BACHELOR'S DEGREE THESIS

Degree in Mathematics

Title: A mathematical proof of the soundness of the Fibonacci retracement rule in technical analysis of asset prices
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A mathematical proof of the soundness of the Fibonacci retracement rule in technical analysis of asset prices

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First of all, I would like to thank Professor Argimiro Arratia for his guidance, help and passion for the topic. I would also like to thank my family and friends for their unconditional love and support.

Abstract

This project covers the idea of the Fibonacci retracement rule and its mathematical proof. Firstly, a thorough analysis of stochastic processes, in particular, diffusion processes such as the Brownian motion, the CEV model or Bessel processes, will be done. Then it follows with a study of Technical Analysis and the Fibonacci and golden numbers, introducing the concept of the Fibonacci retracement rule. Finally, a detailed proof of the soundness of the rule will be given. At the end, we encounter codes for simulating diffusion process and real financial stocks applying the Fibonacci retracement rule.

Keywords

Stochastic processes, diffusion processes, stopping time, Brownian Motion, CEV model, Bessel processes, Technical Analysis, Fibonacci, golden number, Fibonacci retracement rule, Mathematical finance

Contents

1. Introduction

In the world of financial analysis for stock markets prices there are two main branches: fundamental analysis and technical analysis. The latter refers to the study of historical information, such as price movements and volume, and it uses different techniques to achieve so. In particular, there is the Fibonacci retracement rule for determining support and resistance levels.

This rule is based on the idea that the financial markets will retrace a predictable portion of a move, after which it will continue to advance in its original direction.

Such technique from technical analysis has been used for a long time. However, it was not until Glover, Hulley, and Peskir published the article of "Three-Dimensional Brownian Motion and the Golden Ratio Rule" [3] that a rigorous mathematical proof for the rule was given.

1.1 Motivation and goals of the thesis

The central aim of this project is to revise the mathematical proof behind the acknowledged Fibonacci retracement rule used in technical analysis of asset prices. The motivation for such topic comes from my interest in blending the world of mathematics and finance. The first time I learned that the Fibonacci numbers were reflected even in the financial field, it awakened a keen interest in me: how two concepts such as the Fibonacci sequence and an asset's price, apparently unconnected, were so closely related.

As for this thesis, I set two essential goals to be achieved throughout the study.

The first one would be to introduce myself into the area of mathematical finance. Since it is such a broad topic, this project will particularly focus on diffusion processes, studying in further depth those needed for the second part, and a brief introduction of the history and uses of technical analysis.

The second goal would be to describe and formally analyse the Fibonacci retracement rule. For such purpose, I will follow the guidance and steps of the authors in the previously mentioned paper, "Three-dimensional Brownian Motion and the Golden Ratio rule". In that paper, the Fibonacci retracement rule is mathematically described as an optimal stopping problem for a certain diffusion process. Their proof will then be followed and most of the details that the authors omitted will be completed.

1.2 Guidelines of the thesis

Given the aforesaid motivations and goals, this thesis will follow the subsequent structure:

- **Stochastic processes**: this section brings us the basic core concepts about stochastic processes and stochastic differential equations.
- **Brownian motion, CEV model and Bessel processes**: the two succeeding chapters will focus on showing the details of these stochastic processes that will be latter used to solved the central question.
- **Technical analysis**: an introduction to the financial field of technical analysis.

- **Fibonacci sequence and the golden number**: a brief description of the Fibonacci sequence, some properties and its correlation to the golden number. In this chapter it will also be presented the main concepts behind the Fibonacci retracement rule.

- **Optimal stopping problem**: a demonstration for the rule will be shown; solving, step by step, a particular optimal stopping problem.

At the end of the thesis there is also an Appendix. There, I attach the codes and explain further details for the simulation of various stochastic process (in Appendix A) and for a visual analysis of the Fibonacci retracement rule, both with simulated paths and real financial stocks (in Appendix B).

2. Stochastic Processes

A *stochastic process* $X = \{X(t) : t \in T\}$ is a collection of random variables, defined in a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where T is known as the *index set*. As usual, Ω is a sample space; $\mathcal{A}$ is a σ-algebra on Ω; and $\mathbb{P}$ a probability measure on $\mathcal{A}$. If the index set is finite or countable, X is called a discrete-time stochastic process, as opposed to a continuous-time process for which the index variable takes a continuous set of values [6, 10].

A stochastic process may have different outcomes, due to its randomness, and a realization of such is called a *sample path*.

Definition 2.1. *A* **filtration** $(\mathcal{F}_t)_{t\geq 0}$ *in the probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *is an increasing family of* σ*-algebras included in* $\mathcal{A}$*.*

The σ-algebra $\mathcal{F}_t$ represents the information available at time t.

2.1 Martingales

The idea of a martingale in a stochastic process is that the process is "constant on average": at a specific time, the conditional expected value of a future time in the sequence, regardless of all prior values, is equal to the present value.

Definition 2.2. *Let* $(\Omega, \mathcal{A}, \mathbb{P})$ *be a probability space and* $(\mathcal{F}_t)$ *be a filtration on this space. A martingale* M_t *is a stochastic process such that:*

i. $\mathbb{E}[|M_t|] < \infty \; \forall t \in T$

ii. $\mathbb{E}[M_t \mid \mathcal{F}_t] = M_t$

If in the prior definition for the property (*ii*) we had instead $\mathbb{E}[M_t \mid \mathcal{F}_t] \leq M_t$ the process would be named a **supermartingale**; whereas if we had $\mathbb{E}[M_t \mid \mathcal{F}_t] \geq M_t$ the process would be called a **submartingale**.

2.2 Brownian Motion or Wiener Process

The idea behind the Brownian Motion is that such process is the limit of simple random walks. Consider a symmetric random walk which at each time is equally likely to take a step either to the left or to the right. Then go to the limit, taking smaller steps in smaller time intervals, and thus obtaining a Brownian motion. This mathematical model is widely used in science, for instance in physics to study the diffusion of particles suspended in a fluid [2, 6, 10]. This process is a clear example of a martingale.

Definition 2.3. *A* Wiener process *or Brownian Motion $B(t)$, $t \in \mathbb{R}$ is a continuous-time stochastic process defined by the following properties:*

i. $B(0) = 0$

ii. $B(t)$ *is continuous in* t

iii. *The process has stationary and independent increments* $B(t) - B(s) : t \leq s$

iv. *Such increments have a mean of 0 and a variance of* $\mid t - s \mid$

One of the consequences of the aforementioned properties is that $B(t) - B(s) \sim N(0, t-s)$ (property (v)). It is also important to note that the Brownian motion is not necessarily differentiable at all times and a one-dimensional motion would eventually visit every point in $\mathbb{R}$.

Lemma 2.4. *A standard Brownian Motion is a martingale.*
Proof:

$$\begin{aligned} \mathbb{E}[B_t \mid \mathcal{F}_s] &= \mathbb{E}[B_t - B_s + B_s \mid \mathcal{F}_s] \\ &= \mathbb{E}[B_t - B_s \mid \mathcal{F}_s] + \mathbb{E}[B_s \mid \mathcal{F}_s] = 0 + B_s = B_s \end{aligned}$$

where it is used the fact that the increments are independent ($B_t - B_s$ is independent from $\mathcal{F}_s$) and the fact that $B_s \in \mathcal{F}_s$. □

2.3 Markov Time and Stopping time

Definition 2.5. *Let S_t be a stochastic process. A Markov time for $\{S_t\}$ is a non-negative integer-valued random variable τ such that for every $t \geq 0$, the event $\{\tau = t\}$ depends only on $\{S_0, S_1, ..., S_t\}$ and does not depend on S_{t+s} for $s \geq 1$.*

Definition 2.6. *Let S_t be a stochastic process in the filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}_t, \mathbb{P})$. A stopping time is a Markov time τ such that $\mathbb{P}(\tau < \infty) = 1$.*

Some useful properties of stopping times are [11]:

- If τ and σ are stopping times, then $\tau \wedge \sigma = min(\tau, \sigma)$, $\tau \vee \sigma = max(\tau, \sigma)$, and $\tau + \sigma$ are also stopping times.
- Let τ_n for $n = 1, 2, ...$ be a sequence of stopping times. Then $\sup_n \tau_n$ is a stopping time. And if $\mathcal{F}_t$ is right-continuous then $\inf \tau_n$, $\underline{\lim}\, \tau_n$ and $\overline{\lim}\, \tau_n$ are also stopping times.

- Any stopping time τ is an $\mathcal{F}_t$-measurable random variable.
- Let τ and σ be two stopping times. Then each of the events $\{\tau < \sigma\}$, $\{\tau > \sigma\}$, $\{\tau \leq \sigma\}$, $\{\tau \geq \sigma\}$ and $\{\tau = \sigma\}$ belong to $\mathcal{F}_\tau$ and $\mathcal{F}_\sigma$.

The intuitive idea is that a stopping time is completely determined by (at most) the total information known up to time τ. One basic example would be in the context of gambling, where X_τ denotes the total gain at τ^{th} gamble. Playing a specific number of games or also playing until you either run out of money or you have doubled your initial amount are stopping rules as they have a positive probability of occurring and can be known with the information available in the present. On the contrary, playing until you have earned the maximum amount or until you have doubled your initial amount would not be examples of stopping times (there is no way to know whether it is the maximum amount without future information and doubling the initial amount does not have a certain probability since you may ruin first).

Optimal stopping time

The theory of optimal stopping or early stopping is concerned with the problem of choosing a time to take a particular action, in order to maximise an expected payoff or minimise an expected cost [6, 7, 11].

Definition 2.7. *Consider a stochastic process S_t, defined on a filtered space $(\Omega, \mathcal{A}, \mathcal{F}_t, \mathbb{P})$. The optimal stopping problem is defined by finding the stopping time τ_* that maximizes the expected value of S_T (the gain):*

$$V_t^T = \mathbb{E}S_{\tau_*} = \mathop{\mathbb{E}}_{0 \leq t \leq T} S_T$$

Hitting time

A hitting time is the first time at which a process reaches a given subset of the space. Any stopping time is a hitting time for a properly chosen target set.

Definition 2.8. *A hitting time in the subset A, H^A, is defined by*

$$H^A = \inf\{t \geq 0 : X_n \in A\}$$

2.4 Markov process

The term Markov property refers to the memoryless property of a stochastic process. A process with such property is called a *Markov process*.

Definition 2.9. *A stochastic process $\{S_t\}$ defined on the filtered space $(\Omega, \mathcal{A}, \mathcal{F}_t, \mathbb{P})$, is said to be an $\mathcal{F}_t$-Markov process, if for all $B \in \mathcal{B}(\mathbb{R})$, the Borel σ-algebra for $\mathbb{R}$, and $t, h \geq 0$ we have,*

$$\mathbb{P}[S_{t+h} \in B \mid \mathcal{F}_t] = \mathbb{P}[S_{t+h} \in B \mid S_t] \text{ a.s.} \tag{1}$$

Another remarkable fact about the Brownian Motion (2.3) is that it is also a Markov process, meaning that the process is memoryless: the future state of the process depends only on the present value (obvious from property (*iii*)). In particular for Brownian motion (and, in general, for any diffusion process) the Markov property is equivalent to

$$\mathbb{E}[f(X_{t+h})|\mathcal{F}_t] = \mathbb{E}[f(X_h)|\sigma(X_t)] \tag{2}$$

for all $t, h > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ bounded and measurable. (For further details and proof of this equivalent formulation of the Markov property see [21, Thm. 7.1.2]).

Definition 2.10. *A stochastic process $\{S_t\}$ defined on the filtered space $(\Omega, \mathcal{A}, \mathcal{F}_t, \mathbb{P})$, is said to be a strong $\mathcal{F}_t$-Markov process, if for all $B \in \mathcal{B}(\mathbb{R})$, the Borel σ-algebra for $\mathbb{R}$, and for any stopping time $\tau \geq 0$ and $h \geq 0$ we have,*

$$\mathbb{P}[S_{\tau+h} \in B \mid \mathcal{F}_\tau] = \mathbb{P}[S_{\tau+h} \in B \mid S_\tau] \text{ a.s.} \tag{3}$$

Basically, the strong Markov property is equivalent to the Markov property but for stopping times. Similarly to (2) we have a formulation of strong Markov property in terms of expectation

$$\mathbb{E}[f(X_{\tau+h})|\mathcal{F}_\tau] = \mathbb{E}[f(X_h)|\sigma(X_\tau)] \tag{4}$$

for all $h > 0$, τ a stopping time and $f : \mathbb{R}^n \to \mathbb{R}$ bounded and measurable.

A central characteristic for Markov processes is that it contains states that may be either transient or recurrent; transience and recurrence describe the likelihood of a process beginning in some state of returning to that particular state. There is some possibility (a nonzero probability) that a process beginning in a **transient state** will never return to that state. There is a guarantee that a process beginning in a **recurrent state** will return to that state.

2.5 Stochastic Differential Equations

Definition 2.11. *A Stochastic Differential Equation or SDE is a differential equation in which one or more of the terms are stochastic processes. Therefore the solution is also a stochastic process and it is called a diffusion.*

It may be expressed as follows:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$

or, in the integral form

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

In a stochastic differential equation the function $\mu(t, X_t)$ is known as the **drift** and the function $\sigma(t, X_t)$ as the **variance**. The term dB_s refers to the differential of the Brownian Motion.

2.6 Scale function and speed measure

All one-dimensional diffusion processes can be transformed, under certain conditions, into a Brownian motion, first by a change of the space variable (through the scale function) and then a time change (through what is known as the speed measure) [12]. Some diffusion process may be hard to study; thereby, through these changes a lot of information can be known since the Brownian motion is a relatively easy and widely studied process.

Definition 2.12. *The* **scale function** *of a diffusion process X_t on (a, b) starting at $x_0 > 0$, with drift μ and variance σ^2 is:*

$$L(x) = \int_{x_0}^{x} e^{(-\int_{\eta}^{y} \frac{2\mu(z)}{\sigma^2(z)}dz)} dy \tag{5}$$

where x_0 and η are arbitrary fixed points in (a, b).

Definition 2.13. *A process for which $L(x) = x$ is said to be on a* **natural scale**.

Lemma 2.14. *Let B_t be a standard Brownian motion. For each $y \in \mathbb{R}$, let T_y denote the random time at which it hits y for the first time, $T_y = inf\{t > 0 : B_t = y\}$. Then, for $a < x < b$ we may derive the following property:*

$$\mathbb{P}[T_a < T_b \mid B_0 = x] = \frac{b-x}{b-a}$$

Proof: *As seen in [12], let $u(x) = \mathbb{P}[T_a < T_b \mid B_0 = x]$ and assume that $\mathbb{P}[T_a \wedge T_b < h \mid B_0 = x] = o(h)$ as $h \to 0$. Suppose u is sufficiently smooth. First, we will use the Markov property for Brownian motion (1) and then we use a Taylor's expansion for $u(x)$, centered in B_h and up to order 2 (2):*

$$\begin{aligned}
u(x) &= \mathbb{P}[T_a < T_b \mid B_0 = x] \\
&\underset{(1)}{=} \mathbb{E}[u(B_h) \mid B_0 = x] + \mathbb{P}[T_a \wedge T_b < h \mid B_0 = x] \\
&= \mathbb{E}[u(B_h) \mid B_0 = x] + o(h) \\
&\underset{(2)}{=} \mathbb{E}[u(x) + (B_h - x)u'(x) + \frac{1}{2}(B_h - x)^2 u''(x)] + o(h) \\
&= \mathbb{E}[u(x)] + \mathbb{E}[B_h - B_0]u'(x) + \frac{1}{2}\mathbb{E}[(B_h - B_0)^2]u''(x) + o(h) \\
&\underset{Def.(2.3)}{\overset{(iv)}{=}} u(x) + \frac{1}{2}hu''(x) + o(h)
\end{aligned}$$

Subtracting $u(x)$ from each side, dividing by h, we get $0 = \frac{1}{2}u''(x) + o(1)$, from which we obtain $u''(x) = 0$ (as $o(1) = 1/z$ with $z \to \infty$). We have the boundary conditions $u(a) = 1$ and $u(b) = 0$. Solving this, we find:

$$u(x) = \frac{b-x}{b-a}$$

□

Lemma 2.15. *(Hitting probabilities). Let X_t be a diffusion process in (a, b), with drift μ and variance σ^2. If $a < a_0 < x < b_0 < b$ and $T_y = inf\{t > 0 : X_t = y\}$, then*

$$\mathbb{P}[T_{a0} < T_{b0} \mid B_0 = x] = \frac{L(b_0) - L(x)}{L(b_0) - L(a_0)} \tag{6}$$

$$\mathbb{P}[T_{b0} < T_{a0} \mid B_0 = x] = \frac{L(x) - L(a_0)}{L(b_0) - L(a_0)} \tag{7}$$

Proof: *We shall only need to consider the hitting probabilities for the process $Z_t = L(X_t)$, where L is the scale function. The process $\{Z_t\}_{t \geq 0}$ is a time changed Brownian motion but since we care about the where and not the when the process exits the interval $L(a_0, b_0)$ then we only need to calculate the hitting probabilities for a Brownian motion which are the result of the previous lemma (2.14).* □

Definition 2.16. *Let X_t be a diffusion process. The infinitesimal generator $\mathcal{L}$ is defined by:*

$$\mathcal{L}f(x) = \frac{1}{2}\sigma^2(x)\frac{d^2f}{dx^2} + \mu(x)\frac{df}{dx}$$

for a twice continuously differentiable function f in (a, b).

If L is the scale function of a diffusion, then $\mathcal{L}L = 0$.

Definition 2.17. *The* **speed density** *of a diffusion process X_t is the function $M(z) = \frac{1}{\sigma^2(z)L'(z)}$, where L is the scale function. The* **speed measure** *of X_t, starting at $x_0 > 0$ with variance $\sigma^2(x)$ is:*

$$m(x) = \int_{x_0}^{x} M(z)dz \tag{8}$$

Equivalently the speed measure can be written in differential form

$$m(dx) = M(x)dx = \frac{dx}{\sigma^2(x)L'(x)} \tag{9}$$

Lemma 2.18. *Denoting the scale function and the speed measure by L and m respectively we have*

$$\mathcal{L}g = \frac{1}{2}\frac{1}{dm/dL}\frac{d^2g}{dL^2} = \frac{1}{2}\frac{d}{dm}(\frac{dg}{dL})$$

for a twice continuously differentiable function g in (a, b).

Proof:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dm}(\frac{dg}{dL}) &= \frac{1}{2}\frac{1}{dm/dx}\frac{d}{dx}(\frac{1}{dL/dx}\frac{dg}{dx}) \\
&= \frac{1}{2}\sigma^2(x)L'(x)\frac{d}{dx}(\frac{1}{L'(x)}\frac{dg}{dx}) \\
&= \frac{1}{2}\sigma^2(x)\frac{d^2g}{dx^2} - \frac{1}{2}\sigma^2(x)L'(x)\frac{L''(x)}{(L'(x))^2}\frac{dg}{dx} \\
&= \frac{1}{2}\sigma^2(x)\frac{d^2g}{dx^2} - (\mathcal{L}L - \mu(x)L')\frac{dg}{L'dx} \\
&= \frac{1}{2}\sigma^2(x)\frac{d^2g}{dx^2} + \mu(x)\frac{dg}{dx} = \mathcal{L}g
\end{aligned}$$

since L solves $\mathcal{L}L = 0$. □

2.7 Killing measure and Green function

Another two essential functions to describe a diffusion process are the *killing measure and the Green function*. In this section we will specifically refer to Markov diffusion processes.

Definition 2.19. *The* **killing measure** *is the probability per unit time and unit length to terminate a trajectory at a given point at a given time [13].*

Definition 2.20. *The* **Green function** *of X is given by*

$$\begin{aligned} G_{a,b}(x,y) &= 2\frac{(L(b)-L(y))(L(x)-L(a))}{L(b)-L(a)} \quad \text{if } a \le x \le y \le b \\ &= 2\frac{(L(b)-L(x))(L(y)-L(a))}{L(b)-L(a)} \quad \text{if } a \le y \le x \le b \end{aligned}$$

Remark 2.21. *We will use the notation $\mathbb{E}_x[f(X)]$ for $\mathbb{E}[f(X)|X_0 = x]$. Similarly $\mathbb{P}_x[f(X)] = \mathbb{P}[f(X)|X_0 = x]$.*

Lemma 2.22. *If $f : (0, \infty) \to \mathbb{R}$ is a measurable function, then*

$$\mathbb{E}_x \int_0^{\tau_{a,b}} f(X_t)dt = \int_a^b f(y)G_{a,b}(x,y)m(dy)$$

for $a \le x \le b$ in $(0, \infty)$ and $\tau_{a,b}$ being the first exit time of the interval. This identity holds in the sense that if one of the integrals exists, so does the other one, and they are equal.

Proof:

As seen in [12]. Let us write:

$$w(x) = \mathbb{E}\Big[\int_0^{\tau_{a,b}} f(X_t)dt \mid X_0 = x\Big]$$

Since $w(x)$ denotes the expected first exit time if X_0 begins at x, we note that $w(a) = w(b) = 0$. Now consider a small interval of time h. Since X_t is a Markov process:

$$w(x) = \mathbb{E}\Big[\int_0^{\tau_{a,b}} f(X_t)dt \mid X_0 = x\Big] = \mathbb{E}\Big[\int_h^{\tau_{a,b}} f(X_t)dt \mid X_h = x\Big]$$

and so for $a < x < b$

$$w(x) \approx \mathbb{E}\Big[\int_0^h f(X_t)dt \mid X_0 = x\Big] + \mathbb{E}[w(X_h)|X_0 = x] \tag{10}$$

We have ignored the possibility that $h > \tau_{a,b}$. Now, taking that f is a Lipschitz continuous function (with Lipschitz constant $K > 0$):

$$\Big|\mathbb{E}\Big[\int_0^h f(X_t)dt \mid X_0 = x\Big] - hf(x)\Big| = \mathbb{E}\Big[\Big|\int_0^h f(X_t)dt - hf(x)\Big|\Big|X_0 = x\Big]$$

$$\le \mathbb{E}\Big[\int_0^h K|X_t - x|dt\Big|X_0 = x\Big] \le K\int_0^h \sqrt{\mathbb{E}[|X_t - x|^2|X_0 = x]}dt = \mathcal{O}(h^{3/2})$$

The last inequality is a consequence of: linearity of expectation and the Cauchy-Schwartz inequality. The last equality can be found by properties of diffusion process $\mathbb{E}_x[|X_t - X_0|^2] \le Ct$. Now, if we substitute this estimate in equation (10), we subtract $w(x)$, divide by h and let $h \to 0$:

$$\mu(x)w'(x) + \frac{1}{2}\sigma^2(x)w''(x) = -f(x), \quad w(a) = w(b) = 0 \tag{11}$$

Now, we solve the equation (11) using lemma (2.18), with $w = g$, and so we have

$$\mathcal{L}w(x) = \frac{1}{2}\frac{1}{m(x)}\frac{d}{dx}\Big(\frac{1}{L'(x)}w'(x)\Big) = -f(x)$$

$$\frac{d}{dx}\left(\frac{1}{L'(x)}w'(x)\right) = -2f(x)m(x)$$

$$\left(\frac{1}{L'(x)}w'(x)\right) = -2\int_a^x f(y)m(y)dy + \beta$$

and therefore

$$w(x) = -2\int_a^x L'(y)\int_a^y f(z)m(z)dzdy + \beta(L(x) - L(a)) + \alpha$$

where β and α are both constant values. Using the boundary conditions $w(a) = w(b) = 0$ we find the values for these constants. For $w(a) = 0$ it is immediate that $\alpha = 0$. Next, we reverse the order of integration:

$$w(x) = -2\int_a^x \int_x^z L'(y)dyf(z)m(z)dz + \beta(L(x) - L(a))$$

$$w(x) = -2\int_a^x (L(x) - L(z))f(z)m(z)dz + \beta(L(x) - L(a))$$

Now, for $w(b) = 0$, substituting it in the equation, after some algebra, we find:

$$\beta = \frac{2}{L(b) - L(a)}\int_a^b (L(b) - L(y))f(y)m(y)dy$$

And so we have,

$$\begin{aligned} w(x) &= \frac{2}{L(b) - L(a)}\{(L(x) - L(a))\int_x^b (L(b) - L(y))f(y)m(y)dy \\ &+ (L(b) - L(x))\int_a^x (L(y) - L(a))f(y)m(y)dy\} \end{aligned}$$

Finally then:

$$\mathbb{E}_x \int_0^{\tau_{a,b}} f(X_t)dt = \int_a^b f(y)G_{a,b}(x, y)m(dy)$$

□

This lemma gives a probabilistic interpretation of the Green function [12]: by taking f as $1_{(x_1,x_2)}$, we see that $\int_{x_1}^{x_2} G(x, y)dy$ refers to the mean time spent by the process in the interval (x_1, x_2) before exiting (a, b), supposing that the process begins at $x_0 = x$. Sometimes, the Green's function is called the *sojourn density*.

2.8 Barriers and boundary conditions

Barriers

Definition 2.23. *If d is an* **absorbing barrier** *of the process X_t, then the process stops at the instant it reaches d, and takes the value d thereafter [10].*

Definition 2.24. *If d is an* **reflecting barrier** *of the process X_t, then the process is not permitted to pass the barrier but there is no other interaction between them [10].*

Boundary conditions

Definition 2.25. *Let L be the scale function and m the speed measure of the diffusion process X_t on the interval (a, b). Then*

$$u(x) = \int_{x_0}^{x} mdL, \quad v(x) = \int_{x_0}^{x} Ldm$$

Definition 2.26. *(Feller's boundary classification, [12, 16]) Let L be the scale function and m the speed measure of the diffusion process X_t on the interval (a, b). The boundary b is said to be*

- **Regular** *if $u(b) < \infty$ and $v(b) < \infty$: the process can either reach the boundary from an interior point, or reach an interior point from the boundary.*
- **Exit** *if $u(b) < \infty$ and $v(b) = \infty$: the process can reach the boundary from an interior point but cannot reach an interior point from the boundary.*
- **Entrance** *if $u(b) = \infty$ and $v(b) < \infty$: the process cannot reach the boundary from an interior point but can reach an interior point from the boundary.*
- **Natural** *if $u(b) = \infty$ and $v(b) = \infty$: the process cannot reach the boundary from an interior point nor can reach an interior point the boundary.*

3. Brownian Motion or Wiener process

3.1 The Wiener integral and the arithmetic Brownian motion

Definition 3.1. *The* Wiener integral *of a function f with continuos derivative f' over the interval $[a, b]$ is defined by:*

$$\int_a^b f(s)dB(s) = [f(s)B(s)]_a^b - \int_a^b B(s)f'(s)ds \tag{12}$$

Definition 3.2. *The* generalized Wiener process *or* arithmetic Brownian motion *(ABM)is the stochastic process $X(t)$ defined for constants α and σ, and $B(t)$ a Brownian motion that satisfies:*

$$dX(t) = \alpha dt + \sigma dB(t) \tag{13}$$

This model has two parameters: α is the drift (it represents whether the trend is going up, if positive, or down, if negative, over time) and σ is the volatility or diffusion coefficient.

Lemma 3.3. *The SDE (13) can be solved in closed form with the Wiener integrals and thus obtaining:*

$$X(t) = X(0) + \alpha t + \sigma B(t)$$

Proof:

$$\begin{aligned} dX(t) &= \alpha dt + \sigma dB(t) \\ \int_0^t dX(s) &= \int_0^t \alpha ds + \int_0^t \sigma dB(s) \\ X(t) - X(0) &\underset{(12)}{=} \alpha t + [B(s)]_0^t - \int_0^t 0 ds \\ X(t) &= X(0) + \alpha t + \sigma B(t) \end{aligned}$$

□

3.2 Ito's formula

Ito process or stochastic integral

Definition 3.4. *An Ito process or stochastic integral is a stochastic process that can be written as follows:*

$$X_t = X_0 + \int_0^t U_s ds + \int_0^t V_s dB_s \quad (14)$$

or,

$$dX_t = U_t dt + V_t dB_t \quad (15)$$

where B_t is the Brownian motion, U_t is Lebesgue-integrable and V_t is a B-integrable process.

Lemma 3.5 (Ito's lemma)**.** *Suppose B_t is a Brownian motion. Let X(t) be a continuous time stochastic process satisfying:*

$$dX_t = U_X dt + V_X dB_t$$

If $G = G(X, t)$ is a differentiable function of $X(t)$ and t, then we can define Ito's formula as:

$$dG = \left(U_X \frac{\partial G}{\partial X} + \frac{\partial G}{\partial t} + V_X^2 \frac{\partial^2 G}{2 \partial X^2} \right) dt + V_X \frac{\partial G}{\partial X} dB_t \quad (16)$$

3.3 Geometric Brownian motion

The use of the ABM to model the price of the stock has an inconvenient: the process may take negative values with positive probability, since the support of the normal distribution is all the reals. Therefore, it is fixed considering prices log normally distributed instead.

Definition 3.6. *Let X(t) be an arithmetic Brownian motion. The following continuous model is called geometric Brownian motion or GBM:*

$$P(t) = e^{X(t)} = e^{X(0) + \alpha t + \sigma B(t)}$$

Lemma 3.7. *The GBM satisfies the SDE*

$$dP(t) = \mu P(t)dt + \sigma P(t)dB \tag{17}$$

where we have that $\mu = \alpha + \frac{1}{2}\sigma^2$ *and* $lnP(t) = X(t)$ *follows an arithmetic Brownian motion.*

Proof:
Applying Ito's lemma, if $G(X,t) = P(t)$*, then* $U_X = \alpha$ *and* $V_X = \sigma$*:*

$$\begin{aligned} dP(t) &= d(e^{X(0)+\alpha t+\sigma B(t)}) \\ &= (\alpha\frac{\partial P}{\partial X} + \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial X^2})dt + \sigma\frac{\partial P}{\partial X}dB_t \\ &= (\alpha P(t) + 0 + \frac{1}{2}\sigma^2 P(t))dt + \sigma P(t)dB_t \\ &= \alpha + \frac{1}{2}\sigma^2 P(t)dt + \sigma P(t)dB \\ &= \mu P(t)dt + \sigma P(t)dB \end{aligned}$$

□

Properties of the GBM

Lemma 3.8. *The GBM has the following properties:*

- *It is a Markov process.*
- *The expectation is given by* $\mathbb{E}[S(t)] = \mathbb{E}[S_0 e^{X(t)}] = S_0 e^{(\mu+\frac{\sigma^2}{2})t}$
- *The variance is given* $Var(S_t) = S_0^2 e^{2\mu t+\sigma^2 t}(e^{\sigma^2 t} - 1)$

3.4 Brownian motion in finance

The Brownian Motion is commonly used in finance for modelling the behaviour of financial markets, for instance, the asset prices in the stock exchange market. The use of this process to simulate a stock's price complies with the weak form of the Efficient Market Hypothesis (which states that the past prices, historical values and trends can't predict future prices).

A simulation of the Geometric Brownian Motion allows us to reproduce a possible path for a stock's price, since it is always non-negative, with an initial price of P_0, an expected return of μ and a volatility of σ. A very famous use of such model in finance is in order to derive the Black-Scholes formula for options pricing, where the underlying asset is assumed to follow the aforementioned model.

Estimating μ and σ

These values may be approximated from the sample data of a stock's price [2]. Assume $r = \{r_1, ..., r_n\}$ is a sample of n log-returns, taken at an equally spaced time period τ. If $m_r = \frac{1}{n}\sum_{t=1}^{n} r_t$ and $s_r^2 = \frac{1}{n-1}\sum t = 1n(r_t - m_r)^2$, the parameters may be estimated as:

$$\begin{aligned}\hat{\mu} &= \frac{1}{\tau} m_r + \frac{\hat{\sigma}^2}{2} \\ \hat{\sigma} &= \frac{s_r}{\sqrt{\tau}}\end{aligned}$$

3.5 Simulations of the Brownian Motion

Simulation of a BW

The Brownian motion can be simulated in an interval of time $[0, T]$ with a recursive formula. The interval is divided in n (finite) times such that $0 = t_0 < t_1 < \ldots < t_n = T$. Then, we generate for $i = 0, 1, \ldots, n : Z_i \sim N(0, 1)$, which are independent and identically distributed (iid). From the property (v) of the Wiener process we may differ that $\Delta B(t_i) = B(t_i) - B(t_{i-1}) = Z_i \sqrt{\Delta t_i}$. Finally, we may define $B(t_1), B(t_2), \ldots, B(t_n)$ the following way:

$$\begin{aligned}B(0) &= 0 \\ B(t_1) &= Z_1 \sqrt{\Delta t_1} \\ B(t_2) &= B(t_1) + Z_2 \sqrt{\Delta t_2} \\ \ldots & \\ B(t_i) &= \sum_{j=0}^{i} Z_j \sqrt{\Delta t_j}\end{aligned}$$

The code of the simulation in R is explained in Appendix A.

Simulation of an ABM

The simulation of an ABM is similar to the Brownian motion, with the addition of the drift and volatility. The recursive formula is the following:

$$X(i) = X(i-1) + \alpha \frac{T}{n} + \sigma Z_i \sqrt{\frac{T}{n}}$$

The code of the simulation in R is explained in the Appendix A.

Simulation of a GBM

In order to define a simulation for the GBM it is important to note that $\frac{dP}{P}$ behaves like an ABM. After some calculations, it is found that the required formula is the following:

$$P(t) = P(0) \exp\left((\mu - \frac{\sigma^2}{2}) T + \sigma Z_t \sqrt{\Delta t} \right)$$

where $\Delta t = T - t$ and $Z_t \sim N(0, 1)$

The code of the simulation in R is explained in the Appendix A.

4. CEV model and Bessel processes

4.1 CEV model

Definition 4.1. *The constant elasticity of variance or CEV is a one-dimensional diffusion model commonly used in finance to capture stochastic volatility and the leverage effect [8, 15]. It is described by the following SDE:*

$$dS_t = \mu S_t dt + \sigma S_t^{1+\beta} dW_t \tag{18}$$

in which S is for the spot price, t the time, μ is the parameter describing the drift, β $(-1 \leq \beta < 1)$ is the elasticity parameter which characterizes the leverage effect, σ is the volatility coefficient and W is the Brownian motion.

For $\beta = 0$, the CEV returns to the well-known Black-Scholes model for options pricing in which the variance rate is independent of the stock price. If $\beta > 0$ then the asset price volatility is an increasing function of the asset price (*inverse leverage effect*, often observed in commodity markets); whereas if $\beta < 0$ it is a decreasing function (*leverage effect*, common in equity markets) and it incorporates the possibility of bankruptcy - the stock's price hitting 0 with positive probability. For example, in [19] it is shown that the β coefficient for gold on the London Bulion Market in the interval period between 2000 and 2007 was approximately $0,49$ or in [20] $\beta = 1$ and $\sigma = 0,004$ for the Cheung Kong Holdings Limited (CKH) in the period of June 1996 to July 1997.

In the case where $\mu = 0$ and $\beta > 0$, it is known that the process S is a strict local martingale. This implies that S does not admit an equivalent martingale measure so that the CEV process may admit arbitrage opportunities.

Various studies conclude that the CEV diffusion process could be a better candidate for describing the actual stock price behaviour than the Black–Scholes model because it allows for the elasticity of return variance to be non-stationary (for example, if $\beta < 0$ the greater the price, the smaller the volatility or variance of the stock; whereas in Black-Scholes the volatility is constant); specially in out-of-the-money and longer time to expiration options. However it is remarkably more difficult to compute.

Properties of the CEV model

Lemma 4.2. *Depending on the value of β, the CEV model has the following properties [14]:*

- *If $1 + \beta < 1$, $\{0\}$ is reached a.s.*
- *If $1 + \beta \leq \frac{1}{2}$, $\{0\}$ is an instantaneously reflecting barrier.*
- *If $\frac{1}{2} < 1 + \beta < 1$, $\{0\}$ is an absorbing barrier.*
- *If $1 + \beta > 1$, $\{0\}$ is an unreachable boundary.*

4.2 Bessel processes

Definition 4.3. *The Bessel process of order n is the process X given by:*

$$X_t = ||B_t||$$

where $\|\cdot\|$ denotes the Euclidean norm for $\mathbb{R}^n$ and B is the n-dimensional Brownian motion started from the origin.

The Bessel process may also be described through an SDE:

$$dX_t = \frac{n-1}{2}\frac{dt}{X_t} + dB_t \tag{19}$$

where B_t is the one dimensional Brownian motion.

Properties of the Bessel process

Lemma 4.4. *Let X_t^n be a n-dimensional Bessel process. Then we have [17]:*

- *for all real n, ∞ is a natural boundary*
- *for $n \leq 0$, 0 is an exit boundary*
- *for $0 < n < 2$, 0 is a regular boundary*
- *for $n \geq 2$, 0 is an entrance boundary*

Bessel processes and Brownian Motion

As seen in detail in [3, 4], when the dimension n of the Bessel process is $n \in \{3, 4, ...\}$ it is the radial part of an n-dimensional standard Brownian motion. Similar interpretations are valid for $n = 1$ (with an addition of the local time at zero) and $n = 2$ but X is not transient in these cases (but recurrent).

Scale function and speed measure of a Bessel process

Lemma 4.5. *The scale function of a Bessel process, for $n > 2$, in interval (a, b) is:*

$$L(x) = -\frac{1}{x^{n-2}} + C$$

Proof:

$$\begin{aligned} L(x) &= \int_{x_0}^{x} e^{\left(-\int_\eta^y \frac{\mu(z)}{\sigma^2/2(z)}dz\right)}dy \quad \textit{with } x_0, \eta \in (a,b) \\ &= \int_{x_0}^{x} e^{\left(-\int_\eta^y \frac{\frac{n-1}{2z}}{1/2}dz\right)}dy \\ &= \int_{x_0}^{x} e^{\left(-\int_\eta^y \frac{n-1}{z}dz\right)}dy \\ &= \int_{x_0}^{x} e^{-((n-1)\ln(y)-C_0)}dy \\ &= \int_{x_0}^{x} y^{-(n-1)}e^{-C_0}dy \\ &= -\frac{1}{x^{n-2}} + C \end{aligned}$$

**using that $e^{-C_0} = (n-2)$*

□

Lemma 4.6. *The speed measure of a Bessel process is:*

$$m(dx) = \frac{1}{n-2}x^{n-1}dx$$

Proof:

$$\begin{aligned} m(dx) &= \frac{dx}{(\sigma^2(x)L'(x))} \\ &= \frac{dx}{L'(x)} = \frac{1}{\frac{(n-2)}{x^{n-1}}}dx \\ &= \frac{1}{n-2}x^{n-1}dx \end{aligned}$$

□

4.3 Bessel processes and the CEV model

Let $n > 2$ and consider the $n-2$ dimensional Bessel process X_t solving equation (19). Consider the scale function L for the Bessel process computed previously. For $c_\sigma > 0$ given and fixed, define $S_t := -c_\sigma L(X_t) = \frac{c_\sigma}{X_t^{n-2}}$. The process is on a natural scale and we shall show that it solves a stochastic differential equation that defines a CEV.

Lemma 4.7. *Applying Ito's formula (16) S_t solves the following equation:*

$$dS_t = \sigma S_t^{1+\frac{1}{n-2}} d\widetilde{B_t} \tag{20}$$

where $\sigma = \frac{n-2}{c_\sigma^{1/(n-2)}}$ and $\widetilde{B} = -B$ with B is a standard Brownian motion. (Observe that this is equation (18) with $\mu = 0$ and $\beta = 1/(n-2)$.)

Proof:

$$\begin{aligned}
dS_t &= d(\frac{c_\sigma}{X_t^{n-2}}) = c_\sigma d(\frac{1}{X_t^{n-2}}) =^{Eq.(16)}= \\
&= c_\sigma \left(\left(\frac{-(n-1)}{2X_t} \frac{(n-2)}{X_t^{n-1}} + 0 + \frac{(n-2)(n-1)}{X_t^n} \right) dt + (\frac{-(n-2)}{X_t^{n-1}}) dB_t \right) = \\
&= c_\sigma \left(\left(\frac{-(n-2)(n-1)}{X_t^n} + \frac{(n-2)(n-1)}{X_t^n} \right) dt + (\frac{-(n-2)}{X_t^{n-1}}) dB_t \right) = \\
&= c_\sigma \frac{-(n-2)}{X_t^{n-1}} dB_t = \\
&= \frac{-(n-2)}{c_\sigma^{\frac{1}{n-2}}} \frac{c_\sigma c_\sigma^{\frac{1}{n-2}}}{(X_t^{n-2})^{\frac{n-1}{n-2}}} dB_t = \\
&= \frac{-(n-2)}{c_\sigma^{\frac{1}{n-2}}} (\frac{c_\sigma}{X_t^{n-2}})^{1+\frac{1}{n-2}} dB_t = \\
&= \sigma S_t^{1+\frac{1}{n-2}} d\widetilde{B_t}
\end{aligned}$$

□

5. Technical Analysis

Technical analysis is a financial engineering strategy which seeks to predict price movements by examination of historical data, essentially price and volume: it seeks to find what is most likely to happen given past information. Hence, Technical Analysis is based in statistical analysis, pattern recognition and related data mining techniques [1],[2, Ch. 6].

5.1 Dow's Theory

Dow's Theory aims to summarize the basic principles for Technical Analysis.

(1) *The averages discount everything*: the stock prices incorporate new information as soon as it is available. This theory operates within the Efficient Market Hypothesis.

(2) *The market has three trends*: primary, secondary and minor. Primary trends have three subphases: accumulation phase (only informed investors trade), public participation phase (common investors participate and the speculation begins) and distribution phase (rampant speculation occurs and informed investors distribute their holdings to the market).

(3) *Stock market averages must confirm each other*: the signals that occur on one index must match or correspond with the signals on the other.

(4) *Trends are confirmed by volume*: volume and price should move in the same direction in order to confirm a trend.

(5) *Trends exist until a definitive reversal signal appears*: it advocates for caution. Trends exist despite "market noise".

5.2 Support and resistance levels

A *support level* indicates the price where the majority of investors would be buyers (it acts as a floor by preventing the price of an asset from being pushed downwards), whereas a *resistance level* indicates the price where the majority of investors would be sellers (it prevents traders from pushing the price of an asset upwards).
These two concepts mean that the price is more likely to bounce off these levels rather than break through them: turning points of prices that can serve as entry or exit points for investing. It is important to notice that once the price breaks a resistance level, this becomes a support level and otherwise for support levels.

Trend lines

A *trend line* is defined by considering two local maxima (or minima) points and tracing a line through those. These are used to give the investor an idea of in which direction an investment's value is headed. Formally: let P_a and P_b be the highest local maxima or minima points at times t_a and t_a. Then $T(t)$ is a trend line defined by

$$T(t) = \frac{(P_b - P_a)}{t_b - t_a} t + \frac{P_a t_b - P_b t_a}{t_b - t_a}$$

It is an uptrend if, and only if $P_a < P_b$, a downtrend if $P_a > P_b$ and a support or resistance line if $P_a = P_b$.

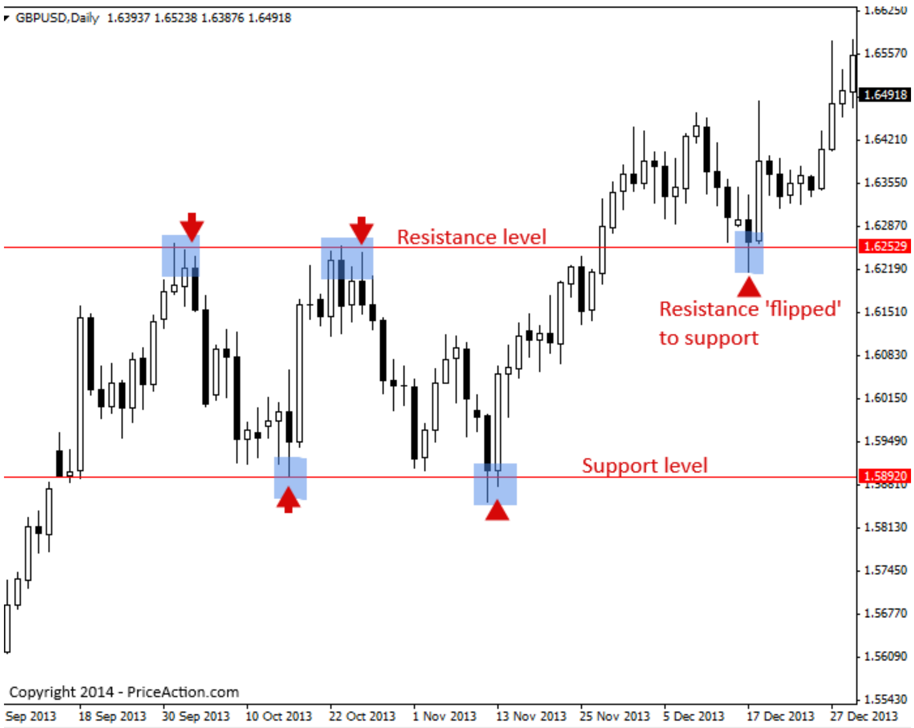

Figure 1: Example of support and resistance levels (source: PriceAction.com)

6. Fibonacci retracement rule

6.1 Fibonacci numbers

The Fibonacci sequence, which was first described by Leonardo Fibonacci in 1202 to calculate the growth of rabbit populations, is an infinite set of numbers which begins with 0 and 1 and any next number is the sum of the two preceding ones. It is formally described recursively as:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \text{ for } n > 2$$

Therefore, the beginning of the sequence is: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377...
Many interesting characteristics can be derived from these numbers [9] such as:
(a) **Sum of sequence**

$$\forall n \in \mathbb{N} \qquad \sum_{i=0}^{n} F_i = F_{n+2} - 1$$

Proof by induction: Base case with n = 0 is seen to hold.

$$\sum_{i=0}^{0} F_i = F_0 = F_2 - 1 = 1 - 1 = 0$$

Assume $n = k$ holds. Now, it will be shown that n = k+1 also holds, proving that

$$\sum_{i=0}^{k+1} F_i = F_{k+3} - 1$$

:

$$\sum_{i=0}^{k+1} F_i = \sum_{i=0}^{k} F_i + F_{k+1} = F_{k+2} - 1 + Fk + 1 = F_{k+3} - 1$$

(b) **Divisibility of Fibonacci Numbers**
$\forall n, m \in \mathbb{N} > 2 : m \mid n \iff F_m \mid F_n$

(c) **'Skipping ahead' in the Fibonacci sequence**
$\forall n, m \in \mathbb{N} \;:\; F_{n+m} = F_m F_{n+1} + F_{m-1} F_n$

Proof by induction:
For $m = 1$, the equation reduces to a trivial identity, so the base case is established. $F_{n+1} = F_1 F_{n+1} + F_0 F_n = 1 * F_{n+1} + 0 * F_n = F_{n+1}$

Now, suppose that the identity is true for $m = k$. Then we aim to show that for $m = k + 1$ is also true:

$$\begin{aligned}
F_{(k+1)+n} &= F_{k+1}F_{n+1} + F_kF_n = (F_k + F_{k-1})F_{n+1} + F_kF_n \\
&= F_{k-1}F_{n+1} + F_k(F_{n+1} + F_n) = F_{k-1}F_{n+1} + F_kF_{n+2} \\
&= F_kF_{n+2} + F_{k-1}F_{n+1} = F_{(n+1)+k} \\
&= F_{n+(k+1)}
\end{aligned}$$

6.2 Fibonacci numbers and the golden ratio

Another very interesting property of the Fibonacci sequence is its relation with the golden number: φ. The golden number is defined by : $\frac{1+\sqrt{5}}{2}$ and it is largely used in mathematics and sciences, as for example to describe predictable patterns on nature.
This number was derived from the Fibonacci sequence in the following manner:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$$

At the same time, the Fibonacci numbers may be determined with the golden ratio [9].

Definition 6.1. *For any real numbers a and b, such that $a > b > 0$, we define φ (golden ratio) as*

$$\frac{a+b}{a} = \frac{a}{b} = \varphi$$

Definition 6.2. *Let ϕ be defined by $\phi = \frac{-1}{\varphi} = 1 - \varphi = \frac{1-\sqrt{5}}{2}$.*

Lemma 6.3. *The subsequent relationship is found (and it is commonly known as* **Binet's formula***):*

$$F_n = \frac{\varphi^n - \phi^n}{\sqrt{5}}$$

Proof: *We will proceed to a proof by induction. First of all, it is important to note that φ and ϕ are the solutions to the quadratic equation $x^2 - x - 1 = 0$. Therefore, it is true that $\varphi^2 = \varphi + 1$ and $\phi^2 = \phi + 1$.*

We shall begin now with the induction. Base case is for n = 0, which is seen to hold. $F_0 = \frac{\varphi^0 - \phi^0}{\sqrt{5} = \frac{1-1}{\sqrt{5}}} = 0$
Now, the induction hypothesis will be proven. Assume the identity is true for n, then it will be shown it checks for n + 1.

$$\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
&= \frac{\varphi^n - \phi^n}{\sqrt{5}} + \frac{\varphi^{(n-1)} - \phi^{(n-1)}}{\sqrt{5}} \\
&= \frac{\varphi^{n-1}(\varphi + 1) - \phi^{n-1}(\phi + 1)}{\sqrt{5}} \\
&= \frac{\varphi^{n-1}\varphi^2 - \phi^{n-1}\phi^2}{\sqrt{5}} \\
&= \frac{\varphi^{n+1} - \phi^{n+1}}{\sqrt{5}}
\end{aligned}$$

□

6.3 The Fibonacci retracement rule

The Fibonacci sequence can be applied to finance as a method of technical analysis in order to determine support and resistance levels. It is based on the idea that the prices on the stock market will retrace a predictable portion of the original move, after which it will proceed in its initial direction.
The support and resistance levels are calculated following some ratios with the Fibonacci sequence. For instance, the most commonly used, which is 61.8%, is found calculating the limit of F_n divided into F_{n+1} when n goes to infinity. Dividing a Fibonacci number by the second number to its right will result in 38.2% and by the third in 23.6%.

This type of analysis can be divided into four specific sections that will below be explained in detail: arcs, fan lines, time zones and retracements.

Arcs

The Fibonacci arcs provide possible support and resistance levels in the price's graphic. First, a trend-line between two extreme points is drawn. Three arcs are then drawn, centered on the second extreme point and at levels 38.2%, 50% and 61.8%. The interpretation of Fibonacci Arcs involves anticipating support and resistance levels as prices approach the arcs.

Fan lines

The first step is to draw a trend line between two extreme points and later a vertical line in the second point. Finally three trend-lines are drawn from the first point so that they pass the vertical line at levels 38.2%, 50% and 61.8%. Fan lines also aims to show support and resistance levels.

Time zones

These are a series of vertical lines spaced with the Fibonacci intervals at 1, 2, 3, 5, 8, 13, 21... The interpretation is that there should be significant changes near the vertical lines.

Retracements

Firstly, a trend line between two extreme points is drawn. Next, a series of 9 horizontal lines intersecting the trend line at levels 0.%, 23.6% (the shallow retracement), 38.2% (the moderate retracement), 50%, 61.8% (the golden retracement), 100%, 161.8%, 261.8% and 423.6% are drawn. After a significant price move (in either direction), prices will typically retrace a significant portion of the original move. Support and resistance levels usually occur near the Fibonacci retracement lines.

Figure 2: Example of Fibonacci retracements (source: Fidelity Investments www.fidelity.com)

7. Optimal Stopping Problem

The aim of this section is to study the optimal stopping time for a Bessel process, following the lines of [3]. First, the main ideas and concepts needed for the resolution will be studied and later the main theorems to proof the Golden Ratio rule will be introduced.

Definition 7.1. *We denote $\mathbb{P}_x$ as the probability measure under which the process X_t starts at $x > 0$.*

Consider a nonnegative diffusion process X_t solving

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t \tag{21}$$

that satisfies:

$$L(0_+) = -\infty, \quad L(\infty_-) = 0$$

$$\int_{0+}^{1} L(dy) = \infty, \quad \int_{0+}^{1} m(dy) < \infty, \quad \int_{0+}^{1} |L(dy)|m(dy) < \infty.$$

where $L(x)$ and $m(x)$ are the scale function and the speed measure respectively. We note that X_t is a transient diffusion process so that $X_t \to \infty$ $\mathbb{P}_x$-a.s as $t \to \infty$ and the process could start at 0 but will never return to it.

Definition 7.2. *Consider the process (21). The* **running minimum process** $I = (I_t)_{t\geq 0}$ *is defined by:*

$$I_t = \inf_{0\leq s\leq t} X_s \tag{22}$$

Due to the fact that X_t is transient (converging to $+\infty$) and 0 is an entrance boundary point for X_t we see that the ultimate infimum $I_\infty = \inf_{t\geq 0} X_t$ is attained at some random time θ ($X_\theta = I_\infty$) with $\mathbb{P}_x$ probability one for $x > 0$ given and fixed.
The random time θ is unknown at any given time. So, the problem is to find a stopping time that is as "close" as possible to θ, that is

$$V(x) = \inf_{\tau} \mathbb{E}_x[|\theta - \tau|] \tag{23}$$

7.1 Groundwork

Lemma 7.3. *For all stopping (random) times τ of X, we have:*

$$|\theta - \tau| = \theta + \int_0^\tau (2I(\theta \leq t) - 1)dt \tag{24}$$

where I is the indicator function.

Proof: *The proof can be seen in [18].*

$$\begin{aligned}
|\theta - \tau| &= (\theta - \tau)^+ + (\tau - \theta)^+ \\
&= \int_0^\theta I(\tau \leq t)dt + \int_0^\tau I(\theta \leq t)dt \\
&= \int_0^\theta (1 - I(\tau > t))dt + \int_0^\tau I(\theta \leq t)dt \\
&= \theta + \int_0^\tau I(\theta > t)dt + \int_0^\tau I(\theta \leq t)dt \\
&= \theta + \int_0^\tau (1 + I(\theta \leq t))dt + \int_0^\tau I(\theta \leq t)dt \\
&= \theta + \int_0^\tau (2I(\theta \leq t) - 1)dt
\end{aligned}$$

□

Recalling problem (23), we will now rewrite it as to 'normalize' and preserve generality. After taking $\mathbb{E}_x$ on both sides of (24),

$$\mathbb{E}_x[|\theta - \tau| - \theta] = \mathbb{E}_x \int_0^\tau (2I(\theta \leq t) - 1)dt$$

Therefore we have defined a new Optimal Stopping Problem (which is clearly equivalent to the former one) to be as follows:

$$V(x) = \inf_{\tau} \mathbb{E}_x[|\theta - \tau| - \theta] \tag{25}$$

Lemma 7.4. *For all stopping times of X_t with finite mean, we have:*

$$\mathbb{E}_x \int_0^\tau (2I(\theta \leq t) - 1)dt = \mathbb{E}_x \int_0^\tau (1 - 2\frac{L(X_t)}{L(I_t)})dt \tag{26}$$

The proof can be found in [18, page 450].

Hence the optimal stopping problem we are studying may also be described as:

$$V(x) = \mathbb{E}_x \int_0^\tau (1 - 2\frac{L(X_t)}{L(I_t)})dt \tag{27}$$

Now, let us introduce some basic concepts for the theory of Optimal Stopping and free boundary problems, as seen in detail in [18]. For the following definitions, we will take $G(X_t)$ to be the gain function of X_t, τ a stopping time of the process, $V(x) = \sup_\tau \mathbb{E}G(X_\tau)$ and E the space where the process is defined.

Definition 7.5. *The continuation set is defined as:*

$$C = \{x \in E : V(x) > G(x)\}$$

Definition 7.6. *The stopping set is defined as:*

$$D = \{x \in E : V(x) = G(x)\}$$

Definition 7.7. *The first entry time into D is defined as:*

$$\tau_D = \inf\{t \geq 0 : X_t \in D\}$$

Lemma 7.8. *The Leibniz rule for differentiating within an integral [22]:*

$$\frac{d}{dt}\Big(\int_{a(t)}^{b(t)} F(x,t)dx\Big) = F(b(t),t)\frac{db(t)}{dt} - F(a(t),t)\frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}F(x,t)dx \tag{28}$$

7.2 Main result

First, for simplicity purposes, let us define

$$c(i,x) = 1 - 2\frac{L(x)}{L(i)}$$

for $i \leq x$, and the set

$$S = \{(i,x) \in (0,\infty) \times (0,\infty) : i \leq x\}$$

Theorem 7.9. *The optimal stopping time in problem (23) is given by*

$$\tau_* = \inf\{t \geq 0 \mid X_t \geq f_*(I_t)\} \tag{29}$$

where the optimal boundary f_* can be characterized as the minimal solution to

$$f'(i) = -\frac{\sigma^2(f(i))L'(f(i))}{c(i,f(i))[L(f(i)) - L(i)]} \int_i^{f(i)} \frac{c_i'(i,y)[L(y) - L(i)]}{\sigma^2(y)L'(y)}dy \tag{30}$$

staying strictly above the curve $h(i) = L^{-1}(L(i)/2)$ *for* $i > 0$ *(in the sense that if the minimal solution does not exist, then there is no optimal stopping time). The value function is given by*

$$V(i,x) = -\int_x^{f_*(i)} \frac{c(i,y)[L(y)-L(x)]}{(\sigma^2/2)(y)L'(y)} dy \tag{31}$$

for $i \le x \le f_*(i)$ *and* $V(i,x) = 0$ *for* $x \ge f_*(i)$ *with* $i > 0$.

Proof:
A detailed procedure of the proof can be seen in [3]. Since it is beyond the level of study of this work, here we will proceed to show a sketch of the solution.
Recalling the general theory of optimal stopping for Markov processes implies that the continuation set in (27) is

$$C = \{(i,x) \in S : V(i,x) < 0\}$$

and the stopping set is

$$D = \{(i,x) \in S : V(i,x) = 0\}$$

Now, we define the set $C_0 = \{(i,x) \in S : c(i,x) < 0\}$. *It is clear that* C_0 *is contained in* C, *and a central question will be to determine the rest of the set* C. *Since* $X_t \to \infty$ $\mathbb{P}_x$*-a.s as* $t \to \infty$, *we see that under these conditions* $L(X_t) \to 0$ *so that the integrand in (27) becomes strictly positive eventually. Therefore, it becomes more and more distant of the stopping set we are looking for. This indicates that there should exist a point,* $f(i)$, *at or above which the process* X_t *should be optimally stopped under the probability* $\mathbb{P}_{i,x}$, *where* $i \le x$ *are given and fixed.*
This yields the following candidate:

$$\tau_f = \inf\{t \ge 0 : X_t \ge f(I_t)\}$$

for a function $f(i)$ *yet to be determined.*

In order to find the values of the functions V *and* f *a free-boundary problem for the optimal stopping will be solved. Further details and the reasoning for choosing this specific boundary conditions and functions are seen in [18, Ch. 13].*

$$(\mathcal{L}_X V)(i,x) = -c(i,x) \quad \text{for } i < x < f(i) \tag{32}$$

$$\frac{\partial}{\partial i} V(i,x) = V_i'(i,x)|_{x=i+} = 0 \quad (\text{normal reflection}) \tag{33}$$

$$V(i,x)|_{x=f(i)-} = 0 \quad (\text{instantaneous stopping}) \tag{34}$$

$$\frac{\partial}{\partial x} V(i,x) = V_x'(i,x)|_{x=f(i)-} = 0 \quad (\text{smooth fit}) \tag{35}$$

The resulting function considering all of the conditions above is:

$$V_f(i,x) = \mathbb{E}_{i,x} \int_0^{\tau_f} c(I_t, X_t) dt \tag{36}$$

for $i \le x \le f(i)$ *in* $(0,\infty)$.
Applying the strong Markov property (3) of (I,X) *at* $\tau_{i,f(i)} = \inf\{t \ge 0 : X_t \notin (i, f(i))\}$ *so that* $\tau_{i,f(i)} = \tau_f \wedge \tau_i$. *Then, using the Green function theorem (2.22) and equations (6, 7), we shall find the following results.*

Lemma 7.10. *Suppose we have $V_f(i,x)$ to be the one defined in (36). Then,*

$$V_f(i,x) = \int_i^{f(i)} c(i,y) G_{i,f(i)}(x,y) m(dy) + V_f(i,i) \frac{L(f(i)) - L(x)}{L(f(i)) - L(i)} \tag{37}$$

Proof:

$$\begin{aligned} V_f(i,x) &\underset{definition(36)}{=} \mathbb{E}_{i,x} \int_0^{\tau_f} c(I_t, X_t) dt \\ &= \mathbb{E}_{i,x} \int_0^{\tau_i, f(i)} c(I_t, X_t) dt + \mathbb{E}_{i,x} \int_{\tau_i, f(i)}^{\tau_f} c(I_t, X_t) dt \end{aligned}$$

For the first summand we use Green's formula:

$$\mathbb{E}_{i,x} \int_0^{\tau_i, f(i)} c(I_t, X_t) dt = \int_i^{f(i)} c(i,y) G_{i,f(i)}(x,y) m(dy)$$

Next, we proceed to resolve the second summand. We rewrite such integral part doing a change of variables with $t \to t + \tau_{i,f(i)}$ and then using the Strong Markov property (3):

$$\begin{aligned} \int_{\tau_{i,f(i)}}^{\tau_f} c(I_t, X_t) dt &= \int_0^{\tau_f} c(I_{t+\tau_{i,f(i)}}, X_{t+\tau_{i,f(i)}}) dt \\ &\underset{property\ \ of\ \ Cond.\ Expect.}{=} \mathbb{E}_{i,x}[\int_0^{\tau_f} c(I_{t+\tau_{i,f(i)}}, X_{t+\tau_{i,f(i)}}) dt \mid \mathcal{F}_{\tau_{i,f(i)}}] \\ &\underset{Strong\ \ Markov}{=} \mathbb{E}_{I_{t+\tau_{i,f(i)}}, X_{t+\tau_{i,f(i)}}}[\int_0^{\tau_f} c(I_t, X_t) dt] \\ &\underset{definition}{=} V_f(I_{\tau_{i,f(i)}}, X_{\tau_{i,f(i)}}) \end{aligned}$$

Then the second summand is (recall that the domain of $c(i,x)$ is $i \leq x$ in $(0,\infty)$):

$$\begin{aligned} \mathbb{E}_{i,x}[V_f(I_{\tau_{i,f(i)}}, X_{\tau_{i,f(i)}})] &= V_f(i,x) \cdot \mathbb{P}_{i,x}(X_{\tau_{i,f(i)}} \geq i) \\ &= V_f(i,i) \cdot \mathbb{P}_{i,x}(X_{\tau_{i,f(i)}} = i) + V_f(i,x) \cdot \mathbb{P}_{i,x}(X_{\tau_{i,f(i)}} > i) \\ &= V_f(i,i) \cdot \mathbb{P}_{i,x}(X_{\tau_{i,f(i)}} = i) \\ &= V_f(i,i) \frac{L(f(i)) - L(x)}{L(f(i)) - L(i)} \end{aligned}$$

Using that $\mathbb{P}_{i,x}(X_{\tau_{i,f(i)}} > i) = 0$ since $X_{\tau_{i,f(i)}} \notin (i, f(i))$ in the penultimate step.

Therefore, adding the two terms that have been studied, we have:

$$V_f(i,x) = \int_i^{f(i)} c(i,y) G_{i,f(i)}(x,y) m(dy) + V_f(i,i) \frac{L(f(i)) - L(x)}{L(f(i)) - L(i)}$$

□

Now, we shall apply lemma (7.10) to equation (36) and rearranging such:

$$\begin{aligned} V_f(i,i) &= \frac{L(f(i))-L(i)}{L(f(i))-L(x)} V_f(i,x) \\ &- \frac{L(f(i))-L(i)}{L(f(i))-L(x)} \int_i^{f(i)} c(i,y) G_{i,f(i)}(x,y) m(dy) \end{aligned} \tag{38}$$

We will proceed to calculate each term when $x \to f(i)$

1. Using the boundary terms of instantaneous stopping and smooth fit (32):

$$\begin{aligned} \lim_{x\to f(i)} \frac{V_f(i,x)}{L(f(i))-L(x)} &= \lim_{x\to f(i)} \frac{V_f(i,x)}{L(f(i))-L(x)} \frac{x-f(i)}{x-f(i)} \\ &= -\frac{1}{L'(f(i))} \frac{\partial V_f}{\partial x}(i,x) \mid_{x=f(i)-} \\ &= 0 \end{aligned}$$

2. Using the definition of Green function:

$$\lim_{x\to f(i)} \frac{L(f(i))-L(i)}{L(f(i))-L(x)} \int_i^{f(i)} c(i,y) G_{i,f(i)}(x,y) m(dy) =$$

$$\lim_{x\to f(i)} \frac{L(f(i))-L(i)}{L(f(i))-L(x)} \int_i^{f(i)} 2c(i,y) \frac{(L(f(i))-L(x))(L(y)-L(i))}{L(f(i))-L(i)} m(dy)$$

$$= \int_i^{f(i)} 2c(i,y)[L(y)-L(i)] m(dy)$$

3. Combining steps (1) and (2)

$$V_f(i,i) = -\int_i^{f(i)} 2c(i,y)[L(y)-L(i)] m(dy)$$

4. Inserting this in equation (37) we can conclude that

$$
\begin{aligned}
V_f(i,x) &= V_f(i,i)\frac{L(f(i))-L(x)}{L(f(i))-L(i)} + \int_i^{f(i)} c(i,y)G_{i,f(i)}(x,y)m(dy) \\
&\underset{(step\ 3.)}{=} -\int_i^{f(i)} 2c(i,y)[L(y)-L(i)]m(dy)\frac{L(f(i))-L(x)}{L(f(i))-L(i)} \\
&+ \int_i^{f(i)} c(i,y)G_{i,f(i)}(x,y)m(dy) \\
&= -\int_i^{f(i)} 2c(i,y)[L(y)-L(i)]m(dy)\frac{L(f(i))-L(x)}{L(f(i))-L(i)} \\
&+ \int_i^{f(i)} c(i,y)\frac{2(L(f(i))-L(y))(L(x)-L(i))}{L(f(i))-L((i))}m(dy) \\
&= \int_i^{f(i)} 2c(i,y)\frac{-L(y)L(f(i)-L(x)L(i)+L(y)L(i)-L(x)L(f(i))}{L(f(i))-L((i))}m(dy) \\
&\quad \text{since } c(i,x) \text{ is defined for } i \le x \le f_*(i) \qquad (39) \\
&= -\int_x^{f_*(i)} 2c(i,y)[L(y)-L(x)]m(dy) \\
&= -\int_x^{f_*(i)} \frac{2c(i,y)[L(y)-L(x)]}{\sigma^2(y)L'(y)}dy \qquad (40)
\end{aligned}
$$

5. *Finally, applying the Leibniz integral rule and then using the normal reflection (28):*

$$
\begin{aligned}
\frac{\partial}{\partial i}V(i,x) &\underset{Leibniz}{=} -\frac{2c(i,y)[L(y)-L(x)]}{\sigma^2(y)L'(y)} \times \frac{\partial}{\partial i}f(i) - \frac{2c(i,y)[L(y)-L(x)]}{\sigma^2(y)L'(y)} \times \frac{\partial}{\partial i}x \\
&- \int_x^{f_*(i)} \frac{\partial}{\partial i}\frac{2c(i,y)[L(y)-L(x)]}{\sigma^2(y)L'(y)}dy \\
0 &\underset{normal\ reflection}{=} -\frac{2c(i,y)[L(y)-L(x)]}{\sigma^2(y)L'(y)}f'(i) - 0 - \int_x^{f_*(i)} \frac{2c_i'(i,y)[L(y)-L(x)]}{\sigma^2(y)L'(y)}dy \\
f'(i) &= \frac{\sigma^2(f(i))L'(f(i))}{c(i,f(i))[L(f(i))-L(i)]}\int_i^{f(i)} \frac{c_i'(i,y)[L(y)-L(i)]}{\sigma^2(y)L'(y)}dy \qquad (41)
\end{aligned}
$$

Denoting $\frac{\partial}{\partial i}c(i,y) = c_i'(i,y)$.

Now, we shall define

$$h(i) = L^{-1}(\frac{1}{2}L(i)) \qquad (42)$$

for $i > 0$.

Lemma 7.11. *We see that $c(i,x) < 0$ for $x < h(i)$ and $c(i,x) > 0$ for $x > h(i)$ whenever $i \leq x$.*
Proof:

$$
\begin{aligned}
c(i,x) &< 0 \quad \Leftrightarrow \\
1 - 2\frac{L(x)}{L(i)} &= 1 - 2\frac{\frac{-1}{x^{n-2}}}{\frac{-1}{i^{n-2}}} < 0 \quad \Leftrightarrow \\
1 - 2\frac{i^{n-2}}{x^{n-2}} &< 0 \quad \Leftrightarrow \\
2(i)^{n-2} &> x^{n-2} \quad \Leftrightarrow \\
(\frac{1}{2(i)^{n-2}})^{n-2} &> x \quad \Leftrightarrow \\
(\frac{-1}{(\frac{-1}{2(i)^{n-2}})^{n-2}})^{-1} &> x \quad \Leftrightarrow \\
L(\frac{-1}{2(i)^{n-2}})^{-1} &> x \quad \Leftrightarrow \\
L(\frac{1}{2}L(i))^{-1} = h(i) &> x
\end{aligned}
$$

It is reciprocal for $c(i,x) > 0$ for $x > h(i)$. □

Also, see that the set C_0 may now be defined as $C_0 = \{(i,x) \in S : i \leq x < h(i)\}$, and, in particular, C_0 contains the diagonal $\{(i,x) \in S : i = x\}$. Now, recalling the candidate function f solving (41), there is no restriction to assume that each candidate function would satisfy $f(i) > h(i)$ for all $i > 0$. The central question will become how to select the optimal boundary f among all admissible candidates (those that satisfy (41)).

The further steps in order to formally finish this demonstration will not be explained in detail here. However they can be found in [3].

The remaining of this argument will focus on finding the optimal minimal solution for f that shall admit a stochastic representation and is non-positive. It is shown that f is a minimal solution and it admits a stochastic representation using classical techniques of differential calculus and stochastic processes. Then, the non-positivity is shown by proving that for every admissible solution f such that $f \geq f_*$ on $(0,\infty)$ the we have $V_f(i,x) \leq 0$ for all $i \leq x \in (0,\infty)$. Finally it is proven that f is optimal by seeing that "selecting the minimal solution f_* staying strictly above h is equivalent to invoking the subharmonic characterization of the value function (according to which the value function is the largest subharmonic function lying below the loss function)" [3] separating it in the case that X has a finite mean or it is not finite valued. □

7.3 Consequences

We shall proceed now with the last steps to be completed: show the correlation between all of the above, more abstract, results and the Fibonacci retracement rule. Thus, first we will shall find those results explicitly for the Bessel process, specifically for dimension $n = 3$, and then transform the solution so as to adapt it for the studied CEV model.

Theorem 7.12. *If X is the n-dimensional Bessel process solving (19) with $n > 2$, then the optimal stopping time in (27) is given by*

$$\tau_* = \inf\{t \geq 0 \mid X_t \geq \lambda I_t\} \tag{43}$$

where λ is the unique solution to

$$\lambda^n - (1+n)\lambda^2 + \frac{4}{4-n}\lambda^{4-n} - \frac{(n-2)^2}{4-n} = 0 \qquad \text{if} \quad n \neq 4, \tag{44}$$

$$\lambda^4 - 5\lambda^2 + 4\log\lambda + 4 = 0 \qquad \text{if} \quad n = 4 \tag{45}$$

belonging to $(2^{1(n-2)}, \infty)$.

Therefore, the value function is given explicitly by

$$\begin{aligned} V(i,x) &= \frac{2}{n-2}\left[x^2\left(\frac{1}{2}+\left(\frac{i}{x}\right)^{n-2}\right)\left(\left(\frac{\lambda i}{x}\right)^2-1\right)-\frac{x^2}{n}\left(\left(\frac{\lambda i}{x}\right)^n-1\right)\right. \\ &- \left.\frac{2\lambda^{4-n}}{n-4}i^2\left(\left(\frac{\lambda i}{x}^{n-4}\right)-1\right)\right] \qquad \text{if } \ n \neq 4 \\ &= \left[x^2\left(\frac{1}{2}+\left(\frac{i}{x}\right)^2\right)\left(\left(\frac{\lambda i}{x}\right)^2-1\right)-\frac{x^2}{4}\left(\left(\frac{\lambda i}{x}\right)^4-1\right)\right. \\ &- \left.2i^2\log(\frac{\lambda i}{x})\right] \qquad \text{if } n = 4 \end{aligned} \tag{46, 47}$$

for $i \leq x \leq \lambda i$ and $V(i,x) = 0$ for $x \geq \lambda i$ with $i > 0$.

Proof: *Using the results seen in (7.9), we know the optimal stopping time for any given problem that satisfies the mentioned conditions is determined by (29) staying strictly above the curve $h(i) = L(\frac{1}{2}L(i))^{-1}$.*

First, we substitute in (30) the specific values for a Bessel process:

$$
\begin{aligned}
f'(i) &= \frac{\sigma^2(f(i))L'(f(i))}{c(i,f(i))[L(f(i))-L(i)]}\int_i^{f(i)}\frac{c_i'(i,y)[L(y)-L(i)]}{\sigma^2(y)L'(y)}dy \\
&= -\frac{\frac{n-2}{f(i)^{n-1}}}{(1-2\frac{L(f(i))}{L(i)})[L(f(i))-L(i)]} \\
&\times \int_i^{f(i)}\frac{\frac{-2(n-2)i^{n-3}}{y^{n-2}}[L(y)-L(i)]}{\frac{n-2}{y^{n-1}}}dy \\
&= -\frac{\frac{n-2}{f(i)^{n-1}}}{(1-2\frac{\frac{-1}{f(i)^{n-2}}}{\frac{-1}{i^{n-2}}})[\frac{-1}{f(i)^{n-2}}-\frac{-1}{i^{n-2}}]} \\
&\times \int_i^{f(i)}\frac{\frac{-2(n-2)i^{n-3}}{y^{n-2}}[\frac{-1}{y^{n-2}}-\frac{-1}{i^{n-2}}]}{\frac{n-2}{y^n-1}}dy \\
&= \frac{\frac{(n-2)}{f(i)^{n-1}}}{(1-2(\frac{i}{f(i)})^{n-2})[\frac{f(i)^{n-2}-i^{n-2}}{i^{n-2}f(i)^{n-2}}]} \\
&\times -2(i)^{n-3}\int_i^{f(i)}y(\frac{-1}{y^{n-2}}+\frac{1}{i^{n-2}})dy \\
&= -\frac{\frac{(n-2)}{f(i)^{n-1}}}{(\frac{f(i)^{n-2}-2(i^{n-2})}{f(i)^{n-2}})(\frac{f(i)^{n-2}-(i^{n-2})}{f(i)^{n-2}i^{n-2}})} \\
&\times -2(i)^{n-3}\int_i^{f(i)}(\frac{-1}{y^{n-3}}+\frac{y}{i^{n-2}})dy \\
&= -\frac{\frac{(n-2)}{f(i)}(\frac{f(i)}{i})^{n-2}}{\left(((\frac{f(i)}{i})^{n-2}-1)((\frac{f(i)}{i})^{n-2}-2)\right)} \\
&\times -2(i)^{n-3}\int_i^{f(i)}(\frac{-1}{y^{n-3}}+\frac{y}{i^{n-2}})dy \\
&= -\frac{(n-2)\frac{f(i)}{i}(f(i))^{n-4}}{\left(((\frac{f(i)}{i})^{n-2}-1)((\frac{f(i)}{i})^{n-2}-2)\right)} \\
&\times -2\int_i^{f(i)}(\frac{-1}{y^{n-3}}+\frac{y}{i^{n-2}})dy
\end{aligned}
$$

Then, we separate depending whether $n = 4$ or $n \neq 4$ in order to calculate the integral.

1. *If $n \neq 4$:*

$$
\begin{aligned}
f'(i) &= -\frac{(n-2)\frac{f(i)}{i}(f(i))^{n-4}}{\Big(\big((\frac{f(i)}{i})^{n-2}-1\big)\big((\frac{f(i)}{i})^{n-2}-2\big)\Big)} \cdot (-2)[(\frac{-y^{-n+4}}{-n+4}+\frac{y^2}{2(i)^{n-2}})] \mid_i^{f(i)} \\
&= -\frac{(n-2)\frac{f(i)}{i}(f(i))^{n-4}}{\Big(\big((\frac{f(i)}{i})^{n-2}-1\big)\big((\frac{f(i)}{i})^{n-2}-2\big)\Big)} \cdot (-2)(\frac{-f(i)^{-n+4}+i^{-n+4}}{-n+4}+\frac{f(i)^2-i^2}{2(i)^{n-2}}) \\
&= \frac{(n-2)\frac{f(i)}{i}}{\Big(\big((\frac{f(i)}{i})^{n-2}-1\big)\big((\frac{f(i)}{i})^{n-2}-2\big)\Big)} \\
&\times (\frac{-2}{-n+4}+\frac{2}{-n+4}(\frac{f(i)}{i})^{n-4}+(\frac{f(i)}{i})^{n-2}-(\frac{f(i)}{i})^{n-4}) \\
&= \frac{\frac{(n-2)}{(4-n)}\frac{f(i)}{i}}{\Big(\big((\frac{f(i)}{i})^{n-2}-1\big)\big((\frac{f(i)}{i})^{n-2}-2\big)\Big)} \\
&\times \big(-2+(n-2)(\frac{f(i)}{i})^{n-4}+(4-n)(\frac{f(i)}{i})^{n-2}\big)
\end{aligned}
$$

2. *If $n = 4$:*

$$
\begin{aligned}
f'(i) &= -\frac{(2)\frac{f(i)}{i}}{\Big(\big((\frac{f(i)}{i})^{2}-1\big)\big((\frac{f(i)}{i})^{2}-2\big)\Big)} \\
&\times -2\int_i^{f(i)}(\frac{-1}{y}+\frac{y}{i^2})dy \\
&= -\frac{(2)\frac{f(i)}{i}}{\Big(\big((\frac{f(i)}{i})^{2}-1\big)\big((\frac{f(i)}{i})^{2}-2\big)\Big)} \\
&\times -2[-\log y+\frac{y^2}{2i^2}]_i^{f(i)} \\
&= -\frac{(2)\frac{f(i)}{i}}{\Big(\big((\frac{f(i)}{i})^{2}-1\big)\big((\frac{f(i)}{i})^{2}-2\big)\Big)} \\
&\times -2[-\log(f(i)/i)+\frac{f(i)^2}{2i^2}-\frac{1}{2}] \\
&= \frac{2(f(i)/i)[(f(i)/i)^2-2\log(f(i)/i)-1]}{((f(i)/i)^2-1)((f(i)/i)^2-2)}
\end{aligned}
$$

Therefore, rearranging the results, we have:

$$
\begin{aligned}
f'(i) &= (\frac{n-2}{4-n}(\frac{f(i)}{i})\big[(4-n)(\frac{f(i)}{i})^{n-2}+(n-2)(\frac{f(i)}{i})^{n-4}-2\big]) \\
&\times \Big(\big((\frac{f(i)}{i})^{n-2}-1\big)\big((\frac{f(i)}{i})^{n-2}-2\big)\Big)^{-1} \quad \text{if } n \neq 4 \qquad (48) \\
&= \frac{2(f(i)/i)[(f(i)/i)^2-2\log(f(i)/i)-1]}{((f(i)/i)^2-1)((f(i)/i)^2-2)} \quad \text{if } n = 4 \qquad (49)
\end{aligned}
$$

and $h(i) = 2^{\frac{1}{n-2}} i$. *Hence, it is enough to show that* $f_* = \lambda i$ *complies with all the above characteristics. To prove so we will impose that* $f(i) = \lambda i$ *for a* $\lambda > 0$ *to be determined. The resulting equation is multiplied on both sides by* λ^{4-n} *and then we see that it yields to the equation* $F(\lambda) = 0$.

$$
\begin{aligned}
\lambda &= \left(\frac{n-2}{4-n}\left(\frac{\lambda i}{i}\right)\left[(4-n)\left(\frac{\lambda i}{i}\right)^{n-2} + (n-2)\left(\frac{\lambda i}{i}\right)^{n-4} - 2\right]\right) \\
&\times \left(\left(\left(\frac{\lambda i}{i}\right)^{n-2} - 1\right)\left(\left(\frac{\lambda i}{i}\right)^{n-2} - 2\right)\right)^{-1} \\
&= \left(\frac{n-2}{4-n}\right)\lambda\left[(4-n)\lambda^{n-2} + (n-2)\lambda^{n-4} - 2\right]\left(\left(\lambda^{n-2} - 1\right)\left(\lambda^{n-2} - 2\right)\right)^{-1} \\
1 &= \left((n-2)\lambda^{n-2} + \frac{(n-2)^2\lambda^{n-4}}{(4-n)} - 2\frac{n-2}{4-n}\right)\left(\left(\lambda^{n-2} - 1\right)\left(\lambda^{n-2} - 2\right)\right)^{-1} \\
\lambda^{4-n} &= \left((n-2)\lambda^2 + \frac{(n-2)^2}{(4-n)} - 2\frac{n-2}{4-n}\lambda^{4-n}\right)\left(\lambda^{2n-4} - 3\lambda^{n-2} + 2\right)^{-1} \\
\lambda^n - 3\lambda^2 + 2\lambda^{4-n} &= (n-2)\lambda^2 + \frac{(n-2)^2}{(4-n)} - 2\frac{n-2}{4-n}\lambda^{4-n} \\
F(\lambda) &= \lambda^n - (n+1)\lambda^2 + \frac{4}{4-n}\lambda^{4-n} - \frac{(n-2)^2}{4-n} \quad \text{for } n \neq 4 \\
&= \lambda^4 - 5\lambda^2 + 4\log\lambda + 4 \quad \text{for } n = 4
\end{aligned}
$$

for $\lambda > 0$. *Now,*

$$
\begin{aligned}
F'(\lambda) &= n\lambda^{n-1} - 2(1+n)\lambda + 4\lambda^{3-n} \\
&= (n\lambda - 2\lambda^{3-n})(\lambda^{n-2} - 2) \\
&= n\lambda^{3-n}\left(\lambda^{n-2} - \frac{2}{n}\right)(\lambda^{n-2} - 2)
\end{aligned}
$$

for $\lambda > 0$ *and* $n > 2$. *Hence, we see that* $F'(\lambda)$ *has two roots:* $\lambda_1 = (\frac{2}{n})^{1/(n-2)}$ *and* $\lambda_2 = 2^{1/(n-2)}$ *and, since* $F''(\lambda_1) < 0$ *and* $F''(\lambda_2) > 0$ *we can check that* F *has a local maxima at* λ_1 *and a local minimum at* λ_2. *Note that:*

(i) F *is strictly increasing on* $(0, \lambda_1)$ *with* $F(0+) < 0$ *and* $F(\lambda_1) > 0$

(ii) F *is strictly decreasing on* (λ_1, λ_2) *with* $F(1) = 0$ *and* $F(\lambda_2) < 0$

(iii) F *is strictly increasing on* (λ_1, ∞) *with* $F(-\infty) = \infty$

It follows therefore that the equation $F(\lambda) = 0$ *has exactly three roots* $\lambda_1^* < 1 < \lambda_2^*$ *where* $\lambda_1^* \in (0, \lambda_1)$ *and* $\lambda_2^* \in (\lambda_2, \infty)$. *Then, setting* $\lambda = \lambda_2^*$, *this shows that* $f_*(i) = \lambda i$ *is a solution to (48) staying strictly above the curve* $h(i) = 2^{1/(n-2)} i$ *for* $i > 0$.

Finally, we would need to prove that f_* *is a minimal solution satisfying this property and the proof will be completed. Set* $k(i) = f(i)/i$ *and so we rewrite (48) as follows:*

$$
ik'(i) = -\frac{F(k(i))}{k^{3-n}(i)(k^{n-2}(i) - 1)(k^{n-2}(i) - 2)} \tag{50}
$$

for $i > 0$. Then, since $F(k(i)) < 0$ for $k(i) \in (2^{1/(n-2)}, \lambda)$ we see that $i \to k(i)$ is increasing and such implies that (50):

$$-k'(i)\frac{k^{3-n}(i)(k^{n-2}(i)-1)(k^{n-2}(i)-2)}{F(k(i))} = \frac{1}{i}$$

$$-\frac{k^{3-n}(i)(k^{n-2}(i)-1)(k^{n-2}(i)-2)}{F(k(i))}dk = \frac{1}{i}di$$

$$-\int_{k(i)}^{k(i_0)} \frac{k^{3-n}(k^{n-2}-1)(k^{n-2}-2)}{F(k)}dk = \int_{i}^{i_0} \frac{1}{i}di = \log(\frac{i_0}{i})$$

it follows therefore that the integrand on the left side is bounded by a constant (not dependent on i) as long as $k(i) \in (2^{1/(n-2)}, \lambda)$ for $i \in (0, i_0)$ with any $i_0 > 0$ given and fixed. However, letting $i \to 0$ the right side tends to ∞, leading to a contradiction. Note that $k(i) \in (2^{1/(n-2)}, \lambda)$ if and only if $f(i) \in (h(i), f_*(i))$, we can therefore conclude that there is no solution f to (48) satisfying such condition for $i > 0$.

Thus f_* is the minimal solution to (48) staying strictly above h and the proof is complete.

□

7.4 The golden ratio rule

Corollary 7.13 (The golden ratio rule.). *If X is a three-dimensional Bessel process then the optimal stopping time in (27) is given by*

$$\tau_* = \inf\{t \geq 0 \mid \frac{X_t - I_t}{I_t} \geq \varphi\} \tag{51}$$

where $\varphi = (1+\sqrt{5})/2 = 1,61...$ is the golden ratio.

Proof: *We solve (43) for $d = 3$. Since $d \neq 4$, we use the equation (44). Therefore, we have:*

$$\lambda^3 - 4\lambda^2 + 4\lambda - 1 = (\lambda - 1)(\lambda^2 - 3\lambda + 1) = 0 \tag{52}$$

for $\lambda > 0$. Solving the latter quadratic equation it shows that:

$$(\lambda - 1)(\lambda^2 - \frac{3-\sqrt{5}}{2})(\lambda^2 + \frac{3+\sqrt{5}}{2}) = 0$$

Then, if we choose the root to be strictly greater than 1, we find that necessarily $\lambda = \frac{3+\sqrt{5}}{2} = 1 + \varphi$. The optimality of (51) then follows from (43) and the proof is complete. □

7.5 Relationship with the CEV process

The CEV model (20) is used to simulate asset's prices movements, especially for modeling prices of equities and commodities. Due to the properties of this model, it is associated with asset price bubbles: after soaring to a finite ultimate maximum (bubble) at a finite time, the asset price will tend to zero as time goes to infinity [3]. Therefore, the essential question for traders is to sell the asset at a time as close as possible to when the ultimate maximum is attained.

For the remainder of this section we will assume that $\mu = 0$ and $\beta > 0$.

Definition 7.14. *Consider Z_t to be a diffusion process. The* **running maximum process** $S = (S_t)_{t\geq 0}$ *is defined by:*

$$S_t = \sup_{0\leq s\leq t} Z_s \tag{53}$$

Suppose Z_t to be the process that solves the CEV equation (20). Since $Z_t \to 0$ as $t \to \infty$, it is clear that the ultimate supremum is attained at some random time θ, S_θ; and hence the optimal selling problem is

$$V(x) = \inf_\tau \mathbb{E}_x[|\theta - \tau| - \theta]$$

We have seen in previous sections that the CEV model and the Bessel processes are closely related. From the (4.7) definition we see that the ultimate minimum θ for the Bessel process X_t coincides with the time for the ultimate maximum of the CEV model Z_t and thus they have the same solution.

Recalling theorem (7.12), we intend that $X_t \geq \lambda I_t$, and that is, if and only if,

$$\frac{Z_t}{c_\sigma} = \frac{1}{X_t^{n-2}} \leq \frac{1}{\lambda^{n-2} I_t^{n-2}} = \frac{S_t}{\lambda^{n-2} c_\sigma}$$

Therefore, the optimal stopping time for the problem addressed with the CEV model is

$$\tau_* = \inf\{t \geq 0 | S_t \geq \lambda^{n-2} Z_t\} \tag{54}$$

where we take λ to be the unique solution to (44). In particular, we have seen that if $n = 3$ then $\lambda = 1 + \varphi$. Thus, the **golden ratio rule** for the CEV process reads as:

$$\tau_* = \inf\{t \geq 0 \mid \frac{S_t - Z_t}{Z_t} \geq \varphi\} \tag{55}$$

7.6 Fibonacci retracement rule

The final idea is to relate the results found in (55) to the Fibonacci retracement rule described in (6.3). Using the definition (6.1), we shall take $a = S_{\tau_*} - Z_{\tau_*}$ and $b = Z_{\tau_*}$.

Now we calculate the percentage of a in $a + b$:

$$\begin{aligned} \frac{a}{a+b} &= \frac{S_{\tau_*} - Z_{\tau_*}}{S_{\tau_*}} \\ &= 1 - \frac{Z_{\tau_*}}{S_{\tau_*}} = 1 - \frac{1}{1+\varphi} \\ &= \frac{\varphi}{1+\varphi} = \frac{1}{\varphi} \end{aligned}$$

Multiplying the above equation by 100, we shall get the percentage of $61,8\%$ which corresponds exactly to the Fibonacci retracement prior explained.

8. Conclusions

Above all, I would like to begin by saying that the whole work and dedication for this thesis has served its initial purposes. I have been able to immerse myself in the world of stochastic processes with a thorough study of them and their applications in mathematical finance.

First of all, I began with the basis: the definition and main concepts and properties related to stochastic processes; and then proceeded to study some relevant theorems and significant notions for the posterior work. Furthermore, I studied a broad change of topics that would later proof useful, such as the Brownian motion, the CEV model and the Bessel processes, an introduction to Technical Analysis and the golden and Fibonacci numbers. Secondly, following the guidelines of Glover, Hulley, and Peskir in the "Three-Dimensional Brownian Motion and the Golden Ratio Rule" article I was able to reproduce their rigorous mathematical explanation to the widely used Fibonacci retracement rule.

In conclusion, this project has been a truly interesting introduction into financial stochastic calculus .

Moreover, besides the academic conclusions, for me this project has been remarkably important to introduce myself into the world of research, and how to read and understand academic papers.

Finally, I would like to emphasize the fact that basic knowledge of stochastic process and an introduction to technical analysis techniques has not been covered in any mandatory course of the Mathematics degree syllabus and I hope this thesis might find itself useful for anyone that could need it.

9. Appendix A. Rstudio package Sim.DiffProc

The different stochastic processes aforementioned can be modelled through the programming language R, used for statistical computing and graphics. To achieve so a very powerful tool is the Rstudio package "Sim.DiffProc" which stands for "Simulation of difussion processes" [5]. It provides users with a wide range of tools to simulate, estimate, analyze, and visualize the dynamics of stochastic differential systems in both forms Ito and Stratonovich. In our case, only Ito's processes will be studied.
Since the Brownian Motion (and its variations) is one of the most used processes it has its own function, whose specific parameters are described below.

N number of simulation steps	**M** number of trajectories
t0 initial time	**x0** initial value of the process at time t_0
T final time	**Dt** time step of the simulation (discretization)
theta the interest rate of the ABM and GBM	**sigma** the volatility of the ABM and GBM

9.1 Simulation of a BM

First, a discrete modelization shown step by step as explained in section (3.5) is written in code. Then we can find the code using the functions in the Sim.DiffProc package. All codes are accompanied by an example image; some with more than one path where the red line indicates the mean.

```
## Brownian Motion
#####################################################
## Step by step
alpha=0; sigma=1; T=1; n=2^(12); X0=0.1;
######### Generate 1 trajectory
dt=T/n
t=seq(0,T,by=dt)
x=c(X0,alpha*dt+sigma*sqrt(dt)*rnorm(n,mean=0,sd=1))
Xt=cumsum(x) ##cumulative sum of the elements of the argument
plot(t,Xt,type='l',xlab="time")


## SimDiffProc commands
######### BM(N, ...)
set.seed(1234);
Nb = 1000; Mb = 3; x0b = 0; t0b = 0; Tb = 1;
Dtb = NULL; ## t - t0 / N
X <- BM(N = Nb, M = Mb, x0 = x0b, t0 = t0b, T = Tb, Dt = Dtb)
summary(X)
plot(X, plot.type = "single")
```

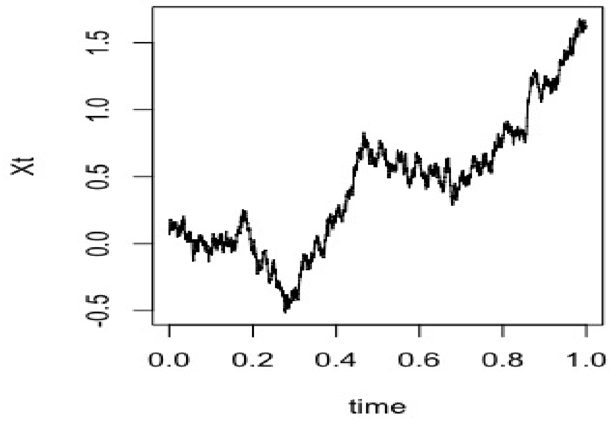

Figure 3: Example of a simulation (simple path) of a Brownian Motion

```
lines(as.vector(time(X)), rowMeans(X),col="red")
########################################################
```

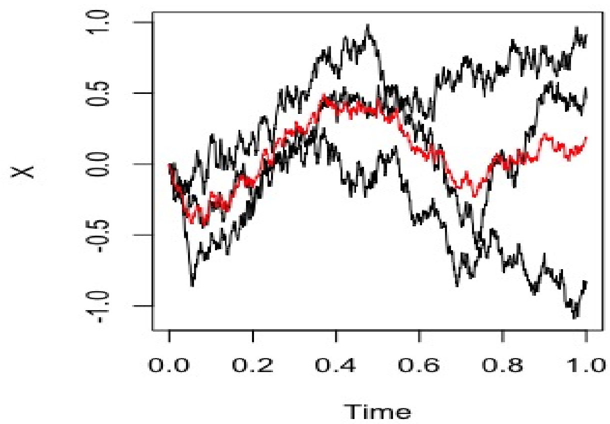

Figure 4: Example of a simulation (simple path) of a Brownian Motion with the Sim.DiffProc package

```
## Arithmetic Brownian Motion
########################################################
## SimDiffProc commands
set.seed(1234);
Nb = 1000; Mb = 2; x0b = 0; t0b = 0; Tb = 5;
Dtb = NULL; ## t - t0 / N
thetab = 2; sigmab = 5;
X <- ABM(N = Nb, M = Mb, x0 = x0b, t0 = t0b, T = Tb,
```

```
Dt = Dtb, theta = thetab, sigmab = sigma)
summary(X)
plot(X, plot.type = "single")
lines(as.vector(time(X)), rowMeans(X),col="red")
##########################################################
```

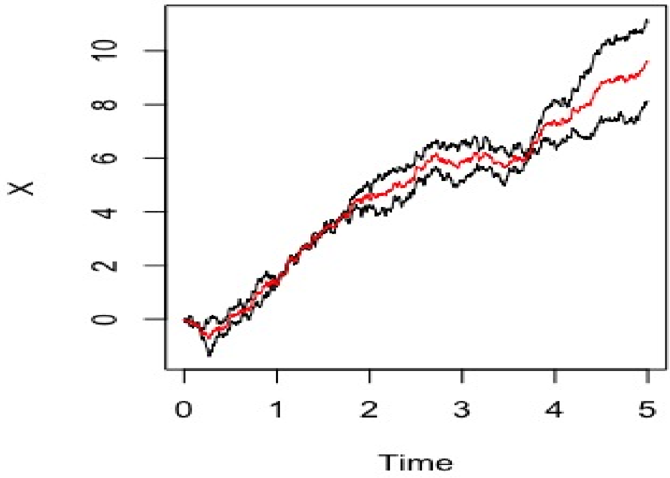

Figure 5: Example of a simulation (simple path) of an Arithmetic Brownian Motion

9.2 Simulation of a GBM

In this section a code for a simulation of a Geometric Brownian Motion is shown. There are two images two show the results: figure (6) draws M simple paths with a red line indicating the mean, whereas figure (7) draws M simple paths each one in a different color.

```
## Geometric Brownian Motion
##########################################################
## SimDiffProc commands
########## GBM(N, ...)
set.seed(1234);
Nb = 1000; Mb = 5; x0b = 0.5; t0b = 0; Tb = 5; Dtb = NULL;
## t - t0 / N
thetab = 2; sigmab = 5;
X <- GBM(N = Nb, M = Mb, x0 = x0b, t0 = t0b, T = Tb, Dt = Dtb)
summary(X)

plot(X, plot.type = "single")
lines(as.vector(time(X)), rowMeans(X),col="red")
```

```
ymax=max(X); ymin=min(X) #bounds for simulated prices
plot(X[1,],t='l',ylim=c(ymin, ymax), col=1,
ylab="Price P(t)",xlab="time t")
for(i in 2:Mb){lines(X[,i], t='l',ylim=c(ymin, ymax),col=i)}
#####################################################
```

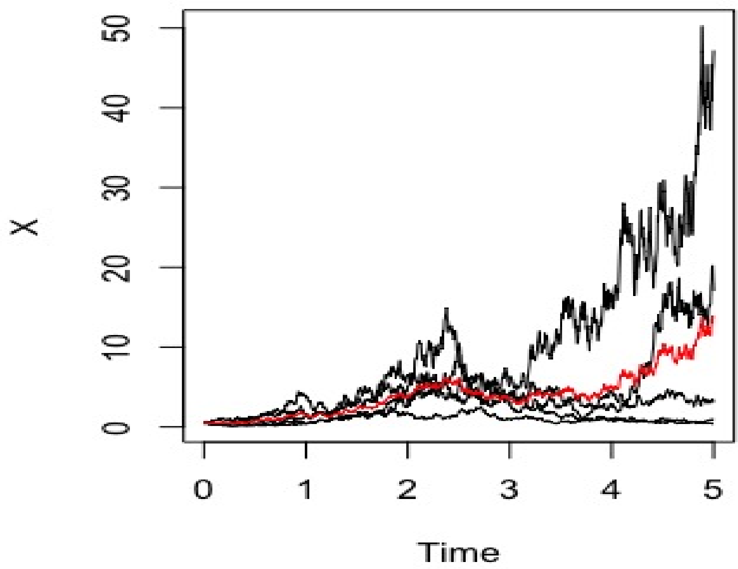

Figure 6: Example of a simulation (simple path) of a Geometric Brownian Motion

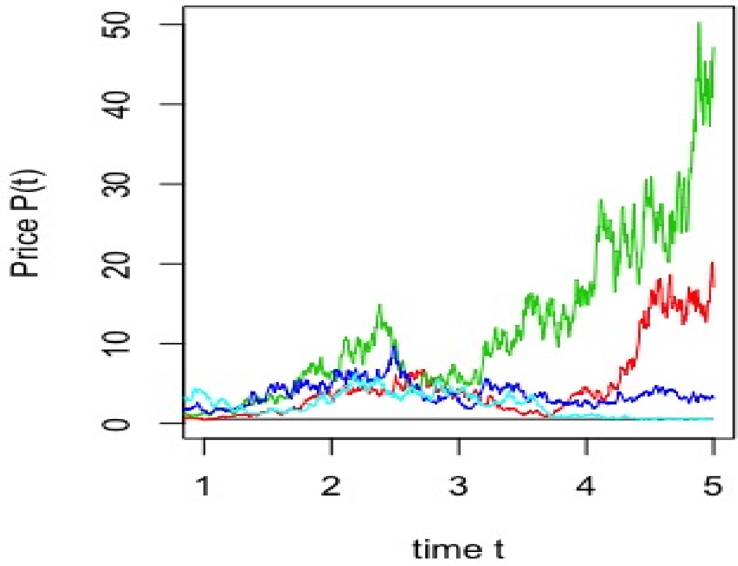

Figure 7: Example of a simulation (simple path) of a Geometric Brownian Motion 2 with the mean in red

9.3 Simulation of CEV

Two different aproaches will be followed in oder to simulate the CEV model. The first one will be using its classical definition (4.7) and the former uses the result found in the lemma (18).
The main function used for this codes is *snssde1d(...)*. Different parameters can be added to the function to adapt it to your model (if not listed below, they are equal to the aforementioned parameters at the beginning of the section):

drift	**diffusion**
drift coefficient: an expression of two variables t and x	diffusion coefficient: an expression of two variables t and x
type	**method**
Ito or Stratonovich type the default type="ito"	numerical methods of simulation, the default method = "euler".

```
## CEV model
#################################################
##Simulation of a CEV (a first order Ito sde). Use snssde1d
## CEV: dSt = mu*St*dt + sigma*St^(1+beta)*dWt

set.seed(1234)
k=1/2  #1/2, 3/2, 3
mu = 1; sigma = 1;
```

```
f <- expression(mu)       #mu*St
g <- expression(sigma*x^k) #sigmaSt^(1+beta)
mod1 <- snssde1d(drift=f, diffusion=g,M=12,x0=10,Dt=0.01)


mod1
summary(mod1)

plot(mod1)
lines(time(mod1),apply(mod1$X,1,mean),col=2,lwd=2)
##mean of trajectories
lines(time(mod1),apply(mod1$X,1,bconfint,level=0.95)[1,],col=4,lwd=2)
##95% confidence intervals
lines(time(mod1),apply(mod1$X,1,bconfint,level=0.95)[2,],col=4,lwd=2)
legend("topright",c("mean path",paste("bound of", 95,
" percent confidence")),inset = .01,col=c(2,4),lwd=2,cex=0.8)



## Ito Sde 1D: CEV de la Eq 5.8
## dX(t) = 0 * dt + sigma * X(t)^(1 + 1/(d - 2)) * dW(t)
d <- 3  # d> 2
cs <- .5    #1,2
sigma <- (d-2)/cs^(1/(d-2))
f <- expression(0*x)  ##revisar que esto corresponda a 0 drift
g <- expression(-sigma*x^(1+1/(d-2)))
mod1 <- snssde1d(drift=f, diffusion=g,N=2000,M=10,x0=.5, Dt=0.01)   #t0=0,T=100)

mod1
summary(mod1)
plot(mod1)
##Hasta M=10 trayectorias se pueden plotear simultaneamente :
plot(mod1$X[,10]) ##plot la trayectoria 10

#####################################################
```

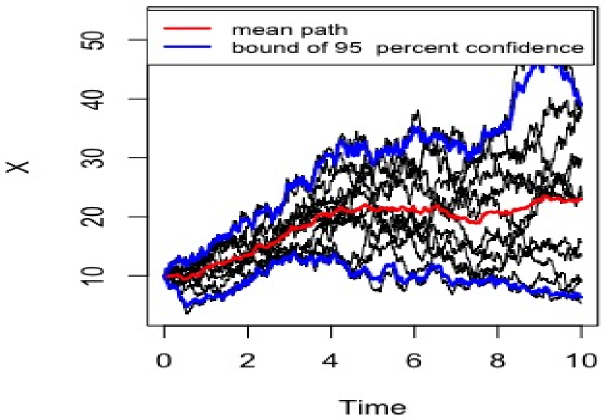

Figure 8: Example of a simulation (simple path) of a CEV using its definition

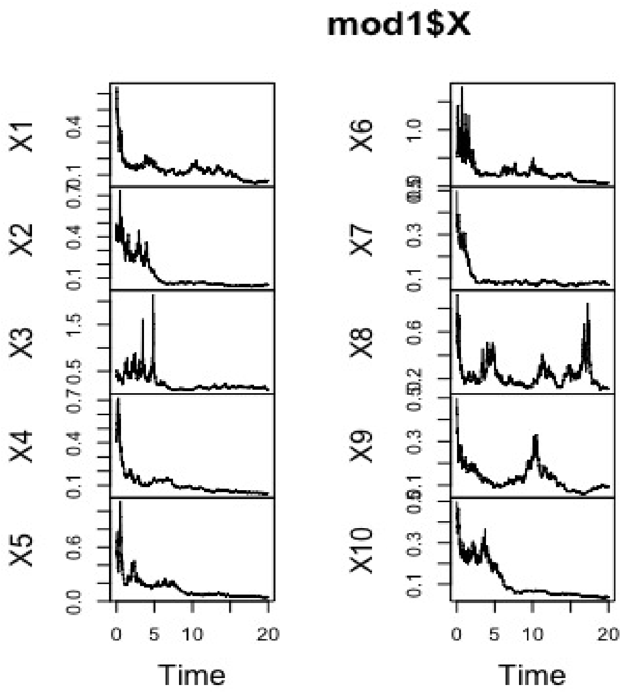

Figure 9: Example of a simulation (simple path) of 10 different models of CEV using the lemma 20

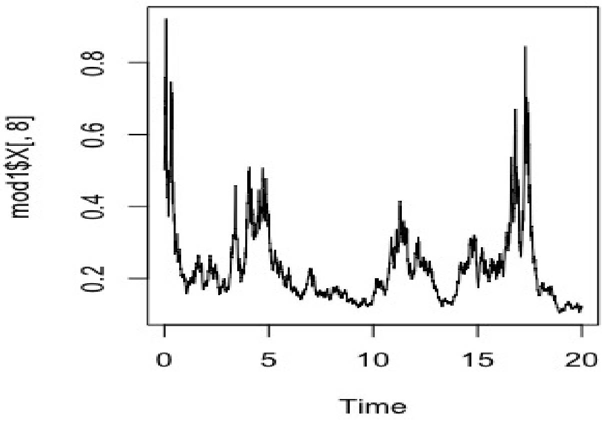

Figure 10: Simulation number 8 in detail of the prior example

10. Appendix B. Empirical Proof of the Fibonacci retracement rule

The aim of this section is to show a visual analysis of the Fibonacci retracement rule, both with a simulation of a the CEV process (using the code explained in the previous section) and with real Financial stocks. In order to achieve so, the following procedure is done:

1. Using R code, a plot of the process is drawn, either by a CEV simulation or retrieving data from financial stocks.

2. Two local extreme points are taken.

3. The support or resistance lines are calculated (depending if it is an upward or downward trend).

4. Those lines are drawn: shallow retracement in orange, moderate retracement in green and golden retracement in red.

5. We can visually check if the support or resistance lines adjust properly to the trends of the process.

```
# PART I: Fibonacci Levels Visual Analysis with CEV
##########################################################
Y<- mod1$X[,UsedModel]
plot(Y,type="l")

## we consider a sample where i_min<i_max for uptrend analysis
Y<- mod1$X[600:1500,UsedModel]
plot(Y,type="l")
```

```
i_max = which.max(Y)  ##index for the max
i_max
price_max = as.numeric(Y[i_max])
B<- 50
i_min=which.min(Y[(i_max-B):i_max])  ##index for the min B days behind
##OJO: en la subserie Y[(i_max-B):i_max] i_max-B pasa a ser el 1.
price_min =  as.numeric(Y[i_max-B + i_min -1])

diff = price_max - price_min

# Fibonacci Levels considering original trend as upward move
#(support lines)
#i_min < i_max
level1 = price_max - 0.236 * diff  ##shallow support  GR^{-3}
level2 = price_max - 0.382 * diff  ##moderate support GR^{-2}
level3 = price_max - 0.618 * diff  ##golden ratio  GR^{-1}

plot(Y, type = "l")
abline(v=i_max+1, col="black") ##draw the vertical line (time)
#up to which we are looking back
abline(h=price_max, col="blue")
abline(h=price_min, col="blue")
abline(h=level1 , col="orange")
abline(h=level2 , col="green")
abline(h=level3 , col="red")
```

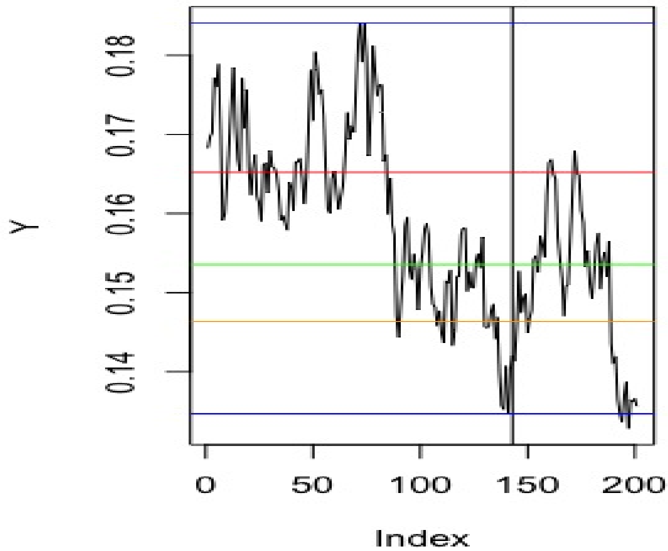

Figure 11: The vertical black line marks the moment of analysis (a local minimum), we look at the trend behind from a previous local maximum to this local minimum and draw forward the Fibonacci resistance lines, the red one being at 61.8% level. We see that the price reaches that level and does not exceeds it. Hence the Fibonacci retracement rule holds in this case.

```
# Fibonacci Levels considering original trend as downward move
#(resistance lines)
#i_max < i_min
price_max <- Y[i_max]
F<- 30
i_min =which.min(Y[i_max:(i_max+F)])
##OJO: en la subserie Y[i_max:(i_max+F)] i_max pasa a ser el 1.
price_min =  as.numeric(Y[i_max+i_min-1])
diff = price_max - price_min

level1 = price_min + 0.236 * diff
level2 = price_min + 0.382 * diff
level3 = price_min + 0.618 * diff

plot(Y,type = "l")
abline(v=i_max+i_min+1, col="black") ##draw the vertical line (time)
#up to which we are looking back
abline(h=price_max, col="blue")
abline(h=price_min, col="blue")
abline(h=level1 , col="orange")
abline(h=level2 , col="green")
```

```
abline(h=level3, col="red")
```

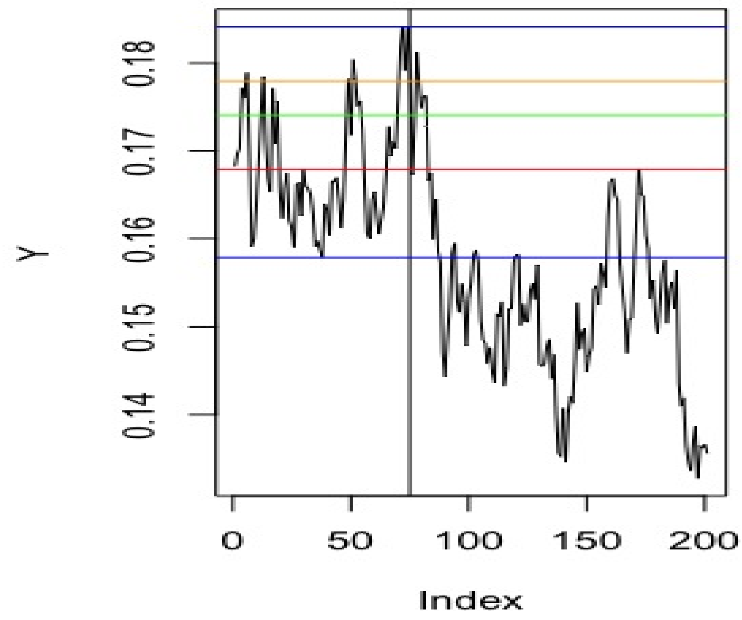

Figure 12: The vertical black line marks the moment of analysis (a local maximum), we look at the trend behind from a previous local minimum to this local maximum and draw forward the Fibonacci resistance lines, the red one being at 61.8% level. However, in this occasion we see that the price trespasses the red support line at 61.8%, which is a critical level and thus the price will remain to fall, it would have been better to sell before.

```
# PART II: Fibonacci Levels Visual Analysis with Financial Stocks
##################################################
library(quantmod)
##retrieve stock data from yahoo
getSymbols("JPM", from="2009-01-01", to="2015-01-01", src="yahoo")
stock <- as.ts(JPM$JPM.Close)
plot(stock)

Y=stock[450:600]
plot(Y,type="l")

i_max = which.max(Y)  ##index for the max
i_max
price_max = as.numeric(Y[i_max])
B<- 50
i_min=which.min(Y[(i_max-B):i_max])  ##index for the min B days behind
##OJO: en la subserie Y[(i_max-B):i_max] i_max-B pasa a ser el 1.
price_min =  as.numeric(Y[i_max-B + i_min -1])

diff = price_max - price_min

# Fibonacci Levels considering original trend as upward move
#(support lines)
#i_min < i_max
level1 = price_max - 0.236 * diff  ##shallow support  GR^{-3}
level2 = price_max - 0.382 * diff  ##moderate support GR^{-2}
level3 = price_max - 0.618 * diff  ##golden ratio  GR^{-1}

plot(Y,type = "l")
abline(v=i_max+1, col="black") ##draw the vertical line (time)
#up to which we are looking back
abline(h=price_max, col="blue")
abline(h=price_min, col="blue")
abline(h=level1, col="orange")
abline(h=level2, col="green")
abline(h=level3, col="red")

# Fibonacci Levels considering original trend as downward move
#(resistance lines)
#i_max < i_min
price_max <- Y[i_max]
F<- 30
i_min =which.min(Y[i_max:(i_max+F)])
##OJO: en la subserie Y[i_max:(i_max+F)] i_max pasa a ser el 1.
price_min =  as.numeric(Y[i_max+i_min-1])
diff = price_max - price_min
```

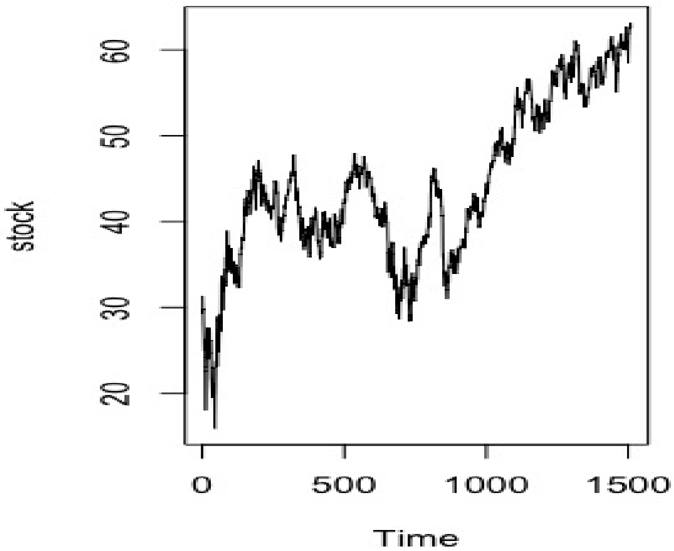

Figure 13: JP Morgan Bank's graphic of its stock prices from 01/01/2009 to 01/01/2015.

```
level1 = price_min + 0.236 * diff
level2 = price_min + 0.382 * diff
level3 = price_min + 0.618 * diff

plot(Y, type = "l")
abline(v=i_max+i_min+1, col="black") ##draw the vertical line
(time) up to which we are looking back
abline(h=price_max, col="blue")
abline(h=price_min, col="blue")
abline(h=level1 , col="orange")
abline(h=level2 , col="green")
abline(h=level3 , col="red")
```

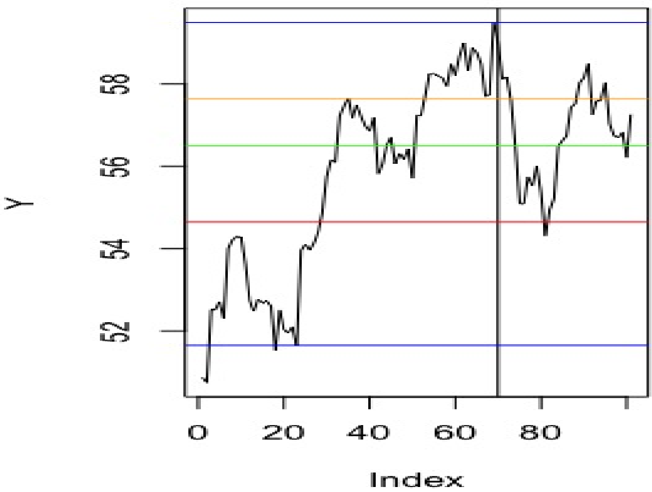

Figure 14: Section of JP Morgan Bank's graphic of its stock prices (interval 1200:1400) where the consequent Fibonacci Retracements are calculated in an upward move. The meaning of the vertical and colored horizontal lines remains the same as in the CEV model. We see that at 61,8 %, the red line, there is a support level, the price reaches that level but does not fall behind it.

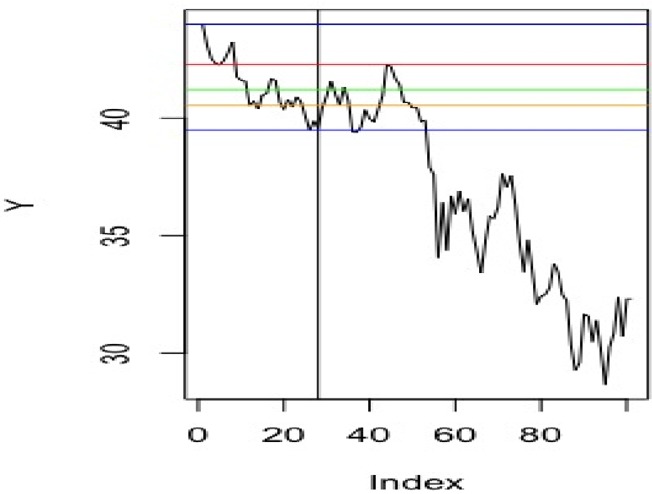

Figure 15: Section of JP Morgan Bank's graphic of its stock prices (interval 400:600) where the consequent Fibonacci Retracements are calculated in a downward move. The meaning of the vertical and colored horizontal lines remains the same as in the CEV model. We see that at 61,8 %, the red line, there is a resistance level, the price reaches that level but does not exceed it.

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