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Title: The Rayleigh-Faber-Krahn inequality and applications

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To do this work, I got the support of very significant people, that without them the TFG would have been incredibly more difficult.

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Abstract

This work aims to go in-depth in the study of Rayleigh-Faber-Krahn inequality and its proof. This inequality solves the shape optimization problem for the first Dirichlet eigenvalue under a volume constraint. Its proof will be addressed using Rayleigh quotient and Pólya-Szegö inequality, that need from Sobolev spaces to be rigorously understood. Besides, there is an enormous attention to applications of Rayleigh-Faber-Krahn inequality. Specifically, branches such as music, finance, fluids' transport, and quantum mechanics will be studied. Finally, some concrete calculus will be seen: the wave equation will be compared in 2 dimensions and in n dimensions over a generic parallelepiped and a ball through MATLAB. With the several examples and the concrete calculus, we aim to determine which domain has the smallest first Dirichlet eigenvalue among various of the same volume, and hence, motivate the Rayleigh-Faber-Krahn inequality.

Keywords

Rayleigh-Faber-Krahn inequality, Eigenvalue problem, Dirichlet homogeneous eigenvalue problem, Sobolev spaces, Pólya-Szegö inequality, Rayleigh quotient, Drums, Finance, Reactive substance, Quantum mechanics, n-dimensional calculus.

Contents

1. Introduction

Eigenvalues and eigenvectors can give us a lot of information about many mathematical models used in science and technology. For example, they are used in Google's Page Rank algorithm or in Markov Processes. In this work, we will focus on the first eigenvalue of the Dirichlet Laplacian. Precisely, we will face the general eigenvalue problem for the Laplacian with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = \lambda u & in\ \Omega \\ u = 0 & in\ \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded set in $\mathbb{R}^n$.

Our purpose is to find which domain Ω minimizes the first eigenvalue under a volume constraint. The solution to this shape optimization problem follows by the well known Rayleigh–Faber–Krahn inequality, that was conjectured by Lord Rayleigh in 1877 and proved independently, at the same time, by Georg Faber and Edgar Krahn. It states the following:

Theorem 1.1. (Rayleigh-Faber-Krahn inequality) *Let Ω be an open set of finite volume in $\mathbb{R}^n$ and let $\Omega^* \subset \mathbb{R}^n$ be the ball of the same volume as Ω. Let $\lambda_1(\Omega)$ be the principal eigenvalue of the Dirichlet Laplacian on Ω, i.e., the smallest value of λ such that the problem*

$$\begin{cases} \Delta u = \lambda u & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}$$

has a non-trivial solution. Then,

$$\lambda_1(\Omega) \geqslant \lambda_1(\Omega^*). \qquad [2]\S 1$$

Generally speaking, Faber-Krahn inequality shows that the smallest first Dirichlet eigenvalue of the Laplace operator on an open bounded domain $\Omega \subset \mathbb{R}^n$ with prescribed volume is the one corresponding to an Euclidean ball, denoted as Ω^*. Moreover, if the equality is satisfied, the domain must be the ball. It means, that Faber-Krahn inequality is rigid.

This result was motivated in the paper *The Shape of the Drum [6]*. Its author considers *natural* and logic to ask for the sound that a drum makes, although he was focused on solving its inverse problem: he wanted to get interference of the shape of the drum by just hearing its sound *[6]*. This research could be applied clearly to seismology, but also to medicine in ultra-sound scans. Nowadays, the knowledge of the leading eigenvalue can be useful in branches such as music, quantum mechanics or finance, as we will see promptly.

1.1 Procedures and methods

This work is subdivided in three main parts that are strongly connected:

- *Mathematical models*: In this first section, we introduce some motivating models that are all from different scientific branches, such as music, finance or quantum mechanics. They show us the importance of the first eigenvalue in unrelated fields.

  In particular, we start studying a musical toy example that shows us that a flute sounds two times higher than a clarinet, only because it has both open ends, while clarinet has one closed and one open end. This means, that if we want them to sound at the same pitch, flute's size must be the double.

Then, we study drums' membranes to understand the importance of its geometric shape. After that, we jump to a model related to the diffusion of a reactive substance that should be stored through the safest container. Finally, we consider a problem about finding an optimal tax and one concerning quantum mechanics that illustrates the minimum kinetic energy system for non-relativistic particles.

- *Explicit calculus in concrete domains*: This section aims to compute the first eigenvalue of the drums' model and it is based on the calculus of the previous section. It is subdivided into two principal parts:

  First of all, we exemplify the eigenvalue problem in two 2-dimensional concrete domains. In this case, we compare a rectangular membrane and a circular one. We start finding which rectangular membrane has the smallest first eigenvalue (a square, a narrow rectangle, ...). Then, we jump to discover if the circular or the best rectangular membrane has the smallest first eigenvalue when both domains have the same area.

  In the second part, we extend previous calculus to n-dimensional domains. In particular, we compare a parallelepiped and a ball. As in the previous subsection, we end up by the appropiate parallelepiped to have the smallest first eigenvalue among all parallelepipeds. We compute also the smallest first eigenvalue between the ball and the optimized parallelepiped when they have the same volume. Since the resulting expressions depend on n, we need to fix it to compute reliable eigenvalues on both domains. To do this, we use MATLAB. The calculus codes are attached in appendix B.

- *Faber-Krahn inequality*: This is the theoretical section of the work. Our purpose is to go deeper in the academical study of some results that will help us to extend, in a rigorous way, the concrete calculus done in the previous section for parallelepipeds and balls to general domains, as well as to interpret the results from the model's section. Every definition, lemma or observation is focused on understanding Faber-Krahn inequality, that will be proved through Pólya-Szegö inequality.

  Since Faber-Krahn's statement is very specific, we need to introduce a large theoretical background to understand every concept that there appears. In consequence, we start with a heuristic reasoning to have an overview of the steps that we will follow to prove the theorem. Then, we see that the Sobolev space $H_0^1(\Omega)$ is the natural Hilbert space to work with. It is essentially the completeness of $\mathcal{C}^1$ functions compactly supported in Ω with respect to a L^2-based scalar product defined in terms of derivatives. Once in Sobolev spaces, we define Rayleigh quotient to characterize the first eigenvalue of the Dirichlet Laplacian. On this spot, we introduce co-area formula, Jensen's inequality, and isoperimetric inequality, since they are the base of Pólya-Szegö inequality's proof. Finally, through this last result, we finish the Faber-Krahn inequality's proof.

2. Mathematical models

In this section, we will see some mathematical models to exemplify some uses and applications of recognizing and optimizing the first eigenvalue of the Laplacian operator. Firstly, we will see an example in $\mathbb{R}$ regarding musical instruments. It has not only Dirichlet boundary conditions, but it will show us the importance and the utility of the first eigenvalue. Then, we will move to Dirichlet Laplacian models in $\mathbb{R}^n$, that use Faber-Krahn inequality and arrive at curious and amazing results.

To compute solutions it will be also useful a well-known method in PDEs: separation of variables. This method splits solutions in time part and spatial part. It will be extensively and repeatedly used; therefore, it will be conscientiously detailed in the firsts models, and from there on, we will only cite its results.

2.1 A toy example

In this example, we analyze the 1-dimensional wave equation through the comparison of two wind musical instruments. It allows us to understand either why some musical instruments are larger or why some instruments can sound deeper than others.

Wind instruments create sound by resonating the sound wave inside them. By itself, the wave produces a very low level of sound, but we can hear it loud and clear due to the resonating effect of the acoustic structure of the instrument. The lowest frequency in which this physical effect effectively occurs is called the fundamental frequency. Its multiples are called higher frequencies. These frequencies determined the pitch of the instrument and we will see that they are, actually, the eigenvalues of an eigenvalue problem.

Since we are working in $\mathbb{R}$, instruments are simply a tube, that we suppose of length L. We can model the pressure of the air inside it by the wave equation. We note it $u(x,t)$ and it depends on the position $x \in (0,L)$ and the time $t \geqslant 0$ of its measurement.

Specifically, we will study the frequency spectrum of two wind instruments: the flute and the clarinet. The first one has open ends, while the second one has one open and one closed end. As will be explained, in the clarinet's case it is suitable to consider mixed boundary conditions. Notice that this differs from Faber-Krahn inequality's statement 1.1, since it is announced with Dirichlet boundary conditions. However, we study this model because it shows the importance of the first eigenvalue and some extremely visual and paradigmatic applications.

Flute

Suppose a flute of length L, i.e., a tube with both open ends. The constraint regarding the ends can be modelled simply using Dirichlet boundary conditions. To simplify the calculus we consider them homogeneous. The PDE system with general initial conditions is the following:

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & x \in (0,L), t > 0 \\ u(0,t) = u(L,t) = 0 & t > 0 \\ u(x,0) = g(x) & x \in (0,L) \\ u_t(x,0) = h(x) & x \in (0,L), \end{cases} \tag{2}$$

where $c > 0$ is the wave velocity.

A separable solution to this wave equation is one of the form $u(x,t) = X(x)T(t)$. Substituting into the wave equation, we obtain

$$X(x)T''(t) - c^2X''(x)T(t) = 0.$$

Rearranging,

$$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{c^2T(t)}.$$

Since both sides are functions of different independent variables, they must be constant:

$$\frac{-X''(x)}{X(x)} = \frac{-T''(t)}{c^2T(t)} = \lambda.$$

Therefore, we obtain two differential equations:

$$X''(x) = -\lambda X(x),$$
$$T''(t) = -\lambda c^2 T(t).$$

Imposing the boundary conditions, we have

$$X''(x) = -\lambda X(x), \quad x \in (0, L), \tag{3a}$$
$$X(0) = 0, \tag{3b}$$
$$X(L) = 0. \tag{3c}$$

Notice that we have an open bounded domain, $(0, L)$, an eigenvalue equation, (3a), and two Dirichlet boundary conditions, (3b) and (3c). Consequently, it corresponds to a Dirichlet eigenvalue problem.

It is important to percive, that λ is a positive constant: take (3a) and multiply by $X(x)$ on both sides. Then, integrating by parts and applying the boundary conditions (3b) and (3c), yields to

$$\lambda \int_0^L X^2(x)dx = -\int_0^L X''X(x)dx = -X(x)X'(x)\Big|_0^L + \int_0^L (X'(x))^2dx = \int_0^L (X'(x))^2dx \geqslant 0. \tag{4}$$

Therefore, $\lambda \geqslant 0$ and the general solution to $X''(x) = -\lambda X(x)$ is

$$X(x) = A\sin\left(\frac{\sqrt{\lambda}\pi}{L}x\right) + B\cos\left(\frac{\sqrt{\lambda}\pi}{L}x\right).$$

Applying the boundary conditions we have

$$X(0) = 0 \Rightarrow B \equiv 0,$$
$$X(L) = 0 \Rightarrow A\sin(\sqrt{\lambda}\pi) = 0.$$

As we want to avoid trivial null solution, A can not be 0. As a result:

$$\sin(\sqrt{\lambda}\pi) = 0 \Rightarrow \lambda = k^2, \quad k \in \mathbb{N}.$$

Consequently,

$$X(x) = A\sin\left(\frac{k\pi}{L}x\right).$$

Solving now the ODE for temporal part, $T''(t) = -\lambda c^2 T(t)$, we obtain that the general solution is

$$T(t) = C \sin\left(\frac{ck\pi}{L} t\right) + D \cos\left(\frac{ck\pi}{L} t\right), \quad k \in \mathbb{N}.$$

By the linearity of the equation and the boundary conditions, it is natural to consider solutions of the form

$$u(x,t) = \sum_{k \geqslant 1} \left(C_k \sin\left(\frac{ck\pi}{L} t\right) + D_k \cos\left(\frac{ck\pi}{L} t\right)\right) \sin\left(\frac{k\pi}{L} x\right). \tag{5}$$

Only remains to compel the initial conditions to determine coefficients C_k and D_k:

$$u(x,0) = g(x) = \sum_{k \geqslant 1} D_k \sin\left(\frac{k\pi}{L} x\right),$$

$$u_t(x,0) = h(x) = \sum_{k \geqslant 1} \left(\frac{ck\pi}{L}\right) C_k \sin\left(\frac{k\pi}{L} x\right).$$

Notice that the behaviour of the acoustic pressure of the air inside a flute is determined by exactly the same boundary conditions as in the case of a vibrating string. Therefore, is reasonable to consider that the solution (5) is $\widehat{T}$ periodic regarding time. Hence,

$$\frac{ck\pi}{L} \widehat{T} = 2\pi \Rightarrow \widehat{T} = \frac{2\pi}{ck}, \quad k \geqslant 1 \Rightarrow f_k = \frac{2\pi}{\widehat{T}} = \frac{ck\pi}{L},$$

where f_k is the k-th wave frequency. By definition, f_1 is the fundamental frequency and any f_k, $k \geqslant 2$, is a higher frequency. Thereby, the smallest frequency is the fundamental one, that is $f_1 = \frac{\pi c}{L}$.

Notice that frequencies and eigenvalues solve the same Laplacian equation. Then, the smallest first eigenvalue is the same as the smallest frequency.

Clarinet

Consider a clarinet of length L, so a tube with one open and one closed end. Since we are modelling a wave equation, we should have Dirichlet boundary condition concerning the open end and Neumann boundary condition referring to the closed one. Therefore, we consider mixed boundary conditions. To simplify the calculus, we take them homogeneous. The PDE system with general initial conditions is the following:

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & x \in (0,L), t > 0 \\ u(0,t) = 0 & t > 0 \\ u_x(L,t) = 0 & t > 0 \\ u(x,0) = g(x) & x \in (0,L) \\ u_t(x,0) = h(x) & x \in (0,L), \end{cases} \tag{6}$$

where $c > 0$ is the wave velocity. We have chosen, without loss of generality, $x = L$ to be the closed end.

Paralleling the previous case, we obtain two ODEs and a constant $\lambda \geqslant 0$:

$$X''(x) = -\lambda X(x),$$
$$T''(t) = -\lambda c^2 T(t).$$

As we have done before, we impose the boundary conditions to obtain an eigenvalue problem:

$$X''(x) = -\lambda X(x), \quad x \in (0, L), \tag{7a}$$
$$X(0) = 0, \tag{7b}$$
$$X'(L) = 0. \tag{7c}$$

Notice that in this case, we have a mixed eigenvalue problem. Nevertheless, we can calculate its solution in the same way. Solving firstly (7a), we obtain

$$X(x) = A \sin\left(\frac{\sqrt{\lambda}\pi}{L}x\right) + B\cos\left(\frac{\sqrt{\lambda}\pi}{L}x\right).$$

Applying the Dirichlet boundary condition on $x = 0$ (7b) and the Neumann boundary condition on $x = L$ (7c), we can simplify it to the following:

$$X(0) = 0 \Rightarrow B \equiv 0,$$
$$X'(L) = 0 \Rightarrow \frac{A\sqrt{\lambda}\pi}{L}\cos(\sqrt{\lambda}\pi) = 0.$$

Since we want to avoid trivial null solution, A can not be 0. Consequently,

$$cos(\sqrt{\lambda}\pi) = 0 \Rightarrow \lambda = \left(\frac{2k+1}{2}\right)^2, \quad k \in \mathbb{N} \cup \{0\}.$$

Hence, we obtain that the general solution to the spatial part is

$$X(x) = A\sin\left(\frac{2k+1}{2L}\pi x\right).$$

In the same line, the general solution to the temporal part, $T''(t) = -\lambda c^2 T(t)$, is

$$T(t) = C\sin\left(\frac{2k+1}{2L}c\pi t\right) + D\cos\left(\frac{2k+1}{2L}c\pi t\right), \quad k \in \mathbb{N} \cup \{0\}.$$

As in the flute's section, regarding the linearity of the equation and the boundary conditions, it is natural to consider solutions of the form

$$u(x,t) = \sum_{k\geqslant 0}\left(C_k \sin\left(\frac{2k+1}{2L}c\pi t\right) + D_k\cos\left(\frac{2k+1}{2L}c\pi t\right)\right)\sin\left(\frac{2k+1}{2L}\pi x\right). \tag{8}$$

Enforcing the initial conditions, we obtain coefficients C_k and D_k:

$$u(x,0) = g(x) = \sum_{k\geqslant 0} D_k \sin\left(\frac{2k+1}{2L}\pi x\right),$$
$$u_t(x,0) = h(x) = \sum_{k\geqslant 0}\left(\frac{2k+1}{2L}c\pi\right) C_k \sin\left(\frac{2k+1}{2L}\pi x\right).$$

As in the flute analysis, it is reasonable to consider that the solution is $\widehat{T}$ periodic regarding time. Hence,

$$\frac{2k+1}{2L}\pi c\widehat{T} = 2\pi \Rightarrow \widehat{T} = \frac{4L}{(2k+1)c}, \quad k \geqslant 0 \Rightarrow f_k = \frac{2\pi}{\widehat{T}} = \frac{2k+1}{2L}c\pi,$$

where f_k is the k-th frequency. Notice that now, the fundamental frequency is f_0, while in the flute's case was f_1. All in all, the smallest clarinet's frequency is $f_0 = \dfrac{\pi c}{2L}$.
Once more, we can assert that the smallest eigenvalue is the same as the smallest frequency.

Comparison and analysis

Finally, it is possible to compare the first eigenvalues of the flute and the clarinet. Observe that the frequencies of vibration correspond to the eigenvalues of the homogeneous eigenvalue problem to the Laplace operator. Specifically, the fundamental frequency is the smallest one.
In the previous calculus, we obtained the fundamental frequency of both instruments. Comparing them,

$$\left.\begin{aligned} f_{flute} &= \frac{\pi c}{L} \\ f_{clarinet} &= \frac{\pi c}{2L} \end{aligned}\right\} \qquad \Rightarrow \qquad f_{flute} = 2f_{clarinet}.$$

This implies that if both instruments have the same length, the clarinet can sound one octave deeper. We can also observe for both instruments, that when the length is reduced the pitch goes up and when it is extended the pitch goes down. One good example of it is when a flautist covers some holes (she is *enlarging* flute's length), then she plays lower notes.
In closing, it is interesting to remark that in steady vibration mode, flute's frequencies are any multiple of its fundamental frequency. Besides, clarinet's frequencies are only odd multiples of it.

2.2 Dirichlet Laplacian Models

In this subsection, we focus on Dirichlet Laplacian models, for which undoubtedly Faber-Krahn inequality can be applied. We will introduce properly each model and do the necessary calculus to analyse the result through Faber-Krahn. Intending not to expand on computation part and trying to focus on the models' physical implications, the calculus will be brief and concise.
We will see four different problems from very different branches: the first one is regarding drums and music. It shows the relation between the drum's membrane shapes and the fundamental frequency. The second model shows how a reactive fluid can be diffused. The third problem speaks about the choice of a finance tax. Finally, the fourth proposal refers to quantum mechanics and aims to model the movement and energy levels of a non-relativistic particle.

First of all, it will be useful to find a basis to express our solutions. In particular, there exist an orthogonal one:

Theorem 2.1. *Let Ω be an open bounded set of $\mathbb{R}^n$. There exists an orthonormal Hilbert basis $(\varphi_k)_{k\geqslant 1}$ of $L^2(\Omega)$ and a sequence of positive numbers $(\lambda_k)_{k\geqslant 1}$ such that*

$$\begin{cases} \lambda_k \xrightarrow[k\longrightarrow+\infty]{} +\infty \\ \varphi_k \in H_0^1(\Omega) \\ -\Delta\varphi_k = \lambda_k\varphi_k \quad \text{in } \Omega. \end{cases} \qquad [3]\S 9$$

$H_0^1(\Omega)$ will be rigorously defined in Section 4.6, but let us say here that, essentially, a function u belongs to $H_0^1(\Omega)$ if u and $\nabla u \in L^2(\Omega)$, and $u = 0$ on $\partial\Omega$.

This result helps us to express solutions in terms of eigenvectors and eigenvalues, and consequently, to identify them. It will be useful to write results as series, where calculus and comparisons are easier.

As it was announced at the beginning of this section, we will use separation of variables. This method will be detailed in the first model since it is set in $\mathbb{R}^n$ and for the moment we have only seen it in $\mathbb{R}$. From there on, given that all models are n-dimensional and quite similar, the explanations will be superficial.

2.2 Drums

This is the historical model that motivated Faber-Krahn inequality. It is suggested in *[1]* and shows which drum geometric membrane shape is the one that produces lower notes. The sound that a drum makes is bound to its frequency spectrum. Since we want to analyse the lowest possible note that this instrument can produce, we are only caring about its fundamental frequency. Due to the fact that this value corresponds to the smallest eigenvalue, we are likewise analysing which membrane shape has the smallest first eigenvalue. The rigorous general model is the following:

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary,

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \times (0, +\infty) \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ u = g & \text{on } \Omega \times \{0\} \\ u_t = h & \text{on } \Omega \times \{0\}, \end{cases} \tag{9}$$

where $u(x, t)$ is the height of the n-dimensional membrane at position x and at time t. Since we are modelling sound waves, we consider the wave equation $u_{tt} - \Delta u = 0$. We have homogeneous boundary condition, as the membrane is fixed on its edges. Finally, we take into account general initial conditions.

To solve this model, we look for separable solutions. Setting u(x,t) = X(x)T(t) we obtain

$$X(x)T''(t) = \Delta(X(x))T(t).$$

Dividing by $X(x)T(t)$ and rearranging,

$$-\frac{\Delta(X(x))}{X(x)} = -\frac{T''(t)}{T(t)}.$$

Since the left-hand side does not depend on t and the right-hand side does not depend on x, both sides are constant. We name it λ. By the same procedure as (4), $\lambda \geqslant 0$. Then, we have two equations: one regarding space and another regarding time:

$$-\Delta(X(x)) = \lambda X(x),$$
$$-T''(t) = \lambda T(t).$$

Notice that the first equation corresponds to an eigenvalue equation for the Dirichlet Laplacian. Moreover, observe that we have Dirichlet boundary conditions and that Ω is an open bounded domain, so we have a Dirichlet eigenvalue problem.

Using theorem 2.1, our solution is

$$u(x,t) = \sum_{k\geqslant 1} \left(C_k \sin\left(\sqrt{\lambda_k} t\right) + D_k \cos\left(\sqrt{\lambda_k} t\right)\right) \varphi_k(x), \tag{10}$$

where

$$\begin{cases} -\Delta\varphi_k = \lambda_k \varphi_k & \forall k \\ \varphi_k = 0 & on\ \partial\Omega \\ g = \sum_{k\geqslant 1} D_k \varphi_k & \\ h = \sum_{k\geqslant 1} C_k \sqrt{\lambda_k} \varphi_k. & \end{cases}$$

When a drum is struck, it vibrates at different frequencies. The fundamental tone is the lowest one, and the others are called overtones. As we have explained in the introduction, we want to investigate how the shape of the drum affects its sound and, in particular, its fundamental tone. Consequently, we focus our study to characterize the first eigenvalue, since its square corresponds to the fundamental tone.

The domain where the minimum first eigenvalue is attained is given by Faber-Krahn inequality. Therefore, the one corresponding to an Euclidean ball is the smallest. Then, we can assert that from all drums with a fixed membrane volume, the one with circular membrane has the lowest fundamental tone.

2.2 Diffusion and reaction

This model shows the spread of a substance inside a general domain $\Omega \subset \mathbb{R}^n$. To make it more interesting we consider a reactive substance, i.e., a substance that generates more substance by itself depending on the current amount of matter in a determined position and time:

$$\begin{cases} u_t - \Delta u = \lambda u & in\ \Omega \times (0,+\infty) \\ u = 0 & on\ \partial\Omega \times (0,+\infty) \\ u = g & on\ \Omega \times \{0\}. \end{cases} \tag{11}$$

The left-hand side of the equation indicates that there is a quantity of substance $u(x,t)$ at certain time $t > 0$ and position $x \in \Omega$, that diffuses along the domain Ω. Whereas the right-hand side of the equation models the substance generation. The parameter λ is called the *reaction parameter* and is a fixed value given by the substance and its ambient conditions. The bigger gets λ, the quicker the substance reacts. Hence, more substance is created in little time.

We also consider homogeneous Dirichlet boundary conditions, which can be seen as the domain sides are absorbent. Finally, we suppose general initial conditions.

To solve the problem we consider a variable change: $v(x,t) := e^{-\lambda t}u(x,t)$. So

$$u = e^{\lambda t}v, \qquad u_t = e^{\lambda t}(\lambda v + v_t), \quad \text{and} \qquad -\Delta u = e^{\lambda t}(-\Delta v).$$

Imposing it to (11):

$$\begin{cases} u_t - \Delta u = \lambda u \iff e^{\lambda t}(\lambda v + v_t) - (\Delta v)e^{\lambda t} = \lambda e^{\lambda t} v & \text{in } \Omega \times (0,+\infty) \\ u = 0 \iff e^{\lambda t} v = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ u = g \iff v = g(x) & \text{on } \Omega \times \{0\}. \end{cases}$$

Simplifying, we obtain an homogeneous problem:

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \Omega \times (0,+\infty) \\ v = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ v = g & \text{on } \Omega \times \{0\}. \end{cases}$$

Through separation of variables and using theorem 2.1, the solution to the above Dirichlet problem is

$$v(x,t) = \sum_{k\geqslant 1} e^{-\lambda_k t} C_k \varphi_k(x),$$

where

$$\begin{cases} -\Delta \varphi_k = \lambda_k \varphi_k & \forall k \\ \varphi_k = 0 & \text{on } \partial\Omega \\ g = \sum_{k\geqslant 1} C_k \varphi_k. \end{cases}$$

By just undoing the variable change, our solution is

$$u(x,t) = \sum_{k\geqslant 1} e^{(\lambda - \lambda_k)t} C_k \varphi_k(x). \tag{12}$$

Observe that λ_k are the eigenvalues of the Dirichlet Laplacian on Ω. By theorem 2.1 we know that $(\lambda_k)_{k\geqslant 1}$ is a monotonous growing sequence that goes to infinity. Then, the smallest eigenvalue is λ_1.

Another fundamental observation lies in the solution form, that has an exponential term. Then, it can easily go to zero or infinity. A natural question is to ask for what does this mean in our model in a physical sense and which consequences does it have.

To clear up all these logical questions, we study the total substance quantity when time is extremely big:

$$\lim_{t\to+\infty} \int_\Omega |u(x,t)|^2 dx = \lim_{t\to+\infty} \sum_{k\geqslant 1} e^{2(\lambda-\lambda_k)t} |C_k|^2,$$

where the equality is consequence of the solution form and the fact that $(\varphi_k)_{k\geqslant 1}$ is an orthonormal basis of L^2. Notice that this limit can go to infinity, to zero or can be a real value depending on the result of $\lambda - \lambda_k$. In fact, the first eigenvalue can approach by itself the whole outcome to any possible solution. Then, we should study it: suppose Ω a connected domain and that exists some initial quantity of the substance in it. So g must be a positive distribution $g \geqslant 0$ with $\int_\Omega g > 0$. Then, its first coefficient $C_1 = \int_\Omega g\varphi_1 > 0$, as the first eigenvalue is simple and $\varphi_1 > 0$, see *[3]§9*. Therefore,

$$\lim_{t\to+\infty} \int_\Omega |u(x,t)|^2 dx = \begin{cases} 0 & \text{if } \lambda < \lambda_1 \\ 0 < C_1 < +\infty & \text{if } \lambda = \lambda_1 \\ +\infty & \text{if } \lambda > \lambda_1. \end{cases}$$

This indicates that we have a substance that fights between diffusion and reaction. In order to avoid its extinction (given if the limit is 0) or its saturation (given if the limit is $+\infty$), the above limit should be equal to C_1. Consequently, reaction coefficient must be $\lambda = \lambda_1(\Omega)$, that corresponds to the first eigenvalue. This means that choosing properly the domain, we can decide the reaction coefficient.

Besides, using Faber-Krahn inequality, we know that $\lambda(\Omega) \geqslant \lambda(\Omega^*)$. This result physically indicates that the domain with the smallest reaction coefficient is an Euclidean ball.

One possible application for this model could be the storage of some radioactive chemical material. Suppose a dangerous radioactive substance that we want to safely preserve for the future; hence, we want to avoid its extinction and its collapse. This substance is reactive, so it creates more matter following the previous model. As more chemical matter we have, more dangerous is the situation, since this substance is risky and at every time step, we have more quantity of it. By Faber-Krahn, we can ensure that the safest container to store this chemical is a one with a ball shape. Then, the reaction coefficient would be the smallest one and the fewest substance would be created. Moreover, it guarantees that the radioactive chemical will remain and will not collapse either disappear.

2.2 Finance

This model shows the initial amount of money that is needed to face several payments in order to *escape* of a domain $\Omega \subset \mathbb{R}^n$. To make it more captivating we consider an interest rate. Then, each payment is proportional to the current capital in that position:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

Take $\Omega \subset \mathbb{R}^n$ an open bounded domain such that it is possible to perform random walks inside. We can discretize Ω and assume that a particle, which is initially inside, moves with steps of length $h > 0$ in any of the cardinal directions with uniform probability. For each step, some fee must be paid. It is proportional to the current amount of money, as the right-hand side of the equation shows. $u(x)$ models the capital needed in order to start a random walk at position x and guarantees that the expected money at the end of the walk is zero, since the boundary condition is homogeneous.

Notice that this is the Poisson equation and it has non-trivial solution if and only if λ is a Dirichlet Laplacian eigenvalue of Ω. In addition, by theorem 2.1 the minimum λ is $\lambda_1(\Omega)$. If we consider Faber-Krahn inequality, we know that the smallest first eigenvalue is the one corresponding to an Euclidean ball domain. All these results illustrate that the optimized first eigenvalue is the minimum interest rate such that we are expected to go out of the domain with no money left.

This model can be more useful if time is considered:

$$\begin{cases} u_t - \Delta u = \lambda u & \text{in } \Omega \times (0, +\infty) \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ u = g & \text{on } \Omega \times \{0\}. \end{cases} \tag{14}$$

Quite similar to the model without time dependency, $u(x, t)$ is the expected amount of money necessary to go out of Ω starting at position x when time is t. For each step, a proportional capital quantity must be paid and no money should remain when the domain is left. Consider $g(x)$ the initial funds.

The basic difference from the previous model is that the time is now a variable and plays a role. Specifically, the payment for each step cares about time, in the sense that in a fixed position, the amount of money that should be paid is different as time goes on.

Notice that this equation is the same as the one in diffusion and reaction model 2.2.2. For that reason, we can do identically the same calculus to obtain the eigenvalues: we define a new function $v := ue^{-\lambda t}$ and solve its problem using separation of variables and theorem 2.1. Undoing the variable change, our solution is

$$u(x,t) = \sum_{k\geqslant 1} e^{(\lambda-\lambda_k)t} C_k \varphi_k(x), \tag{15}$$

where $(\varphi_k)_{k\geqslant 1}$ is an orthonormal basis and $(\lambda_k)_{k\geqslant 1}$ its eigenvalues, that perform a monotonous growing sequence that approaches to infinity.

Then, we study the expected amount of money needed as time increases:

$$\lim_{t\to+\infty} \int_\Omega |u(x,t)|^2 dx = \lim_{t\to+\infty} \sum_{k\geqslant 1} e^{2(\lambda-\lambda_k)t} |C_k|^2.$$

Once more, we perceive that this limit can go straight to zero, infinity or a different concrete real value by just the behaviour of the first eigenvalue $\lambda_1(\Omega)$. To study it, let Ω be a connected domain and suppose that exists some initial capital, otherwise any step could be done. Then, $g \geqslant 0$ and $\int_\Omega g > 0$. As we have already deduced, $C_1 = \int_\Omega g\varphi_1 > 0$ and we can ensure that the first coefficient is not null nor infinite. Therefore,

$$\lim_{t\to+\infty} \int_\Omega |u(x,t)|^2 dx = \begin{cases} 0 & \text{if } \lambda < \lambda_1 \\ 0 < C_1 < +\infty & \text{if } \lambda = \lambda_1 \\ +\infty & \text{if } \lambda > \lambda_1. \end{cases}$$

This shows that if the interest rate is too low, at the end the customer will pay a negligible capital. Whereas if the interest rate is too high, the customer will be ruined because she will have paid a huge amount of money. Only the process can be balanced if the interest rate is $\lambda = \lambda_1(\Omega)$, that corresponds to the first Dirichlet Laplacian eigenvalue. By Faber-Krahn, this interest rate is minimal (and, thus, more attractive to the costumer) when Ω is a ball.

This could be applied, for example, to fix a tax in some process. This process can be seen as the event of going out of the domain, *escaping* from it. On the one hand, customers want to have the smallest possible tax. On the other hand, the company considers fundamental that purchasers do not ruin before finishing the process; otherwise, they could not pay more. Also, the business' owners want to avoid customers from becoming *infinitely* rich. Then, $\lambda_1(\Omega)$ is the suitable value.

2.2 Quantum mechanics

This model shows the movement of a non-relativistic particle trapped in $\Omega \subset \mathbb{R}^n$, such as a proton or an electron:

$$\begin{cases} -iu_t - \Delta u = 0 & \text{in } \Omega \times (0,+\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ u = g & \text{on } \Omega \times \{0\}. \end{cases} \tag{16}$$

By construction in quantum mechanics, every function is normalized, since they are used to model probabilities. Let $u(x,t) \in \mathbb{C}$ be the *wave function*, i.e., a mathematical description of the quantum state of an isolated quantum system *[5]§1*. In particular, it can be used to describe the expected space where it is more likely to find the studied particle. The above-mentioned normalization is $\|u(\cdot,t)\|_{L^2} = 1$.

The equation is the *Schrödinger equation*, based on the energy equations of the system and a well-known postulate of quantum mechanics: *the wave function is a description of the system [5]§1*. It contains the imaginary unit i that models rotations in the complex plane. Then, as i multiplies u_t, the wave function rotates in the complex plane over time.

We consider homogeneous Dirichlet boundary condition and initial condition g, that gives us the expected initial region where the particle can be found. We also have $\|g\|_{L^2} = 1$.

To solve the problem we proceed as in the previous models: we do separation of variables and use the theorem 2.1. Then, our solution is

$$u = \sum_{k\geqslant 1} e^{-i\lambda_k t} C_k \varphi_k, \tag{17}$$

where

$$\begin{cases} -\Delta\varphi_k = \lambda_k \varphi_k & \forall k \\ \varphi_k = 0 & on\ \partial\Omega \\ g = \sum\limits_{k\geqslant 1} C_k \varphi_k. \end{cases}$$

Observe that $e^{-i\lambda_k} = \cos\lambda_k - i\sin\lambda_k$. Then, its behaviour is similar to the wave equation. By this fact, we call u the *wave function*.

We are interested in the energy that the system should have to hold a particle in a concrete position. Consider the expected kinetic energy and integrate by parts:

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega -\Delta u \cdot \overline{u}.$$

Imposing solution (17) and using that $(\varphi_k)_{k\geqslant 1}$ is an orthonormal basis of L^2,

$$\int_\Omega -\Delta u \cdot \overline{u} = \int_\Omega \sum_{k\geqslant 1} |C_k|^2 (-\Delta\varphi_k)\varphi_k = \sum_{k\geqslant 1} \lambda_k |C_k|^2.$$

One the one hand, eigenvalues are a growing sequence that goes to infinity, then $\lambda_1 < \lambda_k,\ \forall k$. On the other hand, $1 = \|g\|_{L^2} = \sum\limits_{k\geqslant 1} |C_k|^2$. As a result,

$$\sum_{k\geqslant 1} \lambda_k |C_k|^2 \geqslant \lambda_1 \sum_{k\geqslant 1} |C_k|^2 = \lambda_1(\Omega).$$

Notice that kinetic energy is preserved over time. Another observation is that $(\varphi_k)_{k\geqslant 1}$ satisfy the *time independent Schrödinger equation*, $\Delta\varphi(x) = E\varphi(x)$, where E is a constant equal to the energy level of the system. It is an eigenvalue equation with eigenvalues E of the Hamiltonian operator. Thereby, as our eigenvalues meet the same equation, we can call them *energy levels*. *[8]*

The energy levels provide the oscillations of the wave function u, as in the wave equation. By Faber-Krahn inequality, the smallest possible kinetic energy of a quantum state among all sets with given volume is provided by the ball and the quantum state $u = \varphi_1(\Omega^*)$. Therefore, for any system, its minimum kinetic energy would be, at least, the one provided by this calculus.

3. Explicit calculus in concrete domains

In this section, we analyze and optimize the first eigenvalue in concrete domains in order to have an idea of its values' differences among them. We consider the drums' model explained in section 2.2.1 with the membrane area fixed. Then, we choose some geometric membrane shapes to perform explicit calculus over them. Specifically, we compare a rectangular and a circular membrane in 2 dimensions and in general higher dimensions.

3.1 Analysis of two membrane shapes in 2 dimensions

Suppose the 2-dimensional problem described before. To model it, let the drum membrane area be parametrised with variables x and y. Name the vertical shift as $u(x, y, t)$, where $x \in (0, a)$, $y \in (0, b)$ with $a,\ b \in \mathbb{R}^+$ and $t \geqslant 0$. Since the membrane is attached to its boundary, the vertical laterals shift is 0, and hence, we consider Dirichlet homogeneous boundary conditions.

3.1 Rectangular membrane

Suppose Ω an open bounded rectangular domain which can be described by $\Omega = (0, a) \times (0, b)$, $a, b \in \mathbb{R}^+$. Since we consider Dirichlet homogeneous boundary conditions and general initial conditions, we have the following problem:

$$\begin{cases} u_{tt}(x, y, t) - \Delta u(x, y, t) = 0 & (x, y) \in \Omega, t > 0 \\ u(0, y, t) = u(a, y, t) = 0 & y \in (0, b), t > 0 \\ u(x, 0, t) = u(x, b, t) = 0 & x \in (0, a), t > 0 \\ u(x, y, 0) = g(x, y) & (x, y) \in \Omega \\ u_t(x, y, 0) = h(x, y) & (x, y) \in \Omega. \end{cases} \tag{18}$$

We look for separable solutions, so let be $u(x, y, t) = X(x)Y(y)T(t)$. Applying the same procedure as in section 2.2.1 and using theorem 2.1, our solution is

$$u(x, t) = \sum_{k \geqslant 1} (C_k \sin(\sqrt{\lambda_k} t) + D_k \cos(\sqrt{\lambda_k} t))\, \varphi_k(x, y), \tag{19}$$

where

$$\begin{cases} -\Delta \varphi_k = \lambda_k \varphi_k & \forall k \\ \varphi_k = 0 & on\ \partial\Omega \\ g = \sum\limits_{k \geqslant 1} D_k \varphi_k \\ h = \sum\limits_{k \geqslant 1} C_k \sqrt{\lambda_k} \varphi_k. \end{cases}$$

Notice that $\varphi_k(x, y)$ is not anything more than $\varphi_k(x, y) = X_k(x)Y_k(y)$. To simplify briefly calculus and notation we fix k and set $\lambda_k = \lambda$, $\varphi_k = \varphi$, $X_k = X$, and $Y_k = Y$. Then, we can re-write its equation as

$$-X''(x)Y(y) - X(x)Y''(y) = \lambda X(x)Y(y).$$

Dividing by $X(x)Y(y)$ and rearranging:

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \lambda.$$

Since each side depends on different independent variables, they should be constant. Let α and β be these positives constants, that satisfy $\alpha = -\frac{X''(x)}{X(x)} = -\beta + \lambda$. Therefore, we obtain two new eigenvalue problems:

$$\begin{cases} X''(x) = -\alpha X(x) \\ X(0) = X(a) = 0, \end{cases} \qquad \begin{cases} Y''(y) = -\beta Y(y) \\ Y(0) = Y(b) = 0, \end{cases}$$

whose general solutions are

$$X(x) = A_1 \sin(\sqrt{\alpha} x) + B_1 \cos(\sqrt{\alpha} x),$$
$$Y(y) = A_2 \sin(\sqrt{\beta} y) + B_2 \cos(\sqrt{\beta} y).$$

By the boundary conditions $X(0) = 0$ and $Y(0) = 0$, we have $B_1 = B_2 = 0$. To avoid trivial null solution and to satisfy the remaining Dirichlet conditions $X(a) = 0$ and $Y(b) = 0$, we require

$$\sqrt{\alpha} a = \pi q_1 \Longrightarrow \alpha = \left(\frac{\pi q_1}{a}\right)^2, \quad q_1 \in \mathbb{N}, \qquad \sqrt{\beta} b = \pi q_2 \Longrightarrow \beta = \left(\frac{\pi q_2}{b}\right)^2, \quad q_2 \in \mathbb{N}.$$

Notice that we need $a > 0$ and $b > 0$ to have well defined fractions.
Hence, $\lambda = \alpha + \beta = \left(\frac{\pi q_1}{a}\right)^2 + \left(\frac{\pi q_2}{b}\right)^2$. Observe that λ depend now of two variables: q_1 and q_2.
The general solution is

$$u(x, y, t) = \sum_{q_1, q_2 \geqslant 1} \sin\left(\frac{\pi q_1}{a} x\right) \sin\left(\frac{\pi q_2}{b} y\right) \left[C_{q_1,q_2} \sin(\sqrt{\lambda_{q_1,q_2}} t) + D_{q_1,q_2} \cos(\sqrt{\lambda_{q_1,q_2}} t)\right]. \tag{20}$$

Imposing the initial conditions we can obtain C_{q_1,q_2} and D_{q_1,q_2}.

On the whole, eigenvalues are

$$\lambda = \lambda_{q_1,q_2} = \pi^2 \left(\frac{q_1^2}{a^2} + \frac{q_2^2}{b^2}\right), \quad q_1,\ q_2 \in \mathbb{N}, \quad a,\ b \in \mathbb{R}^+.$$

Then, the first eigenvalue has the form

$$\lambda_{1,1} = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right), \quad a,\ b \in \mathbb{R}^+. \tag{21}$$

3.1 Circular membrane

Suppose Ω with the same area as the open bounded rectangular domain fixed before. However, it has now a circular shape that can be described in polar coordinates as $\Omega = [0, R) \times [0, 2\pi)$, $R \in \mathbb{R}^+$. Similarly as we have deduced earlier, we have Dirichlet homogeneous boundary conditions at $\partial\Omega$. Hence, we have the following problem:

$$\begin{cases} u_{tt}(r,\theta,t) - \Delta u(r,\theta,t) = 0 & (r,\theta) \in \Omega, t > 0 \\ u(R,\theta,t) = 0 & \theta \in [0, 2\pi), t > 0 \\ u(r,\theta,0) = g(r,\theta) & (r,\theta) \in \Omega \\ u_t(r,\theta,0) = h(r,\theta) & (r,\theta) \in \Omega. \end{cases} \tag{22}$$

By separation of variables we look for a solution $u(r,\theta,t) = V(r)W(\theta)T(t)$. Using the calculus from section 2.2.1 and theorem 2.1, our solution is

$$u(x,t) = \sum_{k\geqslant 1}(C_k \sin(\sqrt{\lambda_k}t) + D_k \cos(\sqrt{\lambda_k}t))\,\varphi_k(r,\theta), \tag{23}$$

where

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \forall k \\ \varphi_k = 0 & on\ \partial\Omega \\ g = \sum\limits_{k\geqslant 1} D_k\varphi_k \\ h = \sum\limits_{k\geqslant 1} C_k\sqrt{\lambda_k}\varphi_k. \end{cases}$$

Note that now $\varphi_k(r,\theta) = V_k(r)W_k(\theta)$. As we have done in the previous example, we fix k and write $\lambda_k = \lambda$, $\varphi_k = \varphi$, $V_k = V$, and $W_k = W$. To solve its eigenvalue equation, we use the polar form of the Laplacian (see appendix A.1). Dividing by $V(r)W(\theta)$ we obtain

$$\left(\frac{V''(r)}{V(r)} + \frac{1}{r}\frac{V'(r)}{V(r)} + \frac{1}{r^2}\frac{W''(\theta)}{W(\theta)}\right) = -\lambda.$$

Rearranging and multiplying each side by r^2,

$$r^2\frac{V''(r)}{V(r)} + r\frac{V'(r)}{V(r)} + \lambda r^2 = -\frac{W''(\theta)}{W(\theta)}. \tag{24}$$

Once more, it is easy to see that each side depends on different independent variables; thus, they must be constant. We call it β, and following the same procedure as (4), $\beta \geqslant 0$. For the right-hand side, we have $W''(\theta) = -\beta W(\theta)$, whose general solution is

$$W(\theta) = A_1 \sin(\sqrt{\beta}\theta) + B_1 \cos(\sqrt{\beta}\theta).$$

Since $W(\theta)$ is an angle function, it must be 2π periodic. Then, $\sqrt{\beta} = m,\ m \in \mathbb{N} \cup \{0\}$.
Regarding the left-hand side equation of (24), it is useful to rearrange it as

$$r^2V''(r) + rV'(r) + (r^2\lambda - m^2)V(r) = 0.$$

This equation is a Bessel ODE; therefore, its solution is

$$V(r) = A_2 J_m(\sqrt{\lambda}r) + B_2 Y_m(\sqrt{\lambda}r),$$

where J_m and Y_m are respectively the first and second kind Bessel functions.
Since $Y_m(z) \xrightarrow[z\longrightarrow 0]{} -\infty$, $B_2 \equiv 0$. Besides, thanks to boundary conditions and avoiding trivial null solution, we have $V(R) = 0 \Rightarrow J_m(\sqrt{\lambda}R) = 0,\ \forall m \in \mathbb{N}\cup\{0\}$. Therefore, $\sqrt{\lambda}R$ must be a zero of Bessel function of the second kind. Let $\xi_{k,m} = \sqrt{\lambda}R$ express the k-th zero of the J_m function.

To sum up,

$$u(r,\theta,t) = \sum_{m\geqslant 0}(A_m^2 J_m(\sqrt{\lambda}r))(A_m^1 \sin(m\theta) + B_m^1 \cos(m\theta))(C_m \sin(\sqrt{\lambda}t) + D_m \cos(\sqrt{\lambda}t)), \tag{25}$$

where A_m^1 indicates the m-th A_1 coefficient. In the same way, A_m^2 and B_m^1.
Setting ϕ and ψ as constants given by the initial conditions, we can simplify the solution as

$$u(r,\theta,t) = \sum_{m\geqslant 0} A_m J_m(\sqrt{\lambda} r)\sin(m\theta+\phi)\sin(\sqrt{\lambda}t+\psi). \tag{26}$$

The first eigenvalue is λ_1, which can be obtained using the zeros of the Bessel function:

$$\xi_{1,0} = \sqrt{\lambda_1}R \Rightarrow \lambda_1 = \frac{\xi_{1,0}^2}{R^2}. \tag{27}$$

3.1 Optimization of the first eigenvalue

In the first place, we want to optimize the first eigenvalue of an area with a rectangular geometric shape. Let $\Omega = (0,a)\times(0,b)$ and S be its area. Then, $S = ab$. By (21) we know that its first eigenvalue is

$$\lambda = \pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = \pi^2\left(\frac{1}{a^2}+\frac{a^2}{S^2}\right).$$

Notice that λ depends only on the value of a. To find the minimum value of $\lambda \equiv \lambda(a)$, we compute the solution to $\lambda'(a) = 0$, and we obtain

$$\lambda'(a) = \pi^2\left(\frac{-2}{a^3}+\frac{2a}{S^2}\right) = 0 \iff \frac{-2}{a^3}+\frac{2a}{S^2} = 0 \iff -2S^2+2a^4 = 0 \iff a^4 = S^2 \iff a = \sqrt{S}.$$

It is essential to keep in mind that $a > 0$, and consequently, to take only the positive value of the square root of S.
To sum up, since $a = \sqrt{S}$ and $S = ab \Rightarrow a = \sqrt{S} = b$. For this reason, the rectangle with smallest first eigenvalue is a square.

It is possible to see this result by inspection: we can draw a mesh of eigenvalues of a rectangle and restrict it to the ones corresponding to our fixed $S = \pi$ area. Immediately, our minimum eigenvalue can be easily seen and notice that it corresponds to a point with the same value of a and b, as expected:

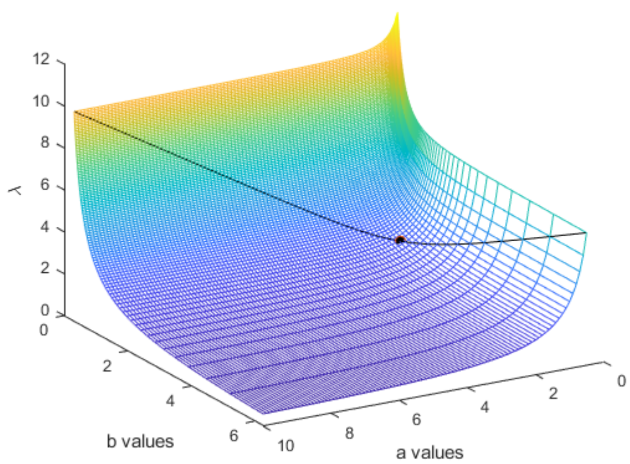

Figure 1: Eigenvalues of a rectangle with restricted π area *black curve* and the minimum *black circle*.

Now we are able to compare the optimized rectangular shape with the circular one. We want to see which one will have the smallest first eigenvalue if they have the same area.
Assume for simplicity that $R = 1$. Then, $S = \pi R^2 = \pi = ab = a^2 \Rightarrow a = \sqrt{\pi}$.
Therefore, using (21) and (27):

$$\lambda_{square} = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \pi^2 \left(\frac{2}{\pi}\right) = 2\pi \approx 2.5066^2 = 6.2832,$$
$$\lambda_{circle} = \frac{\xi_{1,0}^2}{R^2} = \xi_{1,0}^2 \approx 2.4048^2 = 5.7832.$$

Consequently, $\lambda_{square} > \lambda_{circle}$. Then, we can state that, between a 2-dimensional rectangle and circle, the shape with the smallest first eigenvalue is the last one. Observe that this result is in accordance with the Rayleigh-Faber-Krahn inequality 4.15.

3.2 A higher dimensional analysis

In this section, we want to compute the drum's model over a parallelepiped and a ball domain. We parametrise the model as in the 2-dimensional case, although we are now in $\Omega \subset \mathbb{R}^n$. We will use the calculus done in section 2.2.1.

3.2 The parallelepiped

Suppose Ω an open bounded domain in $\mathbb{R}^n$ that describes a parallelepiped. It can be written as $\Omega = (0, a_1) \times \cdots \times (0, a_n)$, where $a_i > 0$, $\forall i = 0 \dots n$. Then, notice that $u = u(x_1, \dots, x_n, t)$. To simplify notation, we denote $\mathbf{x} := (x_1, \dots, x_n) \in \mathbb{R}^n$.
Therefore, the wave problem with Dirichlet homogeneous boundary conditions can be written as:

$$\begin{cases} u_{tt}(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = 0 & & \mathbf{x} \in \Omega, t > 0 \\ u(x_1, \dots, 0, \dots, x_n, t) = 0, & \forall i = 0 \dots n & x_j \in (0, a_j), \forall j = 0 \dots n,\ t > 0 \\ u(x_1, \dots, a_i, \dots, x_n, t) = 0, & \forall i = 0 \dots n & x_j \in (0, a_j), \forall j = 0 \dots n,\ t > 0 \\ u(\mathbf{x}, 0) = g(\mathbf{x}) & & \mathbf{x} \in \Omega \\ u_t(\mathbf{x}, 0) = h(\mathbf{x}) & & \mathbf{x} \in \Omega. \end{cases} \tag{28}$$

By separation of variables we look for a solution $u(\mathbf{x}, t) = X(\mathbf{x})T(t)$, where $X(\mathbf{x}) = X_1(x_1) \cdots X_n(x_n)$. As we have done in the previous calculus and using theorem 2.1, our solution is

$$u(x, t) = \sum_{k \geqslant 1} (C_k \sin(\sqrt{\lambda_k} t) + D_k \cos(\sqrt{\lambda_k} t))\, \varphi_k(\mathbf{x}),$$

where

$$\begin{cases} -\Delta \varphi_k = \lambda_k \varphi_k & \forall k \\ \varphi_k = 0 & on\ \partial\Omega \\ g = \sum_{k \geqslant 1} D_k \varphi_k \\ h = \sum_{k \geqslant 1} C_k \sqrt{\lambda_k} \varphi_k. \end{cases}$$

Note that $\varphi_k(\mathbf{x}) = X_1^k(x_1)\cdots X_n^k(x_n)$. To simplify calculus and notation, we fix k and write $\lambda_k = \lambda$, $\varphi_k = \varphi$, and $X_q^k = X$, $\forall q = 1, \ldots, n$. Rearranging the eigenvalue equation and dividing by $X(x_1)\cdots X(x_n)$,

$$-\frac{X_1''(x_1)}{X_1(x_1)} = \frac{X_2''(x_2)}{X_2(x_2)} + \cdots + \frac{X_n''(x_n)}{X_n(x_n)} + \lambda.$$

Since each side of the equation depends on different independent variables, they must be constant. Iterating this process n times and naming the constants $\mu_1, \ldots, \mu_n$, where $\lambda = \mu_1 + \cdots + \mu_n$ and $\mu_i \geqslant 0$, $\forall i = 0, \ldots, n$, we get n ODE independent systems:

$$\begin{cases} X_1''(x_1) = -\mu_1 X_1(x_1) \\ X(0) = X(a_1) = 0 \end{cases} \quad \cdots \quad \begin{cases} X_n''(x_n) = -\mu_n X_n(x_n) \\ X(0) = X(a_n) = 0. \end{cases}$$

Notice that each of them is a Dirichlet eigenvalue problem by itself. Besides, it is clear that the solution to each system is

$$X_q(\mathbf{x}) = A^q \sin(\sqrt{\mu_q}\mathbf{x}) + B^q \cos(\sqrt{\mu_q}\mathbf{x}), \quad q = 1, \ldots, n.$$

Since $X_q(0) = 0$, $B^q \equiv 0$, $\forall q = 1, \ldots, n$. Taking into account that $X_q(a_q) = 0$, $\forall q = 1, \ldots, n$ and to avoid trivial null solution, we have $\sqrt{\mu_q}a_q = \pi q \Rightarrow \sqrt{\mu_q} = \dfrac{\pi q}{a_q}$, $\forall q = 1, \ldots, n$.

To sum up,

$$\lambda = \mu_1 + \cdots + \mu_n = \frac{\pi^2 q_1^2}{a_1^2} + \cdots + \frac{\pi^2 q_n^2}{a_n^2} = \pi^2\left(\frac{q_1^2}{a_1^2} + \cdots + \frac{q_n^2}{a_n^2}\right), \quad \text{where } q_1, \ldots, q_n = 1, \ldots, n.$$

Then, the first eigenvalue form is

$$\lambda_{1,\ldots,1} = \pi^2\left(\frac{1}{a_1^2} + \cdots + \frac{1}{a_n^2}\right). \tag{29}$$

3.2 The ball

Suppose Ω an open bounded domain in $\mathbb{R}^n$ that describes a ball and has the same volume as the previous rectangular domain. We can note it $\mathcal{B} = \{x \in \mathbb{R}^n \,\big|\, |x| < R\}$, where $R \in \mathbb{R}^+$. Setting $u \equiv u(\mathbf{x}, t)$, where $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$, the wave problem with Dirichlet homogeneous boundary conditions and general initial conditions is

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathcal{B} \times (0, +\infty) \\ u = 0 & \text{on } \partial\mathcal{B} \times (0, +\infty) \\ u(\mathbf{x}, 0) = g(\mathbf{x}) & \text{on } \mathcal{B} \\ u_t(\mathbf{x}, 0) = h(\mathbf{x}) & \text{on } \mathcal{B}. \end{cases} \tag{30}$$

Consider a change of variables $\mathbf{x} = r\theta$, where $r \in (0, R)$ and $\theta \in \mathbb{S}^{n-1}$.

By separation of variables we look for a solution $u(r, \Theta, t) = V(r)W(\theta)T(t)$. As we have done in previous calculus, we use theorem 2.1 and our solution is

$$u(x, t) = \sum_{k \geqslant 1} (C_k \sin(\sqrt{\lambda_k} t) + D_k \cos(\sqrt{\lambda_k} t))\, \varphi_k(r, \theta),$$

where

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \forall k \\ \varphi_k = 0 & on\ \partial\Omega \\ g = \sum\limits_{k\geqslant 1} D_k\varphi_k \\ h = \sum\limits_{k\geqslant 1} C_k\sqrt{\lambda_k}\varphi_k. \end{cases}$$

Notice that $\varphi_k(r,\theta) = V_k(r)W_k(\theta)$. In order to simplify calculus and notation, we fix k and write $\lambda_k = \lambda$, $V_k = V$, $W_k = W$, and $\varphi_k = \varphi$. This last variable satisfies a Dirichlet eigenvalue equation and taking into account the polar form of the N-Laplacian (see appendix A.2) that uses the Laplace-Beltrami operator Λ, which only cares about the angles, we can rearrange the eigenvalue equation as

$$\left(V'' + \frac{n-1}{r}V' + \lambda V\right)W + \frac{\Lambda W}{r^2}V = 0.$$

Dividing by $V(r)W(\theta)$,

$$\frac{V'' + \frac{n-1}{r}V' + \lambda V}{V} = -\frac{\Lambda W}{W}.$$

We define $\mu \geqslant 0$ as the constant that every fraction must equal, since they depend on different independent variables. Because we consider an sphere, W is a spherical harmonic and $\mu = -l(l+(n-2))$, $l \in \mathbb{N}$. Therefore, we have

$$\frac{V'' + \frac{n-1}{r}V' + \lambda V}{V} = \mu.$$

Rearranging,

$$r^2V'' + (n-1)rV' + (r^2\lambda - \mu)V = 0.$$

To simplify notation we define $x := r\sqrt{\lambda}$. All in all,

$$x^2V'' + (n-1)xV' + (x^2 - \mu)V = 0.$$

Notice the similarity of this equation with the Bessel's formula. We should do another variable change: $V := x^{\frac{-n+2}{2}}P$. Then, we compute the first and second derivatives to substitute them into the equation and get the Bessel's one.

$$V' = x^{\frac{-n+2}{2}}P' + \frac{-n+2}{2}x^{\frac{-n}{2}}P,$$

$$V'' = x^{\frac{-n+2}{2}}P'' + (-n+2)x^{\frac{-n}{2}}P' - \frac{(-n+2)n}{4}x^{\frac{-n-2}{2}}P.$$

Accordingly,

$$x^2\left(x^{\frac{-n+2}{2}}P'' + (-n+2)x^{\frac{-n}{2}}P' - \frac{(-n+2)n}{4}x^{\frac{-n-2}{2}}P\right) + (n-1)x\left(x^{\frac{-n+2}{2}}P' + \frac{-n+2}{2}x^{\frac{-n}{2}}P\right) + (x^2-\mu)x^{\frac{-n+2}{2}}P = 0.$$

Simplifying by $x^{\frac{-n+2}{2}}$,

$$x^2\left(P'' + (-n+2)x^{-1}P' - \frac{(-n+2)n}{4}x^{-2}P\right) + (n-1)x\left(P' + \frac{-n+2}{2}x^{-1}P\right) + (x^2-\mu)x^{\frac{-n+2}{2}}P = 0,$$

and successively rearranging,

$$x^2P'' + xP(-n+2+n-1) + P\left(\frac{-n+2}{2} - \frac{(-n+2)n}{4} + x^2 - \mu\right) = 0,$$
$$x^2P'' + xP + P[x^2 - (\frac{1}{4}(n-2)^2)] = 0.$$

As we have explained in section 3.1.2, this is a Bessel equation. It is known that its solution is

$$P(x) = AJ_{\frac{n-2}{2}}(x) + BY_{\frac{n-2}{2}}(x),$$

where $J_{\frac{n-2}{2}}$ and $Y_{\frac{n-2}{2}}$ are the first and second kind Bessel functions.
Like the calculus in section 3.1.2, $Y_m(x) \xrightarrow[x\to 0]{} -\infty$. Since our domain is a ball, we are looking for bounded solutions at 0; hence, $B \equiv 0$.

Plugging all the variable changes that we have done, $J_{\frac{n-2}{2}}(x) = J_{\frac{n-2}{2}}(r\sqrt{\lambda})$. Since $V(R) = 0$ for the Dirichlet boundary conditions, $P(R) = 0$; thus, $J_{\frac{n-2}{2}}(R\sqrt{\lambda})$ must be a zero of the Bessel function $J_{\frac{n-2}{2}}$.
Notice that $n \in \mathbb{N}$ and $n \geqslant 2$; therefore, the subscripts of J are $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2} \dots\}$. It is important to distinguish between integer and rational subscripts. The first group are zeros of the already seen Bessel function. Besides, the second group are zeros of the spherical Bessel function, that are explicit: $\{j_0(x) = \frac{sin(x)}{x},\ j_1(x) = \frac{sin(x)}{x^2} - \frac{cos(x)}{x}, \dots\}$, where $j_n(x) = \sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x)$.
Therefore, eigenvalues of an n-dimensional ball are

$$\lambda_k = \begin{cases} \frac{\xi^2_{k,\frac{n-2}{2}}}{R^2} & \text{if } \frac{n-2}{2} \in \mathbb{Z},\ \text{i.e., } n \text{ is even} \\ \frac{\hat{\xi}^2_{k,\frac{n-3}{2}}}{R^2} & \text{if } \frac{n-2}{2} \notin \mathbb{Z},\ \text{i.e., } n \text{ is odd,} \end{cases} \tag{31}$$

where $\xi_{k,\frac{n-2}{2}}$ is the k-th zero of $J_{\frac{n-2}{2}}$ and $\hat{\xi}_{k,\frac{n-3}{2}}$ is the k-th zero of $j_{\frac{n-3}{2}}$. Recall that $n \geqslant 2$.

3.2 Optimization of the first eigenvalue

Firstly we want to know which parallelepiped has the smallest first eigenvalue. Then, consider $\Omega = (0, a_1) \times \cdots \times (0, a_n)$, where $a_i > 0$, $\forall i = 0, \dots, n$ and fix its volume: $Vol(\Omega) = |\Omega|$. Following the volume definition, we get

$$a_1 \cdots a_n = |\Omega| \Longrightarrow a_n = \frac{|\Omega|}{a_1 \cdots a_{n-1}}. \tag{32}$$

As we have observed in (29), the first parallelepiped eigenvalue is

$$\lambda_{1,\dots,1} = \pi^2\left(\frac{1}{a_1^2} + \cdots + \frac{1}{a_n^2}\right).$$

We are interested in finding the side lengths suitable to minimize it: set $a := (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$ and

$$f(a) := \lambda_{1\dots 1}(a_1, \dots, a_n) = \pi^2 \left(\frac{1}{a_1^2} + \cdots + \frac{1}{a_{n-1}^2} + \frac{1}{a_n^2} \right) = \pi^2 \left(\frac{1}{a_1^2} + \cdots + \frac{1}{a_{n-1}^2} + \frac{a_1^2 \dots a_{n-1}^2}{|\Omega|^2} \right),$$

where we have used (32) for the last equality. Our problem is now reduced to find the variables that minimize this function f:

$$\nabla f(a) = 0 \iff \nabla f(a) = \pi^2 \begin{pmatrix} \frac{-2}{a_1^3} + 2\frac{a_1^2 \cdots \widehat{1} \cdots a_{n-1}^2 a_1}{|\Omega|^2} \\ \vdots \\ \frac{-2}{a_{n-1}^3} + 2\frac{a_1^2 \cdots \widehat{n-1} \cdots a_{n-1}^2 a_{n-1}}{|\Omega|^2} \end{pmatrix} = 2\pi^2 \begin{pmatrix} \frac{-1}{a_1^3} + \frac{a_1^2 \cdots \widehat{1} \cdots a_{n-1}^2 a_1}{|\Omega|^2} \\ \vdots \\ \frac{-1}{a_{n-1}^3} + \frac{a_1^2 \cdots \widehat{n-1} \cdots a_{n-1}^2 a_{n-1}}{|\Omega|^2} \end{pmatrix} = 0,$$

where $\hat{q}$ denotes that the q-th corresponding variable of the multiplying terms is missing. In this specific usage, a_q^2.

Therefore, $\frac{-1}{a_q^3} + \frac{a_1^2 \cdots \widehat{q} \cdots a_{n-1}^2 a_q}{|\Omega|^2}$ must vanish for all $q = 1, \dots, n-1$. Rearranging, we get

$$|\Omega|^2 = a_1^2 \cdots \widehat{q} \cdots a_{n-1}^2 a_q^4, \quad \forall q = 1, \dots, n-1.$$

Joining equations for any chosen subscripts i and j, where $1 \leqslant i < j \leqslant n-1$,

$$\begin{matrix} |\Omega|^2 = a_1^2 \cdots \widehat{i} \cdots a_{n-1}^2 a_i^4 \\ |\Omega|^2 = a_1^2 \cdots \widehat{j} \cdots a_{n-1}^2 a_j^4 \end{matrix} \implies a_1^2 \cdots \widehat{i} \cdots a_{n-1}^2 a_i^4 = a_1^2 \cdots \widehat{j} \cdots a_{n-1}^2 a_j^4 \implies a_i^2 = a_j^2.$$

Physically a_k denotes a side length of the parallelepiped, then $a_k > 0$, $\forall k = 1, \dots, n$. Consequently, $a_i = a_j$, $\forall i \neq j$. Then, by $|\Omega|$ definition, $|\Omega| = a_1 \cdots a_n = a_k \cdots a_k \implies a_k = |\Omega|^{1/n}$, $\forall k = 1 \dots n$. In conclusion, as in the 2-dimensional case, the parallelepiped with the smallest first eigenvalue is the one with all its sides of the same length, i.e., the cube.

Now we are ready to compare the cube with the ball. First of all, we fix the domains' volume: $Vol(\Omega) = |\Omega|$. Then, we consider a cube with side length $|\Omega|^{\frac{1}{n}}$ and the ball of the same volume and radius R. Bear in mind the formula of the volume of an n-dimensional ball,

$$|\Omega| = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} R^n, \tag{33}$$

where Γ is the Gamma function, we can define the volume by just choosing R.

In order to simplify the calculus, we set $R = 1$; hence, volume $|\Omega|$ is given depending on the current dimension. As we don't know the general expression of the zeros of the Bessel function, it is impossible to obtain exact numerical eigenvalues for the generalized n-dimensional case. Consequently, consider the following table of the first eigenvalues of the cube and the ball for some concrete dimensions, computed through MATLAB (see appendix B to find the script):

Dimension	Cube	Ball
2	6.2832	5.7832
3	11.3945	9.8696
4	17.7715	14.6819
5	25.3951	20.1907
6	34.2519	26.3746
10	81.8439	57.5829
100	$6.1208 \cdot 10^3$	$3.8110 \cdot 10^3$
200	$2.3872 \cdot 10^4$	$1.5824 \cdot 10^4$
325	$6.2352 \cdot 10^4$	$4.1336 \cdot 10^4$

Table 1: Cube and ball first eigenvalues for concrete dimensions.

In accordance with the table tendency, we can expect that $\lambda_{cube} > \lambda_{ball}$ for any given dimension. This result is in accordance with Faber-Krahn inequality.

As an additional information, we will compute the relation of eigenvalues when rescaling the domain Ω. This can give us a generalisation of the above results, as they have been done with $R = 1$ but they could be done with any $R \in \mathbb{R}^+$.
Let us consider a rescaled domain $\Omega' := S\Omega,\ S > 0$ and compare its eigenvalues with the ones in Ω.

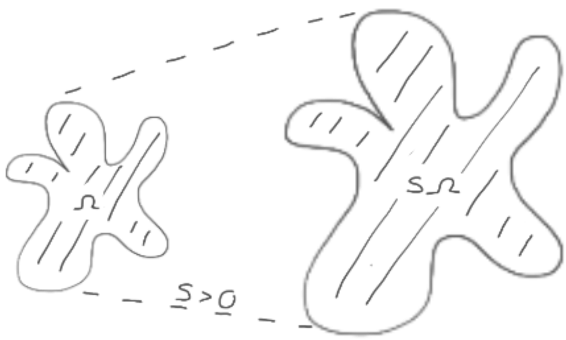

Figure 2: Diagram of Ω domain and its rescaled domain $S\Omega$, $S > 0$.

Start setting the following eigenvalue problem in Ω:

$$\begin{cases} -\Delta u = \lambda u & in\ \Omega \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{34}$$

Set $v(Sx) = u(x),\ \forall x \in \Omega$. Then, $Sx \in S\Omega$ and we can do the following calculus:

$$\Delta u(x) = \Delta(v(Sx)) = S^2(\Delta v)(Sx) \Longrightarrow \Delta v(Sx) = \frac{1}{S^2}\Delta u(x).$$

Thus,

$$-\Delta v(Sx) = -\frac{1}{S^2}\Delta u(x) = \frac{\lambda}{S^2}u(x) = \frac{\lambda}{S^2}v(Sx).$$

The resulting problem for $\Omega' = S\Omega$ is

$$\begin{cases} -\Delta v = \dfrac{\lambda}{S^2}v & \textit{in } \Omega' \\ v = 0 & \textit{on } \partial\Omega'. \end{cases} \tag{35}$$

In conclusion, if $\lambda(\Omega)$ is an eigenvalue of Ω, then $\lambda(S\Omega) = \dfrac{\lambda(\Omega)}{S^2}$ is the corresponding eigenvalue in the rescaled domain $S\Omega$.

This result is coherent with the parallelepiped eigenvalues' form seen in (29): consider the rescaled domain $\Omega' = S\Omega = (0, Sa_1) \times \cdots \times (0, Sa_n)$. Hence, the parallelepiped eigenvalues' form is

$$\lambda = \pi^2 \left(\frac{q_1^2}{(Sa_1)^2} + \cdots + \frac{q_n^2}{(Sa_n)^2} \right) = \frac{\pi^2}{S^2} \left(\frac{q_1^2}{a_1^2} + \cdots + \frac{q_n^2}{a_n^2} \right), \quad q_1, \dots, q_n = 1, \dots, n,$$

and $\lambda(S\Omega) = \dfrac{\lambda(\Omega)}{S^2}$, as desired.

This result is also consistent with the balls eigenvalues' form since $R\sqrt{\lambda}$ are the zeros of the Bessel function $J_{\frac{n-2}{2}}$. Consider the rescaled domain $\Omega' = S\Omega = [0, SR) \times \mathbb{S}^{n-1}$. Then, following the calculus from (31), the Euclidean ball eigenvalues' form are

$$\lambda_k = \begin{cases} \dfrac{\xi^2_{k,\frac{n-2}{2}}}{(SR)^2} & \textit{if } \frac{n-2}{2} \in \mathbb{Z}, \textit{ i.e., n is even} \\ \dfrac{\hat{\xi}^2_{k,\frac{n-3}{2}}}{(SR)^2} & \textit{if } \frac{n-2}{2} \notin \mathbb{Z}, \textit{ i.e., n is odd,} \end{cases}$$

and $\lambda(S\Omega) = \dfrac{\lambda(\Omega)}{S^2}$, as expected.

To sum up, the rescaling process can extend the explicit calculus in concrete domains. Hence, all the results regarding parallelepiped and ball domains are valid for any chosen volume of them. Furthermore, the table 1, that summarize some explicit eigenvalues of cubes and balls and it is computed with $R = 1$, can be easily extended to any selected volume.

4. Faber-Krahn inequality

In this section, we are interested in the proof of Faber-Krahn inequality. We have already seen some motivations and applications of it, but how works its general statement? From which theorems and lemmas it is based? We will answer all these questions promptly.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary. Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega. \end{cases} \tag{36}$$

Our aim is to describe the smallest λ such that (36) has non trivial null solution, i.e. $u \not\equiv 0$ in Ω.

This problem is solved by Faber-Krahn inequality 4.15, that states that, under a volume constraint, the smallest value of λ that meets the equation is the one corresponding to a ball domain. However, it contains some elements that need to be introduced beforehand. Hence, we will begin with a heuristic overview to start to get familiarized with these ideas. Then, we will go in-depth in the theoretical concepts, and finally, we will see the proof of the inequality based on Rayleigh quotient and Pólya-Szegö inequality.

4.1 Heuristic characterization of the first eigenvalue of the Dirichlet Laplacian by minimizing the Rayleigh quotient

For simplicity, assume $u \in \mathcal{C}^2(\overline{\Omega})$ that satisfies the eigenvalue problem (36); therefore, $u \not\equiv 0$ in Ω. If we multiply both sides of the equation by u:

$$\lambda \int_\Omega |u|^2 = -\int_\Omega \Delta u \cdot u = -u \cdot \Delta u\Big|_{\partial\Omega} + \int_\Omega |\nabla u|^2 = \int_\Omega |\nabla u|^2,$$

where we have used integration by parts and the boundary condition. Rearranging,

$$\lambda = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} \geqslant 0.$$

Since we look for the smallest possible λ it is natural to consider the quantity

$$J^* := \operatorname*{Inf}_{\substack{v \in \mathcal{C}^1(\Omega) \\ v \not\equiv 0,\ v|_{\partial\Omega} = 0}} J(v), \qquad \text{where} \quad J(v) := \frac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2}. \tag{37}$$

By the previous argument, it is clear that any eigenvalue λ of the eigenvalue problem (36) satisfies $\lambda \geqslant J^*$.

Definition 4.1. Set $\Omega \subset \mathbb{R}^n$ an open bounded domain, $v \in \mathcal{C}^1(\Omega)$, and $v \not\equiv 0$. The quotient

$$J(v) := \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}$$

is called the **Rayleigh quotient**. *[3]§9*

Assume that J^* is attained by some function u, that is, $J^* = J(u)$. It is natural to ask oneself if this u is indeed an eigenfunction of (36) with eigenvalue J^*. In such case, J^* would be the smallest one and we would have characterized the lowest eigenvalue.

Consequently, it is reasonable to consider some perturbation of the function u: set $\varphi \in \mathcal{C}_c^1(\Omega)$, $\varepsilon \in \mathbb{R}$ and define $f(\varepsilon) := J(u + \varepsilon\varphi)$. Notice that $\varphi \in \mathcal{C}_c^1(\Omega)$, where $\mathcal{C}_c^1(\Omega)$ denote the function space $\mathcal{C}^1(\Omega)$ with compact support.

With this notation, as $u \in \mathcal{C}^1(\overline{\Omega})$, $u|_{\partial\Omega} = 0$, and $\varphi \in \mathcal{C}_c^1(\Omega)$, then: $u + \varepsilon\varphi \in \mathcal{C}^1(\overline{\Omega})$ and $u + \varepsilon\varphi\,|_{\partial\Omega} = 0$.

Together with the previous results follow some implications:

$$f(0) = J(u) = J^* \leqslant J(u + \varepsilon\varphi) = f(\varepsilon), \quad \forall \varepsilon \in \mathbb{R}.$$

Consequently, $f(0)$ is a minimum and $f'(0) = 0$. Hence,

$$f'(0) = \frac{d}{d\varepsilon} \frac{\int_\Omega |\nabla u|^2 + 2\varepsilon \int_\Omega \nabla u \nabla \varphi + \varepsilon^2 \int_\Omega |\nabla \varphi|^2}{\int_\Omega |u|^2 + 2\varepsilon \int_\Omega |u\varphi| + \varepsilon^2 \int_\Omega |\varphi|^2}(0) = 0 \iff$$

$$\iff \left(2\int_\Omega \nabla u \nabla \varphi\right)\left(\int_\Omega |u|^2\right) - \left(\int_\Omega |\nabla u|^2\right)\left(2\int_\Omega u\varphi\right) = 0.$$

Simplifying and rearranging,

$$\int_\Omega \nabla u \nabla \varphi = J^* \int_\Omega u\varphi, \quad \forall \varphi \in \mathcal{C}_c^1(\Omega). \tag{38}$$

Once more, we wonder if we can get from here that $-\Delta u = J^* u$ in Ω. We need to keep working on it: assume for simplicity that $u \in \mathcal{C}^3(\overline{\Omega})$; then, integrating by parts, using that φ has compact support on the boundary and applying (38),

$$-\int_\Omega \Delta u \cdot \varphi = \int_\Omega \nabla u \nabla \varphi = J^* \int_\Omega u\varphi \Longrightarrow \int_\Omega (\Delta u + J^* u)\varphi = 0, \quad \forall \varphi \in \mathcal{C}_c^1(\Omega). \tag{39}$$

To see that $\Delta u + J^* u = 0$, take Ω' such that $\overline{\Omega'} \subset \Omega$, and $\varphi = (\Delta u + J^* u)\chi$, where χ is a piece-wise function such that $\chi \in \mathcal{C}_c^1(\Omega)$ and

$$\begin{cases} \chi \geqslant 0 & \textit{in } \Omega \\ \chi \equiv 1 & \textit{in } \Omega'. \end{cases}$$

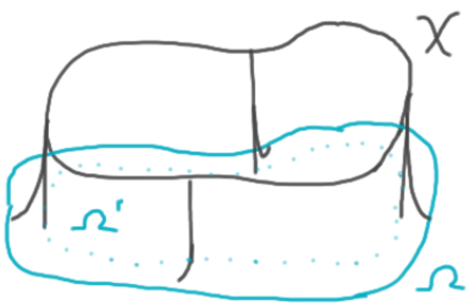

Figure 3: Diagram of χ function with respect to Ω and Ω'.

Therefore,

$$\varphi \in \mathcal{C}_c^1(\Omega), \text{ as desired, and } 0 \leqslant \int_{\Omega'} |\Delta u + J^* u|^2 \leqslant \int_{\Omega} |\Delta u + J^* u|^2 \chi = \int_{\Omega} (\Delta u + J^* u)\varphi = 0,$$

where the last equality is due to (39). As it must be satisfied $\forall \varphi \in \mathcal{C}_c^1(\Omega)$ and by the squeeze lemma,

$$\Delta u + J^* u = 0 \text{ on } \Omega', \quad \forall \overline{\Omega'} \subset \Omega \Longrightarrow -\Delta u = J^* u \text{ on } \Omega, \text{ as expected.}$$

It seems that everything works properly and we can define the smallest λ for our Dirichlet eigenvalue problem (36). However, we should take care of some aspects:

First of all, observe that J^* exists, since $J(v) \geqslant 0$, $\forall v \in \mathcal{C}^1(\Omega)$, $v \not\equiv 0$. But, how can we ensure that such infimum is attained? In other words, is it a minimum?

Notice that the norm $\|v\|_{1,2} := \left(\int_\Omega |v|^2 + \int_\Omega |\nabla v|^2\right)^{1/2}$ is suitable for $J(v)$. To show that the infimum J^* is attained, one possible approach could be to take $\{v_k\}_{k\in\mathbb{N}}$ such that $J(v_k) \xrightarrow[k\longrightarrow+\infty]{} J^*$ and to show that indeed v_k, or a sub-sequence of it, converges to some u in the following way:

$$\begin{cases} \int_\Omega |u|^2 & = & \lim\limits_{k\to+\infty} \int_\Omega |v_k|^2 \\ \int_\Omega |\nabla u|^2 & \leqslant & \liminf\limits_{k\to+\infty} \int_\Omega |\nabla v_k|^2. \end{cases} \tag{40}$$

However, $\mathcal{C}^1(\overline{\Omega})$ is not a complete space with respect to the norm $\|\cdot\|_{1,2}$. Therefore, we need to *enlarge* $\mathcal{C}^1(\overline{\Omega})$ to get a Banach space with respect to this norm.

Remark that $\|\cdot\|_{1,2}$ uses Lebesgue integral. Then, if we enlarge $\mathcal{C}^1(\overline{\Omega})$ to show the existence of a minimizer u of J, what is the meaning of the homogeneous Dirichlet boundary condition "$u = 0$ on $\partial\Omega$"? And how do we make sense of $-\Delta u = J^* u$ pointwise?

These problems are addressed using Sobolev spaces.

4.2 Sobolev spaces

To focus more on our purpose, after setting definitions and results, we will remark them on our study case. In this way, the text will be easier to understand and we will go faster to the useful elements that proof Faber-Krahn inequality.

There are a lot of lemmas and theorems that won't be proved. They belong to the functional analysis branch and we are not interested in their proves, just in their applications. In fact, only Pólya-Szegö inequality and Faber-Krahn inequality will be proved.

Notation 4.2. We write $v \in L^k_{loc}(\Omega)$, $k \geqslant 1$ to state that v is locally an $L^k(\Omega)$ function.

It is necessary to define a derivative for functions that are integrable but maybe not differentiable:

Definition 4.3. Suppose $\Omega \subset \mathbb{R}^n$ an open set. A function $v \in L^1_{loc}(\Omega)$ is **weakly differentiable** with respect to x_i if there exists a function $u_i \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega v \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega u_i \phi dx, \quad \forall \phi \in \mathcal{C}_c^\infty(\Omega).$$

The function u is called the **weak i-th partial derivative** of v and denoted $\dfrac{\partial f}{\partial x_i}$.

In more general sense, suppose that $\alpha \in \mathbb{N}^n$ is a multiindex. A function $v \in L^1_{loc}(\Omega)$ has a **weak derivative** $D^\alpha v := \dfrac{\partial^\alpha v}{\partial x_i^\alpha} \in L^1_{loc}(\Omega)$ if

$$\int_\Omega v \frac{\partial^\alpha \phi}{\partial x_i^\alpha} dx = (-1)^{|\alpha|} \int_\Omega \frac{\partial^\alpha v}{\partial x_i^\alpha} \phi dx, \quad \forall \phi \in \mathcal{C}_c^\infty(\Omega). \qquad [4]§5$$

Since $\mathcal{C}_c^\infty(\Omega)$ is dense in $L^1_{loc}(\Omega)$, the weak derivative of a function, if it exists, is unique up to a set of measure zero. Observe that the weak derivative of a differentiable function coincides with the pointwise derivative. However, the existence of pointwise derivative is not equivalent to the existence of weak derivative. For example, the absolute value function has weak-derivative but is not differentiable; hence, it has not *strong* derivative.

This concept is widely used in Sobolev spaces:

Definition 4.4. For $k \in \mathbb{N}$ and $1 \leqslant p \leqslant +\infty$, the **Sobolev space** $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega) : \exists D^\alpha v \in L^p(\Omega) \text{ in the weak sense derivative } \forall |\alpha| \leqslant k\}. \tag{41}$$

In addition, we call $k \in \mathbb{N}$ the **order of Sobolev space** $W^{k,p}$.

In other words, the Sobolev space is a function space whose members are locally summable functions $u : \Omega \longrightarrow \mathbb{R}$ such that for each multiindex α that satisfies $|\alpha| \leqslant k$, $D^\alpha v$ exists in the weak sense and belongs to $L^p(\Omega)$. *[4]§5*

Definition 4.5. Let $u \in W^{k,p}(\Omega)$. Its norm is

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leqslant k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p} & 1 \leqslant p < +\infty \\ \max_{|\alpha| \leqslant k} \|D^\alpha u\|_{L^\infty(\Omega)} & p = +\infty. \end{cases} \qquad [4]§5$$

We have special interest in Hilbert spaces, denoted as $H^k(\Omega) := W^{k,2}(\Omega)$. Taking into consideration that we will use $H^1(\Omega)$ a lot, remark that

$$H^1(\Omega) = \left\{v \in L^2(\Omega) : \exists D^\alpha v \in L^2(\Omega) \text{ in the weak sense derivative } \forall |\alpha| \leqslant 1\right\}. \tag{42}$$

It is a Hilbert space, in particular, complete, with respect to

$$\|v\|_{H^1(\Omega)} = \left(\int_\Omega |v|^2 + \int_\Omega |\nabla v|^2 \right)^{1/2}. \tag{43}$$

Definition 4.6. We denote by $W_0^{k,p}$ the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ with respect to the norm defined in 4.5.

Observe that $H_0^1(\Omega)$ is the closure of $\mathcal{C}_c^\infty(\Omega)$ with respect to the $H^1(\Omega)$ norm (43). Then, $H_0^1(\Omega) \subset H^1(\Omega)$. Since $H^1(\Omega)$ is a Hilbert space, the closure $H_0^1(\Omega)$ too. Owing to the fact that $H_0^1(\Omega)$ is constructed via compactly supported functions, it is the suitable space to make sense of the Dirichlet boundary condition "$v = 0$ on $\partial\Omega$" for functions such that $\|v\|_{H^1(\Omega)} < +\infty$.

Therefore, our aim is now to re-write the eigenvalue problem (36) using $H_0^1(\Omega)$.

Let us see a theorem that provides us with a tool to bound functions regarding its derivative bounds and the geometry of its domain.

Theorem 4.7. (Poincaré inequality) *Let p such that $1 \leqslant p < +\infty$ and Ω a bounded subset, at least in one direction. Then, there exists some constant $C_{\Omega,p}$, such that for every function $v \in W_0^{1,p}(\Omega)$,*

$$\|v\|_{L^p(\Omega)} \leqslant C_{\Omega,p}\|\nabla v\|_{L^p(\Omega)}.$$ *[3]§2*

In particular, for $v \in H_0^1(\Omega)$ we have $\|v\|_{L^2(\Omega)} \leqslant C_{\Omega,2}\|\nabla v\|_{L^2(\Omega)}$. To simplify notation, we write $C_{\Omega,2} = C_\Omega$. It can be useful sometimes to write the inequality as

$$\int_\Omega |v|^2 \leqslant C_\Omega \int_\Omega |\nabla v|^2.$$

Now we have another theorem that implies that any uniformly bounded sequence in $W^{1,p}(\Omega)$ has a subsequence that converges in $L^q(\Omega)$, whose superscripts p and q should meet some conditions:

Theorem 4.8. (Rellich-Kondrachov) *Assume $\Omega \subset \mathbb{R}^n$ an open bounded domain and $\partial\Omega$ is $\mathcal{C}^1$. Suppose $1 \leqslant p < n$ and $p^* := \dfrac{np}{n-p}$. Then, the inclusion*

$$i_{p^*} : W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

is a continuous operator for every p^* and

$$i_q : W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is a compact operator for every $1 \leqslant q < p^*$. *[4]§5*

Observe that since $p^* > p$ and $p^* \xrightarrow[p \longrightarrow n]{} +\infty$, we have

$$W^{1,p}(\Omega) \subset\subset L^p(\Omega), \quad \forall 1 \leqslant p \leqslant +\infty,$$

that means that $W^{1,p}(\Omega)$ has a compact embedding in $L^p(\Omega)$.
Note also that $W_0^{1,p}(\Omega) \subset\subset L^p(\Omega)$, since $W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$, even if we do not assume $\partial\Omega$ to be $\mathcal{C}^1$. Moreover, we can state that $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is a compact operator, i.e., it is a continuous lineal operator that maps $H_0^1(\Omega)$ in a subset with a compact closure of $L^2(\Omega)$.

In particular, notice that $\|v\|_{H^1(\Omega)}$ and $\|\nabla v\|_{L^2(\Omega)}$ are equivalent norms on $H_0^1(\Omega)$. Together with the Poincaré inequality 4.7 we get that

$$\text{if } \|\nabla v_k\|_{L^2(\Omega)} \leqslant C_\Omega,\ \forall k \in \mathbb{N} \Longrightarrow \exists\, u = \lim_{j\to+\infty} v_{k_j} \in L^2(\Omega).$$

Now we are ready to reformulate the eigenvalue problem (36) using the weak formulation:

$$\begin{cases} u \in H_0^1(\Omega) \\ \int_\Omega \nabla u \nabla v = \lambda \int_\Omega uv, \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{44}$$

Although technically we only know that $u \in H_0^1(\Omega)$, regularity theory ensure that indeed, $u \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}(\overline{\Omega})$.

4.3 Characterization of the first eigenvalue of the Dirichlet Laplacian by minimizing the Rayleigh quotient through Sobolev spaces

After the brief overview of Sobolev spaces, it is clear that to have $\int_\Omega |v|^2$, $\int_\Omega |\nabla v|^2 < +\infty$ and to make sense of the homogeneous Dirichlet boundary condition, it is enough to consider $v \in H_0^1(\Omega)$.
Then, it is natural to redefine also

$$J^* := \operatorname*{Inf}_{\substack{v \in H_0^1(\Omega) \\ v \not\equiv 0}} J(v), \qquad \text{where} \quad J(v) := \frac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2}. \tag{45}$$

Note that $J^* \geqslant \frac{1}{C_\Omega} > 0$ by Poincaré inequality.

Now it is time to see that J^* is attained for some $u \in H_0^1(\Omega)$, that is to say, that J^* is a minimum. Let $v_k \in H_0^1(\Omega)$ such that $J(v_k) \xrightarrow[k \longrightarrow +\infty]{} J^*$. Observe that $J(\lambda v) = J(v)$, $\forall \lambda \neq 0$. In fact, that is true for every not null function, because its integration is superfluous and hence it is simplified. Then, $J(v_k) = J\left(\frac{v_k}{\int_\Omega |v_k|^2}\right)$, and we can assume $\int_\Omega |v_k|^2 = 1$, $\forall k \in \mathbb{N}$. Therefore, by Rellich-Kondrachov 4.8, as

$$J(v_k) \xrightarrow[k \longrightarrow +\infty]{} J^* \qquad \text{and} \qquad J(v_k) = \int |\nabla v_k|^2 \leqslant J^* + 1 < +\infty, \quad \forall k \text{ big enough},$$

we can assert that

$$\exists\, u := \lim_{j \to +\infty} v_{k_j} \in L^2(\Omega) \ \text{ such that } \int_\Omega |u|^2 = 1. \tag{46}$$

For the limit regarding the gradient function of u, desired in equation (40), we can use Banach-Alaoglu theorem. It states that a closed unit ball of $H_0^1(\Omega)^*$, i.e., the dual space of $H_0^1(\Omega)$, is compact concerning weak* topology. So, if $u \in H_0^1(\Omega)^*$,

$$\int_\Omega \nabla v_k \phi \xrightarrow[k \longrightarrow +\infty]{} \int_\Omega \nabla u \phi, \quad \forall \phi \in H_0^1(\Omega).$$

Observation 4.9. Notice that $u \in H_0^1(\Omega)^* \equiv H_0^1(\Omega)$, so $H_0^1(\Omega)^*$ is also a Hilbert space. Then, using weak* topology and Banach-Alaoglu theorem, we obtain that if $u_k \in H_0^1(\Omega)^*$, then $\exists T \in H_0^1(\Omega)^*$ a linear continuous operator such that

$$\|T\| \leqslant C \qquad \text{and} \qquad u_{k_j}(v) = \int_\Omega \nabla u_{k_j} \nabla v \xrightarrow[j \longrightarrow +\infty]{} T(v), \quad \forall v \in H_0^1(\Omega).$$

Using Riesz-Frechet to re-write it, $T(v) =< u, v >_{H_0^1(\Omega)} = \int_\Omega \nabla u \nabla v$, and through Hölder inequality,

$$\int_\Omega |\nabla u|^2 \leqslant \liminf_{k \to +\infty} \int_\Omega |\nabla v_k|^2. \tag{47}$$

Therefore, using J^* definition and that we have assumed $\int_\Omega |u|^2 = 1$, we have

$$J^* \leqslant \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} = \int_\Omega |\nabla u|^2 \leqslant \liminf_{k \to +\infty} \int_\Omega |\nabla v_k|^2. \tag{48}$$

Applying the same transformations and assumptions on v_k but in the other way,

$$\liminf_{k\to+\infty}\int_\Omega |\nabla v_k|^2 = \liminf_{k\to+\infty}\frac{\int_\Omega |\nabla v_k|^2}{\int_\Omega |v_k|^2} = \liminf_{k\to+\infty}\int_\Omega J(v_k) = J^*. \tag{49}$$

Consequently, putting together (48) and (49), $J^* = J(u)$ for $u \in H_0^1(\Omega)$, $u \not\equiv 0$ in Ω. In other words, we can confirm that J^* is attained by some $u \in H_0^1(\Omega)$, an then, it is a minimum.

To sum up, we can briefly observe that the interpretation of the homogeneous Dirichlet boundary condition is given by $H_0^1(\Omega)$. Also, Poincaré inequality gives $J^* > 0$. Moreover, the heuristic argument that yields (38) can be carried out with $\phi \in H_0^1(\Omega)$ and we get

$$\int_\Omega \nabla u \nabla \phi = J^* \int_\Omega u\phi, \quad \forall \phi \in H_0^1(\Omega).$$

Then, regularity theory ensures that $u \in \mathcal{C}^\infty(\Omega)\cap\mathcal{C}(\overline{\Omega})$; thus, we can get (39) and conclude that

$$\begin{cases} -\Delta u = J^* u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Taking all into account, the final conclusion is that if $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{then} \quad \lambda_1(\Omega) = \min_{\substack{v\in H_0^1(\Omega)\\ v\not\equiv 0}} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2}. \tag{50}$$

So, the smallest eigenvalue is characterized.

4.4 From the Rayleigh quotient to Faber-Krahn inequality through Pólya-Szegö inequality

Finally, we are close to the Faber-Krahn statement's proof. We only need to introduce Pólya-Szegö inequality. In order to handle it, we need to present some notations and definitions.

Definition 4.10. Let A a measurable set of finite volume in $\mathbb{R}^n$. Its **symmetric rearrangement** A^* is the open centered ball whose volume agrees with A,

$$A^* = \{x \in \mathbb{R}^n \,|\, \omega_n |x|^n < Vol(A)\}, \qquad [2]\S 1$$

where ω_n is the volume of the unit ball.

It is easy to have an intuition of the concept through the following diagram:

Figure 4: Diagram of A and its symmetric rearrangement A^*, that corresponds to an Euclidean ball.

Definition 4.11. Let f be a non-negative measurable function that vanishes at infinity, in the sense that all its positive level sets have finite measure:

$$Vol(\{x \,|\, f(x) > t\}) < +\infty, \quad \forall t > 0.$$

We define the **symmetric decreasing rearrangement** f^* of f by symmetrizing its level sets,

$$f^*(x) = \int_0^\infty \chi_{\{f(x)>t\}^*} dt, \qquad [2]§1$$

where $\chi_{\{f(x)>t\}^*}$ denotes the characteristic function of the symmetric rearrangement of the set $\{f(x) > t\}$. Notice that f^* is lower semicontinuous, since its level sets are open, and is uniquely determined by the distribution function

$$\mu_f(t) = Vol(\{x|f(x) > t\}) < +\infty.$$

By construction, f^* is equimeasurable with f. In other words, the corresponding level sets of both functions have the same volume:

$$\mu_f(t) = \mu_{f^*}(t), \quad \forall t > 0.$$

In order to see graphically these concepts, it is helpful the following draw. At first sight, we can see that the volume of the striped part is the same, which means that they are equimeasurable. Besides, f^* is radially decreasing and without local critical points, what shows that is lower semicontinuous.

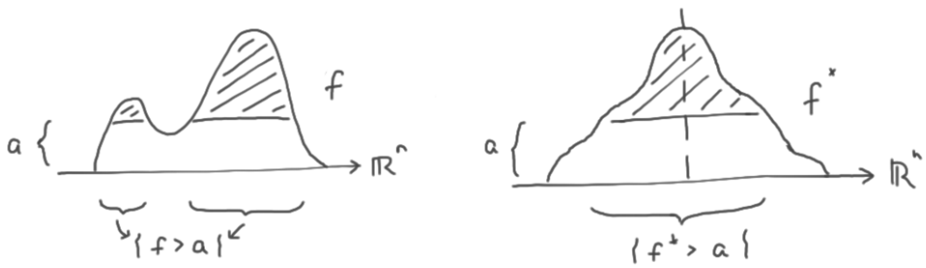

Figure 5: Diagram of f and its symmetric decreasing rearrangement f^*, that is symmetric and equimeasurable.[2]§1

It is useful to take into account the following lemma, that asserts that rearrangement preserves L^p-norms.

Lemma 4.12. *For every non-negative function f in $L^p(\mathbb{R}^n)$, $\|f\|_p = \|f^*\|_p$, $1 \leqslant p \leqslant +\infty$.* [2]§1

Now is the moment to announce Pólya-Szegö inequality, that will be proven at the end of this section:

Theorem 4.13. (Pólya-Szegö inequality) *If $f \in W^{1,p}(\mathbb{R}^n)$ for some $1 \leqslant p \leqslant +\infty$, then,*

$$\|\nabla f\|_p \geqslant \|\nabla f^*\|_p. \qquad [2]§4$$

In our study case, we can apply it as follows:

Observation 4.14. If $u \in H_0^1(\Omega) \Longrightarrow u^* \in H_0^1(\Omega)$ and $\int_\Omega |\nabla u|^2 \geqslant \int_{\Omega^*} |\nabla u^*|^2$, $\forall u \in H_0^1(\Omega)$.

Notice that here u^* is the rearrangement of $|u|$. In addition it uses the fact that $|\nabla u| = |\,\nabla |u|\,|$ almost everywhere in Ω, which we consider as a statement in this work.

Finally, we can proof Faber-Krahn inequality through Pólya-Szegö and all results that we have already seen:

Theorem 4.15. (Rayleigh-Faber-Krahn inequality) *Let Ω be an open set of finite volume in $\mathbb{R}^n$. Let $\lambda_1(\Omega)$ be the principal eigenvalue of the Dirichlet Laplacian on Ω, i.e., the smallest value of λ for which the problem*

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a non-trivial solution. Then,

$$\lambda_1(\Omega) \geqslant \lambda_1(\Omega^*). \qquad [2]\S 1$$

Proof. For simplicity, consider $\Omega \subset \mathbb{R}^n$ open, bounded, and smooth. Let Ω^* be the symmetric rearrangement of Ω, i.e., the ball such that $|\Omega^*| = |\Omega|$. Then,

$$\lambda_1(\Omega) = \operatorname*{Inf}_{\substack{v \in H_0^1(\Omega) \\ v \not\equiv 0}} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2} \geqslant \operatorname*{Inf}_{\substack{v \in H_0^1(\Omega) \\ v \not\equiv 0}} \frac{\int_{\Omega^*} |\nabla v^*|^2}{\int_{\Omega^*} |v^*|^2} \geqslant \operatorname*{Inf}_{\substack{w \in H_0^1(\Omega^*) \\ w \not\equiv 0}} \frac{\int_{\Omega^*} |\nabla w|^2}{\int_{\Omega^*} |w|^2} = \lambda_1(\Omega^*). \tag{51}$$

where the first inequality is clear since denominator keeps equal thanks to lemma 4.12, and numerator has an inequality given by Pólya-Szegö theorem 4.13. On the other hand, the second inequality is due to the fact that the infimum is taken in a bigger set.

Consequently, the first eigenvalue of an open, bounded, and smooth domain Ω achieves its minimum when the shape of Ω corresponds to its symmetric rearrangement, that is, an Euclidean ball. ☐

Lastly, for the sake of completeness, we sketch the main ideas of the proof of Pólya-Szegö inequality 4.13:

Proof from Pólya-Szegö through isoperimetric inequality, based on [2]§4.

In order to do this proof, we need to be familiarized with:

- *Co-area formula:* Suppose that $f \in W^{1,p}$. Then,

  $$\int g(x)|\nabla f(x)|^p dx = \int_0^\infty \left(\int_{f^{-1}(t)} g(\sigma)|\nabla f(\sigma)|^{p-1} d\sigma \right) dt$$

  for every measurable function g such that the left-hand side is well-defined. Notice that we express the Lebesgue integral of f in an open set in terms of integrals over the level sets of another function. Besides, the integration $d\sigma$ is with respect to $(n-1)$-dimensional Hausdorff measure.

  Notice that if co-area formula is applied to *identity* function,

  $$\int_{t_1}^{t_2} \int_{f^{-1}(t)} |\nabla f(\sigma)|^{-1} d\sigma dt = Vol\left(\{x \mid t_1 < f(x) \leqslant t_2, |\nabla f(x)| \neq 0\}\right), \quad \text{for every interval } (t_1, t_2]. \tag{52}$$

- *Jensen's inequality:* Suppose f real-valued function and $\Omega \subset \mathbb{R}$. For every real-valued convex function g,

  $$g\left(\int_\Omega \frac{f(x)}{\|f(x)\|} dx \right) \leqslant \int_\Omega g(x) \circ \frac{f(x)}{\|f(x)\|} dx.$$

- *Isoperimetric inequality:* If $A \subset \mathbb{R}^n$ has finite perimeter, then,
$$Per(A) \geqslant Per(A^*),$$
with equality if and only if A differs from a ball by a set of measure zero.

Now we can start the proof in essence. We begin with some equalities and inequalities that will be useful afterwards:

By co-area formula, applied to $f \in W^{1,p}(\mathbb{R}^n)$ and *identity* function,
$$\int |\nabla f(x)|^p dx = \int_0^\infty \int_{f^{-1}(t)} |\nabla f(\sigma)|^{p-1} d\sigma dt. \tag{53}$$

Consider the inner integral on the right-hand side. Jensen's inequality applied to convex function $s \mapsto s^{-(p-1)}$ over it gives
$$\int_{f^{-1}(t)} |\nabla f|^{p-1} \frac{d\sigma}{Per\left(\{f > t\}\right)} \geqslant \left(\int_{f^{-1}(t)} |\nabla f|^{-1} \frac{d\sigma}{Per\left(\{f > t\}\right)} \right)^{-(p-1)}. \tag{54}$$

It is clear by isoperimetric inequality that if we replace f with f^*, the perimeter of the level set decreases. Therefore,
$$Per\left(\{f > t\}\right) > Per\left(\{f^* > t\}\right). \tag{55}$$

Notice that the volume of the set of critical points decreases under symmetric decreasing rearrangement since every function f can have some local critical points, but symmetric decreasing rearrangement f^* gets rid of all of them and keeps only the absolute critical points. Then,
$$Vol\left(\{x \,|\, t_1 < f(x) \leqslant t_2, |\nabla f(x)| \neq 0\}\right) \leqslant Vol\left(\{x \,|\, t_1 < f^*(x) \leqslant t_2, |\nabla f^*(x)| \neq 0\}\right)$$
$$\Longrightarrow \int_{f^{-1}(t)} |\nabla f|^{-1} d\sigma \leqslant \int_{(f^*)^{-1}(t)} |\nabla f^*|^{-1} d\sigma. \tag{56}$$

Given all these results, we can rearrange (53) as
$$\int_{f^{-1}(t)} |\nabla f|^{p-1} d\sigma \geqslant Per\left(\{f > t\}\right)^p \cdot \left(\int_{f^{-1}(t)} |\nabla f|^{p-1} d\sigma \right)^{-(p-1)}.$$

Focusing on the right-hand part and using isoperimetric inequality, as it is shown in (56), we get
$$Per\left(\{f > t\}\right)^p \cdot \left(\int_{f^{-1}(t)} |\nabla f|^{p-1} d\sigma \right)^{-(p-1)} \geqslant Per\left(\{f^* > t\}\right)^p \cdot \left(\int_{(f^*)^{-1}(t)} |\nabla f^*|^{-1} d\sigma \right)^{-(p-1)}.$$

Now is the moment to apply Jensen's inequality with $s \mapsto s^{-(p-1)}$, as (55). Notice that Jensen's inequality holds with equality when $f = f^*$, because $|\nabla f^*|$ is constant on the level surface. Therefore,
$$Per\left(\{f^* > t\}\right)^p \cdot \left(\int_{(f^*)^{-1}(t)} |\nabla f^*|^{-1} d\sigma \right)^{-(p-1)} = \int_{(f^*)^{-1}(t)} |\nabla f^*|^{p-1} d\sigma$$

Taking this last three results into account, we can conclude that
$$\int_{f^{-1}(t)} |\nabla f|^{p-1} d\sigma \geqslant \int_{(f^*)^{-1}(t)} |\nabla f^*|^{p-1} d\sigma, \tag{57}$$
as desired. □

5. Conclusions

Through this work, we have done an overview of the Rayleigh-Faber-Krahn inequality. We have seen its theoretical side and also some of its possible applications. We have also analyzed some calculus in concrete domains and used numerical code to perform some comparisons between them.

Regarding the theoretical part, we aimed to understand Faber-Krahn inequality, and hence, we started setting the homogeneous Dirichlet Laplacian problem. We needed completeness to show existence of a minimizer of the Rayleigh quotient; therefore, we jumped to Sobolev spaces, specifically to H_0^1. We discovered that Sobolev spaces enlarge spaces of differentiable functions; consequently, our convergence was affordable. Moreover, we wanted to face the homogeneous boundary conditions, and H_0^1 was excellent since it had them implicit as it is constructed using functions with compact support. Then, thanks to Pólya-Szegö inequality and its previous lemmas, we obtained the ultimate ingredients to prove rigorously Faber-Krahn inequality.

On the other hand, we analyzed some applications in hugely different branches such as music, finance or quantum mechanics. Every model showed us the importance of the first Dirichlet Laplacian eigenvalue. It can give us information such as the best tax to attract new clients and make them spend the most money without ruining them, as well as it can show us the proper drum membrane shape to obtain the lowest possible notes. Among other examples.

Additionally, we had the opportunity to see that Faber-Krahn inequality is satisfied in some concrete domains. We computed the first eigenvalue of a parallelepiped and a ball in a wide range of high dimensional domains. After proving that the smallest first eigenvalue of a parallelepiped refers to a cube, we saw the numerical difference of the first eigenvalue of a ball and a cube. As higher the dimension of the domain was, more distinct were its values.

With all these results in mind, we can assert that Rayleigh-Faber-Krahn inequality can give us a large amount of information about a model and can help us to its optimization. This inequality shows how the symmetric rearrangement of any domain has the smallest first eigenvalue. In other words, under a volume constraint, the Euclidean ball is the domain that minimises it. Maybe, Plato had intuited it some centuries earlier as Lord Rayleigh did since he wrote in *Timaeus* that "the sphere is the most perfect and beautiful shape" *[7]*.

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A. Apendix I

Consider $W(r,\theta) \in \mathcal{C}^k,\ k \geqslant 2$.

A.1 2-dimensional Laplacian in polar coordinates

$$\Delta W(r,\theta) = \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}(r,\theta) + \frac{1}{r}\frac{\partial W}{\partial r}(r,\theta) + \frac{\partial^2 W}{\partial r^2}(r,\theta), \quad r \in \mathbb{N},\ \Theta \in [0, 2\pi).$$

A.2 n-dimensional Laplacian in polar coordinates

$$\Delta W(r,\Theta) = \frac{\partial^2 W}{\partial r^2}(r,\Theta) + \frac{n-1}{r}\frac{\partial W}{\partial r} + \frac{\Lambda v}{r^2}, \quad r \in \mathbb{N},\ \Theta \in \mathbb{S}^{n-1}.$$

B. Apendix II

MATLAB script for computing the first eigenvalues of parallelepiped and ball n-dimensional:

```
% Set initial values:
n = 3 % dimension
i = 1; % number of zeros that we want (1, because we only want the first root)
alpha = pi^(n/2)/gamma(n/2 + 1); % area

% Compute the approximate root to start the calculus of real Bessels' root and
% compute it:
guess = 2.5505 + 1.2474*(n-2)/2 + (i-1)*pi;
Bzero = fzero(@(z) besselj((n-2)/2, z),guess);

% Compute the eigenvalues of both domains:
lambdaParallelepiped = pi^2 * n / alpha^(2/n)
lambdaBall = Bzero*Bzero
```