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Moore-Gibson-Thompson theory for thermoelastic dielectrics*

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Abstract In this note we consider the system of equations determining the linear thermoelastic deformations of dielectrics within the recently called Moore-Gibson-Thompson theory. First, we obtain the system of equations for such a case. Second, we consider the case of a rigid solid and we show the existence and the exponential decay of solutions. Third, we consider the thermoelastic case and we obtain the existence and the stability of the solutions. Exponential decay of solutions in the one-dimensional case is also recalled.

Key words Moore-Gibson-Thompson thermoelastic dielectric, existence, semigroups, exponential decay.

2000 Mathematics Subject Classification 37N15, 74F05.

1 Introduction

The interaction of electromagnetic fields with thermoelastic dielectrics have investigated for a long time ago. Several works has been devoted to this theory. During the last years a major interest arose to understand the so-called Moore-Gibson-Thompson (MGT) thermoelasticity and several contributions have been proposed for this recent theory. Our work is concerned with the linear theory of thermoelastic dielectrics based on the MGT theory. That is, the equations for the heat conduction and electric field are based on the MGT theory. To this end, our initial point is the work of Ieşan and Ciarletta [1] concerning thermoviscoelastic dielectrics which is also based on the idea of the invariance of the entropy under time reversal [2].

The invariance of the infinitesimal entropy production under time reversal was studied by Borghesani and Morro [3,4], but we here start with the equations proposed by Ciarletta and Ieşan [1], also including the elastic deformations. Taking them as the initial point, we obtain the system of equations for the thermoelastic dielectrics of the Moore-Gibson-Thompson type. It is worth saying that recently a significant interest has been developed to understand the Moore-Gibson-Thompson thermoelastic theories [5–17]; however, we center our attention on materials with a center of symmetry and, therefore, the tensors of odd order are not considered[1]. In our case, we will obtain that the electric displacement

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[1]It is clear that the general case could be also obtained; however, in this note we want to emphasize the new consequences proposed by the MGT-structure in the case of dielectrics which is different from the usual one.

is present in the equations of the heat and electric field, but not of the displacement. Nevertheless, the thermo-electric coupling leads to a nice problem to be understood. It is the coupling of a hyperbolic partial differential equation with an ordinary differential equation. The contribution of this paper is double. On one side, we extend to dielectric materials the problems of Moore-Gibson-Thompson type [16] and, on the other side, we propose, from the mathematical point of view, an energy for the coupling in such a way that it defines a norm which is equivalent to the classical one in the Sobolev space $W^{1,2}$.

The plan of this note is the following. Section 2 is devoted to obtain the system of equations that we are going to work in this paper. The rigid solid case is considered in Section 3. Existence and exponential decay of solutions are obtained. The general system of MGT thermoelasticity of dielectric materials is studied in Section 4. Existence of solutions and stability are also shown.

2 Basic equations

The system of equations for the thermoviscoelastic dielectrics for materials with a center of symmetry is determined by the evolution equations (see [1]):

$$\begin{aligned}
&\rho\ddot{u}_i = t_{ij,j},\\
&T_0\dot{\eta} = q_{i,i},\\
&d_{i,i} = 0,
\end{aligned}$$

and the constitutive equations[2]:

$$\begin{aligned}
t_{ij} &= \int_{\infty}^{t}\Big[G_{ijmn}(t-s)\dot{u}_{m,n}(s) - B_{ij}(t-s)\dot{\theta}(s)\Big]ds,\\
\eta &= \int_{\infty}^{t}\Big[B_{ij}(t-s)\dot{u}_{i,j}(s) + A(t-s)\dot{\theta}(s)\Big]ds,\\
q_i &= \int_{\infty}^{t}\Big[Q_{ji}(t-s)\dot{E}_j(s) + K_{ij}(t-s)\theta_{,j}(s)\Big]ds,\\
d_i &= \int_{\infty}^{t}\Big[\gamma_{ji}(t-s)\dot{E}_j(s) + Q_{ij}(t-s)\theta_{,j}(s)\Big]ds,
\end{aligned}$$

where ρ is the mass density, (u_i) is the displacement vector, (t_{ij}) is the stress tensor, T_0 is the reference temperature that we will assume equal to one to simplify the calculations, η is the entropy, q_i is the heat flux vector, (d_i) is the electric displacement, $E_i = -\phi_{,i}$ is the electric intensity, ϕ is the electric potential, θ is the temperature shift and $G_{ijmn}(\boldsymbol{x},s), B_{ij}(\boldsymbol{x},s), A(\boldsymbol{x},s), Q_{ij}(\boldsymbol{x},s), K_{ij}(\boldsymbol{x},s)$ and $\gamma_{ij}(\boldsymbol{x},s)$ are the constitutive functions. It is known that

$$G_{ijmn} = G_{mnij},\ \ K_{ij} = K_{ji},\ \ \gamma_{ij} = \gamma_{ji}.$$

We consider the following constitutive functions

$$\begin{aligned}
&G_{ijmn}(\boldsymbol{x},s) = G^*_{ijmn}(\boldsymbol{x}),\ \ B_{ij}(\boldsymbol{x},s) = B^*_{ij}(\boldsymbol{x}),\ \ A(\boldsymbol{x},s) = A^*(\boldsymbol{x}),\\
&K_{ij}(\boldsymbol{x},s) = K^*_{ij}(\boldsymbol{x}) + (\tau^{-1}K_{ij}(\mathbf{x}) - K^*_{ij}(\boldsymbol{x}))\exp(-\tau^{-1}s),\\
&\gamma_{ij}(\boldsymbol{x},s) = \gamma^*_{ij}(\boldsymbol{x}) + (\tau^{-1}\gamma_{ij}(\mathbf{x}) - \gamma^*_{ij}(\boldsymbol{x}))\exp(-\tau^{-1}s),\\
&Q_{ij}(\boldsymbol{x},s) = Q^*_{ij}(\boldsymbol{x}) + (\tau^{-1}Q_{ij}(\boldsymbol{x}) - Q^*_{ij}(\boldsymbol{x}))\exp(-\tau^{-1}s),
\end{aligned}$$

where τ is a positive and constant parameter.

We remark that G^*_{ijmn} is usually called the elasticity tensor, B^*_{ij} is related to the thermo-mechanical expansion, A^* is the thermal capacity, K_{ij} is the thermal conductivity, K^*_{ij} is usually called rate

[2]We recall that in general the tensor multiplying the history of the electric displacement in the heat flux vector and the tensor multiplying the history of the gradient of temperature in the last constitutive equation are equal except for a constant tensor. However, as a first approximation to this problem we assume that they agree.

conductivity, γ_{ij} and γ^*_{ij} are related to the electric permittivity, Q^*_{ij} and Q_{ij} determine the thermo-electric coupling and τ is a relaxation parameter.

From the previous assumptions we see that

$$\dot{\eta}+\tau\ddot{\eta}=B^*_{ij}(\dot{u}_{i,j}+\tau\ddot{u}_{i,j})+A^*(\dot{\theta}+\tau\ddot{\theta}),$$
$$q_i+\tau\dot{q}_i=Q^*_{ji}E_j+Q_{ji}\dot{E}_j+K^*_{ij}\alpha_{,j}+K_{ij}\theta_{,j}.$$

In a similar way, we also find that

$$d_i+\tau\dot{d}_i=\gamma^*_{ji}E_j+\gamma_{ji}\dot{E}_j+Q^*_{ij}\alpha_{,j}+Q_{ij}\theta_{,i}.$$

If we substitute these expressions into the evolution equations we obtain the following system of field equations:

$$\begin{aligned}
&\rho\ddot{u}_i=\Big(G^*_{ijmn}u_{m,n}-B^*_{ij}\theta\Big)_{,j},\\
&A^*(\dot{\theta}+\tau\ddot{\theta})=-B^*_{ij}(\dot{u}_{i,j}+\tau\ddot{u}_{i,j})+\Big(Q^*_{ji}E_j+Q_{ji}\dot{E}_j+K^*_{ij}\alpha_{,j}+K_{ij}\theta_{,j}\Big)_{,i},\\
&\Big(\gamma^*_{ji}E_j+\gamma_{ji}\dot{E}_j+Q^*_{ij}\alpha_{,j}+Q_{ij}\theta_{,j}\Big)_{,i}=0,
\end{aligned}$$

where

$$\alpha(\boldsymbol{x},t)=\alpha_0(\boldsymbol{x})+\int_0^t\theta(\boldsymbol{x},s)ds$$

is the thermal displacement.

In the case that we assume that the electric potential vanishes on the boundary, the system is written as follows:

$$\begin{aligned}
&\rho\ddot{u}_i=\Big(G^*_{ijmn}u_{m,n}-B^*_{ij}(\theta+\tau\dot{\theta})\Big)_{,j},\\
&A^*(\dot{\theta}+\tau\ddot{\theta})=-B^*_{ij}\dot{u}_{i,j}+\Big(K^*_{ij}\alpha_{,j}+K_{ij}\theta_{,j}-Q^*_{ji}\phi_{,j}-Q_{ji}\dot{\phi}_{,j}\Big)_{,i},\\
&\dot{\phi}=\Phi^{-1}\Big[\Big(Q^*_{ij}\alpha_{,j}+Q_{ij}\theta_{,j}\Big)_{,i}-\gamma^*_{ji}\phi_j\Big)_{,i}\Big],
\end{aligned}$$

where Φ is the isomorphism between $W_0^{1,2}\cap W^{2,2}$ and L^2 determined by $\Phi(f)=(\gamma_{ij}f_{,j})_{,i}$.[3]

To work in this general case is a little bit cumbersome. Therefore, in order to make the analysis clearest and transparent, we focus our attention on the isotropic and homogeneous case, but we want to emphasize that the analysis could be done in a similar way. In this situation, our system of equations can be written as:

$$\rho\ddot{u}_i=\mu^*u_{i,jj}+(\lambda^*+\mu^*)u_{j,ji}-\beta^*(\theta_{,i}+\tau\dot{\theta}_{,i}), \quad (1)$$
$$A^*(\dot{\theta}+\tau\ddot{\theta})=-\beta^*\dot{u}_{i,i}+k^*\Delta\alpha+k\Delta\theta-Q^*\Delta\phi-Q\Delta F(\alpha,\theta,\phi), \quad (2)$$
$$\dot{\phi}=F(\alpha,\theta,\phi), \quad (3)$$

where

$$F(\alpha,\theta,\phi)=\gamma^{-1}(Q^*\alpha+Q\theta-\gamma^*\phi).$$

It is worth noting that the energy equation in this case is

$$E(t)+\int_0^t D(s)ds=E(0),$$

where

$$\begin{aligned}
E(t)=&\frac{1}{2}\int_B\big(\rho\dot{u}_i\dot{u}_i+\mu^*u_{i,j}u_{i,j}+(\lambda^*+\mu^*)u_{i,i}u_{j,j}\big)dv\\
&+\frac{1}{2}\int_B\big(A^*(\theta+\tau\dot{\theta})^2+k^*|\nabla(\alpha+\tau\theta)|^2+\tau\bar{k}|\nabla\theta|^2+\gamma^*|\nabla(\phi+\tau F)|^2+\tau\bar{\gamma}|\nabla F|^2\big)dv\\
&-\int_B\big(Q^*\nabla(\alpha+\tau\theta)\nabla(\phi+\tau F)+\tau\bar{Q}\nabla\theta\nabla F\big)dv,
\end{aligned}$$

[3]The existence of this isomorphism is guaranteed whenever γ_{ij} is positive definite and assuming suitable boundary conditions.

and
$$D(t)=\int_B\left(\bar{k}|\nabla\theta|^2+\bar{\gamma}|\nabla F|^2-2\bar{Q}\nabla\theta\nabla F\right)dv.$$
Here, we have used the notation
$$\bar{k}=k-\tau k^*>0,\quad \bar{\gamma}=\gamma-\tau\gamma^*>0,\quad \bar{Q}=Q-\tau Q^*.$$
From now on, we will assume that
$$\rho>0,\quad \mu^*>0,\quad \lambda^*+\mu^*>0,\quad k^*>0,\quad \gamma^*>0,\quad \bar{k}>0,\quad \bar{\gamma}>0,$$
$$k^*\gamma^*>(Q^*)^2,\quad \bar{k}\bar{\gamma}>(\bar{Q})^2.$$

3 Rigid Solid

In this section we study the problem determined on a rigid solid. Our system of equations is
$$A^*(\dot{\theta}+\tau\ddot{\theta})=(k^*-\gamma^{-1}QQ^*)\Delta\alpha+(k-\gamma^{-1}Q^2)\Delta\theta-(Q^*-\gamma^{-1}Q\gamma)\Delta\phi,$$
$$\dot{\phi}=\gamma^{-1}(Q^*\alpha+Q\theta-\gamma^*\phi).$$
We assume that
$$\alpha(\boldsymbol{x},t)=\phi(\boldsymbol{x},t)=0,\quad \boldsymbol{x}\in\partial B,\, t>0, \tag{4}$$
$$\alpha(\boldsymbol{x},0)=\alpha_0(\boldsymbol{x}),\quad \theta(\boldsymbol{x},0)=\theta_0(\boldsymbol{x}),\ \boldsymbol{x}\in B, \tag{5}$$
$$\dot{\theta}(\boldsymbol{x},0)=\xi_0(\boldsymbol{x}),\quad \phi(\boldsymbol{x},0)=\phi_0(\boldsymbol{x}),\ \boldsymbol{x}\in B. \tag{6}$$

We consider our problem on a suitable Hilbert space
$$\mathcal{H}=W_0^{1,2}(B)\times W_0^{1,2}(B)\times L^2(B)\times W_0^{1,2}(B)$$
and, for every $(\alpha,\theta,\xi,\phi),(\alpha^*,\theta^*,\xi^*,\phi^*)\in\mathcal{H}$, we define the inner product
$$\begin{aligned}\langle(\alpha,\theta,\xi,\phi),(\alpha^*,\theta^*,\xi^*,\phi^*)\rangle=\frac{1}{2}\int_B&\left(A^*(\theta+\tau\dot{\theta})\overline{(\theta^*+\tau\dot{\theta}^*)}+k^*\nabla(\alpha+\tau\theta)\nabla\overline{(\alpha^*+\tau\theta^*)}\right.\\&+\tau\bar{k}\nabla\theta\nabla\overline{\theta^*}+\gamma^*\nabla(\phi+\tau G)\nabla\overline{(\phi^*+\tau G^*)}+\tau\bar{\gamma}\nabla G\nabla\overline{G^*}\\&\left.-Q^*[\nabla(\alpha+\tau\theta)\nabla\overline{(\phi^*+\tau G^*)}+\nabla\overline{(\alpha^*+\tau\theta^*)}\nabla(\phi^*+\tau G^*)]-\tau\bar{Q}[\nabla\theta\nabla\overline{G^*}+\nabla\overline{\theta^*}\nabla G]\right)dv,\end{aligned}$$
where the overline over the elements of the Hilbert space means the conjugated complex and
$$G(\alpha,\theta,\phi)=\gamma^{-1}(Q^*\alpha+Q\theta-\gamma^*\phi).$$
We note that, under the assumptions proposed at the end of the previous section, the norm induced by the above inner product is equivalent to the classical one defined in the Hilbert space $\mathcal{H}$. We can write our problem as
$$\frac{dU}{dt}=\mathcal{A}U,\quad U(0)=U_0, \tag{7}$$
where $U=(\alpha,\theta,\xi,\phi)$, $U_0=(\alpha_0,\theta_0,\xi_0,\phi_0)$ and the matrix operator is
$$\mathcal{A}=\begin{pmatrix}0 & I & 0 & 0\\ 0 & 0 & I & 0\\ (\tau A^*)^{-1}(k^*-\gamma^{-1}QQ^*)\Delta & (\tau A^*)^{-1}(k-\gamma^{-1}Q^2)\Delta & -\tau^{-1} & (\tau A^*)^{-1}(\gamma^{-1}Q\gamma^*-Q^*)\Delta\\ \gamma^{-1}Q^* & \gamma^{-1}Q & 0 & -\gamma^{-1}\gamma^*\end{pmatrix}.$$
We note that the domain of the operator is
$$\{(\alpha,\theta,\xi,\phi)\in\mathcal{H},\xi\in W_0^{1,2},(k^*-\gamma^{-1}QQ^*)\Delta\alpha+(k-\gamma^{-1}Q^2)\Delta\theta-(Q^*-\gamma^{-1}Q\gamma^*)\Delta\phi\in L^2\}.$$
Obviously, this is a dense subspace. On the other hand, for every $U=(\alpha,\theta,\xi,\phi)$ in the domain of the operator we have
$$\mathrm{Re}\langle\mathcal{A}U,U\rangle=-\frac{1}{2}\int_B\left(\bar{k}|\nabla\theta|^2+\bar{\gamma}|\nabla G|^2-\bar{Q}(\nabla\theta\nabla\overline{G}+\nabla\overline{\theta}\nabla G)\right)dv. \tag{8}$$

In view of the assumptions we see that this is equal to or less than zero.

Our next step is to prove that zero belongs to the resolvent of the operator. To this end, let us consider $L = (l_1, l_2, l_3, l_4) \in \mathcal{H}$. We will prove that there exists $U = (\alpha, \theta, \xi, \phi)$ in the domain of the operator such that $\mathcal{A}U = L$. Writing this equation in coordinates we see that

$$\theta = l_1, \ \ \xi = l_2, \ \ Q^*\alpha + Q\theta - \gamma^*\phi = \gamma l_4,$$

and

$$(k^* - \gamma^{-1}QQ^*)\Delta\alpha + (k - \gamma^{-1}Q^2)\Delta\theta - A^*\xi - (Q^* - \gamma^{-1}Q\gamma^*)\Delta\phi = \tau A^* l_3.$$

We obtain the expression for θ and ξ. We also have that

$$\phi = (\gamma^*)^{-1}(Q^*\alpha + Ql_1 - \gamma l_4).$$

It then follows that we obtain an equation for the variable α which can be easily solved because $k^*\gamma^* > (Q^*)^2$. Moreover, we can obtain the regularity conditions and the following result is found.

Theorem 3.1. *The operator $\mathcal{A}$ produces a contractive semigroup.*

We note that, using the above result, we conclude the existence, uniqueness and continuous dependence of the solutions to our problem.

In the rest of the note, we will prove the exponential decay of the energy under some additional conditions. In order to show it, we recall the following characterization (see the book of Liu and Zheng [18]).

Theorem 3.2. *Let $S(t) = \{e^{\mathcal{A}t}\}_{t\geqslant 0}$ be a C_0-semigroup of contractions defined in a Hilbert space. Therefore, $S(t)$ is exponentially stable if and only the imaginary axis is contained in the resolvent of $\mathcal{A}$ and*

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{9}$$

Now, we follow the arguments already used in the book of Liu and Zheng ([18], page 25). First, we assume the imaginary axis and the spectrum have a non-empty intersection. We conclude that there exists a sequence of real numbers (of course converging to a real number) λ_n with $\lambda_n \to \varpi$ and $|\lambda_n| < |\varpi|$, and a sequence of corresponding vectors $U_n = (\alpha_n, \theta_n, \xi_n, \phi_n)$, in the domain of $\mathcal{A}$ and with unit norm, such that

$$\|(i\lambda_n\mathcal{I} - \mathcal{A})U_n\| \to 0.$$

It then follows that

$$i\lambda_n\alpha_n - \theta_n \to 0 \text{ in } W^1, \tag{10}$$

$$i\lambda_n\theta_n - \xi_n \to 0 \text{ in } W^1, \tag{11}$$

$$\begin{aligned} &i\tau A^*\lambda_n\xi_n - (k^* - \gamma^{-1}QQ^*)\Delta\alpha_n - (k - \gamma^{-1}Q^2)\Delta + A^*\xi_n \\ &\qquad -(\tau A^*)^{-1}(\gamma^{-1}Q\gamma^* - Q^*)\Delta\phi_n \to 0 \text{ in } L^2, \end{aligned} \tag{12}$$

$$i\gamma\lambda_n\phi_n - Q^*\alpha_n - Q\theta_n + \gamma^*\phi_n \to 0 \text{ in } W^1. \tag{13}$$

In view of the dissipation we see that $\theta_n, \phi_n \to 0$ in W^1. Therefore, we also have that $\alpha_n \to 0$ in W^1. If we now consider convergence (12) multiplied by $\lambda_n^{-1}\xi_n$, after the use of the integration by parts, we obtain that $\xi_n \to 0$ in L^2. This contradicts the condition that the elements of the sequence have unit norm. Therefore, we can conclude that $i\mathbb{R} \subset \rho(\mathcal{A})$.

Now, we want to prove that the asymptotic condition (9) also holds. In the case that this condition does not hold, there exist a sequence of real numbers λ_n with $|\lambda_n| \to \infty$ and another sequence of unit norm vectors $U_n = (\alpha_n, \theta_n, \xi_n, \phi_n)$ in $\mathcal{D}(\mathcal{A})$ in such a way that (10)- (13) hold. Therefore, we can proceed in an analogous way as we shown that the imaginary axis was contained in the resolvent of the operator, because the key point was to note that the sequence λ_n does not tend to zero. Thus, it leads to a contradiction and so, condition (9) is also true.

We have proved the following.

Theorem 3.3. *Let us assume that the previous conditions hold. Then, operator $\mathcal{A}$ produces a semigroup exponentially stable; that is, we can find two positive constants M, ω such that*

$$||U(t)|| \leqslant M\exp(-\omega t)||U(0)||$$

for every $U(0) \in \mathcal{D}(\mathcal{A})$.

4 Thermoelastic case

In this section, we prove the existence of solutions to the problem determined by the general system (1)-(3). Apart from the initial and boundary conditions (4)-(5), we also impose in this section that

$$u_i(\boldsymbol{x},0)=u_{i0}(\boldsymbol{x}),\ \ \dot{u}_i(\mathbf{x},0)=v_{i0}(\boldsymbol{x}), \boldsymbol{x}\in B, \tag{14}$$

and

$$u_i(\boldsymbol{x},t)=0,\quad \boldsymbol{x}\in\partial B,\, t>0. \tag{15}$$

In what follows, we will show an existence theorem for the solutions to the problem determined by system (1)-(3) with conditions (4)-(5) and (14)-(15). The existence will be shown in a suitable Hilbert space. In this section, we will work with the space:

$$\mathcal{H}=\mathbf{W}_0^{1,2}(B)\times\mathbf{L}^2(B)\times W_0^{1,2}(B)\times W_0^{1,2}(B)\times L^2(B)\times W_0^{1,2}(B),$$

and, for every $(\mathbf{u},\mathbf{v},\alpha,\theta,\xi,\phi),(\mathbf{u}^*,\mathbf{v}^*,\alpha^*,\theta^*,\xi^*,\phi^*)\in\mathcal{H}$, we define the inner product

$$\begin{aligned}\langle(\mathbf{u},\mathbf{v},\alpha,\theta,\xi,\phi),(\mathbf{u}^*,\mathbf{v}^*,\alpha^*,\theta^*,\xi^*,\phi^*)\rangle &= \frac{1}{2}\int_B\big(\rho v_i\overline{v_i^*}+\mu^*u_{i,j}\overline{u}^*_{i,j}+(\lambda^*+\mu^*)u_{i,i}\overline{u}^*_{j,j}\\
&+A^*(\theta+\tau\dot{\theta})\overline{(\theta^*+\tau\dot{\theta}^*)}+k^*\nabla(\alpha+\tau\theta)\nabla\overline{(\alpha^*+\tau\theta^*)}\\
&+\tau\bar{k}\nabla\theta\nabla\overline{\theta^*}+\gamma^*\nabla(\phi+\tau G)\nabla\overline{(\phi^*+\tau G^*)}+\tau\bar{\gamma}\nabla G\nabla\overline{G^*}\\
&-Q^*[\nabla(\alpha+\tau\theta)\nabla\overline{(\phi^*+\tau G^*)}+\nabla\overline{(\alpha^*+\tau\theta^*)}\nabla(\phi^*+\tau G^*)]-\tau\bar{Q}[\nabla\theta\nabla\overline{G^*}+\nabla\overline{\theta^*}\nabla G]\big)dv.\end{aligned}$$

Again, our problem can be written in the form of system (7), where $U=(\mathbf{u},\mathbf{v},\alpha,\theta,\xi,\phi)$ and $U_0=(\mathbf{u}_0,\mathbf{v}_0,\alpha_0,\theta_0,\xi_0,\phi_0)$, whenever we define the operator

$$\mathcal{A}\begin{pmatrix}u_i\\ v_i\\ \alpha\\ \theta\\ \xi\\ \phi\end{pmatrix}=\begin{pmatrix}v_i\\ \rho^{-1}(\mu^*u_{i,jj}+(\lambda^*+\mu^*)u_{j,ji}-\beta^*(\theta_{,i}+\tau\xi_{,i}))\\ \theta\\ \xi\\ (A^*)^{-1}(-\beta^*v_{i,i}+\tau^{-1}(M_1\Delta\alpha+M_2\Delta\theta+M_3\Delta\phi))-\tau^{-1}\xi\\ \gamma^{-1}(Q^*\alpha+Q\beta-\gamma^*\phi)\end{pmatrix},$$

where

$$M_1=k^*-\gamma^{-1}QQ^*,\ \ M_2=k-\gamma^{-1}Q^2,\ \ M_3=-(Q^*-\gamma^{-1}Q\gamma^*).$$

The domain of the operator is given by the elements in the Hilbert space $\mathcal{H}$ such that

$$\mathbf{u}\in\mathbf{W}^{2,2},\ \mathbf{v}\in\mathbf{W}_0^{1,2},\ \xi\in W_0^{1,2},\ M_1\Delta\alpha+M_2\Delta\theta+M_3\Delta\phi\in L^2.$$

Therefore, it is a dense subspace. We have that relation (8) also holds in this case. That is, we find that

$$\mathrm{Re}\langle\mathcal{A}U,U\rangle=-\frac{1}{2}\int_B\big(\bar{k}|\nabla\theta|^2+\bar{\gamma}|\nabla G|^2-\bar{Q}(\nabla\theta\nabla\overline{G}+\nabla\overline{\theta}\nabla G)\big)dv.$$

Thus, to prove the existence of a semigroup of linear operators it is sufficient to show that zero belongs to the resolvent of the operator. We consider $L=(\mathbf{n}_1,\mathbf{n_2},l_1,l_2,l_3,l_4)$ in the Hilbert space, and we need to show the existence of an element in the domain of the operator such that $\mathcal{A}U=L$. It leads to the following system:

$$\begin{aligned}&\mathbf{v}=\mathbf{n}_1,\ \ \theta=l_1,\ \ \xi=l_2,\ \ Q^*\alpha+Q\theta-\gamma\phi=\gamma l_4,\\
&-\beta^*v_{i,i}+M_1\Delta\alpha+M_2\Delta\theta+M_3\Delta\phi-A^*\xi=\tau A^*l_3,\\
&\mu^*u_{i,jj}+(\lambda^*+\mu^*)u_{j,ji}-\beta^*(\theta_{,i}+\tau\xi_{,i})=\rho n_{2i}.\end{aligned}$$

As in the case of the rigid solid, we can also obtain ϕ .

We can find the expressions of $\mathbf{v},\theta$ and ξ, and our system reduces to

$$\left(k^*-\frac{(Q^*)^2}{\gamma^*}\right)\Delta\alpha=F_1,\ \ \mu^*u_{i,jj}+(\lambda^*+\mu^*)u_{j,ji}=F_{2i}$$

This system admits a solution in the domain of the operator and we obtain the following.

Theorem 4.1. *The operator $\mathcal{A}$ generates a contractive semigroup.*

We may conclude the stability of solutions as well as the well-posedness in the three-dimensional case.

The exponential decay of solutions in the general case cannot be expected. We should find that the behavior is similar to the usual one for the MGT-thermoelasticity; however, it is obvious that the combination of the arguments proposed in this section, with those used in the previous one, would allow us to prove, in the one-dimensional setting, the exponential decay of solutions. Anyway, we do not give the details in order to shorten the length of the paper.

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