A new class of polynomials from the spectrum of a graph, and its application to bound the k-independence number

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Abstract

The k-independence number of a graph is the maximum size of a set of vertices at pairwise distance greater than k. A graph is called k-partially walk-regular if the number of closed walks of a given length $l \leq k$, rooted at a vertex v, only depends on l. In particular, a distance-regular graph is also k-partially walk-regular for any k. In this paper, we introduce a new family of polynomials obtained from the spectrum of a graph, called minor polynomials. These polynomials, together with the interlacing technique, allow us to give tight spectral bounds on the k-independence number of a k-partially walk-regular graph. With some examples and infinite families of graphs whose bounds are tight, we also show that the odd graph O_ℓ with ℓ odd has no 1-perfect code. Moreover, we show that our bound coincide, in general, with the Delsarte's linear programming bound and the Lovász theta number θ, the best ones to our knowledge. In fact, as a byproduct, it is shown that the given bounds also apply for θ and Θ, the Shannon capacity of a graph. Moreover, apart from the possible interest that the minor polynomials can have, our approach has the advantage of yielding closed formulas and, so, allowing asymptotic analysis.

Keywords: Graph, k-independence number, spectrum, interlacing, minor polynomial, k-partially walk-regular, Delsarte's LP bound, Lovász theta number, Shannon capacity.

Mathematics Subject Classifications: 05C50, 05C69.

1 Introduction

Given a graph G, let $\alpha_k = \alpha_k(G)$ denote the size of the largest set of vertices such that any two vertices in the set are at distance larger than k. Thus, with this notation, α_1 is just the independence number α of G. The parameter $\alpha_k(G)$ therefore represents the largest number of vertices that can be $k+1$ spread out in G. Notice also that $\alpha_k(G) = \alpha(G^k)$, where G^k denotes the power graph, where two vertices u, v are adjacent if and only if they are at distance at most k in G. It is known that determining α_k is a NP-Hard problem in general (see Kong and Zhao [26]).

The k-independence number is an interesting graph parameter with both practical and theoretical implications. For instance, codes and anticodes are k-independent sets in some graphs (for instance, the Hamming graphs). Thus, bounds or exact values of such a parameter yield necessary conditions for the existence of perfect codes. Moreover, the k-independence number is closely related to other combinatorial graph parameters (with the corresponding applications) related to the distance. For instance,

- Distance chromatic number $\chi_k(G) = \chi(G^k)$ (Alon and Mohar [3], and Kang and Pirot [25]); packing chromatic number (Goddard, Hedetniemi, Hedetniemi, Harris, and Rall [19], and injective chromatic number (Hahn, [23]). In this case, upper bounds on the k-independence number give lower bounds on the distance or packing chromatic number;
- Strong chromatic index (Mahdian [28]);
- Average distance (Firby and Haviland [18]);
- d-diameter (Chung, Delorme, and Solé [7]): An h-code in a graph G with distance d is a set of $h \geq 2$ vertices with $\min_{u \neq v}\{\mathrm{dist}(u, v)\} = d$. The d-diameter D_h is the largest possible distance an h-code in G can have. In particular, D_2 is the standard diameter.

In this paper, under some extra assumptions (for instance, that of walk-regularity), we improve the known spectral upper bounds for the k-independence number from F. [13], Abiad, Cioabă, and Tait [1] and Abiad, Coutinho, and F. [2]. Our approach is based on a new family of polynomials, which we call minor polynomials, that have nonnegative values at the eigenvalues of the graph, and minimize a given linear function. For some cases and some infinite families of graphs, we show that our bounds are sharp, and coincide, in general, with the Delsarte's linear programming (LP) bound and the Lovász theta number, the best ones to our knowledge. Moreover, apart from the possible interest that the minor polynomials can have, our approach has the advantage of yielding closed formulas and, thus, allowing asymptotic analysis.

Let $G = (V, E)$ be a graph with $n = |V|$ vertices, $m = |E|$ edges, and adjacency matrix $\boldsymbol{A}$ with spectrum $\mathrm{sp}\, G = \{\theta_0, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}$, where the different eigenvalues are in decreasing order, $\theta_0 > \theta_1 > \cdots > \theta_d$, and the superscripts stand for their multiplicities. When the eigenvalues are presented with possible repetitions, we shall indicate them by $\mathrm{ev}\, G : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

The first known spectral bound for the independence number α of a graph is due to Cvetković [8].

Theorem 1.1 (Cvetković [8]). *Let G be a graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then,*

$$\alpha \leq \min\{|\{i : \lambda_i \geq 0\}|, |\{i : \lambda_i \leq 0\}|\}.$$

Another well-known result is the following bound due to Hoffman (unpublished; see, for instance, Haemers [22]).

Theorem 1.2 (Hoffman [22]). *If G is a regular graph on n vertices with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then*

$$\alpha \leq n \frac{-\lambda_n}{\lambda_1 - \lambda_n}.$$

Regarding the k-independence number, the following three results are known. The first is due to F. [13] and it requires a preliminary definition. Let G be a graph with distinct eigenvalues $\theta_0 > \cdots > \theta_d$. Let $P_k(x)$ be chosen among all polynomials $P(x) \in \mathbb{R}_k(x)$, that is, polynomials of real coefficients and degree at most k, satisfying $|P(\theta_i)| \leq 1$ for all $i = 1, ..., d$, and such that $P(\theta_0)$ is maximized. The polynomial $P_k(x)$ defined above is called the *k-alternating polynomial* of G and it was shown to be unique in F., Garriga, and Yebra[17], where it was used to study the relationship between the spectrum of a graph and its diameter.

Theorem 1.3 (F. [13]). *Let G be a d-regular graph on n vertices, with distinct eigenvalues $\theta_0 > \cdots > \theta_d$, and let $P_k(x)$ be its k-alternating polynomial. Then,*

$$\alpha_k \leq \frac{2n}{P_k(\theta_0) + 1}.$$

More recently, Cvetković-like and Hoffman-like bounds were shown by Abiad, Cioabă, and Tait in [1].

Theorem 1.4 (Abiad, Cioabă, Tait [1]). *Let G be a graph on n vertices with adjacency matrix $\boldsymbol{A}$, with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let w_k and W_k be respectively the smallest and the largest diagonal entries of $\boldsymbol{A}^k$. Then,*

$$\alpha_k \leq \min\{|\{i : \lambda_i^k \geq w_k(G)\}|, |\{i : \lambda_i^k \leq W_k(G)\}|\}.$$

Theorem 1.5 (Abiad, Cioabă, Tait [1]). *Let G be a δ-regular graph on n vertices with adjacency matrix $\boldsymbol{A}$, whose distinct eigenvalues are $\theta_0(= \delta) > \cdots > \theta_d$. Let $\widetilde{W_k}$ be the largest diagonal entry of $\boldsymbol{A} + \boldsymbol{A}^2 + \cdots + \boldsymbol{A}^k$. Let $\theta = \max\{|\theta_1|, |\theta_d|\}$. Then,*

$$\alpha_k \leq n \frac{\widetilde{W_k} + \sum_{j=1}^k \theta^j}{\sum_{j=1}^k \delta^j + \sum_{j=1}^k \theta^j}.$$

Finally, as a consequence of a generalization of the last two theorems, Abiad, Coutinho, and F. [2], proved the following results.

Theorem 1.6 (Abiad, Coutinho, F. [2])**.** *Let G be a δ-regular graph with n vertices and distinct eigenvalues $\theta_0(=\delta) > \theta_1 > \cdots > \theta_d$. Let $W_k = W(p) = \max_{u\in V}\{\sum_{i=1}^{k}(\boldsymbol{A}^k)_{uu}\}$. Then, the k-independence number of G satisfies the following:*

(i) If $k = 2$, then

$$\alpha_2 \leq n\frac{\theta_0 + \theta_i\theta_{i-1}}{(\theta_0 - \theta_i)(\theta_0 - \theta_{i-1})},$$

where θ_i is the largest eigenvalue not greater than -1.

(ii) If $k > 2$ is odd, then

$$\alpha_k(G) \leq n\frac{W_k - \sum_{j=0}^{k}\theta_d^j}{\sum_{j=0}^{k}\delta^j - \sum_{j=0}^{k}\theta_d^j}.$$

(iii) If $k > 2$ is even, then

$$\alpha_k(G) \leq n\frac{W_k + 1/2}{\sum_{j=0}^{k}\delta^j + 1/2}.$$

(iv) If $G = (V, E)$ is a walk-regular graph, then

$$\alpha_k(G) \leq n\frac{1 - \lambda(q_k)}{q_k(\delta) - \lambda(q_k)}$$

for $k = 0, \ldots, d-1$, where $q_k = f_0 + \cdots + f_k$ with the f_i's being the predistance polynomials of G (see the next section), and $\lambda(q_k) = \min_{i\in[2,d]}\{q_k(\theta_i)\}$.

2 Background

For basic notation and results see Biggs [4] and Godsil [20]. Let $G = (V, E)$ be a (simple) graph with $n = |V|$ vertices, $m = |E|$ edges, and adjacency matrix $\boldsymbol{A}$ with spectrum $\operatorname{sp} G = \{\theta_0, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}$. When the eigenvalues are presented with possible repetitions, we shall indicate them by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let us consider the scalar product in $\mathbb{R}_d[x]$:

$$\langle f, g\rangle_G = \frac{1}{n}\operatorname{tr}(f(\boldsymbol{A})g(\boldsymbol{A})) = \frac{1}{n}\sum_{i=0}^{d} m_i f(\theta_i)g(\theta_i).$$

The so-called *predistance polynomials* $p_0(=1), p_1, \ldots, p_d$ are a sequence of orthogonal polynomials with respect to the above product, with degree $p_i = i$, and they are normalized in such a way that $\|p_i\|_G^2 = p_i(\theta_0)$ (this makes sense since it is known that $p_i(\theta_0) > 0$) for $i = 0, \ldots, d$. Therefore, they are uniquely determined, for instance, following the Gram-Schmidt process. These polynomials were introduced by F. and Garriga in [15] to prove the so-called 'spectral excess theorem' for distance-regular graphs, where $p_0(=1), p_1, \ldots, p_d$ coincide with the so-called distance polynomials. See Cámara, Fàbrega, F., and Garriga[6] for further details and applications.

A graph G is called *k-partially walk-regular*, for some integer $k \geq 0$, if the number of closed walks of a given length $\ell \leq k$, rooted at a vertex v, only depends on ℓ. Thus, every (simple) graph is k-partially walk-regular for $k = 0, 1$, and every regular graph is 2-partially walk-regular. More generally, every k-partially distance-regular graph (see Dalfó, van Dam, F., Garriga, Gorissen [9] is also k-partially walk-regular. Moreover G is k-partially walk-regular for any k if and only if G is walk-regular, a concept introduced by Godsil and Mckay in [21]. For example, it is well-known that every distance-regular graph is walk-regular (but the converse does not hold). Other examples of walk-regular graphs are the vertex-transitive graphs and the semi-symmetric graphs (which are edge-transitive, but not vertex-transitive). Moreover, some usual graph operations, such as the complement and the Cartesian product, on walk-regular graphs provides graphs that are also walk-regular, see again [21].

Eigenvalue interlacing is a powerful and old technique that has found countless applications in combinatorics and other fields. This technique will be used in several of our proofs. For more details, historical remarks, and other applications, see Haemers [22] and F. [14]. Given square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with respective eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_m$, with $m < n$, we say that the second sequence *interlaces* the first one if, for all $i = 1, \ldots, m$, it follows that $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$.

Theorem 2.1 ([14, 22]). *Let $\boldsymbol{S}$ be a real $n \times m$ matrix such that $\boldsymbol{S}^T\boldsymbol{S} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix, and let $\boldsymbol{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Define $\boldsymbol{B} = \boldsymbol{S}^T\boldsymbol{A}\boldsymbol{S}$, and call its eigenvalues $\mu_1 \geq \cdots \geq \mu_m$. Then,*

(i) The eigenvalues of $\boldsymbol{B}$ interlace those of $\boldsymbol{A}$.

(ii) If $\mu_i = \lambda_i$ or $\mu_i = \lambda_{n-m+i}$, then there is an eigenvector $\boldsymbol{v}$ of $\boldsymbol{B}$ for μ_i such that $\boldsymbol{S}\boldsymbol{v}$ is eigenvector of $\boldsymbol{A}$ for μ_i.

(iii) If there is an integer $k \in \{0, \ldots, m\}$ such that $\lambda_i = \mu_i$ for $1 \leq i \leq k$, and $\mu_i = \lambda_{n-m+i}$ for $k+1 \leq i \leq m$ (tight interlacing)*, then $\boldsymbol{S}\boldsymbol{B} = \boldsymbol{A}\boldsymbol{S}$.*

Two interesting particular cases where interlacing occurs (obtained by choosing the matrix $\boldsymbol{S}$ appropriately) are the following. First, let $\boldsymbol{A}$ be the adjacency matrix of a graph $G = (V, E)$. First, if $\boldsymbol{B}$ is a principal submatrix of $\boldsymbol{A}$, then $\boldsymbol{B}$ corresponds to the adjacency matrix of an induced subgraph G' of G. Second, when, for a given partition of the vertices of Γ, say $V = U_1 \cup \cdots \cup U_m$, $\boldsymbol{B}$ is the so-called *quotient matrix* of $\boldsymbol{A}$, with elements b_{ij}, for $i, j = 1, \ldots, m$, being the average row sums of the corresponding block $\boldsymbol{A}_{ij}$ of $\boldsymbol{A}$. Actually, the quotient matrix $\boldsymbol{B}$ does not need to be symmetric or equal to $\boldsymbol{S}^\top\boldsymbol{A}\boldsymbol{S}$, but in this case $\boldsymbol{B}$ is similar to (and therefore has the same spectrum as) $\boldsymbol{S}^\top\boldsymbol{A}\boldsymbol{S}$.

3 The minor polynomials

In this section, we introduce a new class of polynomials, called minor polynomials, obtained from the different eigenvalues of a graph, which are used later to derive our main results. More precisely, these polynomials are introduced with the goal of improving existing bounds for the k-independence number of k-partially walk-regular graphs, although their definition applies for any graph.

Let G be a graph with adjacency matrix $\boldsymbol{A}$ and spectrum $\operatorname{sp} G = \{\theta_0, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}$. Let f be a polynomial of degree at most k, satisfying $f(\theta_0) = 1$ and $f(\theta_i) \geq 0$ for $i = 1, \ldots, d$. Then, in Section 4, we prove that, if G is k-partially walk-regular, then its k-independence number satisfies the bound $\alpha_k \leq \operatorname{tr} f(\boldsymbol{A}) = \sum_{i=0}^{d} m_i f(\theta_i)$. So, the search for the best result motivates the following definition.

Definition 3.1. *Let $G = (V, E)$ be a graph with adjacency matrix $\boldsymbol{A}$ with spectrum* $\operatorname{sp} G = \{\theta_0, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}$*. For a given $k = 0, 1, \ldots, d$, let us consider the set of real polynomials $\mathcal{P}_k = \{f \in \mathbb{R}_k[x] : f(\theta_0) = 1, f(\theta_i) \geq 0,$ for $1 \leq i \leq d\}$, and the continuous function $\Psi : \mathcal{P}_k \to \mathbb{R}^+$ defined by $\Psi(f) = \operatorname{tr} f(\boldsymbol{A})$. Then, the k-*minor polynomial *of G is the point f_k where Ψ attains its minimum:*

$$\operatorname{tr} f_k(\boldsymbol{A}) = \min \{\operatorname{tr} f(\boldsymbol{A}) : f \in \mathcal{P}_k\}.$$

An alternative approach to the k-minor polynomials is the following: Let f_k be the polynomial defined by $f_k(\theta_0) = x_0 = 1$ and $f_k(\theta_i) = x_i$, for $i = 1, \ldots, d$, where the vector $(x_1, x_2, \ldots, x_d)$ is a solution of the following linear programming problem:

$$\boxed{\begin{array}{rl} \texttt{minimize} & \sum_{i=0}^{d} m_i x_i \\ \texttt{with constraints} & f[\theta_0, \ldots, \theta_m] = 0, \; m = k+1, \ldots, d \\ & x_i \geq 0, \; i = 1, \ldots, d, \end{array}}$$

where $f[\theta_0, \ldots, \theta_m]$ denote the m-th divided differences of Newton interpolation, recursively defined by $f[\theta_i, \ldots, \theta_j] = \frac{f[\theta_{i+1}, \ldots, \theta_j] - f[\theta_i, \ldots, \theta_{j-1}]}{\theta_j - \theta_i}$, where $j > i$, starting with $f[\theta_i] = f(\theta_i) = x_i$, for $0 \leq i \leq d$.

Thus, we can easily compute the minor polynomials, for instance, by using the simplex method. Moreover, as the problem is in the so-called standard form, with d variables, $x_1, \ldots, x_d$, and $d - (k+1) + 1 = d - k$ equations, the 'basic vectors' have at least $d - (d-k) = k$ zeros. Note also that, from the conditions of the programming problem, the k-minor polynomial turns out to be of the form $f_k(x) = f[\theta_0] + f[\theta_0, \theta_1](x - \theta_0) + \cdots + f[\theta_0, \ldots, \theta_k](x - \theta_0) \cdots (x - \theta_{k-1})$, with degree at most k. Consequently, when we apply the simplex method, we obtain a k-minor polynomial f_k with degree k and exactly k zeros at the mesh $\theta_1, \ldots, \theta_d$. In fact, as shown in the following proposition, a k-minor polynomial has exactly k zeros in the interval $[\theta_d, \theta_0)$.

Proposition 3.2. *Let G be a graph with spectrum $\operatorname{sp} G = \{\theta_0, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}$. Then, for every $k = 0, 1, \ldots, d$, a k-minor polynomial f_k has degree k with its k zeros in $[\theta_d, \theta_0) \subset \mathbb{R}$.*

Proof. We only need to deal with the case $k \geq 1$. Assume that a k-minor polynomial f_k has the zeros $\xi_r \leq \xi_{r-1} \leq \cdots \leq \xi_1$ with $r \leq k$. Then, it can be written as $f_k(x) = \prod_{i=1}^{r} \frac{x-\xi_i}{\theta_0-\xi_i}$. Let us first show that $\xi_r \geq \theta_d$. By contradiction, assume that $\xi_r < \theta_d$, and let θ_i be the smallest eigenvalue that is not a zero of f_k (the existence of such a θ_i is guaranteed from the condition $r \leq k$). Then, the polynomial $g_r(x) = \frac{x-\theta_i}{\theta_0-\theta_i} \prod_{j=1}^{r-1} \frac{x-\xi_i}{\theta_0-\xi_i}$, with degree $r \leq k$ satisfies the conditions $g_k(\theta_0) = 1$, $g_k(\theta_i) \geq 0$ for $i = 1, \ldots, d$, and $\Psi(g_k) < \Psi(f_k)$ since $\frac{\theta_j-\theta_i}{\theta_0-\theta_i} < \frac{\theta_j-\xi_r}{\theta_0-\xi_r}$ for $j > i$, a contradiction with the fact that $\Psi(f_k)$ is minimum. Second, let us prove, again by contradiction, that $\xi_1 > \theta_0$. Otherwise, we could consider the polynomial g_{k-1}, with degree $r-1 < k$, defined as $g_{k-1}(x) = \prod_{i=2}^{r} \frac{x-\theta_i}{\theta_0-\theta_i}$ satisfying again $g_{k-1}(\theta_0) = 1$, and $g_{k-1}k(\theta_i) \geq 0$ for $i = 1, \ldots, d-1$ since $\frac{\theta_i-\xi_1}{\theta_0-\xi_1} > 1$ for all $i = 1, \ldots, d$. But, from the same inequalities, we also have $\Psi(q_{k-1}) < \Psi(f_k)$, a contradiction.

Finally, assume that $r < k$. Since all zeros $\xi_1 \leq \cdots \leq \xi_r$ are in the interval $[\theta_d, \theta_0)$, we can consider again the smallest one θ_i such that $f_k(\theta_i) > 0$. Then, reasoning as before, the polynomial $g_{r+1}(x) = \frac{x-\theta_i}{\theta_0-\theta_i} \prod_{i=1}^{r} \frac{x-\xi_i}{\theta_0-\xi}$, with degree $r+1 \leq k$ leads to the desired contradiction $\Psi(g_{r+1}) < \Psi(f_k)$. □

The above results, together with $f_k(\theta_0) = 1$ and $f_k(\theta_i) \geq 0$ for $i = 1, \ldots, d$ drastically reduce the number of possible candidates for f_k. Thus, if k is even, then there exists some index set $I \subset \{1, \ldots d\}$ of size $|I| = k$ such that

$$f_k(x) = \prod_{i \in I} \frac{x-\theta_i}{\theta_0-\theta_i}. \tag{1}$$

When k is odd, (1) also applies, but we can impose that $d \in I$. Let us see some particular examples:

- The cases $k = 0$ and $k = d$ are easy. Clearly, $f_0 = 1$, and f_d has zeros at all the points θ_i for $i \neq 0$. In fact, $f_d = \frac{1}{n} H$, where H is the Hoffman polynomial [24].
- For $k = 1$, the only zero of f_1 must be θ_d. Hence,

$$f_1(x) = \frac{x-\theta_d}{\theta_0-\theta_d}. \tag{2}$$

  Moreover, since $f_1(\theta_i) < 1$ for every $i = 1, \ldots, d$, we have that

$$(1 =)\Psi(f_d) < \Psi(f_{d-1}) < \Psi(f_{d-2}) < \cdots < \Psi(f_1) < \Psi(f_0) (= n),$$

  since, for $k = 0, \ldots, d-1$, $f_{k+1}(\theta_i) \leq f_k f_1(\theta_i) < f_k(\theta_i)$ for every $i = 1, \ldots, d$.

- For $k = 2$, the two zeros of f_2 must coincide with consecutive eigenvalues θ_i and θ_{i-1}. More precisely, the same reasonings used in Abiad, Coutinho, and F. [2] shows that θ_i must be the largest eigenvalue not greater than -1. Then, with these values,

$$f_2(x) = \frac{(x-\theta_i)(x-\theta_{i-1})}{(\theta_0-\theta_i)(\theta_0-\theta_{i-1})}. \tag{3}$$

- When $k = 3$, the only possible zeros of f_3 are θ_d and the consecutive pair θ_i, θ_{i-1} for some $i \in [2, d-1]$. In this case, empirical results seem to point out that such a pair must be around the 'center' of the mesh (see the examples below). In fact, since, as mentioned above, our results holds with any polynomial $f \in \mathbb{R}_k[x]$ satisfying $f(\theta_0) = 1$ and $f(\theta_i) \geq 0$ for $i = 1, \ldots, d$, a good (and usually optimal) choice is to take as "f_3" the polynomial $f_3 = f_1 f_2$.

- When $k = d-1$, the polynomial f_{d-1} takes only one non-zero value at the mesh, say at θ, which seems to be located at one of the 'extremes' of the mesh. In fact, when G is an r-antipodal distance-regular graph, we show in the last section that either $\theta = \theta_1$ or $\theta = \theta_d$ for odd d yields the tight bound (that is, r) for α_{d-1}, as does Theorem 1.3. Consequently, for such a graph with odd d, we have two different $(d-1)$-minor polynomials, say f and g, and, hence, infinitely many $(d-1)$-minor polynomials of the form $h = \gamma f + (1-\gamma) g$, where $\gamma \in [0, 1]$. (Notice that, if $\gamma \notin \{0, 1\}$, then h must have some zero not belonging to the mesh $\{\theta_1, \ldots, \theta_d\}$.)

Now, let us give all the k-minor polynomials, with $k = 1, \ldots, d$, for two particular distance-regular graphs. Namely, the Hamming graph $H(2, 7)$ and the Johnson graph $J(14, 7)$ (for more details about these graphs, see, for instance, Brouwer, Cohen, and Neumaier [5]). First, we recall that the Hamming graph $H(2, 7)$ has spectrum

$$\operatorname{sp} H(2,7) = \{7^1, 5^7, 3^{21}, 1^{35}, -1^{35}, -3^{21}, -5^7, -7^1\}.$$

Then, the different minor polynomials $f_0, \ldots, f_7$ are shown in Figure 1, and their values $x_i = f_k(\theta_i)$ at the different eigenvalues $\theta_0, \ldots, \theta_7$ are shown in Table 1.

k	x_7	x_6	x_5	x_4	x_3	x_2	x_1	x_0
1	0	1/7	2/7	3/7	4/7	5/7	6/7	1
2	1	1/2	1/6	0	0	1/6	1/2	1
3	0	1/14	1/21	0	0	5/42	3/7	1
4	2/9	0	0	1/45	0	0	2/9	1
5	0	1/35	0	0	0	0	6/35	1
6	1	0	0	0	0	0	0	1
7	0	0	0	0	0	0	0	1

Table 1: Values $x_i = f_k(\theta_i)$ of the k-minor polynomials of the Hamming graph $H(2, 7)$.

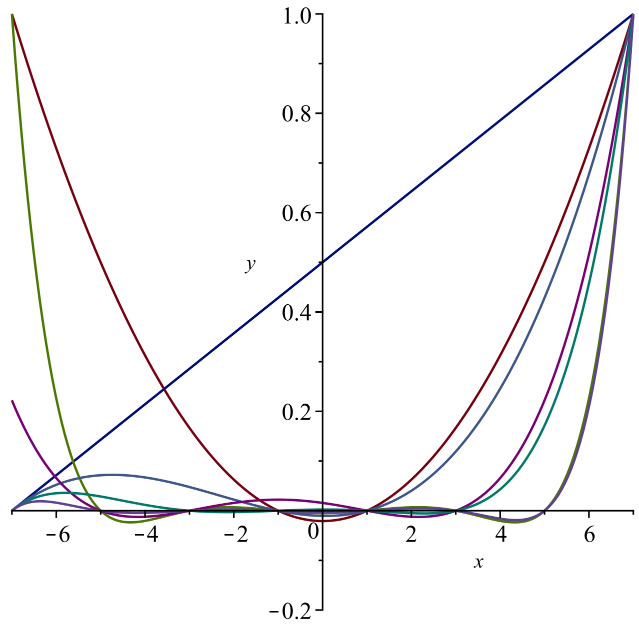

Figure 1: The minor polynomials of the Hamming graph $H(2,7)$.

As another example, consider the the Johnson graph $J(14,7)$ (see, for instance, Brouwer, Cohen, and Neumaier [5] and Godsil [20]). This is an antipodal (but not bipartite) distance-regular graph, with $n = 3432$ vertices, diameter $D = 7$, and spectrum

$$\operatorname{sp} J(14,7) = \{49^1, 35^{13}, 23^{77}, 13^{273}, 5^{637}, -1^{1001}, -5^{1001}, -7^{429}\}.$$

Then, the solutions of the linear programming problem are in Table 2, which correspond to the minor polynomials shown in Figure 2.

k	x_7	x_6	x_5	x_4	x_3	x_2	x_1	x_0
1	0	1/28	3/28	3/14	5/15	15/28	3/4	1
2	9/275	1/55	0	0	14/275	54/275	27/55	1
3	0	5/1232	1/176	0	0	75/1232	5/16	1
4	1/1485	0	0	0	0	14/495	2/9	1
5	0	1/2860	0	0	0	0	27/260	1
6	0	0	0	0	0	0	1/13	1
7	0	0	0	0	0	0	0	1

Table 2: Values $x_i = f_k(\theta_i)$ of the k-minor polynomials of the Johnson graph $J(14,7)$.

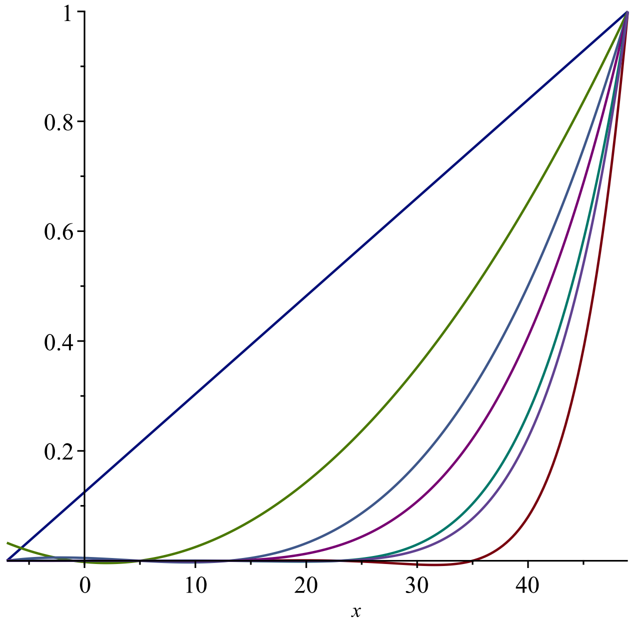

Figure 2: The minor polynomials of the Johnson graph $J(14,7)$.

4 A tight bound for the k-independence number

Now, we are ready to derive our main result about the k-independent number of a k-partially walk-regular graph. The proof is based on the interlacing technique. Then, we require the graph to be k-partially walk-regular to guarantee a constant value of the diagonal entries corresponding to the independent vertices.

Theorem 4.1. *Let G be a k-partially walk-regular graph with n vertices, adjacency matrix $\boldsymbol{A}$, and spectrum $\operatorname{sp} G = \{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$. Let $f_k \in \mathbb{R}_k[x]$ be a k-minor polynomial. Then, for every $k = 0, \ldots, d-1$, the k-independence number α_k of G satisfies*

$$\alpha_k \leq \operatorname{tr} f_k(\boldsymbol{A}) = \sum_{i=0}^{d} m_i f_k(\theta_i). \tag{4}$$

Proof. Let U be a k-independent set of G with $r = |U| = \alpha_k(G)$ vertices. Again, assume the first columns (and rows) of $\boldsymbol{A}$ correspond to the vertices in U. Consider the partition of said columns according to U and its complement. Let $\boldsymbol{S}$ be the normalized characteristic matrix of this partition. The quotient matrix of $f_k(\boldsymbol{A})$ associated with this partition is given by

$$\boldsymbol{S}^T f_k(\boldsymbol{A})\boldsymbol{S} = \boldsymbol{B}_k = \begin{pmatrix} \frac{1}{r}\sum_{u\in U}(f_k(\boldsymbol{A}))_{uu} & f_k(\theta_0) - \frac{1}{r}\sum_{u\in U}(f_k(\boldsymbol{A}))_{uu} \\ \frac{rf_k(\theta_0)-\sum_{u\in U}(f_k(\boldsymbol{A}))_{uu}}{n-r} & f_k(\theta_0) - \frac{rf_k(\theta_0)-\sum_{u\in U}(f_k(\boldsymbol{A}))_{uu}}{n-r} \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} \frac{1}{n}\sum_{i=0}^d m_i f_k(\theta_i) & 1 - \frac{1}{n}\sum_{i=0}^d m_i f_k(\theta_i) \\ \frac{r-\frac{r}{n}\sum_{i=0}^d m_i f_k(\theta_i)}{n-r} & 1 - \frac{r-\frac{r}{n}\sum_{i=0}^d m_i f_k(\theta_i)}{n-r} \end{pmatrix}, \tag{6}$$

with eigenvalues $\mu_1 = f_k(\lambda_1) = 1$ and

$$\mu_2 = \operatorname{tr}\boldsymbol{B}_k - 1 = \Psi(f_k) - \frac{r - r\cdot w(f_k)}{n-r}.$$

where we recall that $\Psi(f_k) = \frac{1}{n}\sum_{i=0}^d m_i f_k(\theta_i)$. Then, by interlacing, we have

$$0 \le \mu_2 \le w(f_k) - \frac{r - r\cdot\Psi(f_k)}{n-r}, \tag{7}$$

whence, solving for r, we get $r \le n\cdot\Psi(f_k)$ and the result follows. □

As mentioned in the previous section, notice that, in fact, this proof works for any polynomial f satisfying $f(\theta_0) = 1$ and $f(\theta_i) \ge 0$ for $i = 1,\dots,d$. By way of example, if G is a distance-regular graph with distance polynomials $p_0,\dots,p_d$, we could take $f(x) = \frac{q_k^2(x)}{q_k^2(\theta_0)}$, with degree $2k$, where the sum polynomial $q_k = p_0 + \cdots + p_k$ satisfies $\|q_k\|_G^2 = q_k(\theta_0)$. Now, recall that $q_k(\theta_0) = n_k$ corresponds to the number of vertices at distance at most k from any vertex of G (see, for instance, Biggs [4]). Thus, we obtain

$$\alpha_{2k} \le \Psi(p) = \sum_{i=0}^d m_i \frac{q_k^2(\theta_i)}{q_k^2(\theta_0)} = \frac{n}{q_k^2(\theta_0)}\|q_k\|_G^2 = \frac{n}{n_k},$$

as expected.

Another possibility is to use the polynomial $f(x) = \frac{P_k(x)+1}{P_k(\theta_0)+1}$, where P_k is the k-alternating polynomial. In this case, when G is an r-antipodal distance-regular graphs and $k = d-1$, it turns out that the d-distance polynomial is $p_d = H - \frac{r}{2}P_{d-1} + \frac{r}{2} - 1$, where H is the Hoffman polynomial (see F. [13]). Then, we get $\Psi(f) = \frac{2n}{P_{d-1}(\theta_0)+1}$, which coincides with the bound for α_{d-1} given in Theorem 1.3.

Let us now consider some particular cases of Theorem 4.1 by using the minor polynomials. As before, in what follows, G has eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$, and θ_i is the largest eigenvalues not greater than -1. Notice that the results for $k = 1, 2$ were already known; see Hoffman [22] and Abiad, Coutinho, and F. [2], respectively.

The case $k = 1$.

As mentioned above, α_1 coincides with the standard independence number α. In this case the minor polynomial is $f_1(x) = \frac{x-\theta_d}{\theta_0-\theta_d}$. Then, (4) gives

$$\alpha_1 = \alpha \leq \operatorname{tr} f_1(\boldsymbol{A}) = \frac{-n\theta_d}{\theta_0 - \theta_d}, \tag{8}$$

which is Hoffman's bound in Theorem 1.2.

The case $k = 2$.

We already stated that $f_2(x) = \frac{(x-\theta_i)(x-\theta_{i-1})}{(\theta_0-\theta_i)(\theta_0-\theta_{i-1})}$. Then, (4) yields

$$\alpha_2 \leq \operatorname{tr} f_2(\boldsymbol{A}) = n\frac{\theta_0 + \theta_i\theta_{i-1}}{(\theta_0 - \theta_i)(\theta_0 - \theta_{i-1})}, \tag{9}$$

in agreement with the result of Abiad, Coutinho, and F. [2] (here in Theorem 1.6(i)). Moreover, in the same paper, two infinite families of (distance-regular) graphs where the bound (9) is tight were provided.

The case $k = 3$.

Assume that G is at least 3-partially walk-regular, and let n_t be the common number of triangles rooted at every vertex. Then, if, as commented above, we take $f_3 = f_1 f_2$, we get

$$\alpha_3 \leq \operatorname{tr} f_3(\boldsymbol{A}) = n\frac{2n_t - \theta_0(\theta_d + \theta_i + \theta_{i-1}) - \theta_d\theta_i\theta_{i-1}}{(\theta_0 - \theta_d)(\theta_0 - \theta_i)(\theta_0 - \theta_{i-1})}. \tag{10}$$

In particular, if G is bipartite, then $n_t = 0$ and, as we will see in the next section, the obtained results for the Hamming graphs are optimal (in the sense that they coincide with the Delsarte's linear programming bound).

The general case $1 \leq k \leq d$

In general, from (1), we can state the following: Let $I \subset \{1, \ldots, d\}$ range over all index sets with k elements (in the case of k odd we can also require that $d \in I$). Then,

$$\alpha_k \leq \operatorname{tr} f_k(\boldsymbol{A}) = \min_I \sum_{j \notin I} m_j \prod_{i \in I} \frac{\theta_j - \theta_i}{\theta_0 - \theta_i}. \tag{11}$$

4.1 The Shannon capacity of G^k

Now, let us see that the upper bound for the k-independence number in (4) also holds for the Shannon capacity Θ of the power graph G^k when G is walk-regular. The parameter Θ was introduced by Shannon [29], the founder of information theory, for a general graph $G=(V,E)$, and it is defined as follows. Let $G^{\boxtimes \ell}$ be the strong product $G\boxtimes \stackrel{(\ell)}{\cdots} \boxtimes G$ of ℓ copies of G, with vertex set the Cartesian product $V\times \stackrel{(\ell)}{\cdots} \times V$, and vertex $(u_1,\ldots,u_\ell)$ is adjacent to vertex $(v_1,\ldots,v_\ell)$ when, for any $1\le i\le \ell$, either $u_i=v_i$ or u_i is adjacent to v_i. The adjacency matrix of $G^{\boxtimes\ell}$ is $(\boldsymbol{A}+\boldsymbol{I})^{\otimes \ell}-\boldsymbol{I}$, where the first term denotes the Kronecker product of ℓ copies of the matrix $\boldsymbol{A}+\boldsymbol{I}$. Then, the *Shannon* (*zero-error*) *capacity* of G is defined as

$$\Theta(G)=\sup_{\ell}\sqrt[\ell]{\alpha(G^{\boxtimes\ell})}=\lim_{\ell\to\infty}\sqrt[\ell]{\alpha(G^{\boxtimes\ell})}.$$

Notice that, since $\alpha(G^{\boxtimes\ell})\ge \alpha(G)^\ell$, the Shannon capacity gives an upper bound for the independence number, $\alpha(G)\le\Theta(G)$. In our context, we have the following result.

Proposition 4.2. *Let G be a walk-regular graph with n vertices, adjacency matrix $\boldsymbol{A}$, and spectrum $\operatorname{sp} G=\{\theta_0^{m_0},\ldots,\theta_d^{m_d}\}$. Let $f_k\in\mathbb{R}_k[x]$ be a k-minor polynomial of G. Then, for every $k=0,\ldots,d$, the Shannon capacity of the power graph $H=G^k$ satisfies*

$$\Theta(G^k)\le \operatorname{tr} f_k(\boldsymbol{A})=\sum_{i=0}^{d} m_i f_k(\theta_i). \tag{12}$$

Proof. First note that, given a regular graph H, the proof of Theorem 4.1 works for any symmetric matrix $\boldsymbol{A}_H$ such that $(\boldsymbol{A}_H)_{uv}=0$ if $\operatorname{dist}_H(u,v)>k$, $\boldsymbol{A}_H$ has constant positive row sum and diagonal, and $\operatorname{tr}\boldsymbol{A}_H>0$. Then, we can apply Theorem 4.1 to the matrix $\boldsymbol{A}_\ell=f_k(\boldsymbol{A})^{\otimes\ell}$, which satisfies the above conditions with respect to the graph $H^{\boxtimes\ell}$ with $H=G^k$. Indeed, $\boldsymbol{A}_\ell$ has constant row sum $f_k(\theta_0)^\ell$, constant diagonal (because $f_k(\boldsymbol{A})$ and its Kronecker product $f_k(\boldsymbol{A})^{\otimes\ell}$ have constant diagonals), and $\operatorname{tr}\boldsymbol{A}_\ell=(\operatorname{tr} f_k(\boldsymbol{A}))^\ell$. So, we get

$$\alpha(H^{\boxtimes\ell})\le \operatorname{tr}\boldsymbol{A}_\ell=\left(\textstyle\sum_{i=0}^{d} m_i f_k(\theta_i)\right)^\ell,$$

and the result follows. □

Form this proposition, and recalling that the Shannon capacity is an upper bound for the independence number, we have the following corollary.

Corollary 4.3. *Let G be a walk-regular graph as above. Let f_k be a minor polynomial of G for a given $k=1,\ldots,d$. Then,*

$$\alpha_k(G)=\alpha(G^k)\le\Theta(G^k)\le\Psi(f_k)=\sum_{i=0}^{d} m_i f_k(\theta_i).$$

Moreover, if $\Psi(f_k)$ is an integer coinciding with the exact value of the k-independence number, then $\Theta_k(G)=\Psi(f_k)$

5 Some examples in distance-regular graphs

In this section, we first deal with distance-regular graphs. (Recall that every distance-regular graph is also walk-regular.) to compare the above bounds with those obtained in F. [13] and Abiad, Cioabă, and Tait [1] (here in Theorems 1.3 and 1.5, respectively), together with the results obtained from the Delsarte's linear programming bound (see Delsarte[10], and Delsarte and Levenshtein [11]).

In our context, Delsarte's results can be formulated in the following way. Let Γ be a distance-regular graph Γ with diameter d, and let C be a subset of r vertices. Then, the *inner distribution* r_k of C, for $0 \le k \le d$, is the mean number of vertices v in C at distance k (in Γ) from a given vertex $u \in C$, that is, $r_k = \frac{1}{|C|}\sum_{u\in C}|\Gamma_k(u)\cap C|$ for $k = 0,\dots,d$. Notice that, as commented by Godsil [20], the numbers r_k determine the probability that a randomly chosen pair of vertices from C are at distance k. By using the (pre)distance polynomials $p_0,\dots,p_d$ of Γ, F. and Garriga [16] showed that the so-called C-multiplicities of each eigenvalues of C are

$$m_C(\theta_i) = \frac{m(\theta_i)}{n}\sum_{k=0}^{d} r_k \frac{p_k(\theta_i)}{p_k(\theta_0)} \qquad (0 \le i \le d). \tag{13}$$

In fact, (13) is essentially equivalent to Delsarte's identity $b = aQ$ which gives rise to the celebrated linear programming bound (see Delsarte [10], and Delsarte and Levenshtein [11]), based on the non-negativity of the C-local eigenvalues. In our case, as C is a k-independent set, we have $r_0 = 1$, $r_1 = \cdots = r_k = 0$, $r = |C| = r_0 + \cdots + r_d$, and the linear programming problem to be solved is the following.

$$\boxed{\begin{array}{r} \texttt{maximize}\ r := 1 + \sum_{i=k+1}^{d} r_i \\ \texttt{subject to}\ m_C(\theta_j) \ge 0,\ j = 0,1,\dots,d; \\ r_i \ge 0,\ i = k+1,\dots,d. \end{array}} \tag{14}$$

5.1 Hamming graphs

Let us first consider the Hamming graph $H(2,7)$ again. Thus, in Table 3 we show the bounds obtained for $\alpha_k(H(2,7))$. Note that, in general, the bounds obtained by Theorem 4.1 constitute a significant improvement with respect to those in F. [13], and Abiad, Cioabă, and Tait [1], and coincide with those obtained from the Delsarte's linear programming bound. Moreover, the bounds for $k \ne 4$ are equal to the exact values. In particular, $\alpha_1 = 64$ and $\alpha_6 = 2$ since the Hamming graphs are bipartite and 2-antipodal; $\alpha_7 = 1$ since $H(2,7)$ has diameter $D = 7$; and the exact value $\alpha_2 = 16$ since $H(2,7)$ contains the Hamming$(7,4)$ perfect code.

For the case $k = 2$, it is possible to get tight bounds for every Hamming graph $H(2,d)$. As commented, in this case, the 3-minor polynomial is $f_3(x) = f_1(x)f_2(x)$ with zeros at

k	1	2	3	4	5	6	7
Bound from Theorem 1.3	109	72	36	19	7	2	–
Bound from Theorem 1.5 ($k > 2$)	-	-	65	67	64	65	64
Bound from Theorem 1.6(i)-(iii)	-	21	56	6	55	3	55
Bound from Delsarte (14)	64	16	8	3	2	2	1
Bound from Theorem 4.1	64	16	8	3	2	2	1
Exact value	64	16	8	2	2	2	1

Table 3: Comparison of the bounds for α_k in the Hamming graph $H(2,7)$.

$\theta_d = -d$, for $\theta_i \in \{-2,-1\}$ (the largest eigenvalues not greater than -1), and $\theta_{i-1} \in \{0,1\}$. Then, we obtain

$$\alpha_3 \leq \frac{2^{d-1}}{d+1} \quad (d \text{ odd}), \qquad \text{and} \qquad \alpha_3 \leq \frac{2^{d-1}}{d} \quad (d \text{ even}).$$

5.2 The Johnson graph $J(14,7)$

As an example of a (non-bipartite) distance-regular graph, consider now the Johnson graph $J(14,7)$. The comparative results are now in Table 4. Note that, in contrast with the case of $H(2,7)$, for small values of k, namely $k = 3,4$, Delsarte's linear programming bound is much better. Again, as in the case of Hamming's graph, the bounds for $k = 6,7$ are equal to the correct values $\alpha_6 = 2$ and $\alpha_7 = 1$, since $J(14,7)$ is also 2-antipodal and it has diameter $D = 7$. In this case, the other exact values are not shown because the computation on a standard laptop failed. (While $G = J(14,7)$, and all their powers, have 3432 vertices, the first values of the table about the graph power G^3 requires to consider 2,942,940 edges!)

k	3	4	5	6	7
Bound from Theorem 1.3	464	125	20	2	–
Bound from Theorem 1.5	935	721	546	408	302
Bound from Theorem 1.6(ii)-(iii)	26	10	5	3	2
Bound from Theorem 1.6(iv)	80	86	25	2	1
Bound from Delsarte (14)	8	3	2	2	1
Bound from Theorem 4.1	19	6	2	2	1

Table 4: Comparison of bounds for α_k in the Johnson graph $J(14,7)$.

5.3 Antipodal distance-regular graphs

Finally, we consider an infinite family of distance-regular graphs where our bound for α_{d-1} is tight. With this aim, we assume that the minor polynomial takes non-zero value only at θ_1. Thus, $f_{d-1}(x) = \frac{1}{\prod_{i=2}^{d}(\theta_0-\theta_i)}\prod_{i=2}^{d}(x-\theta_i)$. Then, the bound (4) of Theorem 4.1 is

$$\sum_{i=0}^{d} m_i f_{d-1}(\theta_i) = m_0 f_{d-1}(\theta_0) + m_1 f_{d-1}(\theta_1) = 1 + m_1\frac{\prod_{i=2}^{d}(\theta_1-\theta_i)}{\prod_{i=2}^{d}(\theta_0-\theta_i)} = 1 + m_1\frac{\pi_1}{\pi_0}$$

where, in general, $\pi_i = \prod_{j=0,j\neq i}|\theta_i - \theta_j|$ for $i = 0, 1, \ldots, d$. Now suppose that G is an r-antipodal distance-regular. Then, in F. [13], it was shown that G is an r-antipodal distance-regular graph if and only if its eigenvalue multiplicities are $m_i = \pi_0/\pi_i$ for i even, and $m_i = (r-1)\pi_0/\pi_i$ for i odd. So, with $m_1 = (r-1)\pi_0/\pi_1$, we get

$$\alpha_{d-1} \leq 1 + m_1\frac{\pi_1}{\pi_0} = r,$$

which is the exact value.

When G is an r-antipodal distance-regular graph with odd d, we can also consider the minor polynomial g_{d-1}, which takes non-zero value only at θ_d, that is $g_{d-1}(x) = \frac{1}{\prod_{i=1}^{d-1}(\theta_0-\theta_i)}\prod_{i=1}^{d-1}(x-\theta_i)$. Then, reasoning as above, we get again the tight bound $\Psi(g_{d-1}) = 1 + m_d\frac{\pi_d}{\pi_0} = r$.

5.4 Odd graphs

For every integer $\ell \geq 2$, the odd graphs O_ℓ constitute a well-known family of distance-regular graphs with interactions between graph theory and other areas of combinatorics, such as coding theory and design theory. The vertices of O_ℓ correspond to the $(\ell-1)$-subsets of a $(2\ell-1)$-set, and adjacency is defined by void intersection. In particular, O_3 is the Petersen graph. In general, the odd O_ℓ is a ℓ-regular graph with order $n = \binom{2\ell-1}{\ell-1} = \frac{1}{2}\binom{2\ell}{\ell}$, diameter $\ell-1$, and its eigenvalues and multiplicities are $\theta_i = (-1)^i(\ell-i)$ and $m(\theta_i) = \binom{2\ell-1}{i} - \binom{2\ell-1}{i-1}$ for $i = 0, 1, \ldots, \ell-1$. For more details, see for instance, Biggs [4] and Godsil [20].

In Table 5 we show the bounds of the k-independence numbers for O_ℓ, $\ell = 2, 3, 4, 5$ given by Theorem 4.1. The numbers in bold faces, 7 and 66, correspond to the known values of 1-perfect codes in O_4 and O_6, respectively. As in the case of Hamming graphs, all the obtained bounds coincide with the Delsarte's LP bounds.

More generally, (8) and (9) allow us to compute the bounds for α_1 and α_2 of every

graph / k	2	3	4	5
O_4	**7**	–	–	–
O_5	13	8	–	–
O_6	**66**	21	11	–
O_7	158	90	17	12

Table 5: Some bounds for α_k in odd graphs O_ℓ for $\ell = 4, 5, 6, 7$.

odd graph O_ℓ, which turn out to be

$$\alpha_1 \leq \frac{\binom{2\ell}{\ell}(\ell-1)}{4\ell-2} \sim \frac{2^{2\ell-2}}{\ell^{\frac{1}{2}}\sqrt{\pi}}, \tag{15}$$

$$\alpha_2 \leq \frac{\binom{2\ell}{\ell}(\ell-2)}{2(\ell+(-1)^\ell(\ell-2(-1^\ell))} \sim \frac{2^{2\ell-1}}{\ell^{\frac{3}{2}}\sqrt{\pi}}, \tag{16}$$

where we have indicated their asymptotic behavior, when $\ell \to \infty$, by using the Stirling's formula. As a consequence, we have the known result that, when ℓ is odd, the odd graph O_ℓ has no 1-perfect code. Indeed, the existence of a 1-perfect code in O_ℓ requires that $\alpha_2 = \frac{n}{\ell+1} = \frac{\binom{2\ell}{\ell}}{2(\ell+1)}$ (since all codewords must be mutually at distance ≥ 3). However, when ℓ is odd, (16) gives $\alpha_2 \leq \frac{\binom{2\ell}{\ell}(\ell-2)}{2(\ell-1)(\ell+2)} < \frac{\binom{2\ell}{\ell}}{2(\ell+1)}$, a contradiction. (In fact, when n is a power of two minus one, $\frac{\binom{2\ell}{\ell}}{2(\ell+1)}$ is not an integer, which also prevents the existence of a 1-perfect code.) Note that this result is in agreement with the fact that a necessary condition for a regular graph to have a 1-perfect code is the existence of the eigenvalue -1, which is not present in O_ℓ when ℓ is odd (see Godsil [20]).

Finally, by using the same polynomial as in Subsection 5.3, we have that the $(d-1)$-independence number of O_ℓ, where $d-1 = \ell-2$, satisfies the bounds

$$\alpha_{\ell-2} \leq 1 + m_1\frac{\pi_1}{\pi_0} = \begin{cases} 2\ell-1, & \ell \text{ even}, \\ 2\ell-2, & \ell \text{ odd}. \end{cases}$$

For instance, for the Petersen graph $P = O_3$, this yields $\alpha_1 \leq 4$, as it is well-known.

6 Some examples in walk-regular graphs

In this section, we consider some walk-regular graphs that are not distance-regular. Thus, in this case, Delsarte's LP bound does not apply. Instead, we compare our results with those obtained with the Lovász theta number, commonly denoted by θ. This is a well-known parameter introduced by Lovász [27] in order to bound the Shannon capacity Θ of a graph G, $\theta(G) \leq \Theta(G)$. It is also known as Lovász theta function and it can be computed by semidefinite programming (SDP). Two possible definitions of this parameter are the following: Let $G=(V,E)$ be a graph on n vertices. Let $\boldsymbol{A}=(a_{uv})$ range over all $n\times n$ symmetric matrices such that $a_{uv}=1$ when $u=v$ or $uv\notin E$. Let $\rho(\boldsymbol{A})$ be the spectral radius of $\boldsymbol{A}$. Then, the *Lovász theta number* of G is

$$\theta(G)=\min_{\boldsymbol{A}}\rho(\boldsymbol{A}). \tag{17}$$

Alternatively, the dual method to this is $\theta(G)=\max_{\boldsymbol{B}}\operatorname{tr}(\boldsymbol{BJ})$, where $\boldsymbol{B}$ ranges over all $n\times n$ symmetric positive semidefinite matrices such that $b_{uv}=0$ for every $uv\in E$ and $tr\boldsymbol{B}=1$, and $\boldsymbol{J}$ is the all-1 matrix (see Lovász [27]).

It is known that the independence number is upper bounded by the Lovász theta number, so that, in our context, $\alpha_k(G)=\alpha_k(G^k)\leq\theta(G^k)$. As a consequence, our spectral bounds for α_k could be all obtained as the values of some feasible solutions to the minimization formulation of the Lovász theta semidefinite programming. Therefore, they are all greater than or equal to the Lovász theta numbers of $\theta(G^k)$. However, for k-partially walk-regular graphs, computing our spectral bound $\Psi(f_k)$ through the minor polynomials is significantly faster than solving an SDP, and, in many cases, we have equality, $\Psi(f_k)=\theta(G^k)$, as shown in Table 6. Moreover, as commented in the Introduction, we think that the minor polynomials have interest on their own, and could be used in other contexts. In fact, as it happened with the Shannon capacity, the upper bound of Theorem 4.1 also applies for the Lovász theta number,

$$\theta(G)\leq\Psi(f_k)=\operatorname{tr}f_k(\boldsymbol{A}).$$

To prove it, just consider the matrix $\boldsymbol{A}_k=\boldsymbol{J}-nf_k(\boldsymbol{A})+\Psi(f_k)\boldsymbol{I}$, which satisfies the conditions in (17) and it has spectral radius $\rho(\boldsymbol{A}_k)=n-nf_k(\theta_0)+\Psi(f_k)=\Psi(f_k)$.

Putting all together (Proposition 4.2 and the above results), if G is walk-regular, the following inequalities hold:

$$\alpha_k(G)\leq\theta(G^k)\leq\Theta(G_k)\leq\operatorname{tr}f_k(\boldsymbol{A}),$$

and, if the exact value of $\alpha_k(G)$ equals $\operatorname{tr}f_k(\boldsymbol{A})$ (so that both numbers are integers), then all inequalities become equalities. This gives a method to find the Shannon capacity $\Theta(G^k)$ being equal to the Lovász theta number $\theta(G^k)$. (For $k=1$ this was already noted by Lovász in [27]).

Apart from some known graphs, we compared our bounds with $GM\times K_2$, where GM is the walk-regular graph of Figure 3 (see Godsil and McKay [21]), and the graph $F20$ of Figure 4 obtained by Farrell [12], which is the first example of a distinct cubic walk-regular graph that is neither vertex-transitive nor distance-regular. The obtained results are in Table 6, where, as we mentioned, we observe that all our bounds coincide with the Lovász theta number.

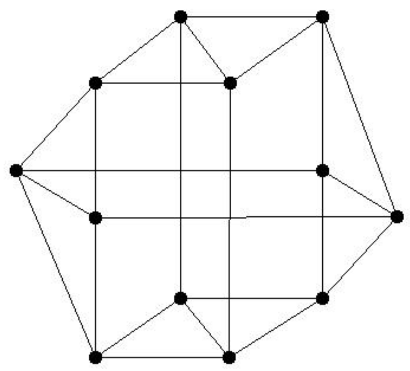

Figure 3: Godsil & McKay graph.

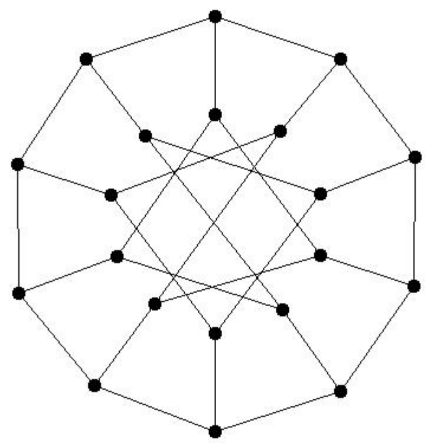

Figure 4: Farrell graph $F20$.

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$GM \times K_2$ [21]	k	2	3	4	5	6
	Bound from Lovász θ number	4	2	1	–	–
	Bound from Theorem 4.1	4	2	1	–	–
	Exact value	4	2	1	–	–
$F20$ [12]	k	2	3	4	5	6
	Bound from Lovász θ number	4	2	2	1	–
	Bound from Theorem 4.1	4	2	2	1	–
	Exact value	4	2	2	1	–
Folkman graph	k	2	3	4	5	6
	Lovász θ number	3	2	1	–	–
	Bound from Theorem 4.1	3	2	1	–	–
	Exact value	3	2	1	–	–
Gray graph	k	2	3	4	5	6
	Bound from Lovász θ number	11	9	3	3	1
	Bound from Theorem 4.1	11	9	3	3	1
	Exact value	11	9	3	3	1
Ljubljana graph	k	2	3	4	5	6
	Bound from Lovász θ number	27	18	10	8	2
	Bound from Theorem 4.1	27	18	10	8	3
	Exact value	26	17	8	8	2

Table 6: Comparison between bounds and exact values of α_k in some walk-regular graphs (not distance-regular).

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