b-STRUCTURES ON LIE GROUPS AND POISSON REDUCTION

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ABSTRACT. We introduce the notion of b-Lie group as a pair (G, H) where G is a Lie group and H is a codimension-one Lie subgroup, and study the associated canonical b-symplectic structure on the b-cotangent bundle ${}^bT^*G$ together with its reduction theory. Namely, we prove that the Poisson reduction under the cotangent lifted action of H by left translations is globally isomorphic to a product of the minus Lie Poisson structure on $\mathfrak{h}^*$ (where $\mathfrak{h}$ is the Lie algebra of H) and the canonical b-symplectic structure on ${}^bT^*(G/H)$, where G/H is viewed as a one-dimensional b-manifold having as critical hypersurface (in the sense of b-manifolds) the identity element.

RÉSUMÉ. Nous introduisons la notion d'un b-groupe de Lie comme une paire (G, H) où G est un groupe de Lie et H est un sous-groupe de Lie de codimension un. Nous étudions la structure b-symplectique canonique du fibré b-cotangent ${}^bT^*G$ et sa réduction. Plus précisément, nous montrons que la réduction Poisson de ${}^bT^*G$ par rapport au relèvement de l'action de H par translation à gauche est isomorphe au produit de l'opposée de la structure Lie-Poisson sur $\mathfrak{h}^*$ (où $\mathfrak{h}$ est l'algèbre de Lie de H) et la structure b-symplectique canonique de ${}^bT^*(G/H)$, où G/H est considéré comme une variété b-symplectique de dimension un ayant comme l'hypersurface critique (dans la términologie des b-variétés) l'élément neutre.

1. INTRODUCTION AND PRELIMINARIES

The study of b-manifolds has its origins in the calculus on manifolds with boundary to give a conceptual approach to the Atiyah-Patodi-Singer index theorem in terms of the classical Atiyah-Singer theorem [Me]. The language of b-tangent bundles was also used in the extension of the deformation quantization scheme to manifolds with boundary [NT]. b-Symplectic structures and normal forms for group actions on them have been intensely studied by several authors (see for instance, [GMP11], [GMP14], [GMPS14b], [GLPR]). However the notion of b-structures on Lie groups has yet to be treated.

In this short article we examine b-structures on Lie groups and prove that they are very rigid, with few examples, due to the restrictive nature of the symmetries. We study the reduction of the structure of the b-cotangent bundle of the Lie group by the lift of the action of a subgroup H identified as the critical set of the b-symplectic structure. These b-cotangent models generalize the models for integrable systems treated in [KM] and will be a starting point to understand the b-symplectic slice theorem as the b-analogue of the Marle-Guillemin-Sternberg normal form in [Ma] and [GS84].

A **b-manifold** is a pair (M, Z), consisting of an oriented manifold M and an oriented hypersurface $Z \subset M$. A **b-vector field** on a b-manifold (M, Z) is a vector field which is tangent to Z at every point $p \in Z$.

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If f is a local defining function for Z on some open set $U \subset M$ and $(f, z_2, \ldots, z_n)$ is a chart on U, then the set of b-vector fields on U is a free $C^\infty(U)$-module with basis

$$\left(f\frac{\partial}{\partial f}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n} \right). \tag{1}$$

We call the vector bundle associated to this locally free C^∞_M-module the b**-tangent bundle** and denote it ${}^bTM$. We define the b**-cotangent bundle** ${}^bT^*M$ of M to be the vector bundle dual to ${}^bTM$. The sheaf of sections of $\Lambda^k({}^bT^*M)$ is denoted ${}^b\Omega^k$ and its elements are called b**-forms of degree** k.

The classical exterior derivative d on the complex of (smooth) k-forms extends to the complex of b-forms in a natural way. Indeed, any b-form ω of degree k can locally be written in the form $\omega = \alpha \wedge \frac{df}{f} + \beta$ where $\alpha \in \Omega^{k-1}, \beta \in \Omega^k$. Here f is a local defining function of Z and $\frac{df}{f}$ is the b-form of degree 1 dual to $f\frac{\partial}{\partial f}$ in a frame of the form (1). We then define the exterior derivative $d\omega := d\alpha \wedge \frac{df}{f} + d\beta$ (see [GMP14] for details). In order to have a Poincaré lemma for b-forms, we enlarge the set of smooth functions and consider the set of b**-functions** ${}^bC^\infty(M)$, which consists of functions with values in $\mathbb{R}\cup\{\infty\}$ of the form $c\log|f|+g$, where $c \in \mathbb{R}$, f is a defining function for Z, and g is a smooth function. The differential operator d on this space is defined as: $d(c\log|f|+g) := c\frac{df}{f} + dg$, where dg is the standard de Rham derivative.

Definition 1. *Let (M, Z) be a b-manifold, with Z the critical hypersurface. We say a a closed b-form of degree 2, $\omega \in {}^b\Omega^2(M)$, is b-**symplectic** if ω_p is of maximal rank as an element of $\Lambda^2({}^bT^*_pM)$ for all $p \in M$.*

A b-symplectic form defines a symplectic form away from Z, implying in particular that M has even dimension. The local structure of b-symplectic forms is well-understood thanks to a Darboux theorem in this context:

Theorem 2 (b**-Darboux theorem,** [GMP14])**.** *Let ω be a b-symplectic form on (M^{2n}, Z). Let $p \in Z$. Then we can find a local coordinate chart*

$$(x_1, y_1, \ldots, x_n, y_n)$$

centered at p such that the hypersurface Z is locally defined by $y_1 = 0$ and

$$\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

b**-Cotangent lifts**. Recall (see for instance, [GS]) that given any group action the cotangent lift of this action is Hamiltonian with respect to the canonical symplectic structure on T^*M with moment map μ:

$$\langle \mu(p), X \rangle := \langle \lambda_p, X^\#|_p \rangle = \langle p, X^\#|_{\pi(p)} \rangle,$$

where $X^\#$ is the fundamental vector field of the action. In [GMP14] it was noted that, analogous to the symplectic case, the b-cotangent bundle comes equipped with a canonical b-symplectic form.

Definition 3. *Let (M, Z) be a b-manifold. Then we define a b-one-form λ on ${}^bT^*M$, considered as a b-manifold with critical hypersurface ${}^bT^*M|_Z$, in the following way:*

$$\langle \lambda_p, v \rangle := \langle p, (\pi_p)_*(v) \rangle, \qquad p \in {}^bT^*M, v \in {}^bT_p({}^bT^*M)$$

*We call λ the b-**Liouville form**. The negative differential*

$$\omega = -d\lambda$$

is the **canonical b-symplectic form** *on ${}^bT^*M$.*

Using this form the canonical b-cotangent lift was defined as follows [KM]:

Definition 4. *Consider the b-cotangent bundle ${}^bT^*M$ endowed with the canonical b-symplectic structure. Assume that the action ρ of G on M preserves the hypersurface Z, i.e. ρ_g is a b-map for all $g \in G$. Then the lift of ρ to an action on ${}^bT^*M$ is well-defined:*

$$\hat{\rho} : G \times {}^bT^*M \to {}^bT^*M : (g, p) \mapsto \rho^*_{g^{-1}}(p).$$

Moreover, it is b-Hamiltonian with respect to the the canonical b-symplectic structure on ${}^bT^*M$. We call this action together with the underlying canonical b-symplectic structure the **canonical b-cotangent lift**.

2. Main results

In the symplectic case, reducing T^*G by the action of G yields the Lie-Poisson structure on $\mathfrak{g}^*$. In this paper we study the analogue in the b-context: In order to lift an action to ${}^bT^*G$ (Definition 4), we have to demand that the action leaves the critical hypersurface invariant. This motivates us to consider the setting where the critical hypersurface of the b-structure is a codimension one Lie subgroup H and we consider the action of H on G by translations.

This is the content of the next definition:

Definition 5. *A b-manifold (G, H), where G is a Lie group and $H \subset G$ is a closed codimension one subgroup*[1] *is called a* **b-Lie group**.

Example 6. *The special Euclidean group of orientation-preserving isometries in the plane is the semidirect product*

$$\mathrm{SE}(2) \cong \mathrm{SO}(2) \ltimes T(2)$$

where $T(2)$ are translations in the plane. Recall that we can identify $\mathrm{SE}(2)$ with matrices of the form

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}, \qquad A \in \mathrm{SO}(2), b \in \mathbb{R}^2$$

Then $T(2)$ (identified with $\{I\} \times T(2) \subset \mathrm{SE}(2)$) is a closed codimension 1 subgroup and the pair $(\mathrm{SE}(2), T(2))$ is a b-Lie group.

Example 7. *The Galilean group G is the group of transformations in space-time $\mathbb{R}^{3+1}$ (the first three dimensions are interpreted as spatial dimensions and the last one is time) whose elements are given by composition of a spatial rotation $A \in SO(3)$, uniform motion with velocity $v \in \mathbb{R}^3$ and translations in space and time by a vector $(a, s) \in \mathbb{R}^{3+1}$. As a matrix group, the elements are given by*

$$\begin{pmatrix} A & v & a \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \qquad A \in \mathrm{SO}(3), v, a \in \mathbb{R}^3, s \in \mathbb{R}$$

The subgroup H given by $s = 0$ (which corresponds to fixing time) is a closed codimension one subgroup and hence the pair (G, H) is a b-Lie group.

Example 8. *We consider the $(2n+1)$-dimensional Heisenberg group $H_{2n+1}(\mathbb{R})$ given by matrices of the form*

$$\begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad a \in \mathbb{R}^{1\times n}, b \in \mathbb{R}^{n \times 1}, c \in \mathbb{R}$$

The subgroup Γ of matrices of the form

$$\begin{pmatrix} 1 & 0 & k \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad k \in \mathbb{Z}$$

[1] This is equivalent to H being an embedded Lie subgroup.

is central, hence normal, and so we can consider $G := H_{2n+1}(\mathbb{R})/\Gamma$*. This is a well-known example of a non-matrix Lie group. Now fixing one component* $a_i = 0$ *or* $b_i = 0$ *yields a closed codimension one subgroup of* G*.*

Let us consider the action of H on G by left translations. This action is obviously free and since H is closed, it is also proper. Therefore, the left coset space G/H can be given the structure of a smooth manifold such that the projection $\pi : G \to G/H$ is a smooth submersion. Moreover, it is well-known that π turns G into a principal H-bundle.

For future reference we summarize these facts in the following lemma:

Lemma 9. *Let* (G, H) *be a b-Lie group. The projection* $\pi : G \to G/H$ *is a principal* H*-bundle; in particular* G *is semilocally around* H *a product*

$$\pi^{-1}(V) \cong V \times H$$

for some open neighborhood V *of* $[e]_\sim$ *in* G/H *and under this diffeomorphism* π *corresponds to the projection onto the first component.*

Note that by taking a coordinate φ on V centered at $[e]_\sim$, we obtain a global defining function $\varphi \circ \pi$ for the critical hypersurface H. Also note that the local trivialization in the lemma gives rise to a natural projection $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ from the Lie algebra $\mathfrak{g}$ of G to the Lie algebra $\mathfrak{h}$ of H.

2.1. **The H-action on ${}^bTG$ and ${}^bT^*G$.** As in the previous section, let (G, H) be a b-Lie group and consider the action of H by left translations.

We can lift this action to TG in the obvious way:

$$H \times TG \to TG : (h, v_g) \mapsto (L_h)_* v_g.$$

This action is again proper and free; therefore the quotient space is a manifold, which we want to describe below.

Let us introduce the subbundle $\mathcal{H}$ of TG whose fibre $\mathcal{H}_g$ at $g \in G$ is given by the corresponding left-shift of the Lie algebra $\mathfrak{h}$ of H, $\mathcal{H}_g = (L_g)_*\mathfrak{h}$. Let $\pi_{\mathcal{H}} : TG \to \mathcal{H}$ be the natural projection onto $\mathcal{H}$ given by conjugating $\pi_{\mathfrak{h}}$ with left-translations. Recall that $\pi : G \to G/H$ induces a surjective bundle morphism $\pi_* : TG \to T(G/H)$ and at each fibre T_gG the kernel is $\mathcal{H}_g$. The content of the next proposition is well-known, we include the proof for the sake of completeness:

Proposition 10. *There is a diffeomorphism*

$$\begin{aligned} (TG)/H &\xrightarrow{\sim} \mathfrak{h} \times T(G/H) \\ [v_g]_\sim &\mapsto \left((L_{g^{-1}})_*(\pi_{\mathcal{H}}(v_g)), \pi_*(v_g)\right). \end{aligned}$$

Proof. The map is well-defined as it does not depend on the representative of $[v_g]_\sim = \{(L_h)_*(v_g) : h \in H\}$. It is obviously smooth and surjective. If $[v_g]_\sim$ and $[v'_{g'}]_\sim$ have the same image, then $\pi_*(v_g) = \pi_*(v'_{g'})$ implies $\pi(g) = \pi(g')$ so by choosing a different representative in $[v'_{g'}]_\sim$ we can assume $g = g'$. Then $v_g - v'_g \in \ker(\pi_*)_g = \mathcal{H}_g$ and combining this with $\pi_{\mathcal{H}}(v_g) = \pi_{\mathcal{H}}(v'_g)$ we see $v_g = v'_g$. □

The analogous result holds for the action of H on the b-tangent bundle,

$$H \times {}^bTG \to {}^bTG : (h, v_g) \mapsto (L_h)_* v_g.$$

Note that this action is well-defined since the left translation by $h \in \mathcal{H}$ preserves H i.e. it is a b-map. Moreover we define the projection $\pi_{\mathcal{H}} : {}^bTG \to \mathcal{H}$ in the analogous way.

Proposition 11. *There is a diffeomorphism*

$$\begin{aligned} ({}^bTG)/H &\xrightarrow{\sim} \mathfrak{h} \times {}^bT(G/H) \\ [v_g]_\sim &\mapsto \left((L_{g^{-1}})_*(\pi_{\mathcal{H}}(v_g)), \pi_*(v_g)\right) \end{aligned}$$

where ${}^bT(G/H)$ *is the b-tangent bundle of the one-dimensional b-manifold* G/H *with critical hypersurface* $[e]_\sim$*. Note that* $\pi : (G, H) \to (G/H, [e]_\sim)$ *is a b-map and therefore* $\pi_* : {}^bTG \to {}^bT(G/H)$ *is well-defined.*

The right hand sides of the diffeomorphisms in Proposition 10 and 11 are vector bundles over G/H. This makes TG/H resp. ${}^bTG/H$ vector bundles over G/H as well with bundle map $[v_g]_\sim \mapsto \pi(g) \in G/H$:

Corollary 12. $(TG)/H$ *(resp.* $({}^bTG)/H$*) is a vector bundle of rank* n *over* G/H *isomorphic to the direct sum of the trivial vector bundle* $\mathfrak{h} \times G/H$ *with* $T(G/H)$ *(resp.* ${}^bT(G/H)$*):*

$$(TG)/H \cong (\mathfrak{h} \times G/H) \oplus T(G/H), \quad ({}^bTG)/H \cong (\mathfrak{h} \times G/H) \oplus {}^bT(G/H).$$

2.2. **The b-cotangent lift.** In Definition 4 we introduced the b-cotangent lift; in the present setting this is given by the following action on the b-cotangent bundle ${}^bT^*G$:

$$H \times {}^bT^*G \to {}^bT^*G : (h, \alpha_g) \mapsto (L_{h^{-1}})^*\alpha_g.$$

The quotient space $({}^bT^*G)/H$ can be viewed as a vector bundle over G/H which is isomorphic to $\big(({}^bTG)/H\big)^*$ via the identification

$$({}^bT^*G)/H \xrightarrow{\sim} \big(({}^bTG)/H\big)^* : [\alpha_g]_\sim \mapsto \big([v_g]_\sim \mapsto \langle \alpha_g, v_g\rangle\big), \quad v_g \in {}^bT_gG.$$

Therefore we can dualize the result for ${}^bTG/H$ of the previous section to obtain an isomorphism of vector bundles

$$({}^bT^*G)/H \cong (\mathfrak{h}^* \times G/H) \oplus {}^bT^*(G/H).$$

As smooth manifolds,

$$({}^bT^*G)/H \cong \mathfrak{h}^* \times {}^bT^*(G/H),$$

where the isomorphism is given by identifying an element of the right hand side $(\alpha, \beta_{[g]_\sim}) \in \mathfrak{h}^* \times {}^bT^*_{[g]_\sim}(G/H)$ with the class of $L^*_{g^{-1}}(\alpha) + \pi^*\beta_{[g]_\sim} \in {}^bT^*_gG$ on the left hand side.

2.3. **Reduction of the canonical b-symplectic structure.** The cotangent bundle T^*G has a canonical symplectic structure, which under the action of G on itself by left translations reduces to the minus Lie-Poisson structure on $T^*G/G \cong \mathfrak{g}^*$.

In Definition 3 we have seen how to endow the b-cotangent bundle ${}^bT^*G$ with a canonical b-symplectic structure (with critical hypersurface ${}^bT^*G|_H$). What is the reduced Poisson structure on $({}^bT^*G)/H$?

Theorem 13. *Let* ${}^bT^*G$ *be endowed with the canonical b-Poisson structure. Then the Poisson reduction under the cotangent lifted action of* H *by left translations is*

$$(({}^bT^*G)/H,\, \Pi_{red}) \cong (\mathfrak{h}^* \times {}^bT^*(G/H),\, \Pi^-_{L\text{-}P} + \Pi_{b\text{-}can})$$

where $\Pi^-_{L\text{-}P}$ *is the minus Lie-Poisson structure on* $\mathfrak{h}^*$ *and* $\Pi_{b\text{-}can}$ *is the canonical b-symplectic structure on* ${}^bT^*(G/H)$*, where* G/H *is viewed as a b-manifold with critical hypersurface the point* $[e]_\sim$*.*

Proof. Let $V \subset G/H$ be open and such that G trivializes as a principal H-bundle over V (cf. Lemma 9), i.e.

$$G \supset U := \pi^{-1}(V) \xrightarrow{\sim} H \times V$$

where the projection onto the second component corresponds to the quotient projection π; in particular the critical hypersurface H gets mapped to $H \times [e]_\sim \subset H \times V$ and the b-cotangent bundle over U splits in the following way:

$${}^bT^*U \cong T^*H \times {}^bT^*V.$$

Then the canonical b-symplectic structure ω_0 on ${}^bT^*U$ is the product of the canonical symplectic structure ω_1 on T^*H and the canonical b-symplectic structure ω_2 on ${}^bT^*V$. Denoting the Poisson tensor corresponding to ω_i by Π_i,

$$\Pi_0 = \Pi_1 + \Pi_2.$$

The action of H on ${}^bT^*U \cong T^*H \times {}^bT^*V$ is given by the standard cotangent lift of left translations by H on T^*H times the identity on ${}^bT^*V$. For the corresponding quotient projections $\pi_0 : {}^bT^*U \to ({}^bT^*U)/H$ and $\pi_0' : T^*H \to (T^*H)/H$ we therefore have $\pi_0 = \pi_0' \times \mathrm{id}_{{}^bT^*V}$. Hence the reduced Poisson structure on $({}^bT^*U)/H$ is

$$\Pi_{\text{red}} = (\pi_0)_*\Pi_0 = (\pi_0)_*(\Pi_1 + \Pi_2) = (\pi_0')_*\Pi_1 + \Pi_2.$$

Now note that $(\pi_0')_*(\Pi_1)$ is the minus Lie Poisson structure on $\mathfrak{h}^*$ if we identify $(T^*H)/H \cong \mathfrak{h}^*$. $\square$

Example 14. *We return to Example 7 of the special Euclidean group* $\mathrm{SE}(2)$*. Since* $T(2)$ *is abelian, the Lie-Poisson structure on the dual of its Lie algebra is zero. Hence* ${}^bT^*(\mathrm{SE}(2))$ *reduces under the action of* $T(2)$ *to*

$$({}^bT^*(\mathrm{SE}(2))/T(2),\, \Pi_{red}) \cong (\mathbb{R}^2 \times {}^bT^*(\mathrm{SO}(2)),\, 0 + \Pi_{b\text{-}can}),$$

where $\Pi_{b\text{-}can}$ *is the canonical b-Poisson structure on* ${}^bT^*(\mathrm{SO}(2))$*, i.e. identifying* $\mathrm{SO}(2) \cong \mathbb{S}^1$ *in the usual way and letting* φ *be the angle,* (φ, p) *a b-canonical chart in a neighborhood of* $\{\varphi = 0\}$*, then in these coordinates*

$$\Pi_{red} = \varphi \frac{\partial}{\partial \varphi} \wedge \frac{\partial}{\partial p}.$$

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