$, $|F|^{-\frac12}\chi_{F\cup F_N}$ is the unique constant weight belonging to $\Omega_\sigma(F\cup F_N)$.

First we show that the function p_F can be seen as a Doob potential associated with constant weights on $F\cup F_N$.

Lemma 4 *Given $\omega\in\Omega(F\cup F_N)$, then $\int_{\bar F}\omega\,(q_\omega+p_F)=0$ and moreover, $q_\omega=-p_F$ iff $\omega=a\chi_{F\cup F_N}$, $a>0$. In particular, $q_\omega+p_F$ takes positive and negative values, except when $\omega=a\chi_{F\cup F_N}$, $a>0$.*

Proof Applying the Second Green Identity to ω and $\chi_{\bar F}$, we get that

$$\int_{\bar F}\omega q_\omega=-\int_F\mathcal{L}(\omega)-\int_{F_N}\frac{\partial\omega}{\partial\mathsf{n}_{\mathsf{F}}}\int_{F_D}\frac{\partial\omega}{\partial\mathsf{n}_{\mathsf{F}}}=$$

$$=-\int_{F_D}\int_F c(x,y)\omega(y)\,dydx=-\int_F\omega(y)p_F(y)dy$$

and the first claim follows bearing in mind that $\mathsf{supp}(p_F)\subset F$. On the other hand, if $\omega=a\chi_{F\cup F_N}$, $a>0$, it is clear that its associated potential coincides with $-p_F$. Conversely, if $q_\omega=-p_F$, then $\mathcal{L}(\omega)=\omega p_F$ on F, $\dfrac{\partial\omega}{\partial\mathsf{n}_F}=0$ on F_N, $\omega=0$ on F_D and hence $\displaystyle\int_{\delta(F)}\omega\frac{\partial\omega}{\partial\mathsf{n}_F}=0$. Applying now the First Green Identity and the Identity (5), we obtain that

$$\int_{F\cup F_N}\omega^2p_F=\int_F\omega^2p_F=\int_F\omega\mathcal{L}(\omega)$$

$$=\frac12\int_{F\cup F_N}\int_{F\cup F_N}c_F(x,y)\big(\omega(x)-\omega(y)\big)^2dxdy+\int_{F\cup F_N}p_F\omega^2$$

which implies that

$$\int_{F\cup F_N}\int_{F\cup F_N} c_F(x,y)\big(\omega(x)-\omega(y)\big)^2\,dxdy = 0$$

and hence that ω is constant on $F\cup F_N$, since $F\cup F_N$ is connected for c_F. □

The motivation to define the class of Doob potentials appears clear after the following key result. As its proof follows the same reasoning than in [3, Identity 2.1], we omit it.

Theorem 1 (Doob Transform) *Given $\omega\in\Omega(F\cup F_N)$, then for any $u\in\mathcal{C}(F\cup F_N)$ the following identities hold:*

$$\mathcal{L}(u)(x) = \frac{1}{\omega(x)}\sum_{y\in F\cup F_N} c(x,y)\omega(x)\omega(y)\left(\frac{u(x)}{\omega(x)}-\frac{u(y)}{\omega(y)}\right) - q_\omega(x)\,u(x),\ x\in F,$$

$$\left(\frac{\partial u}{\partial \mathsf{n}_F}\right)(x) = \frac{1}{\omega(x)}\sum_{y\in F} c(x,y)\omega(x)\omega(y)\left(\frac{u(x)}{\omega(x)}-\frac{u(y)}{\omega(y)}\right) - q_\omega(x)\,u(x), \qquad x\in F_N.$$

In addition, for any $u,v\in\mathcal{C}(F\cup F_N)$ we get that

$$\begin{aligned}\mathcal{E}^F(u,v) =& \frac{1}{2}\int_{F\cup F_N}\int_{F\cup F_N} c_F(x,y)\omega(x)\omega(y)\left(\frac{u(x)}{\omega(x)}-\frac{u(y)}{\omega(y)}\right)\left(\frac{v(x)}{\omega(x)}-\frac{v(y)}{\omega(y)}\right)dxdy\\ &-\int_{F\cup F_N} q_\omega u\,v.\end{aligned}$$

As a by-product of the Doob transform we obtain that if $q\in\mathcal{C}(F\cup F_N)$ is such that $q\ge q_\omega$ for some $\omega\in\Omega(F\cup F_N)$ then $\mathcal{E}_q^F$ is positive semi-definite and positive definite except when $q=q_\omega$ in which case $\mathcal{E}_q^F(u,u)=0$ iff u is a multiple of ω. Then, $\mathcal{E}_q^F$ can be positive semi-definite even when q takes non positive values on $F\cup F_N$. In fact, following the same reasoning as in [3], we could show that there exist weights on $F\cup F_N$ whose associated potentials take always negative values except on a single vertex. Next we analyze the properties of this class of potentials.

First, notice that Lemma 4 characterizes the constant weight from its associated potential. The following result shows that this is true for arbitrary weights and its proof is analogue to the proof for the case $F=V$, see [3, Lemma 2.1] .

Lemma 5 *Given $\omega_1,\omega_2\in\Omega(F\cup F_N)$, then $q_{\omega_1}\ge q_{\omega_2}$, iff $q_{\omega_1}=q_{\omega_2}$ and this occurs iff $\omega_2=a\omega_1$, $a>0$.*

As a by-product of the above Lemma, we obtain that fixed $\sigma\in\Omega_F$, if $\omega_1,\omega_2\in\Omega_\sigma(F\cup F_N)$ satisfy that $q_{\omega_2}=q_{\omega_1}$, then $\omega_2=\omega_1$.

The following result shows that any potential; that is, any function in $\mathcal{C}(F\cup F_N)$ is closely related with a Doob potential. Its proof follows the same arguments than in [3, Proposition 3.3], see also [5,12].

Lemma 6 *Given $q \in \mathcal{C}(F \cup F_N)$, there exist a unique $\omega_q \in \Omega_1(F \cup F_N)$ and a unique $\alpha_q \in \mathbb{R}$ such that $q = q_{\omega_q} + \alpha_q$ on $F \cup F_N$. Moreover, $q > -\kappa_F$ on F_N iff $\alpha_q > -\min\limits_{x \in F_N} \Big\{ \omega_q(x)^{-1} \sum\limits_{y \in F} c(x,y)\omega_q(y) \Big\}$.*

Proof It suffices to apply Lemma 1 to the endomorphism $\mathcal{L}_q^F : \mathcal{C}(F \cup F_N) \longrightarrow \mathcal{C}(F \cup F_N)$ and hence, there exists $\alpha_q \in \mathbb{R}$ and $\omega_q \in \Omega_1(F \cup F_N)$ such that $\mathcal{L}_q^F(\omega_q) = \alpha_q \omega_q$. Therefore, $\alpha_q = \omega_q^{-1}\mathcal{L}_q^F(\omega_q) = q - q_{\omega_q}$. □

Although the above result would allows us to characterize when the energy is positive semi-definite and would maintain the subspace $\mathcal{V}_q^H$ under control, see the above mentioned references or the results below, it still does not fit perfectly with the boundary value problem (3), since when $q = q_{\omega_q} + \alpha_q$, then $\mathcal{L}_q(\omega_q) = \alpha_q \omega_q$ on F and $\dfrac{\partial \omega_q}{\partial \mathsf{n}_F} + q\omega_q = \alpha_q \omega_q$ on F_N. Therefore, $\omega_q \notin \mathcal{V}_q$, except when $\alpha_q = 0$.

We can adequately modify our reasoning to obtain a more accurate representation of any potential $q \in \mathcal{C}(F \cup F_N)$ satisfying $q + \kappa_F > 0$ on F_N as a potential associated to a weight belonging to $\mathcal{V}_q$. As we show next, this representation is closely related with eigenvalues and eigenfunctions for the BVP (3) and it can be interpreted as a discrete version of the *Krein-Rutman* theorem for an elliptic differential operator, see [19, Vol. 3]. We first obtain the claimed representation and then we apply it to the characterization of the positive semi-definiteness of the energy.

Proposition 4 *Given $q \in \mathcal{C}(F \cup F_N)$ such that $q > -\kappa_F$ on F_N, then for each $\sigma \in \Omega_F$ there exist a unique $\omega \in \Omega_\sigma(F \cup F_N)$ and a unique $\lambda \in \mathbb{R}$ such that $q = q_\omega + \lambda\sigma$. Therefore, $\omega \in \mathcal{V}_q$, $\alpha_q \lambda \geq 0$ and $\alpha_q \lambda = 0$ iff $\lambda = \alpha_q = 0$ and then $\omega = ||\omega_q||_\sigma^{-1}\omega_q$.*

Proof Suppose that $q = q_{\omega_1} + \lambda_1\sigma = q_{\omega_2} + \lambda_2\sigma$ on $F \cup F_N$, where $\omega_1, \omega_2 \in \Omega_\sigma(F \cup F_N)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. If $\lambda_2 \geq \lambda_1$, then $q_{\omega_1} \geq q_{\omega_2}$ on $F \cup F_N$ and hence $\omega_2 = a\omega_1$, $a > 0$, applying Lemma 5. Moreover, $a = 1$, since $\omega_1, \omega_2 \in \Omega_\sigma(F \cup F_N)$ and hence $\lambda_1 = \lambda_2$.

On the other hand, since $\sigma = 0$ on F_N, the representation $q = q_\omega + \lambda\sigma$ implies that $q = q_\omega$ on F_N. Therefore, we have $\dfrac{\partial \omega}{\partial \mathsf{n}_F} + q\omega = \dfrac{\partial \omega}{\partial \mathsf{n}_F} + q_\omega \omega = 0$ on F_N, and hence $\omega \in \mathcal{V}_q$.

In addition, taking into account that $q = q_{\omega_q} + \alpha_q = q_\omega + \lambda\sigma$ on $F \cup F_N$ then applying the Second Green Identity we obtain that

$$\lambda \int_F \omega\omega_q\sigma - \alpha_q \int_F \omega\omega_q = \int_F \Big(\omega_q \mathcal{L}(\omega) - \omega\mathcal{L}(\omega_q)\Big)$$
$$= \int_{\delta(F)} \left(\omega \frac{\partial \omega_q}{\partial \mathsf{n}_F} - \omega_q \frac{\partial \omega}{\partial \mathsf{n}_F}\right) = \alpha_q \int_{F_N} \omega\omega_q,$$

which implies that

$$\lambda \int_F \omega\omega_q \sigma = \alpha_q \int_{F\cup F_N} \omega\omega_q.$$

Therefore, α_q and λ have the same sign and, in addition, $\alpha_q = 0$ iff $\lambda = 0$. Moreover, $\alpha_q = \lambda = 0$ iff $q_{\omega_q} = q_\omega$; that is, $\omega = ||\omega_q||_\sigma^{-1}\omega_q$.

Given $\sigma \in \Omega_F$, let us consider the operator $\mathcal{K}_\sigma : \mathcal{C}(F) \longrightarrow \mathcal{C}(F)$, defined for any $u \in \mathcal{C}(F)$ as

$$\mathcal{K}_\sigma(u) = \sigma^{-\frac{1}{2}}\mathcal{L}_q\big(\gamma(\sigma^{-\frac{1}{2}}u)\big) = \sigma^{-\frac{1}{2}}\widehat{\mathcal{L}}_{\widehat{q}}(\sigma^{-\frac{1}{2}}u).$$

If we label the vertices of F, then the kernel K_σ has associated a square matrix K_σ of order $|F|$. Moreover, since

$$K_\sigma(x,y) = \begin{cases} -\widehat{c}(x,y)\sigma^{-\frac{1}{2}}(x)\sigma^{-\frac{1}{2}}(y), \; x \neq y, \\ \big(\widehat{\kappa}(x) + \widehat{q}(x)\big)\sigma^{-1}(x), \quad y = x, \end{cases}$$

we have that K_σ is an irreductible, symmetric Z-matrix, since $\widehat{\Gamma}(F)$ is connected. From Lemma 1, there exists $\widetilde{\omega} \in \Omega(F)$ and $\lambda \in \mathbb{R}$ such that that

$$\sigma^{-\frac{1}{2}}\mathcal{L}\big(\gamma(\sigma^{-\frac{1}{2}}\widetilde{\omega})\big) + q\sigma^{-1}\widetilde{\omega} = \mathcal{K}_\sigma(\widetilde{\omega}) = \lambda\widetilde{\omega} \;\text{ on }\; F.$$

Defining $\omega = \gamma(\sigma^{-\frac{1}{2}}\widetilde{\omega})$, we get that $\omega \in \Omega(F \cup F_N) \cap \mathcal{V}_q$ and that $q = q_\omega + \lambda\sigma$. □

We are now ready to characterize the positive definiteness of the energy on $F \cup F_N$.

Proposition 5 *Given $q \in \mathcal{C}(F \cup F_N)$ the following statements are equivalent:*

(i) *$\mathcal{E}_q^F$ is positive definite on $\mathcal{C}(F \cup F_N)$.*
(ii) $\alpha_q > 0$.
(iii) *There exists $\omega \in \Omega(F \cup F_N)$ such that $q \geq q_\omega$ on $F \cup F_N$ and $q \neq q_\omega$.*
(iv) *$q > -\kappa_F$ on F_N and $\mathcal{E}_q^F$ is positive definite on $\mathcal{V}_q$.*
(v) *For any $\sigma \in \Omega_F$, there exists $\omega \in \Omega(F \cup F_N)$ such that $q = q_\omega + \lambda\sigma$ and $\lambda > 0$.*
(vi) *There exists $\omega \in \Omega(F \cup F_N) \cap \mathcal{V}_q$ such that $q \geq q_\omega$ on F and $q \neq q_\omega$ on F.*

Proof The proof itinerary is

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(i)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(v)} \Longrightarrow \text{(vi)} \Longrightarrow \text{(iii)}.$$

In many of the steps of the proof we use that given $\omega \in \Omega(F \cup F_N)$ and applying the Doob transform we have

$$\begin{aligned}\mathcal{E}_q^F(u,u) =& \frac{1}{2}\int_{F\cup F_N}\int_{F\cup F_N} c_F(x,y)\omega(x)\omega(y)\left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right)^2 dxdy \\ &+ \int_{F\cup F_N}(q-q_\omega)u^2 \geq \int_{F\cup F_N}(q-q_\omega)u^2,\end{aligned}$$

for any $u\in\mathcal{C}(F\cup F_{N})$.

(i) $\Longrightarrow$ (ii): Since $q=q_{\omega_q}+\alpha_q$ applying the Doob transform to $\omega=\omega_q$ we obtain that

$$0<\mathcal{E}_q^F(\omega_q,\omega_q)=\alpha_q\int_{F\cup F_N}\omega_q^2$$

which implies that $\alpha_q>0$.

(ii) $\Longrightarrow$ (iii): It suffices to take $\omega=\omega_q$ to conclude that in fact $q>q_\omega$ on $F\cup F_N$.

(iii) $\Longrightarrow$ (i): When (iii) is in force, then given $u\in\mathcal{C}(F\cup F_N)$, $\mathcal{E}_q^F(u,u)\geq\int_{F\cup F_N}(q-q_\omega)u^2\geq 0$ and moreover $\mathcal{E}_q^F(u,u)=0$ iff $u=a\omega$, $a\in\mathbb{R}$, and $(q-q_\omega)u^2=a^2\omega^2(q-q_\omega)=0$ on $F\cup F_N$, which implies $a=0$.

(i) $\Longrightarrow$ (iv): Since (i) implies (iii), then for some $\omega\in\Omega(F\cup F_N)$, $q\geq q_\omega>-\kappa_F$ on F_N. The last conclusion follows taking into account that $\mathcal{V}_q\subset\mathcal{C}(F\cup F_N)$.

(iv) $\Longrightarrow$ (v): From Proposition 4, given $\sigma\in\Omega_F$ we have $\omega\in\mathcal{C}(F\cup F_N)$ and $\lambda\in\mathbb{R}$ such that $q=q_\omega+\lambda\sigma$, which also implies that $\omega\in\mathcal{V}_q$. On the other hand, applying the Doob transform to ω we obtain that $0<\mathcal{E}_q^F(\omega,\omega)=\lambda\int_F\sigma\omega^2$ and hence $\lambda>0$.

(v) $\Longrightarrow$ (vi): Considering the identity $q=q_\omega+\lambda\sigma$ we have that $\omega\in\mathcal{C}(F\cup F_N)\cap\mathcal{V}_q$ and $q>q_\omega$ on F.

(vi) $\Longrightarrow$ (iii): Since $\omega\in\mathcal{V}_q$ we have that $q=q_\omega$ on F_N and hence $q\geq q_\omega$ on $F\cup F_N$ and, in addition, $q\neq q_\omega$. □

Similarly, we obtain the characterization of positive semi-definiteness.

Proposition 6 *Given $q\in\mathcal{C}(F\cup F_N)$ the following statements are equivalent:*

(i) *$\mathcal{E}_q^F$ is positive semi-definite on $\mathcal{C}(F\cup F_N)$ but not positive definite.*
(ii) *$\alpha_q=0$ or equivalently, $q=q_{\omega_q}$ on $F\cup F_N$.*
(iii) *$q>-\kappa_F$ on F_N and $\mathcal{E}_q^F$ is positive semi-definite on $\mathcal{V}_q$ but not positive definite.*

In addition, $\mathcal{E}_q^F(v)=0$ iff $v=a\omega_q$.

Next we use the positive definiteness of the energy to obtain the variational formulation of the main result about existence and uniqueness of solutions of the BVP (3), see [4].

Proposition 7 (Dirichlet Principle) *Suppose that $q\geq q_\omega$ on $F\cup F_N$ for some $\omega\in\Omega(F\cup F_N)$. Given $f\in\mathcal{C}(F)$, $g\in\mathcal{C}(F_N)$ and $h\in\mathcal{C}(F_D)$, consider the convex set $C_h=\big\{v\in\mathcal{C}(\bar F):v=h\text{ on }F_D\big\}$ and the quadratic functional $\mathcal{J}:\mathcal{C}(\bar F)\longrightarrow\mathbb{R}$ determined by the expression*

$$\mathcal{J}(u)=\frac{1}{2}\int_{\bar F}\int_{\bar F}c_F(x,y)\,\big(u(x)-u(y)\big)^2dx\,dy+\int_{\bar F}q\,u^2\,dx-2\int_F f\,u-2\int_{F_N}g\,u.$$

Then, $u\in\mathcal{C}(\bar F)$ is a solution of the boundary value problem (3) iff u minimizes $\mathcal{J}$ on C_h. Moreover $\mathcal{J}$ has a unique minimum on C_h except when $q=q_\omega$ in *which case $\mathcal{J}$ has a minimum iff* $\displaystyle\int_F \omega f + \int_{F_N} \omega g = \int_{F_D} h\frac{\partial\omega}{\partial \mathsf{n}_F}$ *and moreover for any* $\sigma \in \Omega_F$ *there exists an unique minimum* $u \in C_h$ *such that* $\langle u,\omega\rangle_\sigma = 0$.

Remark: If $\mathcal{Q}(F\cup F_N)$ denotes the set of admissible potentials, then

$$\mathcal{Q}(F\cup F_N) = \Big\{q\in\mathcal{C}(F\cup F_N) : q > -\kappa_F \text{ on } F_N \\ \text{and } \mathcal{E}_q \text{ is positive semi-definite on } \mathcal{V}_q\Big\}$$

and the map

$$\begin{aligned}\mathsf{q}\colon\ \Omega(F\cup F_N)\times\Omega_F\times[0,+\infty) &\longrightarrow \mathcal{Q}(F\cup F_N)\\ (\omega,\sigma,\lambda) &\longrightarrow q_\omega+\lambda\sigma\end{aligned}$$

is surjective. Notice that if $\mathsf{q}(\omega,\sigma,\lambda)=\mathsf{q}(\omega,\hat\sigma,\hat\lambda)$ for some $\omega\in\Omega(F\cup F_N)$ and $\lambda\cdot\hat\lambda>0$, then $\hat\lambda=\lambda$ and $\hat\sigma=\sigma$, since $\sigma,\hat\sigma\in\Omega_F$. However, $\mathsf{q}(\omega,\sigma,0)=q_\omega$ for all $\sigma\in\Omega_F$, so q is never injective.

Fixed $\omega\in\Omega(F\cup F_N)$, the potentials in the set

$$\mathcal{Q}_\omega(F\cup F_N)=\{q_\omega\}\cup\Big\{q_\omega+\lambda\sigma : \sigma\in\Omega_F, \lambda>0\Big\}$$

are called *admissible potentials based on* ω. Therefore, the class of admissible potentials can be described as

$$\mathcal{Q}(F\cup F_N)=\bigcup_{\omega\in\Omega(F\cup F_N)}\Big\{\mathcal{Q}_\omega(F\cup F_N)\Big\}. \tag{7}$$

If $q\in\mathcal{Q}_\omega(F\cup F_N)$, then $q=q_\omega$ on F_N and moreover either $q=q_\omega$ on F or $q>q_\omega$ on F. Conversely, if $q=q_\omega$ on F_N and $q>q_\omega$ on F, then $q\in\mathcal{Q}_\omega(F\cup F_N)$: If $a=\displaystyle\int_F\big(q(x)-q_\omega(x)\big)dx$, then $a>0$ and it suffices to define $\sigma=a^{-1}|F|(q-q_\omega)$ and $\lambda=a|F|^{-1}$ to conclude that $\sigma\in\Omega_F$ and $q=q_\omega+\lambda\sigma$. Therefore,

$$\mathcal{Q}_\omega(F\cup F_N)=\{q_\omega\}\cup\Big\{q\in\mathcal{C}(F\cup F_N) : q=q_\omega \text{ on } F_N \text{ and } q>q_\omega \text{ on } F\Big\}.$$

Motivated by the above identities, fixed $\omega\in\Omega(F\cup F_N)$ and given $q\in\mathcal{Q}_\omega(F\cup F_N)$ we define $\lambda_q\geq 0$ and $\sigma_q\in\Omega_F$ as

$$\begin{cases} \lambda_q=0,\ \sigma_q=\chi_F, & \text{when } q=q_\omega \text{ on } F\cup F_N\\ \lambda_q=|F|^{-1}\displaystyle\int_F(q-q_\omega),\ \sigma_q=\lambda_q^{-1}(q-q_\omega), & \text{when } q>q_\omega \text{ on } F.\end{cases} \tag{8}$$

Therefore, we have that $q=q_\omega+\lambda_q\sigma_q$.

4 Green Functions and Eigenvalues of a Self-Adjoint BVP

The main objective of this section is to obtain the relation between the spectrum of a self-adjoint BVP and its corresponding Green function. For the discrete Poisson Equation, the Green function is closely related with the group inverse of the matrix associated with the Schrödinger operator. In this case, the relation between the spectrum and the Green function is well-known and it is obtained through the so-called *Mercer Theorem*, see [17] for the case of the normalized Laplacian. Moreover, other generalized inverses can be obtained from the group inverse by considering the addition of suitable projectors, see [6].

In this section, we state the Mercer Theorem for general BVP where the eigenvalues and eigenfunctions are computed with respect to an arbitrary weight. In this case, the function obtained from a Mercer-type theorem does not necessarily coincide with the group inverse of the associated operator, but with another generalized inverse. Then, the expression of the group inverse is obtained from the addition of suitable projectors.

If the potential $q \in \mathcal{C}(F \cup F_{_N})$ satisfies that $q > -\kappa_{_F}$ on $F_{_N}$ then the conditions of Lemma 3 hold and hence, for any data, the BVP (3) can be reduced to a semi-homogeneous one. In this section, we assume the above hypothesis on the potential and we only consider semi-homogeneous BVP; that is, for any data $f \in \mathcal{C}(F)$ the semi-homogeneous boundary value problem (4)

$$\mathcal{L}_q(u) = f \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \mathsf{n}_{_F}} + qu = 0 \quad \text{on} \quad F_{_N} \qquad \text{and} \qquad u = 0 \quad \text{on} \quad F_{_D}.$$

Moreover, from Proposition 3 we know that the above problem is equivalent to the Poisson equation $\mathcal{L}_q(\gamma(u)) = f$ on $\mathcal{C}(F)$. In addition, we also assume that the energy $\mathcal{E}_q^F$ is positive semi-definite on $\mathcal{V}_q$ and hence, we can use the Dirichlet Principle to establish the existence and uniqueness of solution for the BVP (4).

Given $\sigma \in \Omega_{_F}$, from Proposition 4 there exists a unique $\omega \in \Omega_\sigma(F \cup F_{_N})$ and a unique $\lambda \in \mathbb{R}$ such that $q = q_\omega + \lambda\sigma$. Moreover, $\lambda \geq 0$ since $\mathcal{E}_q^F$ is positive semi-definite. Under these hypotheses we know that $q = q_\omega$ on $F_{_N}$, $\omega \in \mathcal{V}_q$ and moreover $\mathcal{L}_q(\omega) = \lambda\sigma\omega$ on F. Therefore, from the self-adjointness of the BVP (4), it follows that

$$\langle \omega, f \rangle_{_F} = \int_F \omega\, f = \int_F \omega\, \mathcal{L}_q(u) = \int_F u\, \mathcal{L}_q(\omega) = \lambda \int_F \sigma\omega u = \lambda \langle \omega, u \rangle_\sigma,$$

for any $u \in \mathcal{V}_q$ solution of Problem (4). If we consider now the subspaces

$$\omega^\perp = \big\{ u \in \mathcal{C}(F) : \langle \omega, u \rangle_{_F} = 0 \big\} \quad \text{and} \quad \omega^{\perp_\sigma} = \big\{ u \in \mathcal{V}_q : \langle \omega, u \rangle_\sigma = 0 \big\},$$

then, when $\lambda > 0$, we have $f \in \omega^\perp$ iff $u \in \omega^{\perp_\sigma}$. On the other hand, when $\lambda = 0$ we also know that the BVP (4) is solvable iff $f \in \omega^\perp$ and then there exists a unique solution $u \in \omega^{\perp_\sigma}$. Therefore, $\mathcal{L}_q$ is an isomorphism from $\omega^{\perp_\sigma}$

onto $\omega^{\perp}$. In addition, when $q \neq q_\omega$ on F, then $\mathcal{L}_q$ is an isomorphism from $\mathcal{V}_q$ onto $\mathcal{C}(F)$ whose inverse is denoted by $\mathcal{L}_q^{-1}$ that has L_q^{-1} as kernel.

In this section we consider fixed $\omega \in \Omega(F \cup F_N)$ and the set of admissible potentials based in ω, $\mathcal{Q}_\omega(F\cup F_N)$. If $q \in \mathcal{Q}_\omega(F\cup F_N)$, then according with (8), $q = q_\omega + \lambda_q\sigma_q$ which implies that $q = q_\omega$ on F_N and hence $\mathcal{V}_q = \mathcal{V}_{q_\omega}$, $\omega \in \mathcal{V}_q$ and $\mathcal{L}_q(\omega) = \lambda_q\sigma_q\omega$.

The above reasonings imply that $\mathcal{L}_q$ is an isomorphism from $\omega^{\perp_{\sigma_q}}$ onto $\omega^{\perp}$ whose inverse is denoted by $\mathcal{G}_q$. We call *Green operator* to $\mathcal{G}_q\colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F\cup F_N)$ the extension of $\mathcal{G}_q$ to $\mathcal{C}(F)$ defined for any $f \in \mathcal{C}(F)$ as $\mathcal{G}_q(f) = \mathcal{G}_q\big(f - ||\omega||_{\sigma_q}^{-2}\langle f,\omega\rangle_F\,\sigma_q\omega\big)$. Therefore, for any $f \in \mathcal{C}(F)$, $\mathcal{G}_q(f)$ is the unique solution of the semi-homogeneous BVP (4) with data $f - ||\omega||_{\sigma_q}^{-2}\langle f,\omega\rangle_F\,\sigma_q\omega$ belonging to $\omega^{\perp_{\sigma_q}}$. We remark that when $q > q_\omega$ on F, then $\mathcal{G}_q(f) \in \omega^{\perp_{\sigma_q}}$ for any $f \in \mathcal{C}(F)$. The kernel of $\mathcal{G}_q$ is denoted by G_q and named *Green function.* If we label the vertex set, then the matrix associated with G_q is the group inverse of the matrix associated with the operator $\mathcal{L}^F_{q_\omega}$ on $\mathcal{C}(F\cup F_N)$.

More generally, given $\sigma \in \Omega_F$ we define the *Green operator with respect to* σ as $\mathcal{G}_q^\sigma\colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F\cup F_N)$ the operator that to any $f \in \mathcal{C}(F)$ assigns $\mathcal{G}_q^\sigma(f)$, the unique solution of the semi-homogeneous BVP (4) with data $f - ||\omega||_\sigma^{-2}\langle f,\omega\rangle_F\,\sigma\omega$ belonging to $\omega^{\perp_{\sigma_q}}$. Therefore, when $\sigma = \sigma_q$, then $\mathcal{G}_q^\sigma = \mathcal{G}_q$. We anew notice that when $q > q_\omega$ on F, then $\mathcal{G}_q^\sigma(f) \in \omega^{\perp_{\sigma_q}}$ for any $f \in \mathcal{C}(F)$.

The kernel of $\mathcal{G}_q^\sigma$ is denoted by G_q^σ and named *Green function with respect to* σ or simply *Green function* when $\sigma = \sigma_q$. It is clear that $\mathcal{G}_q^\sigma$ is singular and moreover $\mathcal{G}_q^\sigma(f) = 0$ iff $f = a\sigma\omega$, $a \in \mathbb{R}$.

When $q = q_\omega$ on F, given $\sigma, \widehat{\sigma} \in \Omega_F$ we define *Green operator with respect to* σ *and* $\widehat{\sigma}$ as the operator $\mathcal{G}_{q_\omega}^{\sigma,\widehat{\sigma}}\colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F\cup F_N)$ that to any $f \in \mathcal{C}(F)$ assigns $\mathcal{G}_{q_\omega}^{\sigma,\widehat{\sigma}}(f)$, the unique solution of the semi-homogeneous BVP (4) with data $f - ||\omega||_\sigma^{-2}\langle f,\omega\rangle_F\,\sigma\omega$ belonging to $\omega^{\perp_{\widehat{\sigma}}}$. When $\widehat{\sigma} = \chi_F$, then $\mathcal{G}_{q_\omega}^{\sigma,\widehat{\sigma}} = \mathcal{G}_{q_\omega}^\sigma$ and hence we drop $\widehat{\sigma}$ in the above notation and refer it simply as the *Green operator with respect to* σ.

The kernel of $\mathcal{G}_{q_\omega}^{\sigma,\widehat{\sigma}}$ is denoted by $G_{q_\omega}^{\sigma,\widehat{\sigma}}$ and named *Green function with respect to* σ *and* $\widehat{\sigma}$ or simply $G_{q_\omega}^\sigma$ and *Green function with respect to* σ when $\widehat{\sigma} = \chi_F$. It is clear that $\langle \omega, \mathcal{G}_{q_\omega}^{\sigma,\widehat{\sigma}}(f)\rangle_{\widehat{\sigma}} = 0$ for any $f \in \mathcal{C}(F)$, that $\mathcal{G}_{q_\omega}^{\sigma,\widehat{\sigma}}$ is singular and moreover $\mathcal{G}_{q_\omega}^{\sigma,\widehat{\sigma}}(f) = 0$ iff $f = a\sigma\omega$, $a \in \mathbb{R}$.

From the above definitions, we obtain that if $q \in \mathcal{Q}_\omega(F\cup F_N)$, then for any $\sigma \in \Omega_F$ and any $y \in F$, $v = \big(G_q^\sigma\big)_y$ is characterized as the unique solution of the semi-homogeneous BVP

$$\mathcal{L}_q(v) = \varepsilon_y - \frac{\omega(y)}{||\omega||_\sigma^2}\sigma\omega \text{ on } F, \quad \frac{\partial v}{\partial \mathsf{n}_\mathsf{F}} + q_\omega v = 0 \text{ on } F_N \quad v = 0 \text{ on } F_D$$

such that $\langle v,\omega\rangle_{\sigma_q} = 0$. In addition, for any $\widehat{\sigma} \in \Omega_F$ and any $y \in F$ the function $z = \big(G_{q_\omega}^{\sigma,\widehat{\sigma}}\big)_y$ is characterized as the unique solution of the semi-homogeneous BVP

$$\mathcal{L}_{q_\omega}(z) = \varepsilon_y - \frac{\omega(y)}{||\omega||_\sigma^2}\sigma\omega \text{ on } F, \quad \frac{\partial z}{\partial \mathsf{n}_\mathsf{F}} + q_\omega z = 0 \text{ on } F_N \quad z = 0 \text{ on } F_D$$

such that $\langle z,\omega\rangle_{\widehat{\sigma}}=0$. If in addition, $q\neq q_\omega$, then for any $\sigma\in\Omega_{_F}$ and any $y\in F$, the function $u=\left(L_q^{-1}\right)_y$ is characterized as the unique solution of the semi-homogeneous BVP

$$\mathcal{L}_q(u)=\varepsilon_y \text{ on } F,\quad \frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}}+q_\omega u=0 \text{ on } F_{_N}\quad u=0 \text{ on } F_{_D}.$$

Our next objective is to find the relations between the above operators and kernels. To do this, it is useful to introduce for any $\sigma\in\Omega_{_F}$ the functions

$$\tau_\sigma=||\omega||_\sigma^{-2}\mathcal{G}_q(\sigma\omega)\ \text{ and }\ \zeta_\sigma=||\omega||_{_F}^{-2}\mathcal{G}_{q_\omega}^{\sigma,\sigma}(\omega). \tag{9}$$

Then $\tau_\sigma,\zeta_\sigma\in\mathcal{V}_q$, $\langle\tau_\sigma,\omega\rangle_{\sigma_q}=0$, $\langle\zeta_\sigma,\omega\rangle_\sigma=0$ and moreover

$$\mathcal{L}_q(\tau_\sigma)=\Big(||\omega||_\sigma^{-2}\sigma-||\omega||_{\sigma_q}^{-2}\sigma_q\Big)\omega\quad\text{and}\quad\mathcal{L}_{q_\omega}(\zeta_\sigma)=\Big(||\omega||_{_F}^{-2}-||\omega||_\sigma^{-2}\sigma\Big)\omega,$$

which implies that for any $\widehat{\sigma}\in\Omega_{_F}$

$$\mathcal{E}_q(\tau_\sigma,\tau_{\widehat{\sigma}})=\int_F\mathcal{L}_q(\tau_\sigma)\tau_{\widehat{\sigma}}=||\omega||_\sigma^{-2}\langle\tau_{\widehat{\sigma}},\omega\rangle_\sigma=||\omega||_{\widehat{\sigma}}^{-2}\langle\tau_\sigma,\omega\rangle_{\widehat{\sigma}}$$

and

$$\mathcal{E}_q(\zeta_\sigma,\zeta_\sigma)=\int_F\mathcal{L}_{q_\omega}(\zeta_\sigma)\zeta_\sigma=||\omega||_{_F}^{-2}\langle\zeta_\sigma,\omega\rangle_{_F}.$$

Notice that $\tau_\sigma=0$ iff $\sigma=\sigma_q$. More generally, if $\tau_\sigma=\tau_{\widehat{\sigma}}$ on F, then $\tau_\sigma=\gamma(\tau_\sigma)=\gamma(\tau_{\widehat{\sigma}})=\tau_{\widehat{\sigma}}$ since $\tau_\sigma,\tau_{\widehat{\sigma}}\in\mathcal{V}_q$ and hence,

$$||\omega||_\sigma^{-2}\sigma\omega-||\omega||_{\sigma_q}^{-2}\sigma_q\omega=\mathcal{L}_q(\tau_\sigma)=\mathcal{L}_q(\tau_{\widehat{\sigma}})=||\omega||_{\widehat{\sigma}}^{-2}\widehat{\sigma}\omega-||\omega||_{\sigma_q}^{-2}\sigma_q\omega$$

which in turns implies that $\widehat{\sigma}=\sigma$.

Proposition 8 *The Green operator is singular and self-adjoint on $\mathcal{C}(F)$; that is,*

$$\int_F g\mathcal{G}_q(f)=\int_F f\mathcal{G}_q(g),\quad \textit{for any } f,g\in\mathcal{C}(F)$$

and when $q\neq q_\omega$ then

$$L_q^{-1}(x,y)=G_q(x,y)+\lambda_q^{-1}||\omega||_{\sigma_q}^{-2}\omega(x)\omega(y),\quad x\in V,\ y\in F$$

and hence, $\mathcal{L}_q^{-1}$ is self-adjoint on $\mathcal{C}(F)$. Moreover, given $\sigma\in\Omega_{_F}$ we have that

$$G_q^\sigma(x,y)=G_q(x,y)-\tau_\sigma(x)\omega(y),\quad x\in V,\ y\in F$$

and hence, $\mathcal{G}_q^\sigma$ is self–adjoint on $\mathcal{C}(F)$ iff $\sigma=\sigma_q$; that is, iff $\tau_\sigma=0$ or equivalently iff $\mathcal{G}_q^\sigma=\mathcal{G}_q$. In addition, given $\widehat{\sigma}\in\Omega_{_F}$ we have that

$$G_{q_\omega}^{\sigma,\widehat{\sigma}}(x,y)=G_{q_\omega}(x,y)+\mathcal{E}_{q_\omega}(\tau_\sigma,\tau_{\widehat{\sigma}})\omega(x)\omega(y)-\tau_\sigma(x)\omega(y)-\omega(x)\tau_{\widehat{\sigma}}(y),\quad x\in V,\ y\in F$$

and hence, $\mathcal{G}_{q_\omega}^{\sigma,\widehat{\sigma}}$ is self–adjoint on $\mathcal{C}(F)$ iff $\widehat{\sigma}=\sigma$. Moreover,

$$G_{q_\omega}(x,y)=G_{q_\omega}^{\sigma,\sigma}(x,y)+\mathcal{E}_{q_\omega}(\zeta_\sigma,\zeta_\sigma)\omega(x)\omega(y)-\zeta_\sigma(x)\omega(y)-\omega(x)\zeta_\sigma(y),\quad x\in V,\ y\in F.$$

Proof Given $f,g\in\mathcal{C}(F)$, if $u=\mathcal{G}_q(f)$, $v=\mathcal{G}_q(g)$, then $u,v\in\omega^{\perp_{\sigma_q}}$, $\mathcal{L}_q(u)=f-||\omega||_{\sigma_q}^{-2}\langle f,\omega\rangle_{_F}\sigma_q\omega$, $\mathcal{L}_q(v)=g-||\omega||_{\sigma_q}^{-2}\langle g,\omega\rangle_{_F}\sigma_q\omega$ and from the self-adjointness of the BVP (4) we have

$$\int_F g\mathcal{G}_q(f)=\int_F ug=\int_F u\big(g-||\omega||_{\sigma_q}^{-2}\langle g,\omega\rangle_{_F}\sigma_q\omega\big)=\int_F u\mathcal{L}_q(v)=\int_F v\mathcal{L}_q(u)$$
$$=\int_F v\big(f-||\omega||_{\sigma_q}^{-2}\langle f,\omega\rangle_{_F}\sigma_q\omega\big)=\int_F vf=\int_F f\mathcal{G}_q(g).$$

Moreover, since $\mathcal{L}_q(\omega)=\lambda_q\sigma_q\omega$ on F and $\lambda_q>0$ when $q\neq q_\omega$, then $\mathcal{L}_q^{-1}(\sigma_q\omega)=\lambda_q^{-1}\omega$ and hence

$$\mathcal{G}_q(f)=\mathcal{L}_q^{-1}\big(f-||\omega||_{\sigma_q}^{-2}\langle f,\omega\rangle_{_F}\sigma_q\omega\big)=\mathcal{L}_q^{-1}(f)-\lambda_q^{-1}||\omega||_{\sigma_q}^{-2}\langle f,\omega\rangle_{_F}\omega,$$

which implies the given expression for the kernel L_q^{-1}.

On the other hand, given $\sigma\in\Omega_{_F}$ if $\hat{u}=\mathcal{G}_q^\sigma(f)$, then $\hat{u}\in\omega^{\perp_{\sigma_q}}$, and

$$\mathcal{L}_q(\hat{u})=f-||\omega||_\sigma^{-2}\langle f,\omega\rangle_{_F}\sigma\omega,\quad\text{on } F$$

which implies that

$$\hat{u}=\mathcal{G}_q(f)-||\omega||_\sigma^{-2}\langle f,\omega\rangle_{_F}\mathcal{G}_q(\sigma\omega)=\mathcal{G}_q(f)-\langle f,\omega\rangle_{_F}\tau_\sigma,\quad\text{on } F.$$

We obtain the claimed expression for the kernel G_q^σ, by taking $f=\varepsilon_y$, $y\in F$. Moreover $\mathcal{G}_q^\sigma$ is self-adjoint on $\mathcal{C}(F)$ iff

$$G_q(x,y)-\tau_\sigma(x)\omega(y)=G_q^\sigma(x,y)=G_q^\sigma(y,x)=G_q(y,x)-\tau_\sigma(y)\omega(x),\quad x,y\in F;$$

that is, iff $\tau_\sigma=a\omega$ on F. Since $0=\langle\tau_\sigma,\omega\rangle_{\sigma_q}=a||\omega||_{\sigma_q}^2$, we conclude that $a=0$ and hence $\sigma=\sigma_q$.

Consider now $\tilde{u}=\mathcal{G}_{q_\omega}^{\sigma,\widehat{\sigma}}(f)$. Then $\tilde{u}\in\mathcal{V}_q$, $\langle\tilde{u},\omega\rangle_{\widehat{\sigma}}=0$, and $\mathcal{L}_{q_\omega}(\tilde{u})=f-||\omega||_\sigma^{-2}\langle f,\omega\rangle_{_F}\,\sigma\omega$, which implies that

$$\tilde{u}=\mathcal{G}_{q_\omega}(f)-\langle f,\omega\rangle_{_F}\,\tau_\sigma+\tilde{a}\omega$$

and hence,

$$\tilde{a}=||\omega||_{\widehat{\sigma}}^{-2}\big[\langle f,\omega\rangle_{_F}\,\langle\tau_\sigma,\omega\rangle_{\widehat{\sigma}}-\langle\mathcal{G}_{q_\omega}(f),\widehat{\sigma}\omega\rangle_{_F}\big]=\mathcal{E}_{q_\omega}(\tau_\sigma,\tau_{\widehat{\sigma}})\langle f,\omega\rangle_{_F}-\langle f,\tau_{\widehat{\sigma}}\rangle_{_F}.$$

Finally, $\mathcal{G}_{q_\omega}^{\sigma,\widehat{\sigma}}$ is self–adjoint on $\mathcal{C}(F)$ iff $G_{q_\omega}^{\sigma,\widehat{\sigma}}(x,y)=G_{q_\omega}^{\sigma,\widehat{\sigma}}(y,x)$ for any $x,y\in F$ and this happens iff

$$\big(\tau_\sigma(x)-\tau_{\widehat{\sigma}}(x)\big)\omega(y)=\big(\tau_\sigma(y)-\tau_{\widehat{\sigma}}(y)\big)\omega(x)\quad x,y\in F;$$

that is, iff $\tau_{\widehat{\sigma}}=\tau_\sigma+a\omega$ on F and hence iff $\tau_{\widehat{\sigma}}=\tau_\sigma$ on F since $\tau_{\widehat{\sigma}},\tau_\sigma\in\omega^{\perp_{\sigma_q}}$. Therefore, $\mathcal{G}_{q_\omega}^{\sigma,\widehat{\sigma}}$ is self–adjoint on $\mathcal{C}(F)$ iff $\widehat{\sigma}=\sigma$.

Finally, it suffices to note that

$$\zeta_\sigma=-\tau_\sigma+||\omega||_\sigma^{-2}\langle\tau_\sigma,\omega\rangle_\sigma\omega\quad\text{and}\quad\mathcal{E}_{q_\omega}(\tau_\sigma,\tau_\sigma)=\mathcal{E}_{q_\omega}(\zeta_\sigma,\zeta_\sigma)=||\omega||_\sigma^{-2}\langle\tau_\sigma,\omega\rangle_\sigma.$$

$\square$

Now, we consider eigenvalue problems for boundary value problems with respect to a weight. As far as we know this is the first time that this type of problems is considered in the discrete setting.

A real number $\mu \in \mathbb{R}$ is named *eigenvalue of the BVP* (4), *with respect to* $\sigma \in \Omega_F$, if there exists a non-null function $v \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(v) = \mu\sigma v \quad \text{on} \quad F, \qquad \frac{\partial v}{\partial \mathsf{n}_\mathsf{F}} + qv = 0 \quad \text{on} \quad F_N \qquad \text{and} \qquad v = 0 \quad \text{on} \quad F_D. \quad (10)$$

Equivalently, $\mu \in \mathbb{R}$ is an eigenvalue of (10) iff $\mathcal{V}^H_{q-\mu\sigma}$ is a non trivial subspace; that is, the problem is not regular.

If μ is an eigenvalue we say that v is an *associated eigenfunction* if v satisfies the equalities in (10). In addition, $\mathcal{V}(\mu) \subset \mathcal{V}_q$ denotes the subspace of eigenfunctions associated with μ. Observe that $v \in \mathcal{C}(\bar{F})$ is an eigenfunction associated with μ iff $\mathcal{L}_q\big(\gamma(v\chi_F)\big) = \mu\sigma v$ on F.

The self-adjointness of the BVP (4) leads to the following results.

Lemma 7 *Given $\mu, \widehat{\mu}$ two eigenvalues of the BVP* (4)*, then*

$$\mu\langle u, v\rangle_\sigma = \mathcal{E}^F_q(u, v) = \widehat{\mu}\langle u, v\rangle_\sigma \quad \textit{for any } u \in \mathcal{V}(\mu) \textit{ and } v \in \mathcal{V}(\widehat{\mu}).$$

In particular, $\mathcal{E}^F_q(u, u) = \mu||u||^2_\sigma$ and when $\mu \neq \widehat{\mu}$, then $\mathcal{V}(\mu)$ and $\mathcal{V}(\widehat{\mu})$ are orthogonal each other with respect to σ.

Bearing in mind that the finite dimensionality of $\mathcal{V}_q$ implies that it is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_\sigma$ and moreover that $\mathcal{L}_q$ is a *Hilbert-Schmidt operator*, we can prove that there exist an orthonormal basis of eigenfunctions. Although we could prove this result by using the Rayleigh quotient, see for instance [13] for a specific weight, we give here a direct proof based on the standard Spectral Theorem.

Theorem 2 (Hilbert-Schmidt Theorem) *Given $\sigma \in \Omega_F$, there exist a sequence $\mu_1 \leq \cdots \leq \mu_{|F|}$ and $\{v_j\}_{j=1}^{|F|} \in \mathcal{V}_{q_\omega}$ an orthonormal system, with respect to σ, such that*

(i) $\mathcal{L}_{q_\omega}(v_j) = \mu_j\sigma v_j$, $j = 1, \ldots, |F|$.

(ii) $0 = \mu_1 < \mu_2$*; that is, $\mu_1 = 0$ is a simple eigenvalue and moreover* $v_1 = ||\omega||^{-1}_\sigma\omega$.

(iii) *If μ is an eigenvalue of the BVP* (4) *for $q = q_\omega$ with respect to σ, then $\mu = \mu_j$ for some $j = 1, \ldots, |F|$.*

(iv) $\mathcal{L}_{q_\omega}(u) = \sigma \displaystyle\sum_{j=2}^{|F|} \mu_j\langle u, v_j\rangle_\sigma v_j$ *on F for any $u \in \mathcal{V}_{q_\omega}$.*

Proof As in the proof of Proposition 4, let us consider the operator $\mathcal{K}_\sigma : \mathcal{C}(F) \longrightarrow \mathcal{C}(F)$, defined for any $u \in \mathcal{C}(F)$ as

$$\mathcal{K}_\sigma(u) = \sigma^{-\frac{1}{2}}\mathcal{L}_{q_\omega}\big(\gamma(\sigma^{-\frac{1}{2}}u)\big).$$

Then, $\mathcal{K}_\sigma$ is self-adjoint and positive semi-definite with respect to the standard inner product in $\mathcal{C}(F)$. By applying the Spectral Theorem to $\mathcal{K}_\sigma$ we get that, there exist $0 \le \mu_1 \le \cdots \le \mu_{|F|}$ and $\{u_j\}_{j=1}^{|F|} \in \mathcal{C}(F)$ an orthonormal system such that

(a) $\mathcal{K}_\sigma(u_j) = \mu_j u_j$, $j = 1, \ldots, |F|$.

(b) If μ is an eigenvalue of the $\mathcal{K}_\sigma$, then $\mu = \mu_j$ for some $j = 1, \ldots, |F|$.

On the other hand, if we denote by K_σ the matrix associated with $\mathcal{K}_\sigma$, then K_σ is an irreducible Z-matrix and hence its first eigenvalue, μ_1, is simple and the corresponding eigenvector, u_1, is positive.

Therefore, if we define $v_j = \sigma^{-\frac{1}{2}} u_j$ for each $j = 1, \ldots, |F|$, then $\mathcal{L}_{q_\omega}(\gamma(v_j)) = \mu_j \sigma v_j$ on F and $\{v_j\}_{j=1}^{|F|}$ is an orthonormal basis of eigenfunctions with respect to σ in $\mathcal{C}(F)$. Since $\mathcal{L}_{q_\omega}(\omega) = 0$, necessarily $\mu_1 = 0$ and $v_1 = ||\omega||_\sigma^{-1}\omega$. □

We call *sequence of eigenvalues* and *orthonormal basis of eigenfunctions*, with respect to σ, for the BVP (4) for $q = q_\omega$, to $\{\mu_j^\sigma\}_{j=1}^{|F|}$ and $\{v_j^\sigma\}_{j=1}^{|F|}$ given in the Hilbert-Schmidt Theorem. When σ is constant; that is $\sigma = \chi_{_F}$, then we drop the symbol σ in the above sequences.

Corollary 1 *Let $q \in \mathcal{Q}_\omega(F \cup F_{_N})$ such that $q = q_\omega + \lambda\sigma$ on F, where $\lambda > 0$ and $\sigma \in \Omega_{_F}$ and consider $\{\mu_j^\sigma\}_{j=1}^{|F|}$ and $\{v_j^\sigma\}_{j=1}^{|F|}$ the sequence of eigenvalues and orthonormal basis of eigenfunctions, with respect to σ. Then,*

(i) $\mathcal{L}_q(v_j^\sigma) = (\mu_j^\sigma + \lambda)\sigma v_j^\sigma$, $j = 1, \ldots, |F|$.

(ii) $\mathcal{L}_q(u) = \sigma \displaystyle\sum_{j=1}^{|F|} (\mu_j^\sigma + \lambda)\langle u, v_j^\sigma\rangle_\sigma v_j^\sigma$ *on F for any $u \in \mathcal{V}_q$.*

Our next objective is to get the discrete version of Mercer Theorem; that is, the expressions for Green's functions in terms of eigenvalues and eigenfunctions. We start with a technical Lemma that will be useful later.

Lemma 8 *Let $\sigma \in \Omega_{_F}$ and consider $\{\mu_j^\sigma\}_{j=1}^{|F|}$ and $\{v_j^\sigma\}_{j=1}^{|F|}$ the sequence of eigenvalues and orthonormal basis of eigenfunctions, with respect to σ. Then, $\tau_\sigma = ||\omega||_\sigma^{-2}\mathcal{G}_{q_\omega}(\sigma\omega)$ is given by*

$$\tau_\sigma = \omega||\omega||_{_F}^{-4} \sum_{j=2}^{|F|} (\mu_j^\sigma)^{-1} \langle v_j^\sigma, \omega\rangle_{_F}^2 - ||\omega||_{_F}^{-2} \sum_{j=2}^{|F|} (\mu_j^\sigma)^{-1} \langle v_j^\sigma, \omega\rangle_{_F} v_j^\sigma$$

and hence, $\mathcal{E}_{q_\omega}(\tau_\sigma, \tau_\sigma) = ||\omega||_{_F}^{-4} \sum\limits_{j=2}^{|F|} (\mu_j^\sigma)^{-1} \langle v_j^\sigma, \omega\rangle_{_F}^2$.

Proof Since $\tau_\sigma \in \mathcal{V}_{q_\omega}$ and $\mathcal{L}_{q_\omega}(\tau_\sigma) = ||\omega||_\sigma^{-2}\sigma\omega - ||\omega||_{_F}^{-2}\omega$, we have that

$$\mathcal{E}_{q_\omega}(\tau_\sigma, \tau_\sigma) = ||\omega||_\sigma^{-2} \langle \tau_\sigma, \omega\rangle_\sigma = ||\omega||_\sigma^{-1} \langle \tau_\sigma, v_1^\sigma\rangle_\sigma.$$

On the other hand, if $\tau_\sigma = \sum_{j=1}^{|F|} \langle \tau_\sigma, v_j^\sigma \rangle_\sigma v_j^\sigma$, then

$$||\omega||_\sigma^{-2}\sigma\omega - ||\omega||_F^{-2}\omega = \mathcal{L}_{q_\omega}(\tau_\sigma) = \sum_{j=2}^{|F|} \mu_j^\sigma \langle \tau_\sigma, v_j^\sigma \rangle_\sigma \sigma v_j^\sigma,$$

which for any $k = 2, \dots, |F|$ implies that

$$-||\omega||_F^{-2}\langle v_k^\sigma, \omega\rangle_F = \langle v_k^\sigma, ||\omega||_\sigma^{-2}\sigma\omega - ||\omega||_F^{-2}\omega\rangle_F = \sum_{j=2}^{|F|} \mu_j^\sigma \langle \tau_\sigma, v_j^\sigma\rangle_\sigma \langle v_j^\sigma, v_k^\sigma\rangle_\sigma = \mu_k^\sigma \langle \tau_\sigma, v_k^\sigma\rangle_\sigma.$$

In addition, since $\langle \tau_\sigma, \omega\rangle_F = 0$ we also have that

$$\langle \tau_\sigma, v_1^\sigma\rangle_\sigma = -||\omega||_F^{-2}||\omega||_\sigma \sum_{j=2}^{|F|} \langle \tau_\sigma, v_j^\sigma\rangle_\sigma \langle v_j^\sigma, \omega\rangle_F = ||\omega||_F^{-4}||\omega||_\sigma \sum_{j=2}^{|F|} (\mu_j^\sigma)^{-1}\langle v_j^\sigma, \omega\rangle_F^2$$

and the results follow. □

Corollary 2 (Mercer Theorem) *Let $\sigma \in \Omega_F$ and consider $\{\mu_j^\sigma\}_{j=1}^{|F|}$ and $\{v_j^\sigma\}_{j=1}^{|F|}$ the sequence of eigenvalues and orthonormal basis of eigenfunctions, with respect to σ. Then,*

$$G_{q_\omega}^{\sigma,\sigma}(x,y) = \sum_{j=2}^{|F|} (\mu_j^\sigma)^{-1} v_j^\sigma(x) v_j^\sigma(y), \qquad x \in V,\ y \in F$$

and hence,

$$\begin{aligned} G_{q_\omega}(x,y) =& \sum_{j=2}^{|F|} (\mu_j^\sigma)^{-1} v_j^\sigma(x) v_j^\sigma(y) \\ & - ||\omega||_F^{-2} \sum_{j=2}^{|F|} (\mu_j^\sigma)^{-1} \langle v_j^\sigma, \omega\rangle_F \big(v_j^\sigma(x)\omega(y) + \omega(x) v_j^\sigma(y)\big) \\ & + ||\omega||_F^{-4}\omega(x)\omega(y) \sum_{j=2}^{|F|} (\mu_j^\sigma)^{-1} \langle v_j^\sigma, \omega\rangle_F^2, \qquad x \in V,\ y \in F. \end{aligned}$$

In addition, if for any $\lambda > 0$ we consider $q = q_\omega + \lambda\sigma$, then $q \in \mathcal{Q}_\omega(F \cup F_N)$ and moreover

$$G_q(x,y) = \sum_{j=2}^{|F|} (\mu_j^\sigma + \lambda)^{-1} v_j^\sigma(x) v_j^\sigma(y) \quad \textit{and} \quad L_q^{-1}(x,y) = \sum_{j=1}^{|F|} (\mu_j^\sigma + \lambda)^{-1} v_j^\sigma(x) v_j^\sigma(y),$$

$x \in V,\ y \in F$.

Proof Given $y \in F$, if $\lambda > 0$, $u = (G_q)_y$ is the unique function in $\omega^{\perp_\sigma}$ satisfying that $\mathcal{L}_q(u) = \varepsilon_y - ||\omega||_\sigma^{-2}\omega(y)\sigma\omega$ on F, whereas $\hat{u} = \big(G_{q_\omega}^{\sigma,\sigma}\big)_y$ is the unique function in $\omega^{\perp_\sigma}$ satisfying that $\mathcal{L}_{q_\omega}(u) = \varepsilon_y - ||\omega||_\sigma^{-2}\omega(y)\sigma\omega$ on F.

Since $\{v_j^\sigma\}_{j=1}^{|F|}$ is an orthonormal basis of $\mathcal{V}_q$, we know that $u = \sum_{j=1}^{|F|}\langle u, v_j^\sigma\rangle_\sigma v_j^\sigma$, $\hat{u} = \sum_{j=1}^{|F|}\langle \hat{u}, v_j^\sigma\rangle_\sigma v_j^\sigma$ and moreover, $\langle u, v_1^\sigma\rangle_\sigma = \langle \hat{u}, v_1^\sigma\rangle_\sigma = 0$, because $u, \hat{u} \in \omega^{\perp_\sigma}$ and $v_1^\sigma = ||\omega||_\sigma^{-1}\omega$. In addition,

$$\varepsilon_y - v_1^\sigma(y)\sigma v_1^\sigma = \varepsilon_y - ||\omega||_\sigma^{-2}\omega(y)\sigma\omega = \mathcal{L}_q(u) = \sum_{j=2}^{|F|}(\mu_j^\sigma + \lambda)\langle u, v_j^\sigma\rangle_\sigma \sigma v_j^\sigma,$$
$$\varepsilon_y - v_1^\sigma(y)\sigma v_1^\sigma = \varepsilon_y - ||\omega||_\sigma^{-2}\omega(y)\sigma\omega = \mathcal{L}_{q_\omega}(\hat{u}) = \sum_{j=2}^{|F|}\mu_j^\sigma\langle \hat{u}, v_j^\sigma\rangle_\sigma \sigma v_j^\sigma,$$

which for any $k = 2, \dots, |F|$ implies that

$$v_k^\sigma(y) = \langle v_k^\sigma, \varepsilon_y - v_1^\sigma(y)\sigma v_1^\sigma\rangle_{F} = \sum_{j=2}^{|F|}(\mu_j^\sigma + \lambda)\langle u, v_j^\sigma\rangle_\sigma\langle \sigma v_j^\sigma, v_k^\sigma\rangle_{F} = (\mu_k^\sigma + \lambda)\langle u, v_k^\sigma\rangle_\sigma,$$
$$v_k^\sigma(y) = \langle v_k^\sigma, \varepsilon_y - v_1^\sigma(y)\sigma v_1^\sigma\rangle_{F} = \sum_{j=2}^{|F|}\mu_j^\sigma\langle \hat{u}, v_j^\sigma\rangle_\sigma\langle \sigma v_j^\sigma, v_k^\sigma\rangle_{F} = \mu_k^\sigma\langle \hat{u}, v_k^\sigma\rangle_\sigma,$$

and hence,

$$u = \sum_{j=2}^{|F|}\langle u, v_j^\sigma\rangle_\sigma v_j^\sigma = \sum_{j=2}^{|F|}(\mu_j^\sigma + \lambda)^{-1}v_j^\sigma(y)v_j^\sigma$$

and

$$\hat{u} = \sum_{j=2}^{|F|}\langle \hat{u}, v_j^\sigma\rangle_\sigma v_j^\sigma = \sum_{j=2}^{|F|}(\mu_j^\sigma)^{-1}v_j^\sigma(y)v_j^\sigma.$$

Applying now Proposition 8, for any $x \in V$ and any $y \in F$, we have that

$$\begin{aligned} L_q^{-1}(x,y) =& G_q(x,y) + \lambda^{-1}||\omega||_\sigma^{-2}\omega(x)\omega(y) \\ =& \sum_{j=2}^{|F|}(\mu_k^\sigma + \lambda)^{-1}v_j^\sigma(y)v_j^\sigma(x) + \lambda^{-1}v_1^\sigma(x)v_1^\sigma(y), \end{aligned}$$

and the claimed expression for L_q^{-1} follows taking into account that $\mu_1^\sigma = 0$.

On the other hand, applying again Proposition 8, for any $x \in V$ and any $y \in F$, we have

$$\begin{aligned} G_{q_\omega}(x,y) =& \sum_{j=2}^{|F|}(\mu_j^\sigma)^{-1}v_j^\sigma(x)v_j^\sigma(y) \\ &+\tau_\sigma(x)\omega(y) + \omega(x)\tau_\sigma(y) - \mathcal{E}_{q_\omega}(\tau_\sigma, \tau_\sigma)\omega(x)\omega(y), \end{aligned}$$

and hence, the expression for G_{q_ω} follows from the identities in Lemma 8. $\square$

5 Green functions and spectrum for a Dirichlet-Robin problem in a star-like network

Here we apply the results of the above section for computing the Green functions and the spectrum of a star-like network. Given the network $\Gamma = (V, c)$, for any $x \in V$ we denote by $V(x)$ the set of vertices adjacent to x. We say that *Γ is a star-like network with center* x_0, if $V(x_0)$ is an independent set; that is, no two vertices in $V(x_0)$ are adjacent. A star network with center x_0 is the most obvious example of star-like network with center x_0. More generally, any weighted tree is a star-like network whose center can be any of its vertices. Moreover, any distance-regular graph with diameter at least 2 and parameter $a_1 = 0$ is also a star-like network.

In the rest of this section we assume that Γ is a star-like network with center at x_0 and moreover that $V(x_0) = \{x_1, \dots, x_n\}$. With the notation of the above sections, $F = V(x_0)$, $F_{_N} = \{x_0\}$ and $F_{_D} = \bigcup\limits_{j=1}^{n} \big(V(x_i) \setminus \{x_0\}\big)$. Notice that $F_{_D} = \emptyset$ iff Γ is a star whose center is x_0.

Fixed $\omega \in \Omega(F \cup F_{_N})$ and $\sigma \in \Omega_{_F}$, for any $t \in \mathbb{R}$ we consider the potential $q = q_\omega + t\sigma$ and also the corresponding Schrödinger operator $\mathcal{L}_q$. The Mixed Dirichlet-Robin Boundary Problem we raise is for $j = 1, \dots, n$,

$$\mathcal{L}_q(u)(x_j) = f(x_j), \quad \frac{\partial u}{\partial n_{_F}}(x_0) + q_\omega(x_0)u(x_0) = 0 \text{ and } u = 0 \text{ on } F_{_D}. \tag{11}$$

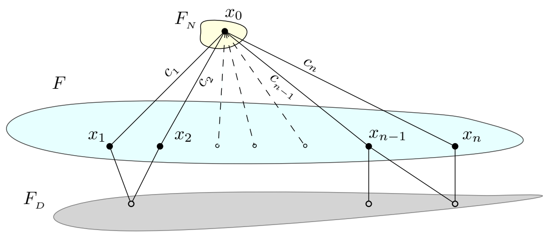

Fig. 2 Example of a star-like network.

Our goal is to obtain the Green functions of the above mixed problem when $t \geq 0$, its eigenvalues and their corresponding orthonormal basis of eigenfunctions, with respect to σ when $t = 0$ and the kernel $(L_q)^{-1}$ when the problem is regular. Recall that the eigenvalues and the eigenfunctions with respect to σ for Problem (11) can be obtained easily from the corresponding to $t = 0$ and that the BVP is regular iff $-t$ is not an eigenvalue of Problem (11) for $q = q_\omega$. In particular, the problem is regular when $t > 0$; that is, the case in which the Schödinger operator $\mathcal{L}_q$ is positive definite. Moreover, when $t > 0$, we apply Proposition 8 to compute $(L_q)^{-1}$ from the Green function, whereas when $t < 0$ and the problem is regular, we compute $(L_q)^{-1}$ directly. Of course, we could use this methodology also in the case $t > 0$.

For any $t \in \mathbb{R}$ the value $t^{\#}$ is defined as t^{-1} when $t \neq 0$ and 0 when $t = 0$. For the sake of simplicity, we denote $c_i = c(x_i, x_0) > 0$, $i = 1, \ldots, n$. Moreover, given an arbitrary function $u \in \mathcal{C}(V)$, we denote $u_i = u(x_i)$, $i = 0, \ldots, n$ and $\langle \mathsf{c}, u\rangle = \sum\limits_{j=1}^{n} c_j u_j$. With this notation, we have

$$q_\omega(x_i) = -\kappa_i + \frac{c_i \omega_0}{\omega_i}, \quad i = 1, \ldots, n \quad \text{and} \quad q_\omega(x_0) = -\langle \mathsf{c}, 1\rangle + \frac{\langle \mathsf{c}, \omega\rangle}{\omega_0}.$$

To determine all the Green functions defined in the above section, it will be useful to define for any $\nu \in \Omega(F)$ the function

$$\eta_\nu = \frac{\sigma}{||\omega||^2_\sigma} - \frac{\nu}{||\omega||^2_\nu} \in \mathcal{C}(F).$$

In addition, we define the functions $Q\colon \mathbb{R} \times \mathcal{C}(F) \times \mathcal{C}(F) \longrightarrow (0, +\infty)$ and $\Psi\colon \mathbb{R} \longrightarrow \mathbb{R}$ as

$$Q(t, u, v) = \sum_{j=1}^{n} (c_j \omega_0 + t\sigma_j \omega_j)^{\#} \omega_j^3 u_j v_j \qquad \text{and} \qquad \Psi(t) = ||\omega||^2_\sigma - tQ(t, \sigma, \sigma).$$

When $\sigma = \chi_{_F}$, function Q was already defined by the authors in [11] to compute the Green function of a star. The new Q here introduced is a generalization of the former to include the mixed boundary conditions. Related with Q, we consider the set $A = \left\{ \dfrac{c_j \omega_0}{\sigma_j \omega_j} \right\}_{j=1}^{n}$ and $m = |A|$. Observe that $1 \le m \le n$, and we can suppose without loss of generality that $A = \big\{a_1, \ldots, a_m\big\}$, where $0 < a_1 < \cdots < a_m$. If for any $j = 1, \ldots, m$, we consider $I_j = \Big\{ i \in \{1, \ldots, n\} : \dfrac{c_i \omega_0}{\sigma_i \omega_i} = a_j \Big\}$ and $m_j = \big|I_j\big|$, then $m_j \ge 1$ and moreover, $\sum\limits_{j=1}^{m} m_j = n$, since $I_1, \ldots, I_m$ is a partition of $\{1, \ldots, n\}$. Moreover, for any $j = 1, \ldots, m$, we define $W_j = \sum\limits_{i \in I_j} \omega_i^2 \sigma_i$. With these notations, we have that $||\omega||^2_\sigma = \sum\limits_{j=1}^{m} W_j$, $Q(t, \sigma, \sigma) = \sum\limits_{j=1}^{m} W_j (a_j + t)^{\#}$ and

$$\Psi(t) = \begin{cases} \displaystyle\sum_{j=1}^{m} \frac{a_j W_j}{a_j + t}, \text{ if } -t \notin A, \\ \displaystyle W_k + \sum_{\substack{j=1 \\ j \neq k}}^{m} \frac{a_j W_j}{a_j - a_k}, \text{ if } t = -a_k, \text{ for some } k = 1, \ldots, m, \end{cases}$$

which implies that Ψ is continuous and decreasing on $\mathbb{R}\setminus\{-a_1,\dots,-a_m\}$, positive for $t>-a_1$ and negative for $t<-a_m$. Moreover, for any $j=1,\dots,m-1$, Ψ has a unique zero, say $-\mu_{j+1}$, in $(-a_{j+1},-a_j)$.

Proposition 9 *Assume that $t\geq 0$. If $f\in\mathcal{C}(F)$ is such that $f\in\omega^{\perp}$, then the unique solution of the Mixed Dirichlet-Robin problem* (11) *such that $u\in\omega^{\perp_\sigma}$ is given by*

$$\begin{aligned}u_0 =& -\frac{\omega_0 Q(t,\omega^{-1}\sigma,f)}{\Psi(t)},\\ u_i =& \frac{\omega_i a_j}{\Psi(t)(a_j+t)}\Big[\frac{f_i}{c_i\omega_0}\Psi(t)-Q(t,\omega^{-1}\sigma,f)\Big], \qquad i\in I_j,\ j=1,\dots,m.\end{aligned}$$

Proof From (11), we get

$$\kappa_i u_i - c_i u_0 + (q_\omega(x_i)+t\sigma_i)u_i = f_i,\ \ i=1,\dots,n \ \ \text{and} \quad -\langle\mathsf{c},u\rangle+\langle\mathsf{c},\omega\rangle\frac{u_0}{\omega_0}=0$$

which implies that

$$u_0=\omega_0\frac{\langle\mathsf{c},u\rangle}{\langle\mathsf{c},\omega\rangle} \text{ and } u_i=\frac{\omega_i}{c_i\omega_0+t\sigma_i\omega_i}\left[f_i+\frac{\langle\mathsf{c},u\rangle}{\langle\mathsf{c},\omega\rangle}c_i\omega_0\right],\ i=1,\dots,n. \tag{12}$$

Therefore,

$$\langle u,\omega\rangle_\sigma=\sum_{j=1}^n u_j\omega_j\sigma_j=\left[\omega_0\frac{\langle\mathsf{c},u\rangle}{\langle\mathsf{c},\omega\rangle}\sum_{j=1}^n\frac{c_j\omega_j^2\sigma_j}{c_j\omega_0+t\sigma_j\omega_j}+\sum_{j=1}^n\frac{f_j\omega_j^2\sigma_j}{c_j\omega_0+t\sigma_j\omega_j}\right],$$

which implies that $u\in\omega^{\perp_\sigma}$ iff

$$\frac{\langle\mathsf{c},u\rangle}{\langle\mathsf{c},\omega\rangle}=-\frac{Q(\lambda,\omega^{-1}\sigma,f)}{\Psi(t)}$$

and hence, the expression for u_i results by substituting the value of u_0 in the above expression. □

The desired expression for the Green function for Problem (11), can be obtained from the above Proposition by taking $f=\varepsilon_\ell-\dfrac{\omega_\ell}{||\omega||_\sigma^2}\sigma\omega$, for any $\ell=1,\dots,n$.

Corollary 3 *If $t\geq 0$, the Green function for the Mixed Dirichlet-Robin Problem* (11) *is given for $i\in I_j,\ \ell\in I_k$ by*

$$\begin{aligned}G_q(x_0,x_\ell)=&\frac{\omega_0\omega_\ell}{||\omega||_\sigma^2(a_k+t)}\left[\frac{a_kQ(t,\sigma,\sigma)}{\Psi(t)}-1\right],\\ G_q(x_i,x_\ell)=&\frac{\omega_i\omega_\ell}{||\omega||_\sigma^2(a_j+t)(a_k+t)}\left[\frac{a_ja_kQ(t,\sigma,\sigma)}{\Psi(t)}-a_j-a_k-t\right]+\frac{\varepsilon_{x_\ell}(x_i)}{\sigma_i(a_j+t)}.\end{aligned}$$

Moreover, for any $\nu \in \Omega(F)$ we have

$$\begin{aligned}\tau_\nu(x_0) &= \frac{\omega_0}{\Psi(t)} Q(t,\sigma,\eta_\nu),\\ \tau_\nu(x_i) &= \frac{\omega_i}{\Psi(t)\big[a_j+t\big]}\Big[\sigma_i^{-1}\eta_\nu(x_i)\Psi(t) + a_j Q(t,\sigma,\eta_\nu)\Big], \qquad i \in I_j.\end{aligned}$$

In addition, when $t>0$, then

$$\begin{aligned}(L_q)^{-1}(x_0,x_\ell) &= \frac{\omega_0\omega_\ell a_k}{t\Psi(t)(a_k+t)}, & \ell \in I_k,\\ (L_q)^{-1}(x_i,x_\ell) &= \frac{\omega_i\omega_\ell a_j a_k}{t\Psi(t)(a_j+t)(a_k+t)} + \frac{1}{\sigma_i(a_j+t)}\varepsilon_{x_\ell}(x_i), & i\in I_j,\ \ell \in I_k,\end{aligned}$$

whereas when $t=0$, given $\widehat{\nu} \in \Omega(F)$, we have

$$\mathcal{E}_q(\tau_\nu, \tau_{\widehat{\nu}}) = Q(0,\eta_\nu,\eta_{\widehat{\nu}})$$

and hence,

$$\begin{aligned}\zeta_\nu(x_0) &= -\frac{\omega_0}{||\omega||_\nu^2} Q(0,\nu,\eta_\nu),\\ \zeta_\nu(x_i) &= \frac{\omega_i\eta_\nu(x_i)}{a_j\sigma_i\omega_0} - \frac{\omega_i}{||\omega||_\nu^2} Q(0,\nu,\eta_\nu), \qquad i \in I_j.\end{aligned}$$

Next we compute the eigenvalues and the corresponding eigenfunctions with respect to the weight σ for the Mixed Dirichlet-Robin problem

$$\mathcal{L}_{q_\omega}(v) = \mu\sigma v \ \text{ on } \ F, \quad \frac{\partial v}{\partial \mathsf{n}_{\mathsf{F}}}(x_0) + q_\omega(x_0)v(x_0) = 0 \quad \text{and} \quad v=0 \ \text{ on } \ F_D. \tag{13}$$

Recall that in this case,

$$q_\omega(x_i) = -\kappa_i + \frac{c_i\omega_0}{\omega_i} \quad \text{and} \quad q_\omega(x_0) = -\kappa_0 + \frac{\langle \omega, \mathsf{c}\rangle}{\omega_0}.$$

From the Robin condition, we get $\dfrac{v_0}{\omega_0} = \dfrac{\langle \mathsf{c}, v\rangle}{\langle \mathsf{c},\omega\rangle}$, whereas from the first equation on Problem (13) we obtain

$$\mathcal{L}_{q_\omega}(v)(x_i) = c_i\Big(-v_0 + \frac{\omega_0}{\omega_i}v_i\Big) = \mu\sigma_i v_i$$

which implies that

$$\Big[c_i\omega_0 - \mu\sigma_i\omega_i\Big]\frac{v_i}{\omega_i} = c_i v_0 = c_i\omega_0\,\frac{\langle \mathsf{c}, v\rangle}{\langle \mathsf{c},\omega\rangle}, \quad i = 1,\dots,n. \tag{14}$$

If $\mu = a_j$ for some $j = 1,\dots,m$ then, from Equation (14), we get that $v_0 = 0$, which implies that $\langle \mathsf{c}, v\rangle = 0$. Moreover, for any $i \notin I_j$, we have that $v_i = 0$.

If $m_j = 1$, then the identity $\langle \mathsf{c}, v\rangle = 0$ determines that $v = 0$, so a_j cannot be an eigenvalue.

If $m_j > 1$, then a_j is an eigenvalue and $\mathcal{V}(a_j) = \operatorname{span}\Big\{\dfrac{\varepsilon_{j_{m_j}}}{c_{j_{m_j}}} - \dfrac{\varepsilon_{j_s}}{c_{j_s}} : s = 1,\dots,m_j-1\Big\}$ is the corresponding space of eigenfunctions. Therefore, the multiplicity of a_j is $m_j - 1$.

In conclusion, we have obtained $\sum\limits_{j=1}^{m}(m_j-1) = n-m$ linearly independent eigenfunctions and hence $n-m$ different eigenvalues at most.

Moreover, an orthonormal basis of the subspace $\mathcal{V}(a_j)$ with respect to σ, is $v^r = \dfrac{u^r}{||u^r||_\sigma}$, where

$$u^r = c_{j_r}\sum_{s=1}^{r-1}\omega_{j_s}\varepsilon_{j_s} - \Big(c_{j_{m_j}}\omega_{j_{m_j}} + \sum_{s=1}^{r-1} c_{j_s}\omega_{j_s}\Big)\varepsilon_{j_r} + c_{j_r}\omega_{j_{m_j}}\varepsilon_{j_{m_j}},$$

$$||u^r||^2_\sigma = \sigma_{j_r}\Big(\sum_{t=1}^{r-1} c_{j_t}\omega_{j_t} + c_{j_{m_j}}\omega_{j_{m_j}}\Big)\Big(\sum_{t=1}^{r} c_{j_t}\omega_{j_t} + c_{j_{m_j}}\omega_{j_{m_j}}\Big).$$

for any $r = 1,\dots,m_j-1$.

To find the remaining m eigenfunctions, let first define the function $\Phi\colon \mathbb{R}\setminus A \longrightarrow \mathbb{R}$ given by

$$\Phi(t) = t\Psi(-t) = t\sum_{j=1}^{m}\frac{a_jW_j}{a_j-t}.$$

Then,

$$\Phi'(t) = \sum_{j=1}^{m}\frac{a_j^2W_j}{(a_j-t)^2}$$

and in particular, $\Phi'(0) = ||\omega||^2_\sigma$.

Consider now an eigenvalue $0 \le \mu \notin A$ of the Problem (13). If v is a non-null eigenfunction associated with μ, from Equation (14), necessarily $v_0 \neq 0$ and hence $\langle \mathsf{c}, v\rangle \neq 0$. Moreover,

$$v_i = \frac{\langle \mathsf{c}, v\rangle c_i\omega_0\omega_i}{\langle \mathsf{c},\omega\rangle\big(c_i\omega_0 - \mu\sigma_i\omega_i\big)} = \frac{\langle \mathsf{c}, v\rangle\omega_i a_j}{\langle \mathsf{c},\omega\rangle(a_j-\mu)}, \qquad i\in I_j,\ j = 1\dots,m$$

which implies that

$$\langle \mathsf{c}, v\rangle = \frac{\omega_0\langle \mathsf{c}, v\rangle}{\langle \mathsf{c},\omega\rangle}\sum_{j=1}^{n}\frac{c_j^2\omega_j}{c_j\omega_0-\mu\sigma_j\omega_j} = \frac{\langle \mathsf{c}, v\rangle}{\langle \mathsf{c},\omega\rangle}\sum_{j=1}^{m}\frac{a_j}{a_j-\mu}\sum_{i\in I_j}\omega_i c_i$$

and hence

$$0 = \sum_{j=1}^{m}\frac{a_j}{a_j-\mu}\sum_{i\in I_j}\omega_i c_i - \langle \mathsf{c},\omega\rangle = \sum_{j=1}^{m}\frac{\mu}{a_j-\mu}\sum_{i\in I_j}\omega_i c_i = \frac{\mu}{\omega_0}\sum_{j=1}^{m}\frac{a_jW_j}{a_j-\mu} = \frac{\Phi(\mu)}{\omega_0}.$$

As a conclusion, $0 \le \mu \notin A$ is an eigenvalue of Problem (13) iff it is a zero of Φ; that is, iff either $\mu = 0 = \mu_1$ or $\mu = \mu_j$, $j = 2, \dots, m$. Since $\dim \mathcal{V}(\mu) \ge 1$, we conclude that $\mu_1, \dots, \mu_m$ are simple eigenvalues. Moreover the normalized eigenfunction corresponding to μ_j is u^j, where

$$\begin{aligned} u_0^j =& \frac{\omega_0}{\sqrt{\Phi'(\mu_j)}}, && j = 1, \dots, m. \\ u_i^j =& \frac{\omega_i a_k}{\sqrt{\Phi'(\mu_j)}\,(a_k - \mu_j)}, && i \in I_k, \quad j, k = 1, \dots, m. \end{aligned}$$

Notice that the above formula for $j = 1$ gives $u_i^1 = \dfrac{\omega_i}{\sqrt{\Phi'(0)}} = ||\omega||_\sigma^{-1}\omega_i$ as expected.

We end this paper considering the case $t < 0$ when $-t$ is not an eigenvalue of the Problem (13) with respect to σ.

Proposition 10 *Let $q = q_\omega + t\sigma$ when $t < 0$ and $\mu_1 = 0$. Then, the boundary value problem (11) is regular iff either $t \notin \{-a_j, -\mu_j\}_{j=1}^m$ or there exists $k = 1, \dots, m$ such that $t = -a_k$ and $m_k = 1$. Moreover, the following identities hold:*

(a) $t \notin \{-a_j, -\mu_j\}_{j=1}^m$, *then for* $\ell \in I_k,\ i \in I_j$

$$\begin{aligned} (L_q)^{-1}(x_0, x_\ell) =& \frac{\omega_0 \omega_\ell a_k}{t\Psi(t)(a_k + t)}, \\ (L_q)^{-1}(x_i, x_\ell) =& \frac{\omega_i \omega_\ell a_j a_k}{t\Psi(t)(a_j + t)(a_k + t)} + \frac{\varepsilon_{x_\ell}(x_i)}{\sigma_i(a_j + t)}. \end{aligned}$$

(b) If $t = -a_k$ *with* $m_k = 1$, *for some* $k = 1, \dots, m$ *and* $I_k = \{r\}$ *then*

$$\begin{aligned} (L_q)^{-1}(x_0, x_r) =& -\frac{1}{c_r}, \\ (L_q)^{-1}(x_r, x_r) =& \frac{a_k}{c_r^2 \omega_0^2}\Big[\Psi(-a_k) - 2\sigma_r \omega_r^2\Big], \\ (L_q)^{-1}(x_i, x_r) =& \frac{\omega_i a_j}{c_r \omega_0 (a_k - a_j)}, && i \in I_j,\ j \ne k \end{aligned}$$

whereas for $\ell \in I_s,\ s \ne k$,

$$\begin{aligned} (L_q)^{-1}(x_0, x_\ell) =& 0, \\ (L_q)^{-1}(x_r, x_\ell) =& \frac{\omega_\ell a_s}{c_r \omega_0 (a_k - a_s)}, \\ (L_q)^{-1}(x_i, x_\ell) =& \frac{\varepsilon_{x_\ell}(x_i)}{\sigma_i (a_k - a_j)},\ i \in I_j,\ j \ne k. \end{aligned}$$

Proof If $f \in \mathcal{C}(F)$, then the unique solution of the Mixed Dirichlet-Robin problem (11) is given by Equation (12)

$$u_0 = \omega_0 \frac{\langle \mathsf{c}, u\rangle}{\langle \mathsf{c}, \omega\rangle} \quad \text{and} \quad (c_i\omega_0 + t\sigma_i\omega_i)u_i = \omega_i\big[f_i + c_i u_0\big] = \omega_i\left[f_i + \frac{\langle \mathsf{c}, u\rangle}{\langle \mathsf{c}, \omega\rangle}\, c_i\omega_0\right],$$

for any $i = 1, \dots, n$.

(a) If $t \notin \{-a_j, -\mu_j\}_{j=1}^m$, then

$$u_0 = \omega_0 \frac{\langle \mathsf{c}, u\rangle}{\langle \mathsf{c}, \omega\rangle} \quad \text{and} \quad u_i = \frac{\omega_i}{a_j + t}\left[\frac{f_i}{\sigma_i\omega_i} + \frac{\langle \mathsf{c}, u\rangle}{\langle \mathsf{c}, \omega\rangle}\, a_j\right], \quad i \in I_j.$$

Therefore,

$$\langle \mathsf{c}, u\rangle = \sum_{s=1}^m \frac{1}{a_s + t} \sum_{r\in I_s} c_r f_r \sigma_r^{-1} + \frac{\langle \mathsf{c}, u\rangle}{\langle \mathsf{c}, \omega\rangle} \sum_{s=1}^m \frac{a_s}{a_s + t} \sum_{r\in I_s} c_i r\omega_r,$$

which implies that

$$\frac{\langle \mathsf{c}, u\rangle}{\langle \mathsf{c}, \omega\rangle}\left[\sum_{s=1}^m \frac{t}{a_s + t} \sum_{r\in I_s} c_r\omega_r\right] = \sum_{s=1}^m \frac{1}{a_s + t} \sum_{r\in I_s} c_r f_r \sigma_r^{-1}$$

and hence

$$\frac{\langle \mathsf{c}, u\rangle}{\langle \mathsf{c}, \omega\rangle} = \frac{\omega_0}{t\Psi(t)} \sum_{s=1}^m \frac{1}{a_s + t} \sum_{r\in I_s} c_r f_r \sigma_r^{-1}.$$

Therefore, we have obtained that for any $j = 1, \dots, m$ and $i \in I_j$,

$$\begin{aligned} u_0 =& \frac{\omega_0^2}{t\Psi(t)} \sum_{s=1}^m \frac{1}{a_s + t} \sum_{r\in I_s} c_r f_r \sigma_r^{-1}, \\ u_i =& \frac{\omega_i}{a_j + t}\left[\frac{f_i}{\sigma_i\omega_i} + \frac{\omega_0 a_j}{t\Psi(t)} \sum_{s=1}^m \frac{1}{a_s + t} \sum_{r\in I_s} c_r f_r \sigma_r^{-1}\right]. \end{aligned}$$

The desired expression for $(L_q)^{-1}$ can be obtained by taking $f = \varepsilon_\ell$, for any $\ell = 1, \dots, n$.

(b) If $t = -a_k$ with $m_k = 1$, for some $k = 1, \dots, \ell$ and $I_k = \{r\}$, then $u_0 = -\dfrac{f_r}{c_r}$ and for $i \in I_j, j = 1, \dots, m, j \neq k$

$$u_i = \frac{\omega_i\big[c_r f_i - c_i f_r\big]}{c_r\big[c_i\omega_0 + t\sigma_i\omega_i\big]} = \frac{f_i}{\sigma_i(a_j - a_k)} - \frac{\omega_i a_j f_r}{c_r\omega_0(a_j - a_k)}.$$

which implies that

$$-\frac{f_r}{c_r} = u_0 = \frac{\omega_0}{\langle \mathsf{c}, \omega\rangle}\left[c_r u_r + \frac{1}{\omega_0} \sum_{\substack{j=1\\ j\neq k}}^m \frac{a_j}{a_j - a_k} \sum_{i\in I_j} \omega_i f_i - \frac{f_r}{c_r\omega_0} \sum_{\substack{j=1\\ j\neq k}}^m \frac{a_j}{a_j - a_k} \sum_{i\in I_j} c_i\omega_i\right]$$

and hence

$$
\begin{aligned}
c_r u_r &= \frac{f_r}{c_r\omega_0}\left[-c_r\omega_r + \frac{a_k}{\omega_0}\sum_{\substack{j=1\\ j\neq k}}^m \frac{a_j W_j}{a_j-a_k}\right] - \frac{1}{\omega_0}\sum_{\substack{j=1\\ j\neq k}}^m \frac{a_j}{a_j-a_k}\sum_{i\in I_j}\omega_i f_i\\
&= \frac{f_r}{c_r\omega_0^2}\left[a_k\Psi(-a_k) - 2c_r\omega_0\omega_r\right] - \frac{1}{\omega_0}\sum_{\substack{j=1\\ j\neq k}}^m \frac{a_j}{a_j-a_k}\sum_{i\in I_j}\omega_i f_i.
\end{aligned}
$$

The desired expression for $(L_q)^{-1}$ can be obtained by taking $f = \varepsilon_\ell$, for any $\ell = 1,\dots,n$. □

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