

A POSTERIORI ERROR ESTIMATION FOR PARTIAL DIFFERENTIAL EQUATIONS WITH SMALL UNCERTAINTIES

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Abstract. Partial differential equations (PDEs) are widely used for modelling problems in many fields such as physics, biology or engineering. In general, any problem is affected by a certain level of uncertainty, due to the intrinsic variability of the system or the inability to adequately characterize all the involved input. Nowadays, it is common to include uncertainty in the mathematical models describing such complex systems, using for instance probability theory characterizing the uncertain input by random variables or more generally by random fields.

The purpose of this work is to perform *a priori* and *a posteriori* error analysis for PDEs with random input data. We consider small uncertainties and adopt a perturbation approach expanding the exact (random) solution u of a given problem up to a certain order as

$$u(\mathbf{x}, \omega) = u_0(\mathbf{x}) + \varepsilon u_1(\mathbf{x}, \omega) + \varepsilon^2 u_2(\mathbf{x}, \omega) + \dots \quad (1)$$

where ε is a parameter that controls the amount of randomness in the input data. Uncoupled deterministic problems can be derived to find each term in the expansion, the previous term being needed to compute the next one. Each of these problems can be solved approximately using for instance the finite element method (FEM). We derive *a*

a priori and *a posteriori* error estimators in various norms for the error between the exact solution and an approximation of a certain order. For instance for the first order approximation, which requires the resolution of only one deterministic problem, we obtain an *a posteriori* error estimator constituted of two computable parts, namely a part due to FE discretization (which depends on the mesh size) and another part due to the uncertainties affecting the input data. This estimator, easy and cheap to compute, can then be used for mesh adaptation to balance the two sources of error.

We apply this method to several classes of problems. We first consider the linear elliptic problem

$$-\operatorname{div}(a(\mathbf{x}, \omega) \nabla u(\mathbf{x}, \omega)) = f(\mathbf{x}) \quad (2)$$

where the random diffusion coefficient a depends in an affine way on a finite number of independent random variables. The derivation of *a priori* and *a posteriori* error estimates in various norms for the first order approximation $u \approx u_{0,h}$, with $u_{0,h}$ the continuous, piecewise linear finite element approximation of u_0 , is detailed in [1]. The analysis is straightforwardly extended to higher order approximations and to some class of nonlinear problems. We then consider the steady Navier-Stokes equations on randomly perturbed domains and, in particular, the flow past a cylinder with perturbation of the center or the outer shape of the cylinder. A stochastic mapping is introduced which transforms the original problem to PDEs on a deterministic reference domain with random coefficients. Finally, we consider the heat equation with random (Robin) boundary conditions. For this time-dependent problem, the *a posteriori* error estimator contains a third term due to time discretization.

REFERENCES

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