A canonical realization of the Weyl BMS symmetry[⋆]

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Abstract

We construct a free field realization of an extension of the BMS algebra in 2+1 dimensional space-time. Besides the supertranslations and superrotations, the extension contains an infinite set of superdilatations. We also comment the difficulties that appear when trying to extend the algebra to that of the full conformal group.

Keywords: BMS symmetry, Conformal symmetry

1. Introduction

There is a renewed interest in the BMS group [1, 2]. One of the interest is to deduce Weinberg's soft graviton theorems [3] as the Ward identities of BMS supertranslations [4, 5, 6, 7]. In this case the BMS symmetry is spontaneously broken. A pedagogical overview of the recent role of BMS symmetries is presented in [8].

In the case of three dimensions the BMS algebra has been studied [9, 10] with supertranslations and superrotations [11]. A canonical realization of the $\mathfrak{bms}_3$ algebra with supertranslations and superrotations associated to a free Klein-Gordon (KG) field in 2+1 dimensions, for both massive and massless fields, was studied in [12]. The canonical approach to BMS symmetry was introduced in [13].

At the quantum level, the Hilbert space of one-particle states supports a unitary irreducible representation of the Poincaré group, and at the same time a unitary reducible representation of the BMS_3 group. In contrast with the gravitational approach, our canonical realization of the supertranslations symmetry is not spontaneously broken.

In this paper we reconsider the canonical realization associated to a massless KG field in terms of the Fourier modes. As is well known the massless KG Lagrangian has an enlarged Poincaré symmetry with dilatations and special conformal transformations.

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Knowing the fact that the Poincaré symmetry can be extended to a Poincaré BMS symmetry, it is natural to look for a possible infinite dimensional extension of dilatations and special conformal transformations.

We perform the analysis in the simple case of $d = 3$. We will see that that there is a natural generalization of dilatations to superdilatations that together with the supertranslations and superrotations close in an algebra that we call Weyl BMS algebra, as it is a generalization of the standard Weyl (Poincaré plus dilatations) algebra.[14, 15, 16] We have not been able to construct a generalization of special conformal transformations that close with supertranslations, superrotations and superdilatations.

The organization of the note is as follows. In Section 3 we construct the generators of the conformal algebra in terms of the Fourier modes of a massless KG field in arbitrary spacetime dimensions. In Section 3 we specialize to the case $d = 3$ and give the expression of the BMS supertranslations and superrotations for the same field. Finally, in Section 4 we introduce the superdilatations and explain the problems that appear when one considers the full BMS plus conformal algebra.

Note added. When this paper was being completed a paper [17] has appeared that constructs a generalization of the BMS symmetry by studying asymptotic symmetries of a gravitational theory. Their algebra, that includes superdilatations, agrees with our canonical results. A more general version of this algebra has also been found in [18], where general boundary symmetries in $d = 2$ and $d = 3$ are studied.

2. Conformal algebra realization in terms of a massless free Klein-Gordon field

In this section we will construct a realization of the conformal group in terms of the Fourier modes of a free massless scalar field. Let us consider a massless scalar field theory with Lagrangian[1]

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi, \tag{2.1}$$

The solution of the equations of motion can be written in terms of Fourier modes as

$$\begin{aligned} \phi(t, \vec{x}) &= \int d\widetilde{k} \, \left(a(\vec{k}, t) e^{i\vec{k}\cdot\vec{x}} + a^*(\vec{k}, t) e^{-i\vec{k}\cdot\vec{x}} \right), \\ d\widetilde{k} &= \frac{d^{d-1}k}{(2\pi)^{(d-1)}(2\omega)}, \quad \omega = k^0 = \sqrt{\vec{k}\cdot\vec{k}}. \end{aligned} \tag{2.2}$$

Following standard procedures, see for example [13, 19, 20], one can construct the generators of the conformal algebra (Poincaré, dilatations and special conformal transformations) as integrals over the space of local densities depending on the field and their space-time derivatives. In terms of the Fourier modes $a(\vec{k}, t)$, $a^*(\vec{k}, t)$ defined in (2.2) they

[1] The signature is $\eta = (-, +, \ldots, +)$ for a d-dimensional spacetime.

are given by

$$P^\mu = \int d\widetilde{k}\, a^*(\vec{k},t) k^\mu a(\vec{k},t), \quad k^\mu = (\omega, \vec{k}), \tag{2.3}$$

$$M^{0j} = tP^j - i\int d\widetilde{k}\, a^*(\vec{k},t)\omega\frac{\partial}{\partial k_j} a(\vec{k},t), \qquad M^{j0} = -M^{0j}, \tag{2.4}$$

$$M^{ij} = -i\int d\widetilde{k}\, a^*(\vec{k},t)\left(k^i\frac{\partial}{\partial k_j} - k^j\frac{\partial}{\partial k_i}\right) a(\vec{k},t), \qquad M^{ji} = -M^{ij}, \tag{2.5}$$

$$D = -tP^0 + i\int d\widetilde{k}\, a^*(\vec{k},t)\left(k^j\frac{\partial}{\partial k^j} + \frac{d-2}{2}\right) a(\vec{k},t), \tag{2.6}$$

$$K^0 = -t^2P^0 - \int d\widetilde{k}\, a^*(\vec{k},t)\left[-\left(\omega\frac{\partial}{\partial k_i} + 2itk^i\right)\frac{\partial}{\partial k^i} - it(d-2)\right] a(\vec{k},t), \tag{2.7}$$

$$K^j = t^2P^j - \int d\widetilde{k}\, a^*(\vec{k},t)\left[-\left(k^j\frac{\partial}{\partial k_i} - 2k^i\frac{\partial}{\partial k_j}\right)\frac{\partial}{\partial k^i} + (2it\omega + (d-2))\frac{\partial}{\partial k_j}\right] a(\vec{k},t). \tag{2.8}$$

The equal-time canonical Poisson brackets between $\phi(t,\vec{x})$ and $\pi(t,\vec{x}) = \dot{\phi}(t,\vec{x})$ lead to the following Poisson brackets for the Fourier modes

$$\{a(\vec{k},t), a^*(\vec{k},t)\} = -i(2\pi)^{(d-1)}(2\omega)\delta^{d-1}(\vec{k}-\vec{q}). \tag{2.9}$$

Using (2.9) one can compute the brackets between the generators of the conformal algebra

$$\{D, P^\mu\} = P^\mu, \tag{2.10}$$
$$\{D, K^\mu\} = -K^\mu, \tag{2.11}$$
$$\{K^\mu, P^\nu\} = 2(\eta^{\mu\nu}D + M^{\mu\nu}), \tag{2.12}$$
$$\{K^\mu, M^{\nu\sigma}\} = \eta^{\mu\sigma}K^\nu - \eta^{\mu\nu}K^\sigma, \tag{2.13}$$
$$\{P^\mu, M^{\nu\sigma}\} = \eta^{\mu\sigma}P^\nu - \eta^{\mu\nu}P^\sigma, \tag{2.14}$$
$$\{M^{\mu\nu}, M^{\sigma\rho}\} = \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}, \tag{2.15}$$

with all the remaining brackets equal to zero. The Poisson brackets are related to the commutators of the differential operators in $\vec{k}$ by means of

$$\{P, Q\} = -i\int d\widetilde{k}\, a^*(\vec{k},t)[\hat{P},\hat{Q}]a(\vec{k},t), \tag{2.16}$$

where $\hat{P}$, $\hat{Q}$ are the operators appearing in the integrals of conserved charges P and Q, respectively, at $t=0$. Explicitly,

$$\hat{P}^\mu = k^\mu, \tag{2.17}$$

$$\hat{M}^{0j} = -i\omega \frac{\partial}{\partial k_j}, \tag{2.18}$$

$$\hat{M}^{ij} = -i\left(k^i \frac{\partial}{\partial k_j} - k^j \frac{\partial}{\partial k_i}\right), \tag{2.19}$$

$$\hat{D} = i\left(k^j \frac{\partial}{\partial k^j} + \frac{d-2}{2}\right), \tag{2.20}$$

$$\hat{K}^0 = \omega \frac{\partial^2}{\partial k_i^2}, \tag{2.21}$$

$$\hat{K}^j = \left(k^j \frac{\partial}{\partial k_i} - 2k^i \frac{\partial}{\partial k_j}\right)\frac{\partial}{\partial k^i} - (d-2)\frac{\partial}{\partial k_j}. \tag{2.22}$$

3. Canonical realization of the BMS algebra

Following the ideas in [13], a canonical realization of the full BMS algebra [21] (supertranslations and superrotations) in terms of Fourier modes of a massless Klein-Gordon field in $2+1$ spacetime was constructed in [12]. Using the notation of the previous section, the generators of supertranslations and superrotations are given respectively by

$$\mathcal{P}_\ell = \int d\widetilde{k}\, a^*(\vec{k},t)\hat{\mathcal{P}}_\ell a(\vec{k},t),\ \ m \in \mathbb{Z}, \tag{3.1}$$

$$\mathcal{R}_m = \int d\widetilde{k}\, a^*(\vec{k},t)\hat{\mathcal{R}}_m a(\vec{k},t),\ \ m \in \mathbb{Z}, \tag{3.2}$$

with

$$\hat{\mathcal{P}}_\ell = \omega_\ell = \omega^{1-\ell}(k^1 + ik^2)^\ell, \quad \omega = \sqrt{k_1^2 + k_2^2}, \tag{3.3}$$

$$\hat{\mathcal{R}}_m = \frac{1}{\omega}\omega_m\left((k^2 + imk^1)\frac{\partial}{\partial k^1} - (k^1 - imk^2)\frac{\partial}{\partial k^2}\right). \tag{3.4}$$

These differential operators obey the BMS algebra

$$[\hat{\mathcal{P}}_m, \hat{\mathcal{P}}_\ell] = 0, \tag{3.5}$$

$$[\hat{\mathcal{R}}_m, \hat{\mathcal{P}}_\ell] = i(m-\ell)\hat{\mathcal{P}}_{m+\ell}, \tag{3.6}$$

$$[\hat{\mathcal{R}}_m, \hat{\mathcal{R}}_n] = i(m-n)\hat{\mathcal{R}}_{m+n}. \tag{3.7}$$

For $m = 0, \pm 1$ one obtains a 6-dimensional closed algebra which corresponds to Poincaré in $2+1$. Corresponding expressions for the massive Klein-Gordon field are also presented in [12], but we will not discuss them here since we are interested in extending this algebra with conformal generators.

4. Superdilatations

We have seen in Section 2 that there is a canonical realization of the conformal algebra in terms of a massless Klein-Gordon, and in Section 3, in the case of $d = 3$, that there is a BMS extension of the Poincaré algebra. It is therefore natural to look, in this simple case, for an extension of BMS Poincaré that also includes dilatations and special conformal transformations.

The differential operators (2.20), (2.21), (2.22), appearing in the canonical realization of the conformal group are, for $d = 3$,

$$\hat{D} = i\left(k^j \frac{\partial}{\partial k^j} + \frac{1}{2}\right), \tag{4.1}$$

$$\hat{K}^0 = \omega \frac{\partial^2}{\partial k_i^2}, \tag{4.2}$$

$$\hat{K}^j = \left(k^j \frac{\partial}{\partial k_i} - 2k^i \frac{\partial}{\partial k_j}\right)\frac{\partial}{\partial k^i} - \frac{\partial}{\partial k_j}. \tag{4.3}$$

For the dilatations the commutations relations with the Poincaré BMS generators are

$$[\hat{D}, \hat{\mathcal{P}}_\ell] = i\hat{\mathcal{P}}_\ell, \tag{4.4}$$

$$[\hat{D}, \hat{\mathcal{R}}_m] = 0, \tag{4.5}$$

but when acting with the special conformal transformations on the supertranslations one gets new operators not present in the algebra (2.17)

$$-i[\hat{K}^1, \hat{P}_\ell] = -(1-\ell)\ell\frac{1}{\omega}\omega_{\ell+1}\hat{D} + (1+\ell)\ell\frac{1}{\omega}\omega_{\ell-1}\hat{D} + (1-\ell)\hat{\mathcal{R}}_{\ell+1} - (1+\ell)\hat{\mathcal{R}}_{\ell-1}, \tag{4.6}$$

$$[\hat{K}^2, \hat{P}_\ell] = -(1-\ell)\ell\frac{1}{\omega}\omega_{\ell+1}\hat{D} - (1+\ell)\ell\frac{1}{\omega}\omega_{\ell-1}\hat{D} + (1-\ell)\hat{\mathcal{R}}_{\ell+1} + (1+\ell)\hat{\mathcal{R}}_{\ell-1}, \tag{4.7}$$

This algebra can be closed if we introduce an infinite family of superdilatations, given by

$$\hat{D}_\ell = \frac{1}{\omega}\omega_\ell \hat{D}, \tag{4.8}$$

so that

$$-i[\hat{K}^1, \hat{P}_\ell] = -(1-\ell)\ell\hat{D}_{\ell+1} + (1+\ell)\ell\hat{D}_{\ell-1} + (1-\ell)\hat{\mathcal{R}}_{\ell+1} - (1+\ell)\hat{\mathcal{R}}_{\ell-1}, \tag{4.9}$$

$$[\hat{K}^2, \hat{P}_\ell] = -(1-\ell)\ell\hat{D}_{\ell+1} - (1+\ell)\ell\hat{D}_{\ell-1} + (1-\ell)\hat{\mathcal{R}}_{\ell+1} + (1+\ell)\hat{\mathcal{R}}_{\ell-1}. \tag{4.10}$$

Using $\hat{D}_\ell$, the commutator of $\hat{K}^0$ with the supertranslations also closes,

$$[\hat{K}^0, \hat{P}_\ell] = -2i\left((1-\ell^2)\hat{D}_\ell + \ell\hat{\mathcal{R}}_\ell\right). \tag{4.11}$$

These commutators yield Poisson brackets, via (2.16), which generalize (2.12) when the supertranslations are considered.

It remains to study the action of the special conformal transformations on the superrotations. Using $\hat{K}^{\pm}$ instead of $\hat{K}^{1,2}$, given by

$$\hat{K}^{\pm} = \frac{1}{2}\left(\hat{K}^1 \pm i\hat{K}^2\right) \tag{4.12}$$

one gets

$$\left[\hat{K}^+, \hat{\mathcal{R}}_\ell\right] = -i\frac{1}{2}\ell(1-\ell)\frac{1}{\omega}\omega_{\ell+1}\hat{K}^0 + i(1-\ell^2)\frac{1}{\omega}\omega_\ell\hat{K}^+ + i\frac{1}{2}\ell(1-\ell^2)\frac{1}{\omega^2}\omega_{\ell+1}k^i\partial_i, \tag{4.13}$$

$$\left[\hat{K}^-, \hat{\mathcal{R}}_\ell\right] = -i\frac{1}{2}\ell(1+\ell)\frac{1}{\omega}\omega_{\ell-1}\hat{K}^0 - i(1-\ell^2)\frac{1}{\omega}\omega_\ell\hat{K}^- + i\frac{1}{2}\ell(1-\ell^2)\frac{1}{\omega^2}\omega_{\ell-1}k^i\partial_i, \tag{4.14}$$

$$\left[\hat{K}^0, \hat{\mathcal{R}}_\ell\right] = -i\ell(1+\ell)\frac{1}{\omega}\omega_{\ell-1}\hat{K}^+ - i\ell(1-\ell)\frac{1}{\omega}\omega_{\ell+1}\hat{K}^- + i\ell(1-\ell^2)\frac{1}{\omega^2}\omega_\ell k^i\partial_i, \tag{4.15}$$

and new operators appear. One can be tempted, as we did for superdilatations, to introduce a family of superspecial conformal transformations in order to take into account the first and second terms in the right-hand side of the above commutators. However, the last term cannot be absorbed, unless one introduces yet another, completely new family of operators, and we have not succeed in obtaining a close algebra which includes both the supertranslations and superrotations and the special conformal transformations. Notice also that one cannot just drop the superrotations, since they appear in the commutator between special conformal transformations and supertranslations. The extra terms in (4.13)-(4.15) disappear for $\ell = 0, \pm 1$ as it must be, since then we have a part of the conformal algebra.

Leaving aside the special conformal transformations, the superdilatations, together with the supertranslations and the superrotations, form a closed algebra, which we call a Weyl BMS algebra, given by

$$[\hat{\mathcal{P}}_m, \hat{\mathcal{P}}_\ell] = 0, \tag{4.16}$$

$$[\hat{\mathcal{R}}_m, \hat{\mathcal{P}}_\ell] = i(m-\ell)\hat{\mathcal{P}}_{m+\ell}, \tag{4.17}$$

$$[\hat{\mathcal{R}}_m, \hat{\mathcal{R}}_n] = i(m-n)\hat{\mathcal{R}}_{m+n}, \tag{4.18}$$

$$[\hat{D}_\ell, \hat{\mathcal{P}}_m] = i\hat{\mathcal{P}}_{\ell+m}, \tag{4.19}$$

$$[\hat{D}_\ell, \hat{D}_m] = 0, \tag{4.20}$$

$$[\hat{D}_\ell, \hat{\mathcal{R}}_m] = i\ell\hat{D}_{\ell+m}. \tag{4.21}$$

5. Conclusions

We have obtained a family of operators, the superdilatations, that appear naturally when considering the action of the special conformal transformations on the supertranslations. The set of supertranslations, superrotations and superdilatations form a closed infinite dimensional algebra which can be considered a BMS extension of the ordinary Weyl algebra (Poincaré plus dilatations). Trying to include the special conformal transformations leads to the appearance of an infinite tower of new kinds of operators. A more detailed description of the difficulties encountered when trying to close the algebra in our approach will be reported in an extended work.

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