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# A model for the simultaneous selection of bus lines and frequency setting problems in the expansion of public transit systems

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## Abstract

When expanding an existing public transport system a common practice consists of defining a set of candidate new lines and from them, choose those that will revert into a greater increase of the system's performance while keeping at a moderate level the associated amount of economical costs. In this work, a mathematical programming model for the selection of candidate lines in extensions of the public transport system is proposed based on the linearization of the model presented by [Codina et al. \(2013\)](#) and its application to situations of medium congestion. The model assumes that the pool of candidate lines is an input externally determined by technical criteria and it is focused in the compatibility constraints that the final set of selected lines must verify. These constraints include the overall throughput capacity of the stops/stations of the model, the availability of space for users to wait at stations, the resulting line capacities for passenger flows at line segments and the waiting times of passengers at stops. Solution procedures are based on the Cutting Plane's algorithm and several small to medium size instances are successfully solved

*Keywords:* Frequency setting; Bus lines selection; Mathematical programming.

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## 1. Introduction

Public transport is a basic component in the social, economic and physical structure of an urban area. In addition, it is considered an important component in the sustainable urban development, since it should allow efficient citizen movements throughout the city. Planning, operating and controlling a public transportation system can be challenging. Several actors with different objectives are involved, mainly users and operators, where the travel needs of users vary significantly in space and time, according to their socio-economic characteristics. In addition, urban congestion and limited vehicular capacity are determinant factors that must be taken into account when planning expansions of these systems in terms of new operating lines and the corresponding (re)assignment of frequencies and units to these lines.

The main characteristics of the models for solving the Line Frequency Setting Problem (LFSP) on a Transit Network are related to the fact that the decision variables correspond to the routes and frequencies; the objective function

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usually consists of a weighting of the economic interests of passengers in terms of some utility measure (minimization of travel times) and the economical interests of operators (such as investment costs and operating costs) and the most common restrictions are those that limit the frequencies, the trip time of the new routes and some measures related to the system's congestion, such as waiting time at stops.

The LFSP on a Transit Network was already stated in the work of [Ceder and Wilson \(1986\)](#). The problem was divided into two consecutive stages. The first considers the users' interests to minimize excessive travel time, expressed as the sum of the excess travel time plus the transshipment time for all origin-destination pairs. The second stage presents both the interests of passengers and operators. The model of [Baaj and Mahmassani \(1991\)](#) uses the same constraints than the second phase of the model of [Ceder and Wilson \(1986\)](#) and adds constraints establishing a maximum occupation value of the buses according to the capacity of seated passengers.

In the model by [Israeli and Ceder \(1995\)](#) the level of occupancy of buses is included, which is of interest both for users (if the bus capacity is exceeded) and for operators (if the bus travels with many empty seats for a long time, the profitability of the route decreases). Factors such as travel time, waiting time and bus idleness are weighted and also included in the objective function. [Tom and Mohan \(2003\)](#) solve a variant of the model by [Baaj and Mahmassani \(1990\)](#). In [Borndörfer et al. \(2007\)](#) a simplified model for a global line planning and frequency setting problem is stated and solved by column generation type algorithms.

A relevant component of these models consists of the inherent modelling of the assignment of passengers to lines. Many of them simply assume that passengers follow a pathfinder type of passenger assignment model for boarding at operating lines. Instead, pure frequency setting models (i.e., no line selection or layout is involved), such as [Constantin and Florian \(1995\)](#) and in [Noriega and Florian \(2003\)](#) have been stated as a bi-level programming problem in which the second level is made up by the uncongested passengers-to-lines assignment model stated in [Spiess and Florian \(1989\)](#). Also, in [Mauttone \(2005\)](#) this passenger transit assignment is followed as a description of the passenger's line choice. [Codina et al. \(2013\)](#) propose a heuristic method for the line selection-frequency setting combined problem in transit networks under high congestion, which includes in the constraints design aspects such as maximum throughput of buses at stops and a piecewise linear approximation for the waiting times of a bulk-arrival queuing model for the arrivals of passengers at stops. Using more sophisticated and advanced transit assignment models, such as, for instance, those in [Nuzzolo and Comi \(2018\)](#) and [Di Gangi et al. \(2019\)](#), has not been considered, since this would result into a much more complex formulation of the LFSP problem. Also, objective function formulations may be single objective or multiobjective, where the interests of users and operators are taken into account by means of weighting coefficients. Computationally, combined line layout-frequency setting problems have been solved generally using heuristics such as genetic algorithms ([Pattnaik et al. \(1998\)](#)). In the case of exact algorithmic approaches, such as in [Borndörfer et al. \(2007\)](#) only small to moderate size instances have been solved.

This paper presents a simplified formulation of the expansion of public transportation systems by performing a simultaneous selection of lines from a pool of candidates and a solution of the frequency setting problem. It is assumed that no previous lines operate on the network. Considering previous lines in operation on the network could be dealt without difficulties in by a larger model. The proposed model is based on a mixed linear integer programming formulation for which three different (exact) solution methods are proposed based on the Dantzig's cutting plane algorithm. The model is addressed to situations of moderate-to-high congestion levels and an intrinsic pathfinder model of passenger assignment to lines is assumed. A distinctive characteristic of the model presented is that it takes into account capacity constraints at bus stops for bus flows of incoming lines and the queueing time of passengers. Finally, this work aims to validate the proposed methods of resolution through its application to realistic case studies, as well as to compare the computational results obtained among the different methods. To achieve this objective, two auxiliary line networks are used for cases of interruption of a railway corridor in Madrid and the line 1 of the Barcelona metro (from Plaça d'Espanya station to Clot station).

The paper is structured as follows. Section 2 introduces the basic notation and elements in the model. The mathematical programming problem is presented in Section 3. In Section 4 the solution methods based on the cutting plane algorithm are described. Section 5 contains the description of the application of this model to two test networks. Finally, a set of conclusions closes the paper.

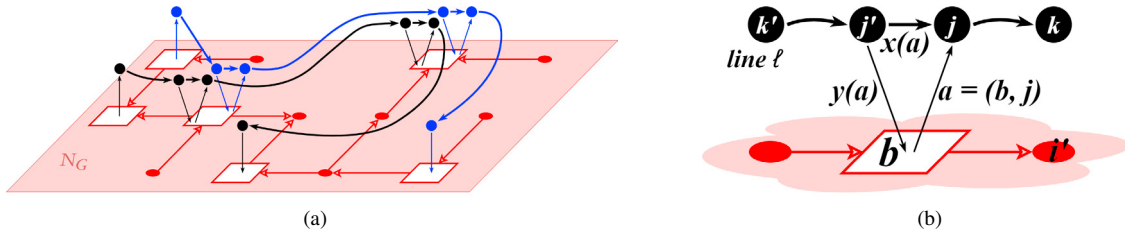


Fig. 1: (a) Network representation by means of the graph  $G = (N, A)$ ; (b) Detail of the expanded graph for modeling bus lines and stops.

## 2. Definitions and notation

Passenger flows go through a directed graph  $G = (N, A)$ , whose structure is sketched in Fig. 1. The set of nodes  $N$  splits into two subsets:  $N_G$  which are the nodes that can be considered on the ground plane and on the other hand,  $N \setminus N_G$  which is the subset of the remaining ones making up the bus lines. Nodes in the set  $N_G$  and links  $a = (i, j)$ , so that  $i \in N_G$  and  $j \in N_G$ , are used to model transfer movements and possibly movements on an alternative mode. A subset of nodes within  $N_G$ , those that will be considered as representing bus stops, will be denoted by  $\hat{N}_G$ .  $L$  designates the set of bus lines and  $\Pi_\ell = \{b_1, \dots, b_{\mu^\ell}\}$  is the ordered set of  $\mu^\ell$  bus stops  $b_j \in \hat{N}_G$  on line  $\ell$ .

Figure 1(b) shows in greater detail the set of links modeling alighting and boarding movements at bus stops for a given line  $\ell \in L$ . Links  $a = (b, j)$ ,  $b \in \Pi_\ell$ ,  $j \notin N_G$  capture boarding and waiting time at a bus stop for passengers willing to board on line  $\ell$ . The boarding links from stop  $b \in \hat{N}_G$  to line  $\ell$  will be designated by  $a(\ell, b)$ . For a boarding link  $a$ , link  $x(a)$  denotes the link on which passengers wait on board the bus without alighting at that bus stop, whereas  $y(a)$  represents the corresponding alighting link. Links  $(k', j')$  and  $(j, k)$  represent the in-vehicle time.

Table 1: Notation

Parameters	
$\rho$	Bus units available
$c^\ell$	Passenger capacity on line $\ell \in L$
$H$	Time period
$C_\ell$	Bus cycles on line $\ell \in L$
$\hat{Z}_b$	Maximum number of services at bus stop $b \in \hat{N}_G$
$h_{max}$	Maximum headway
$[f, \hat{f}]$	Range of allowed frequencies
$\eta_b$	Ratio between the passenger's queue length exceeded only a fraction $1 - \alpha$ of the time and the average queue length for bus stop $b \in \hat{N}_G$
$\hat{N}_b^{pax}$	Maximum number of passengers that the bus stop $b \in \hat{N}_G$ is physically able to accommodate
$T_a$	Travel time on link $a \in A$
$P_a$	Waiting time at the bus stop per service and passenger on link $a = a(\ell, b)$ , $\ell \in L$ , $b \in \hat{N}_G$
$\varsigma^\ell$	Cost of renting a bus for a line $\ell \in L$
$\gamma^\ell$	Operating costs of a service on line $\ell \in L$
$\hat{\beta}_{a,k}, \tilde{\gamma}_{a,k}$	Coefficients for a bulk service queue M/M/1
$n_\varphi$	Number of points at which the equation of the passengers queue delay is discretized
Variables	
$n^\ell$	Number of bus units assigned to line $\ell \in L$
$z^\ell$	Number of services assigned to line $\ell \in L$
$\lambda^\ell$	If $\lambda^\ell = 1$ , a positive number of services will be assigned to line $\ell \in L$
$v_a$	Total flow of passengers on a link $a \in A$
$\zeta_a$	Total passenger waiting time for boarding at a line, $a = a(\ell, b)$ , $\ell \in L$ , $b \in \hat{N}_G$

## 3. Mathematical programming problem

The following model, M0, is based in a mixed linear integer programming problem, which corresponds to a simplification of the model presented by Codina et al. (2013), by fixing the parameters  $C_\ell$ ,  $T_a$ ,  $P_a$  and  $\hat{Z}_b$  (which have an effect in cases of high congestion) to constant and suitable values.

This model requires the following basic input: (a) a set of candidate bus lines, in a urban environment, identified by their segments and bus stops; (b) an origin-destination trip table. The model provides, as basic results, the number of bus units to be assigned to a bus line along with the total number of services to be carried out by these units and the routes to be followed by passengers of each origin-destination pair.

The objective function of the design model will consist of economical costs for fleet recruiting and service assignment plus the economical cost of the passengers total travel time using an estimation  $\theta$  of the social cost of time (for instance, in €/min).

$$\begin{aligned}
 \text{[M0]} \quad & \min_{n,z,\lambda,v,\zeta} \sum_{\ell \in L} (\varsigma^\ell n^\ell + \gamma^\ell z^\ell) + \theta \sum_{a \in A} v_a T_a + \theta \sum_{\ell \in L} \sum_{b \in \Pi_\ell} \zeta_{a(\ell,b)} \\
 \text{s.t.: A1} \quad & \sum_{\ell \in L} n^\ell \leq p \\
 \text{A2} \quad & n^\ell H^\ell \geq z^\ell C_\ell, \quad \ell \in L \\
 \text{A3} \quad & z^\ell \leq \lambda^\ell H \hat{f}, \quad \ell \in L \\
 \text{A4} \quad & z^\ell \geq \lambda^\ell \frac{H}{h_{max}}, \quad \ell \in L \\
 \text{B1} \quad & v \in \mathcal{V} \\
 \text{Qb1} \quad & \sum_{\ell \in L_b} z^\ell \leq \hat{Z}_b, \quad b \in \hat{N}_G \\
 \text{Qp1} \quad & v_a + v_{x(a)} \leq c^\ell z^\ell \\
 \text{Qp2} \quad & \sum_{\ell \in L} \zeta_{a(\ell,b)} \leq \frac{H}{\eta_b} \hat{N}_b^{\text{pax}} \\
 \text{Qp3} \quad & \tilde{\beta}_{a,k} v_{x(a)} + \tilde{\gamma}_{a,k} v_a \leq \frac{\zeta_a}{P_a} + c^\ell \tilde{\beta}_{a,k} z^\ell, \quad 0 \leq k \leq n_\varphi - 1 \\
 & a = a(\ell, b), b \in \Pi_\ell, \ell \in L \text{ in Qp1-Qp3} \\
 & n \in \mathbb{Z}^+, z \in \mathbb{Z}^+, \lambda \in \{0, 1\}, v \geq 0, \zeta \geq 0
 \end{aligned}$$

The constraints in previous model M0 will be explained briefly here. Constraints A1 and A2 relate the availability of bus units, their assignment to bus lines and the number of services that the fleet assigned to a line must perform on that line. The purpose of constraints A3 and A4 is that, if one or more bus units are assigned to a line, then frequencies on that line must be within the interval  $[H/h_{max}, \hat{f}]$ . The feasibility of passenger flows according to demands on the expanded graph is expressed by constraint B1. Constraint Qb1 is a limitation on the total input flow that a bus stop is able to admit. Constraint Qp1 expresses the limitation imposed by the overall capacity  $c z^\ell$  and passenger flow on that line at a station. Constraint Qp2 has the purpose of limiting the queue length of passenger waiting at a bus stop according to the space available there. Finally, constraint Qp3 is an approximation that allows modeling passengers' waiting times at a bus stop.

## 4. Solving the mathematical problem using Lagrangian relaxation

### 4.1. Definition of the dual lagrangian function

The idea of a Lagrangian Relaxation is to relax some of the constraints, considered “difficult”, and “move” them (or dualize them) into the objective function. In this case, the capacity constraints Qp1 and the passengers' waiting time at stops constraints Qp3 will be dualized, since both depend on the passenger flow (continuous variable) and the number of services (integer variable). The Lagrange multipliers for Qp1 and Qp3 will be  $\hat{\mu}_a^\ell \geq 0$  and  $\tilde{\mu}_{a,k}^\ell \geq 0$ ,  $a = a(\ell, b)$ ,  $b \in \Pi_\ell$ ,  $\ell \in L$ ,  $0 \leq k \leq n_\varphi - 1$ , grouped into vectors  $\hat{\mu}$  and  $\tilde{\mu}$ , respectively. For simplicity of notation, the variables of problem M0 can be grouped in turn in a vector  $x = (n, z, \lambda, v, \zeta)$ . Then, the corresponding lagrangian  $\mathcal{L}(x, \hat{\mu}, \tilde{\mu})$  appears in the problem M0<sub>LR</sub> below, defining the corresponding dual lagrangian function  $w(\hat{\mu}, \tilde{\mu})$ .

$$\begin{aligned}
 \text{[M0}_{\text{LR}}] \quad w(\hat{\mu}, \tilde{\mu}) = \min_{x=(n,z,\lambda,v,\zeta)} \mathcal{L}(x, \hat{\mu}, \tilde{\mu}) &= \sum_{\ell \in L} (s^\ell n^\ell + \gamma^\ell z^\ell) + \theta \sum_{a \in A} v_a T_a + \theta \sum_{\ell \in L} \sum_{b \in \Pi_\ell} \zeta_{a(\ell,b)} \\
 &- \sum_{\ell \in L} \sum_{\substack{a=a(\ell,b) \\ b \in \Pi_\ell}} \hat{\mu}_a^\ell (c^\ell z^\ell - v_a - v_{x(a)}) \\
 &- \sum_{\ell \in L} \sum_{\substack{a=a(\ell,b) \\ b \in \Pi_\ell}} \sum_{0 \leq k \leq n_\ell - 1} \tilde{\mu}_{a,k}^\ell \left( \frac{\zeta_a}{P_a} + c^\ell \tilde{\beta}_{a,k} z^\ell - \tilde{\beta}_{a,k} v_{x(a)} - \tilde{\gamma}_{a,k} v_a \right) \\
 \text{s.t.:} \quad & \text{A1, A2, A3, A4, B1, Qb1, Qp2 in M0} \\
 & n \in \mathbb{Z}^+, z \in \mathbb{Z}^+, \lambda \in \{0, 1\}, v \geq 0, \zeta \geq 0
 \end{aligned}$$

It may be observed that the model M0<sub>LR</sub> can be divided into three subproblems:

- SP<sub>z</sub>, an integer problem which contains the integer variables *n*, *z* and *λ* and constraints A1, A2, A3, A4 and Qb1.
- SP<sub>v</sub>, a continuous problems which contains the continuous variables *v* and constraint B1. Moreover, it can be solved by the shortest path algorithm
- SP<sub>ζ</sub>, a continuous problem which contains the continuous variable *ζ* and constraint Qp2.

#### 4.2. Applying the Dantzig’s cutting plane algorithm

The algorithm divides the original problem into a *master problem* (MP) and a *subproblem* (SP), so that the solution of the MP is part of the definition of the SP and the solution of the SP generates cuts that will limit the solution of the MP. Basically, the SP solves the Lagrangian relaxations and the resulting solutions will be added to the MP to update the Lagrange multipliers. The application of the algorithm for solving model M0 is as follows.

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#### Algorithm 1 Dantzig’s cutting plane version for problem M0

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- 1: Determine an initial feasible solution  $x^{(0)} = (n^{(0)}, z^{(0)}, \lambda^{(0)}, v^{(0)}, \zeta^{(0)})$ . Let  $j = 0$
  - 2: Solve the MP and obtain  $Z_{mp}^j, \hat{\mu}^{(j)}$  and  $\tilde{\mu}^{(j)}$ 

$$\begin{aligned}
 \text{[MP]} \quad & \max_{Z_{mp}, \hat{\mu}, \tilde{\mu}} Z_{mp} \\
 \text{s.t.} \quad & Z_{mp} \leq \mathcal{L}(x^{(j)}, \hat{\mu}, \tilde{\mu}), \quad i = 0, \dots, j \\
 & \hat{\mu}, \tilde{\mu} \geq 0
 \end{aligned}$$
  - 3: In order to calculate  $w(\hat{\mu}^{(j)}, \tilde{\mu}^{(j)})$ , solve SP<sub>z</sub>, SP<sub>v</sub>, SP<sub>ζ</sub> and obtain:  $x^{(j+1)} = (n^{(j+1)}, z^{(j+1)}, \lambda^{(j+1)}, v^{(j+1)}, \zeta^{(j+1)})$ .
  - 4: **If**  $\left( \text{relgap} = \left| \frac{Z_{mp}^j - w(\hat{\mu}^{(j)}, \tilde{\mu}^{(j)})}{w(\hat{\mu}^{(j)}, \tilde{\mu}^{(j)})} \right| \leq \varepsilon \right)$  **then** STOP  
**else** Add a new cut to MP.  $j = j + 1$ . Go to step 2.
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In the previous algorithm, the stopping criteria corresponds to calculate the relative gap for a given tolerance  $\varepsilon$ . For simplicity, the initial feasible solution is obtained by assigning all the passengers on the alternative mode network (walking).

#### 4.3. Accelerating the cutting plane algorithm

The master problem has an infinite number of constraints (or cuts). To avoid this, there is an algorithm that purges the master problem and accelerates it. Lawphongpanich and Hearn (1991) said that when the subproblem is easy to solve, it would be advantageous to solve more subproblems in a effort to obtain better cuts, i.e., those which may lead to a reduction in the number of master problems to be solved. The algorithm is presented in Fig. 2.

Another method applied to accelerate the master problem is updating the Lagrange multipliers as follows:

$$\mu^{(j+1)} = \mu^{(j)} + \alpha^{(j)} (\mu^* - \mu^{(j)}) \quad (1) \quad \alpha^{(j)} = \frac{j^d}{\sum_{i=1}^j i^d}, \forall j = 1, 2, \dots, N \quad (2)$$

where  $\mu^*$  is the solution obtained from solving the master problem, the step length  $\alpha$  is obtained using the method of successive weighted averages (MSWA), which was developed by Liu et al. (2009), and  $d \geq 0$  is an integer number. Nevertheless, it is necessary to restart the step length value every few iterations, for instance every 10, in order to avoid the stagnation of the previous iterative procedure.

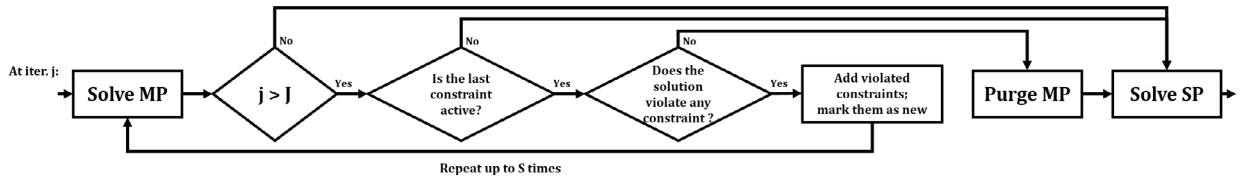


Fig. 2: Algorithm to purge the master problem

4.4. Generating primal solutions

It is known that the Dantzig’s cutting plane algorithm may not generate a primal feasible solution for the original problem. As shown in Shapiro (1979), by using the generalized linear programming problem procedure, a feasible solution close to the optimality can be obtained from the sequence of points yielded at each subproblem step of the cutting plane’s algorithm. Since this is strictly valid for linear programming problems and problem M0 contains also integer variables, the following heuristic procedure has been applied:

- Obtain a sequence of points  $x^j = (n^j, z^j, \lambda^j, v^j, \zeta^j)$ ,  $j = 1, 2, \dots, N$ , using Dantzig’s cutting plane as shown in subsection 4.2.
- Using the generalized linear programming procedure, suitable values for the continuous variables have been obtained  $(\bar{v}, \bar{\zeta})$ , which in turn have been set fixed in problem M0 in order to calculate appropriate integer values  $(\bar{n}, \bar{z}, \bar{\lambda})$ .

5. Computational tests

To illustrate the model’s performance and the proposed methods for solving it, two test networks have been used: a railway corridor in Madrid comprising stations of Atocha, Recoletos, Nuevos Ministerios and Chamartín and a set of metro stations in Barcelona’s line 1 (from station Plaça d’Espanya to station Clot). Table 1 shows their characteristics. Fig. 3 depicts a schematic representation of the main arterials where the system may operate in Barcelona. The expanded bus network contains a set of links  $A_G$ , depicted in Fig. 3(b), for the movements of passengers carried out outside the bus network (access from the stations to bus stops, transfers between bus lines, portions of the trip carried out on foot). The schematic representation of Madrid is similar.

In both test networks, (a) bus units with a capacity of 100 passengers have been assumed; (b) the time period under consideration is  $H = 180$  minutes; (c) the value of time  $\theta$  has been assumed to be 0.081 €/min. It has been also assumed that appropriate bus lanes exist to keep buses running at an average speed of 26 km/h between bus stops. There are three different typologies of bus lines that make up the set of candidate lines in the computational tests: symmetric (all bus stops are visited in both directions), symmetric express and asymmetric lines.

Table 2: Parameters of the test networks.

Network	Candidate lines	Bus stops	Nodes	Links	O-D pairs	Centroids	Trips	Variables	Constraints
Madrid	20	6	118	240	12	4	37000	1318	1449
Barcelona	48	17	310	640	88	10	37992	7328	5599

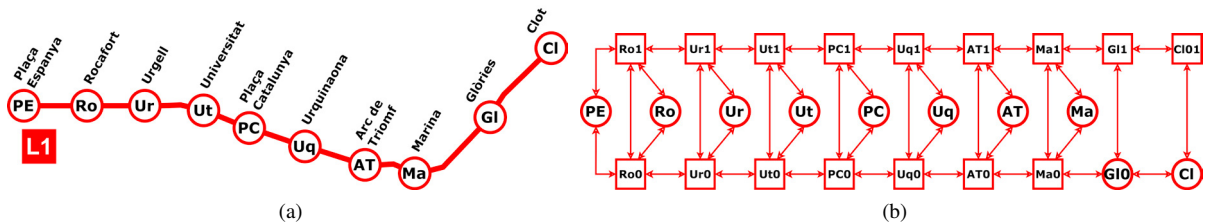


Fig. 3: (a) Schematic representation of metro line 1 in Barcelona; (b) Representation of the subgraph  $(N_G, A_G)$  for movements of passengers outside the bus network.

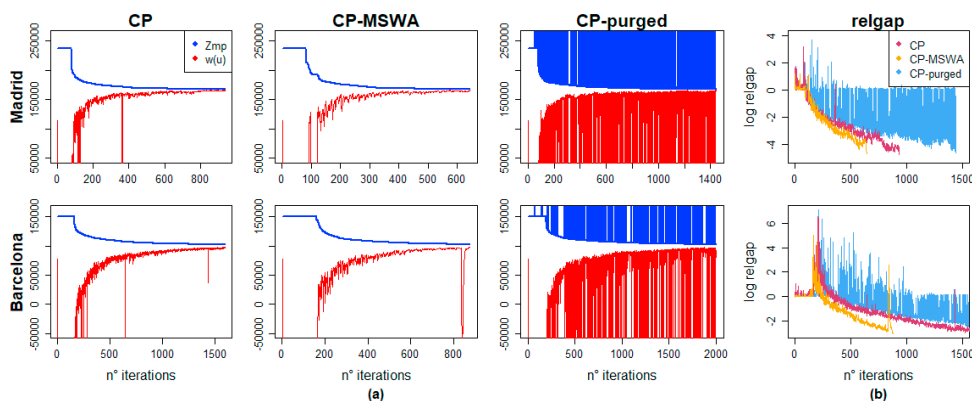


Fig. 4: (a) Convergence of the master problem ( $Z_{mp}$ ) and subproblem ( $w(\mu)$ ) for the three methods; (b)  $relgap$  values (in logarithmic scale) of the three methods, for both test networks

Table 3: Main results for both test networks.

Network	Method	# iter.	Time	$Z_{mp}$	$w(\mu)$	$relgap$	$f^k$	$f_p^*$	% walk
Madrid	CP	937	24 min.	167518.60	166067.30	0.0087	159820.21	175897.29	56.84
	CP-MSWA	644	11 min.	167538.39	165983.94	0.0094	147844.44	176321.89	56.61
	CP-purged	1441	1.33 h.	167608.09	165956.78	0.01	138523.12	176150.79	57.92
Barcelona	CP	1598	4.5 h	102412.97	97806.21	0.0471	122372.28	103866.20	39.47
	CP-MSWA	878	1.33 h	102591.52	97754.18	0.0495	138382.84	104573.30	39.32
	CP-purged	2000	11.33 h	102513.02	91894.26	0.1156	159148.92	103876.12	39.78

The solution methods have been implemented in AMPL, using CPLEX 12.5.0.0 as the solver. Runs have been performed on a 2.4 GHz computer, 48 Gb RAM.

Table 3 shows the main results of applying the Dantzig's cutting plane algorithm and the methods to accelerate it. Rows **CP**, **CP-MSWA** and **CP-purged** are the results without acceleration, updating the Lagrange multipliers and purging the master problem, respectively. Column **# iter.** reports the required number of iterations to achieve  $relgap \leq \varepsilon$ . Column **Time** reports the total run time. Columns  $Z_{mp}$  and  $w(\mu)$  report the master problem and subproblem values at the last iteration, respectively. Column  $relgap$  reports the value of the relative gap at the last iteration. Columns  $f^k$  and  $f_p^*$  report the objective function values of the last iteration and the generated primal solution, respectively. Finally, column **% walk** reports the percentage of the total trips made by walking according to the generated primal solution.

The required tolerance  $\varepsilon$  for the relative gap is 0.01 and 0.05 for Madrid and Barcelona, respectively. Regarding the initial feasible solution, the total cost of sending all passengers on foot is  $f^0 = 237323.68$  and  $f^0 = 150330.61$ , respectively. For both cases, in the purging method  $J = 40$  and  $S = 3$ . In addition, a limit of 2000 iteration was established for all runs.

Fig. 4(a) shows the convergence for the three methods. As can be seen, over 300 iterations the  $Z_{mp}$  and  $w(\mu)$  values begin to approach, but the convergence to be equals or have a  $relgap \leq \varepsilon$  is slow. Whereas for Barcelona it takes longer to approach each other and even, for CP and CP-MSWA methods, the subproblem values have a significant drop almost at the final iterations. Fig. 4(b) shows the  $relgap$  performance (in logarithmic scale) and it complements the fact that the best method to accelerate the cutting plane algorithm is updating the Lagrange multipliers using the MSWA method in (1) and (2). As can be noted from Table 3, updating the Lagrange multipliers through the MSWA is the most efficient method, since it is the only one that accelerates the cutting plane algorithm.

Despite the purging method seeks to reduce the number of cuts and the master problem runtime at some iterations, the master problem may be resolved up to four times, for this reason the runtime increases. In addition, it has the highest number of iterations and the relative gap value is very unstable, as shown in Fig. 4.

Analysing the primal solutions obtained of applying the updating method for the Lagrange multipliers stated in (1) and (2), in the case of Madrid, 11 lines (out of 20 candidates) are selected and three of them have been assigned



a significant number of vehicles and services with small headways. It must be remarked that several of the selected lines have similar routes. For the case of the Barcelona test network almost 90% of the candidate lines are selected, of which 20 have been assigned a single bus unit and 10 with two units, with less than five services. A factor that contributes to the large number of selected lines obtained in the computational tests is the different typologies in the set of candidates.

## 6. Conclusions and further research

In this paper we have presented a model for the selection of candidate bus lines in the expansion of public transit system. The model is an adaptation for the case of moderate congestion of the model in Codina et al. (2013). An heuristic method based on the Dantzig's cutting plane algorithm has been proposed, together with two variants for accelerating it. The first variant consists in updating the Lagrange multipliers using the method of successive weighted averages (MSWA) in Liu et al. (2009). The second variant consists of purging the Master Problem step of the cutting plane in order to accelerate it, although the computational tests show that this is not achieved since computational times are higher, and the evolution of the algorithm shows an additional instability. The numeric results show that the Lagrange updating procedure is the most efficient method, since the number of iterations and the runtime are reduced by at least 45% and 50%, respectively. A characteristic of this heuristic method is that it may yield solutions with large number of bus lines with few vehicles and services assigned to them. The authors are currently examining the possibility of modelling passenger's route choice by means of strategies on the transit network, as well as the option to consider not only new lines, but also the existing lines, so that their frequencies may be determined as decision variables of the optimization process itself.

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