Analysis and Control of a Lur'e System with Sector-Bounded, Slope-Restricted Nonlinearities Using Linear and Bilinear Matrix Inequalities

Joint Bachelor's Thesis

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Abstract

ENG – The aim of this thesis is to analyze and control a Lur'e system, i.e., a system with a linear time-invariant forward path and a nonlinear static feedback with sector and slope constraints on its nonlinearities. This description encompasses a broad class of systems that commonly arise in a wide range of engineering disciplines. The approach of this work is based on Lyapunov stability theory and uses linear matrix inequalities to propose criteria for the system's absolute stability and ℓ^2- and RMS-gains, and bilinear matrix inequalities to propose criteria for state feedback control and state estimation. These types of conditions, in particular linear matrix inequalities, are conveniently treated computationally, as the optimization problems that they give rise to are convex. Numerical results are obtained for several examples, showing a significant improvement with respect to previous conditions in the literature.

CAT – L'objectiu d'aquesta tesi és l'anàlisi i el control d'un sistema de Lur'e, és a dir, un sistema amb un camí directe lineal i invariant en el temps i amb una realimentació no lineal estàtica, amb restriccions de sector i de pendent en les seves no-linealitats. Aquesta descripció engloba una àmplia classe de sistemes que apareixen comunament en una àmplia gamma de disciplines d'enginyeria. L'enfocament d'aquest treball es basa en la teoria d'estabilitat de Lyapunov i utilitza desigualtats de matrius lineals per proposar criteris per a l'estabilitat absoluta del sistema i els guanys ℓ^2 i RMS, i desigualtats de matrius bilineals per proposar criteris per al seu control per realimentació d'estat i estimació d'estat. Aquest tipus de condicions, en particular les desigualtats de matrius lineals, són convenients de tractar computacionalment, ja que els problemes d'optimització a que donen lloc són convexos. S'obtenen resultats numèrics per a diversos exemples, els quals mostren una millora significativa respecte a les condicions anteriors de la literatura.

ESP – El objetivo de esta tesis es el análisis y el control de un sistema de Lur'e, es decir, un sistema con un camino directo lineal e invariante en el tiempo y con una realimentación no lineal estática, con restricciones de sector y de pendiente en sus no-linealidades. Esta descripción engloba una amplia clase de sistemas que aparecen comúnmente en una amplia gama de disciplinas de ingeniería. El enfoque de este trabajo se basa en la teoría de estabilidad de Lyapunov y utiliza desigualdades de matrices lineales para proponer criterios para la estabilidad absoluta del sistema y las ganancias ℓ^2 y RMS, y desigualdades de matrices bilineales para proponer criterios para su control por realimentación de estado y estimación de estado. Este tipo de condiciones, en particular las desigualdades de matrices lineales, son convenientes tratar computacionalmente, ya que los problemas de optimización a que dan lugar son convexos. Se obtienen resultados numéricos para varios ejemplos, los cuales muestran una mejora significativa respecto a las condiciones anteriores de la literatura.

Keywords: Nonlinear control, Lyapunov stability, Lur'e systems, LMIs, BMIs.
Paraules clau: Control no lineal, estabilitat de Lyapunov, sistemes de Lur'e, LMIs, BMIs.
Palabras clave: Control no lineal, estabilidad de Lyapunov, sistemas de Lur'e, LMIs, BMIs.

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Contents

List of Abbreviations

LTI: Linear time invariant

LMI: Linear matrix inequality

BMI: Bilinear matrix inequality

1 Introduction

This thesis was conducted from February 2020 to July 2020 at the Massachusetts Institute of Technology under the supervision of Prof. Richard D. Braatz, as part of the requirements for graduation from CFIS, Universitat Politècnica de Catalunya – BarcelonaTech, with a double Bachelor's degree in Physics Engineering and in Mathematics.

The thesis studies a specific class of nonlinear systems known as Lur'e systems, essentially a combination of a linear time-invariant forward path and a nonlinear static feedback. In particular, the thesis deals with a subclass of these systems, which have sector bounds and slope restrictions on their nonlinearities.

Lur'e systems arise very commonly in a wide range of disciplines including chemical engineering, mechanical engineering, and aerospace engineering, among others. The sector and slope constraints on the nonlinearities do not largely reduce the applicability of the problem. A Lur'e system with these constraints can describe broad classes of nonlinear feedback behaviors, included systems described by dynamic neural networks. The neural network model of the nonlinearities will consist of functions such as hyperbolic tangents, which satisfy these constraints.

The aim of this thesis is to find conditions to ensure global asymptotic stability, compute input-output gains, and design state feedback control and state estimation for Lur'e systems with sector-bounded, slope-restricted nonlinearities. The approach in this thesis for the study of these systems is based on linear and bilinear matrix inequalities.

1.1 Mathematical Preliminaries

1.1.1 Linear and Bilinear Matrix Inequalities

This section summarizes the most important aspects of linear and bilinear matrix inequalities that are used in this thesis, and are obtained from the more detailed tutorial [1]. Another more complete study is available in the book [2].

A linear matrix inequality (LMI) has the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \prec 0, \tag{1}$$

where $x \in \mathbb{R}^m$ and the matrices $F_i \in \mathbb{R}^{n\times n},\ i = 0, 1, \ldots, m$ are symmetric and known. The matrix $F(x)$ is an affine function of the elements of the variable x. The inequality (1) denotes that $F(x)$ is a negative definite matrix, that is,

$$z^T F(x) z < 0,\ \ \forall z \neq 0,\ \ z \in \mathbb{R}^n.$$

LMIs can be defined analogously for positive definite, negative semidefinite, and positive semidefinite matrices. The first two cases are called *strict* LMIs, while the last two are called *nonstrict* LMIs.

Linear inequalities, convex quadratic inequalities, matrix norm inequalities, and various constraints from control theory such as Lyapunov and Riccati inequalities can all be written as LMIs. Thus, LMIs are a useful tool for solving a wide variety of optimization and control problems.

The LMI (1) is said to be *feasible* if the set $\{x \mid F(x) \prec 0\}$ is nonempty, and the analogous sets can be defined similarly for the rest of cases. An important property of LMIs is that the set $\{x \mid F(x) \prec 0\}$ is convex, i.e., (1) forms a convex constraint on x. To see this, let x, y be two vectors such that $F(x) \prec 0$ and $F(y) \prec 0$, and let $\lambda \in (0, 1)$. Then,

$$\begin{aligned}
F(\lambda x + (1-\lambda)y) &= F_0 + \sum_{i=1}^{m} (\lambda x_i + (1-\lambda)y_i)\, F_i \\
&= \lambda F_0 + (1-\lambda)F_0 + \lambda \sum_{i=1}^{m} x_i F_i + (1-\lambda) \sum_{i=1}^{m} y_i F_i \\
&= \lambda F(x) + (1-\lambda)F(y) \\
&\prec 0.
\end{aligned}$$

The advantage of formulating control problems in terms of convex optimizations (when possible) is that wide classes of convex optimizations can be solved in polynomial time. Convex optimizations often arise in engineering practice and many can be written as LMIs, which is the strength of using LMI formulations: convex optimizations over LMIs are solvable in polynomial time.

A bilinear matrix inequality (BMI) has the form

$$F(x, y) = F_0 + \sum_{i=1}^{m} x_i F_i + \sum_{j=1}^{l} y_j G_j + \sum_{i=1}^{m} \sum_{j=1}^{l} x_i y_j H_{ij} \prec 0, \tag{2}$$

where $F_i, G_j \in \mathbb{R}^{n\times n},\ i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, l$, are symmetric known matrices, $H_{ij} \in \mathbb{R}^{n\times n},\ i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, l$ are known matrices and $x \in \mathbb{R}^m$, $y \in \mathbb{R}^l$ are the variables.

As for LMIs, BMIs can be negative or positive definite and negative or positive definite, and the same definitions of *strict* and *nonstrict* apply. BMI feasibility is defined analogously to that of LMIs.

BMIs commonly arise when formulating control design procedures for those uncertain and/or nonlinear systems in which an LMI formulation is not available. In fact, nearly every problem of interest in control can be formulated in terms of optimizations over BMIs.

If x is fixed, a BMI is an LMI for y and, therefore, convex in y. Analogously, if y is fixed, a BMI is an LMI for x and convex in x. However, BMIs are not *jointly convex* in x and y, and control problems expressed as BMIs are not convex optimizations. BMI optimizations are NP-hard and cannot be ensured to be solvable in polynomial time. In more practical terms, this classification implies that algorithms for finding global solutions to optimizations over BMIs are not efficient for large-scale problems.

1.1.2 The S-Procedure for Quadratic Forms

The S-procedure is useful for reformulating mathematical structures that commonly arise in Lyapunov control as LMIs. Its statement and proof are extracted from [1] and shown below.

Lemma (S-Lemma or S-Procedure): Let $f_i(x)$, $i = 0, \dots, p$ be quadratic forms with respect to $x \in \mathbb{R}^n$: $f_i(x) = x^T T_i x$, where T_i are symmetric matrices. If there exist $\tau_1 \geq 0, \dots, \tau_p \geq 0$ such that

$$f_0 - \sum_{i=1}^{p} \tau_i f_i \geq 0, \quad \forall x, \tag{3}$$

or, equivalently,

$$T_0 - \sum_{i=1}^{p} \tau_i T_i \succcurlyeq 0,$$

then

$$x^T T_0 x \geq 0 \quad \forall x \text{ such that } x^T T_i x \geq 0,\ i = 1, \dots, p.$$

Proof: If there exist $\tau_1 \geq 0, \dots, \tau_p \geq 0$ such that (3) holds for all x then (3) also holds for all x such that $f_i(x) \geq 0, \forall i = 1, \dots, p$. Then, for all such x, it must hold that

$$f_0(x) \geq \sum_{i=1}^{p} \tau_i f_i(x) \geq 0$$

since the summation is over terms that are all nonnegative.

□

The S-procedure also holds, and is proved similarly, for the case where the main inequality is strict. If there exist $\tau_1 \geq 0, \dots, \tau_p \geq 0$ such that

$$T_0 - \sum_{i=1}^{p} \tau_i T_i \succ 0,$$

then

$$x^T T_0 x > 0 \quad \forall x \neq 0 \text{ such that } x^T T_i x \geq 0,\; i = 1, \dots, p.$$

1.1.3 The Schur Complement Lemma

Like the S-procedure, the Schur complement lemma is a useful method for formulating commonly arising control problems as LMIs. Its statement and proof are extracted from [1] and shown below.

Lemma (Schur Complement): Let $Q(x) = Q(x)^T, S(x) = S(x)^T$ and $R(x) = R(x)^T$ be functions of $x \in \mathbb{R}^n$. Then

$$\left.\begin{array}{c} Q(x) - S(x)R(x)^{-1}S(x)^T \succ 0 \\ R(x) \succ 0 \end{array}\right\} \iff \left[\begin{array}{cc} Q(x) & S(x) \\ S(x)^T & R(x) \end{array}\right] \succ 0.$$

Proof: $(\implies)$ Assume that

$$\left[\begin{array}{cc} Q(x) & S(x) \\ S(x)^T & R(x) \end{array}\right] \succ 0$$

and define

$$F(u,v) = \left[\begin{array}{c} u \\ v \end{array}\right]^T \left[\begin{array}{cc} Q(x) & S(x) \\ S(x)^T & R(x) \end{array}\right] \left[\begin{array}{c} u \\ v \end{array}\right]. \tag{4}$$

Then,

$$F(u,v) > 0,\;\; \forall [u\; v] \neq 0.$$

First, consider $u = 0$. Then,

$$F(0,v) = v^T R(x) v > 0,\;\; \forall v \neq 0 \implies R(x) \succ 0.$$

Next, consider

$$v = -R(x)^{-1}S(x)^T u, \text{ with } u \neq 0.$$

Then,

$$\begin{aligned} F(u,v) &= u^T \left(Q(x) - S(x)R(x)^{-1}S(x)^T\right) u > 0, \forall u \neq 0 \\ &\implies Q(x) - S(x)R(x)^{-1}S(x)^T \succ 0. \end{aligned}$$

$(\impliedby)$ Now assume

$$Q(x) - S(x)R(x)^{-1}S(x)^T \succ 0, \quad R(x) \succ 0$$

with $F(u,v)$ defined as in (4). Fix u and optimize over v, i.e.,

$$\nabla_v F^T = 2R(x)v + 2S(x)^T u = 0. \tag{5}$$

Since $R(x) \succ 0$, (5) gives a single extrema $v^* := -R(x)^{-1}S(x)^T u$. Plugging this expression into (4) gives $F(u, v^*) = u^T \left(Q(x) - S(x)R(x)^{-1}S(x)^T\right) u$. Since $Q(x) - S(x)R(x)^{-1}S(x)^T \succ 0$ the minimum of $F(u, v^*)$ occurs for $u = 0$, which also implies that $v^* = 0$. Thus, the minimum of $F(u, v)$ occurs at (0,0) and is equal to zero. Therefore, $F(u, v)$ is positive definite.

□

1.1.4 Initial Definitions and Notation

This thesis frequently refers to certain classes of static nonlinear functions, namely sector-bounded and slope-restricted nonlinearities, whose definitions are

$$\Phi_{sb}^{[0,\xi]} := \left\{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q} \mid \phi_i(\sigma)\left[\xi_i^{-1}\phi_i(\sigma) - \sigma\right] \leq 0, \forall \sigma \in \mathbb{R}, i = 1, \ldots, n_q\right\},$$

$$\Phi_{sb}^{[\alpha,\beta]} := \left\{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q} \mid [\phi_i(\sigma) - \alpha_i\sigma][\phi_i(\sigma) - \beta_i\sigma] \leq 0, \forall \sigma \in \mathbb{R},\ i = 1, \ldots, n_q\right\},$$

$$\Phi_{sr}^{[0,\mu]} := \left\{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q} \mid 0 \leq \frac{\phi_i(\sigma) - \phi_i(\hat{\sigma})}{\sigma - \hat{\sigma}} \leq \mu_i,\ \forall \sigma \neq \hat{\sigma} \in \mathbb{R} \text{ and for } i = 1, \ldots, n_q\right\},$$

$$\Phi_{sr}^{[0,\infty]} := \left\{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q} \mid 0 \leq \frac{\phi_i(\sigma) - \phi_i(\hat{\sigma})}{\sigma - \hat{\sigma}},\ \forall \sigma \neq \hat{\sigma} \in \mathbb{R} \text{ and for } i = 1, \ldots, n_q\right\}.$$

Other expressions appearing throughout the thesis are the integrals

$$\int_0^{q_{k,i}} \phi_i(\sigma)d\sigma \quad \text{and} \quad \int_0^{q_{k,i}} [\xi_i\sigma - \phi_i(\sigma)]d\sigma. \tag{6}$$

The variable $q_{k,i}$ can take any real value, including any value smaller than zero. Some properties used in the thesis to account for $q_{k,i} < 0$ are

$$\int_0^{q_{k,i}} \phi_i(\sigma)d\sigma = -\int_{q_{k,i}}^0 \phi_i(\sigma)d\sigma,$$

$$\int_0^{q_{k,i}} [\xi_i\sigma - \phi_i(\sigma)]d\sigma = -\int_{q_{k,i}}^0 [\xi_i\sigma - \phi_i(\sigma)]d\sigma.$$

1.2 Literature Review on the Lur'e Problem

In 1944, Lur'e and Postnikov formulated the Lur'e problem [3], which addressed the stability of a class of nonlinear feedback systems, which today are called Lur'e systems. These systems are described by a linear time-invariant (LTI) forward path together with a nonlinear feedback. The nonlinearities present in the feedback are considered to be memoryless and static, and can be either time-invariant or time-varying.

The forward path of a Lur'e system, being LTI, can be described by four matrices A, B, C, D. The nonlinear feedback ϕ satisfies the sector-boundedness condition

$$0 \leq \phi_i(\sigma)\sigma \leq \xi_i \sigma^2, \quad \forall \sigma \in \mathbb{R}, \tag{7}$$

for each of its components ϕ_i, or, equivalently, $\phi \in \Phi_{sb}^{[0,\xi]}$.

Although Lur'e systems are defined in both continuous and discrete time, this work deals with discrete-time Lur'e systems. For this reason, the rest of this thesis will consider this case exclusively, which has the form

$$\begin{cases} x_{k+1} = Ax_k + Bp_k \\ \quad q_k = Cx_k + Dp_k \\ \quad p_k = -\phi(q_k) \end{cases} \tag{8}$$

where $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^{n_p}$ are the state and nonlinear output vectors, respectively, k is the sampling instance, $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n_p}, C \in \mathbb{R}^{n_q \times n}, D \in \mathbb{R}^{n_q \times n_p}$, and $\phi \in \Phi_{sb}^{[0,\xi]}$. Each component i of ϕ is assumed to act uniquely on component i of the vector q and, thus, $n_p = n_q$.

The results of this thesis are also applicable to Lur'e systems with nonlinearities satisfying the more general sector conditions:

$$\alpha_i \sigma^2 \leq \phi_i(\sigma)\sigma \leq \beta_i \sigma^2, \quad \forall \sigma \in \mathbb{R} \tag{9}$$

or, equivalently, $\phi \in \Phi_{sb}^{[\alpha,\beta]}$ to transform into systems of the form (8) – with a sector condition of the form (7) – through a loop transformation. Details on this transformation can be found in [2].

The Lur'e problem, also called *the absolute stability problem*, aims to find sufficient conditions that ensure that the origin $x = 0$ is a globally asymptotically stable, or absolutely stable, equilibrium of system (8). Since this analysis problem was first posed, several criteria for stability have been developed, the most notable of which are outlined in this section.

The Circle criterion was first developed by Sandberg in 1964 [4] and can be seen as a generalization of the well-known Nyquist stability criterion, which only applies to LTI systems. A discrete-time formulation of the criterion can be found in [5], and is outlined below.

Circle Criterion (Discrete-Time Case): Let $n(z), d(z)$ be two coprime polynomials with $\deg(n(z)) < \deg(d(z))$ and let $G(z) = \frac{n(z)}{d(z)}$ be the frequency-domain transfer function for the system

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k \end{cases} \tag{10}$$

satisfying $G\left(e^{i\omega}\right) \geq 0 \;\; \forall\omega \in \mathbb{R}$ or $G(z) \geq 0$ for $|z| = 1$. Then, for system (8) with $D = 0$ and with a single nonlinearity, if $d(z)$ has no zeros outside the unit circle, then

1. All solutions of the system are bounded if $\phi \in \Phi_{sb}^{[\alpha,\beta]}$ and the Nyquist locus of $G\left(e^{z}\right)$ does not encircle or intersect the open disk which is centered on the negative real axis of the $G\left(e^{z}\right)$ plane and has as a diameter the segment of the negative real axis $(-1/\alpha, -1/\beta)$.

2. All solutions are bounded and approach zero at an exponential rate if there exists some $\epsilon > 0$ such that $\phi \in \Phi_{sb}^{[\alpha+\epsilon,\beta+\epsilon]}$ and the Nyquist locus behaves as in 1.

The Circle criterion has been extended to the multivariable nonlinearity case, e.g., [6].

A similar criterion to the Circle criterion was developed in 1963 by Tsypkin for discrete-time LTI-systems with a single nonlinear element [7] and is outlined below.

Tsypkin Criterion: Let $G(z)$ be the frequency-domain transfer function of the LTI part of system (8) as defined in the Circle criterion. Then, system (8) with $\phi \in \Phi_{sb}^{[0,\xi]}$ is absolutely stable if

$$\operatorname{Re} G(z) + \frac{1}{\xi} > 0, \quad \text{for } |z| = 1. \tag{11}$$

Geometrically, the Tsypkin criterion ressembles the Circle criterion in that absolute stability is ensured by the Nyquist locus of $G(z)$ not intersecting a forbidden region, which is defined on the negative real axis by the sector condition on the system's nonlinearity.

Also in 1964, Jury and Lee [8] considered a single-input single-output system not only with a sector condition $\phi = \Phi_{sb}^{[0,\xi]}$ on the nonlinearity but also a local slope restriction given by $\phi \in \Phi_{sr}^{[0,\mu]}$. This new consideration led to a less conservative absolute stability condition

$$\operatorname{Re} G(z)\left[1 + \lambda(z-1)\right] + \frac{1}{\xi} - \frac{\mu\lambda}{2}\left|(z-1)G(z)\right|^2 \geq 0, \quad \text{for } |z| = 1 \text{ and for some } \lambda \geq 0. \tag{12}$$

The condition (12) is equivalent to the Tsypkin criterion condition (11) for the case $\lambda = 0$. The result has been generalized to the multivariable case, e.g., [6].

The Circle, Tsypkin, and derived criteria are interpreted in the frequency domain and are based on the important result obtained by Kalman and Yakubovich in the early 1960s known as the Positive Real Lemma or the Kalman-Yakubovich Lemma. A simple statement of this lemma can be found in [2]. The lemma consists of a criterion for the transfer function of a system to be positive real and gives conditions for a quadratic Lyapunov function

$$V(x_k) = x_k^T P x_k,$$

where $P \succ 0$, to ensure the absolute stability of a Lur'e system.

More recent results focused on the time domain instead, and have used Lyapunov functions that take properties of the nonlinearities into account. This consideration is relevant because strictly quadratic Lyapunov functions are exact for linear systems but conservative for the analysis of nonlinear systems. These time-domain approaches modify the Lyapunov function to significantly reduce conservatism in the results. In some cases, the Lyapunov functions are the sum of a traditional quadratic term and one or more terms containing the system's nonlinearities, frequently in the form of integrals. These functions are known as Lur'e-Postnikov Lyapunov functions or modified Lur'e-Postnikov Lyapunov functions.

An example of this approach for $\phi = \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\infty]}$ includes a single integral term in the Lyapunov function [9],

$$V(x_{k+1}) = \bar{x}_k^T P \bar{x}_k + 2\sum_{i=1}^{n_q} Q_{ii} \int_0^{q_{k,i}} \phi_i(\sigma) d\sigma,$$

where $\bar{x}_k := \begin{bmatrix} x_{k+1} \\ q_k \end{bmatrix}$, $P \succ 0$, and $Q_{ii} \geq 0 \; \forall i = 1, \ldots, n_q$.

Another, less conservative, approach for this nonlinearity employs a Lyapunov function with additional terms depending on the nonlinearities [10],

$$\begin{aligned} V(x_{k+1}) = \bar{x}_k^T P \bar{x}_k &+ 2\sum_{l=0}^{k}\sum_{i=1}^{n_q} \phi_i^T(q_{k,i}) \left\{ q_{k,i} - \xi_i^{-1} \phi_i\left(q_{k,i}\right) \right\} + 2\sum_{i=1}^{n_q} Q_{ii} \int_0^{q_{k,i}} \phi_i(\sigma) d\sigma \\ &+ 2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_0^{q_{k+1,i}} (\xi_i \sigma - \phi_i(\sigma)) d\sigma, \end{aligned}$$

where $\bar{x}_k := \begin{bmatrix} x_{k+1} \\ q_k \end{bmatrix}$, $P \succ 0$, and Q_{ii} and $\tilde{Q}_{ii} \geq 0 \; \forall i = 1, \ldots, n_q$.

In these approaches, among others in the literature, the quadratic term acts explicitly not only on the state variables x_k but also on q_k and/or the output of the nonlinearities, p_k. Another approach using only this nonlinearity dependence in the Lyapunov function for $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$ used [11]

$$V(x_k) = \bar{x}_k^T P \bar{x}_k,$$

where $\bar{x}_k := \begin{bmatrix} x_k \\ \phi(q_k) \end{bmatrix}$ and $P \succ 0$.

This latter study went beyond analyzing the stability of an uncontrolled Lur'e system, and found sufficient conditions for its stability when a proportional controller is introduced. In other words, sufficient conditions were found for the existence of a matrix K that stabilizes a nominally unstable Lur'e system.

Asymptotic stability is then guaranteed by the Lyapunov stability criterion: using Lyapunov functions that are, by construction, strictly positive for $x_k \neq 0$ (and null for $x_k = 0$) and radially unbounded, and then finding LMIs that ensure that the Lyapunov function decreases at each sampling instance k, i.e.,

$$\Delta V(x_k) = V(x_{k+1}) - V(x_k) < 0, \quad \forall k \geq 0.$$

In the case of a Lyapunov function that includes integrals, finding that this condition is satisfied involves also finding quadratic upper bounds on the integral terms.

1.3 Problem Statement and Methods

This thesis considers the Lur'e system

$$\begin{cases} x_{k+1} = Ax_k + Bp_k \\ q_k = Cx_k + Dp_k \\ p_k = -\phi(q_k) \end{cases} \tag{13}$$

with all variables and matrices as defined in (8), but with $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$. As such, this thesis considers a subclass of the systems described by (8), since $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]} \subset \Phi_{sb}^{[0,\xi]}$.

1.3.1 Analytical Approach

In order to reduce the conservativeness of quadratic Lyapunov functions when used for the analysis of nonlinear systems, a modified Lur'e-Postnikov Lyapunov function is used, of the form

$$V(x_k) = \bar{x}_k^T P \bar{x}_k + 2\sum_{i=1}^{n_q} Q_{ii} \int_0^{q_{k,i}} \phi_i(\sigma) d\sigma + 2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_0^{q_{k,i}} [\xi_i \sigma - \phi_i(\sigma)] d\sigma, \tag{14}$$

where

$$\bar{x}_k := \begin{bmatrix} x_k \\ p_k \\ q_k \end{bmatrix}, \; P^T = P := \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix} \nsucc 0, \; P_{11} \succ 0, \text{ and } Q_{ii} \;\&\; \tilde{Q}_{ii} \geq 0, \;\; \forall i = 1, \ldots, n_q.$$

The dependence of the integral terms of the Lyapunov function (14) on the system's nonlinearities reduces the conservatism in the conditions appearing in the stability and performance analysis. The sign definiteness of these integral terms is implied by the sector-bounded property of the nonlinearities. The Lyapunov function is radially unbounded with respect to $\bar{x}_k$, zero for $x_k = 0$, and strictly positive for all nonzero $x_k \in \mathbb{R}^n$.

Conservatism is reduced by considering the sector boundedness, which is a global condition on the nonlinearities, and even more reduced by considering the slope restriction, which poses a local condition to their behavior inside the bounded conical sector. These properties are introduced by two means:

1. The local slope restriction is used to find the tightest possible quadratic upper bounds for the integral terms in the Lyapunov function. These quadratic upper bounds lead to LMI conditions ensuring that the Lyapunov function decreases for each k. The tighter that the upper bounds are, the less conservative are the derived conditions.

2. The obtained LMIs need to be satisfied for all values of the state variables that satisfy the sector-boundedness and slope-restriction conditions. This step is introduced through the S-procedure, which is applied to the quadratic forms resulting from the LMI formulation explained above.

1.3.2 Numerical Approach

The LMI problems to be solved numerically in this work are LMI feasibility problems, i.e., finding whether a given system of LMIs is feasible. These problems were solved using the LMI Lab [12] which is incorporated into the Matlab Robust Control Toolbox [13, 14].

The commands from this toolbox used in this thesis are:

`setlmis`: Initializes description of LMI system

`lmivar`: Specifies matrix variables in LMI problem

`newlmi`: Attaches identifying tag to LMIs

`lmiterm`: Specifies term content of LMIs

`getlmis`: Calls internal description of LMI system

`feasp`: Computes solution to given system of LMIs

In particular, for a given LMI system `lmisys`, the function call is:

```
[tmin,xfeas] = feasp(lmisys,options,target)
```

The value of `tmin` indicates whether `lmisys` is feasible. If negative, `lmisys` is feasible. If positive, the LMI is infeasible or, if sufficiently small, the system may be feasible but not strictly feasible. For a feasible system, `xfeas` gives a set of solutions to the variables `lmivar` that make the problem feasible.

For a set of examples, Chapters 2 and 3 solve optimizations over LMIs. Specifically, the maximum or minimum value of a parameter is found such that the given system of LMIs is feasible. In particular, these correspond to the maximum lower bound on the stability margin and the minimum upper bound on the ℓ^2- or RMS-gain of the examples of Lur'e systems with sector-bounded, slope-restricted nonlinearities. These optimizations are solved by the procedure:

1. Define the system of LMIs.
2. Define the example system.
3. Find a value of the parameter to optimize such that the LMI system is feasible for the example system, and one for which the LMI system is infeasible, using `feasp`.
4. Use the bisection method to find the maximum or minimum value of the parameter for which `tmin` is negative, with the previously found feasible and infeasible values as initial search points.

The scripts for the described procedure can be found in Appendices A and B.

For a given example, Chapters 4 and 5 find whether a system of LMIs is feasible, which is done through the simpler procedure:

1. Define the example system.
2. Define the system of LMIs.
3. Find whether the system of LMIs is feasible for the example system.

In these chapters, additionally, the example system is iterated to show its behavior. The scripts for the above procedures and for plotting the system behavior are included in Appendices C and D.

1.4 Thesis Outline

Chapter 2 contains the main enabling results by dealing with the stablity analysis of the studied system. The properties of the system's nonlinearities are used with the aim of finding the tightest possible quadratic upper bounds for the integrals in the Lyapunov function (14), which is the most important contribution of this thesis. The approach was used in this thesis was published in a past Master's thesis [15]. The previously published bounds on the integrals are disproved by counterexamples, with a simple example being $\phi(\sigma) = \sigma$ and using $q_k, q_{k+1} > 0$. Through the new, corrected quadratic upper bounds presented in this thesis, LMIs are derived as sufficient conditions for the studied system (13) to be absolutely stable. The obtained LMIs are also used to solve the optimization $\max \xi$ such that the system is absolutely stable, which gives a lower bound on the stability/robustness margin of the system. This margin bound is found for a series of numerical examples and compared with the most relevant results in the literature.

Chapter 3 considers performance analysis of the Lur'e system. The integral bounds are used again to obtain LMIs for computing upper bounds on the system's ℓ_2-gain and RMS-gain. The tightness of the integral bounds is useful in this chapter as well, to reduce conservatism in the result of this value. The upper bound on the gain is obtained for a few of the numerical example systems analyzed in Chapter 2.

Chapter 4 considers the robust stabilization of the system. The bounds and LMIs obtained in Chapter 2 are used to analyze the stability of a system with a proportional controller $u_k = Kx_k$. The addition of the new matrix variable K results in higher order matrix inequalities, which are reduced to BMIs through the Schur complement lemma. A numerical example is given to illustrate the results.

Chapter 5 studies the state estimation of the Lur'e system. The dynamics of the error between the system variables and the estimated variables are analyzed. Because these dynamics are found to have nonlinearities such that $\phi \notin \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$ and $\phi \in \Phi_{sb}^{[0,\mu]}$, new bounds are found for the integrals in the Lyapunov function and, consequently, the Lyapunov function used for the error dynamics system is modified. Further, the corresponding higher order matrix inequalities are found, and again reduced to BMIs through the Schur complement lemma. A numerical example is shown as well to illustrate the results.

2 Stability Analysis

This chapter derives sufficient conditions for a system of the form (13) to be globally asymptotically stable. The stability or robustness margin is defined accordingly and is then found numerically for a set of examples. The numerical results are compared to the results obtained with the criteria presented in the literature review to quantify the improvement given by the presented criterion.

2.1 Analytical LMI Derivation for Stability Analysis

The derivation of this stability criterion requires finding a quadratic upper bound to the variation between sampling instances of the Lyapunov function (14). An essential step for finding this bound is given in Lemma 1 and is later used in the proof of Theorem 1.

Lemma 1: Let q_k, q_{k+1} be any two consecutive sampling instances of q and $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$ as defined in system (13). Let $\phi_k \equiv \phi(q_k)$ and $\phi_{k+1} \equiv \phi(q_{k+1})$. Then, for each $i = 1, \ldots, n_q$,

$$\begin{aligned} &\phi_{k,i}(q_{k+1,i} - q_{k,i}) + \frac{1}{2\mu_i}(\phi_{k+1,i} - \phi_{k,i})^2 \\ &\leq \int_{q_{k,i}}^{q_{k+1,i}} \phi_i(\sigma) d\sigma \\ &\leq \phi_{k+1,i}(q_{k+1,i} - q_{k,i}) - \frac{1}{2\mu_i}(\phi_{k+1,i} - \phi_{k,i})^2, \end{aligned} \tag{15}$$

provided that $\mu_i \neq 0$. For $\mu_i = 0$, the value of the integral is 0.

Proof: The case $\mu_i = 0$ is trivial. The bounds also clearly hold for $q_{k,i} = q_{k+1,i}$. For the nontrivial case, first consider the case where $q_{k,i} < q_{k+1,i}$. Then the local slope restriction property of ϕ gives

$$0 < \frac{\phi_i(\sigma) - \phi_i(\hat{\sigma})}{\sigma - \hat{\sigma}} \leq \mu_i, \forall \sigma \neq \hat{\sigma} \in \mathbb{R} \implies \phi_i(\sigma) \leq \min\{\phi_{k,i} + \mu_i(\sigma - q_{k,i}), \phi_{k+1,i}\}, \quad \forall \sigma \in [q_{k,i}, q_{k+1,i}]. \tag{16}$$

Let $\sigma_k \in [q_{k,i}, q_{k+1,i}]$ be the value of σ at which $\phi_{k,i} + \mu_i(\sigma - q_{k,i}) = \phi_{k+1,i}$, then

$$\sigma_k \equiv q_{k,i} + \frac{\phi_{k+1,i} - \phi_{k,i}}{\mu_i}$$

and, from (16),

$$\begin{cases} \phi_i(\sigma) \leq \phi_{k,i} + \mu_i(\sigma - q_{k,i}), & \forall \sigma \in [q_{k,i}, \sigma_k], \\ \phi_i(\sigma) \leq \phi_{k+1,i}, & \forall \sigma \in [\sigma_k, q_{k+1,i}]. \end{cases}$$

The integral in (15) can then be separated into two parts, satisfying:

$$\begin{aligned}\int_{q_{k,i}}^{q_{k+1,i}} \phi_i(\sigma)d\sigma &= \int_{q_{k,i}}^{\sigma_k} \phi_i(\sigma)d\sigma + \int_{\sigma_k}^{q_{k+1,i}} \phi_i(\sigma)d\sigma \\ &\leq \int_{q_{k,i}}^{\sigma_k} \left(\phi_{k,i} + \mu_i(\sigma - q_{k,i})\right) d\sigma + \int_{\sigma_k}^{q_{k+1,i}} \phi_{k+1,i} d\sigma \\ &= \phi_{k+1,i}(q_{k+1,i} - q_{k,i}) - \frac{1}{2\mu_i}(\phi_{k+1,i} - \phi_{k,i})^2.\end{aligned}$$

The lower bound is obtained analogously using that

$$\phi_i(\sigma) \geq \max\{\phi_{k,i}, \phi_{k+1,i} + \mu_i(\sigma - q_{k+1,i})\}, \;\; \forall \sigma \in [q_{k,i}, q_{k+1,i}], \tag{17}$$

again implied by the slope restriction, leading to

$$\int_{q_{k,i}}^{q_{k+1,i}} \phi_i(\sigma)d\sigma \geq \phi_{k,i}(q_{k+1,i} - q_{k,i}) + \frac{1}{2\mu_i}(\phi_{k+1,i} - \phi_{k,i})^2.$$

The case for $q_{k,i} > q_{k+1,i}$ consequently holds by using that

$$\int_{q_{k,i}}^{q_{k+1,i}} \phi_i(\sigma)d\sigma = -\int_{q_{k+1,i}}^{q_{k,i}} \phi_i(\sigma)d\sigma$$

and applying the bounds to integral on the right-hand side. □

Theorem 1: Given the system (13), a sufficient condition for global asymptotic stability is the existence of a positive semidefinite matrix $P = P^T \in \mathbb{R}^{(n+n_p+n_q)\times(n+n_p+n_q)}$, with a positive definite submatrix $P_{11} = P_{11}^T \in \mathbb{R}^{n\times n}$ and diagonal positive semidefinite matrices $Q, \tilde{Q}, T, \tilde{T}, N \in \mathbb{R}^{n_q\times n_q}$ such that

$$G := \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{12}^T & G_{22} & G_{23} \\ G_{13}^T & G_{23}^T & G_{33} \end{bmatrix} \prec 0,$$

where

$$\begin{aligned}G_{11} = {} & A^T(P_{11} + C^T P_{13}^T + P_{13}C + C^T P_{33}C)A - P_{11} - C^T P_{13}^T - P_{13}C - C^T P_{33}C + \\ & + A^T C^T \tilde{Q} X C A - C^T \tilde{Q} X C,\end{aligned}$$

$$\begin{aligned}G_{12} = {} & A^T(P_{11} + C^T P_{13}^T + P_{13}C + C^T P_{33}C)B - P_{12} - C^T P_{23}^T - P_{13}D - C^T P_{33}D + \\ & + A^T C^T \tilde{Q} X C B - C^T \tilde{Q} X D + (CA - C)^T \tilde{Q} - C^T T + (CA - C)^T N,\end{aligned}$$

$$G_{13} = A^T P_{12} + A^T C^T P_{23}^T + A^T P_{13} D + A^T C^T P_{33} D - (CA - C)^T Q + A^T C^T \tilde{Q} X D - A^T C^T \tilde{T} - \\ - (CA - C)^T N,$$

$$G_{22} = B^T (P_{11} + C^T P_{13}^T + P_{13} C + C^T P_{33} C) B - P_{22} - D^T P_{23}^T - P_{23} D - D^T P_{33} D - Q M^{-1} + \\ + B^T C^T \tilde{Q} X C B - D^T \tilde{Q} X D + \tilde{Q}(CB - D) + (CB - D)^T \tilde{Q} - \tilde{Q} M^{-1} - 2 T X^{-1} - T D - \\ - D^T T - 2 N M^{-1} + N(CB - D) + (CB - D)^T N,$$

$$G_{23} = B^T P_{12} + B^T C^T P_{23}^T + B^T P_{13} D + B^T C^T P_{33} D + M^{-1} Q - (CB - D)^T Q + B^T C^T \tilde{Q} X D + \\ + \tilde{Q} D + \tilde{Q} M^{-1} - B^T C^T \tilde{T} - (CB - D)^T N + 2 N M^{-1} + N D,$$

$$G_{33} = P_{22} + D^T P_{23}^T + P_{23} D + D^T P_{33} D - Q D - D^T Q - Q M^{-1} + D^T \tilde{Q} X D - \tilde{Q} M^{-1} - \\ - 2 \tilde{T} X^{-1} - \tilde{T} D - D^T \tilde{T} - 2 N M^{-1} - N D - D^T N,$$

where $M := \mathrm{diag}\{\mu_1, \ldots, \mu_{n_q}\}$ and $X := \mathrm{diag}\{\xi_1, \ldots, \xi_{n_q}\}$.

Proof: Given the Lyapunov function (14), a sufficient condition for the global asymptotic stability of the system is for the inequality

$$\Delta V(x_k) < 0, \ \ \forall k \geq 0 \tag{18}$$

to be satisfied. The variation between the two sampling instances k and $k+1$ is expressed as

$$\begin{aligned} \Delta V\,(x_k) = {} & \zeta_k^T (A_a^T P A_a - E_a^T P E_a) \zeta_k + 2 \sum_{i=1}^{n_q} Q_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \phi_i(\sigma) d\sigma \\ & + 2 \sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_{q_{k,i}}^{q_{k+1,i}} [\xi_i \sigma - \phi_i(\sigma)] d\sigma, \end{aligned} \tag{19}$$

where

$$\zeta_k := \begin{bmatrix} x_k \\ p_k \\ p_{k+1} \end{bmatrix}, \ A_a := \begin{bmatrix} A & B & 0 \\ 0 & 0 & I \\ CA & CB & D \end{bmatrix}, \ E_a := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ C & D & 0 \end{bmatrix}.$$

In order to find an LMI condition that implies the inequality (18), Lemma 1 is used to find quadratic upper bounds on the two integral terms of $\Delta V(x_k)$:

$$2 \sum_{i=1}^{n_q} Q_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \phi_i(\sigma) d\sigma \leq 2 \sum_{i=1}^{n_q} Q_{ii} \left\{ \phi_{k+1,i}(q_{k+1,i} - q_{k,i}) - \frac{1}{2\mu_i} (\phi_{k+1,i} - \phi_{k,i})^2 \right\} = \zeta_k^T U_1 \zeta_k,$$

where $M := \mathrm{diag}\{\mu_1, \ldots, \mu_{n_q}\} \succcurlyeq 0$, and $U_1 = U_1^T$ is defined as

$$U_1 := \begin{bmatrix} 0 & 0 & -(CA-C)^T Q \\ * & -QM^{-1} & (M^{-1}-(CB-D)^T)Q \\ * & * & -QD - D^T Q - QM^{-1} \end{bmatrix},$$

and for the second integral term,

$$2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_{q_{k,i}}^{q_{k+1,i}} [\xi_i \sigma - \phi_i(\sigma)]\, d\sigma = 2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \left\{ \frac{\xi_i}{2}(q_{k+1,i}^2 - q_{k,i}^2) - \int_{q_{k,i}}^{q_{k+1,i}} \phi_i(\sigma) d\sigma \right\}$$
$$\leq 2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \left\{ \frac{\xi_i}{2}\left(q_{k+1,i}^2 - q_{k,i}^2\right) - \phi_{k,i}(q_{k+1,i} - q_{k,i}) - \frac{1}{2\mu_i}(\phi_{k+1,i} - \phi_{k,i})^2 \right\} = \zeta_k^T U_2 \zeta_k,$$

where $X := \mathrm{diag}\{\xi_1, \ldots, \xi_{n_q}\} \succcurlyeq 0$, and $U_2 = U_2^T$ is defined as

$$U_2 := \begin{bmatrix} A^T C^T \tilde{Q} X C A - C^T \tilde{Q} X C & \begin{pmatrix} A^T C^T \tilde{Q} X C B - C^T \tilde{Q} X D + \\ +(CA-C)^T \tilde{Q} \end{pmatrix} & A^T C^T \tilde{Q} X D \\ * & \begin{pmatrix} B^T C^T \tilde{Q} X C B - D^T \tilde{Q} X D + \\ +\tilde{Q}(CB-D) + \\ +(CB-D)^T \tilde{Q} - \tilde{Q} M^{-1} \end{pmatrix} & B^T C^T \tilde{Q} X D + \tilde{Q} D + Q M^{-1} \\ * & * & D^T \tilde{Q} X D - \tilde{Q} M^{-1} \end{bmatrix}.$$

Thus, $\Delta V(x_k) \leq \zeta_k^T (A_a^T P A_a - E_a^T P E_a + U_1 + U_2)\zeta^k$, $\forall k \geq 0$, which implies that

$$\zeta_k^T (A_a^T P A_a - E_a^T P E_a + U_1 + U_2)\zeta_k < 0 \implies \Delta V(x_k) < 0. \tag{20}$$

Therefore, a sufficient condition for the global asymptotic stability of the system is for the left-hand side of (20) to be satisfied for all ζ_k that satisfy the sector-boundedness and slope-restriction conditions on ϕ. These conditions on ζ_k are introduced via the S-procedure. First, the LMI form of the conditions is found.

For the sector boundedness:

$$\phi \in \Phi_{sb}^{[0,\xi]} \iff \phi_{k,i}[\xi_i^{-1}\phi_{k,i} - q_{k,i}] \leq 0, \;\; i = 1, \ldots, n_q, \;\; \forall k \geq 0. \tag{21}$$

A useful notation for using the S-procedure with condition (40) is

$$\sum_{i=1}^{n_q} 2\tau_i \phi_{k,i}[\xi_i^{-1}\phi_{k,i} - q_{k,i}] = \zeta_k^T S_1 \zeta_k \leq 0,$$

where $T := \mathrm{diag}\{\tau_1, \ldots, \tau_{n_q}\} \succcurlyeq 0$ and $S_1 = S_1^T$ is defined as

$$S_1 := \begin{bmatrix} 0 & C^T T & 0 \\ * & 2TX^{-1} + TD + D^T T & 0 \\ * & * & 0 \end{bmatrix}.$$

Similarly, for the next sampling instance,

$$\sum_{i=1}^{n_q} 2\tilde{\tau}_i \phi_{k+1,i}[\xi_i^{-1}\phi_{k+1,i} - q_{k+1,i}] = \zeta_k^T S_2 \zeta_k \leq 0,$$

where $\tilde{T} := \operatorname{diag}\{\tilde{\tau}_1, \ldots, \tilde{\tau}_{n_q}\} \succcurlyeq 0$, and $S_2 = S_2^T$ is defined as

$$S_2 := \begin{bmatrix} 0 & 0 & A^T C^T \tilde{T} \\ * & 0 & B^T C^T \tilde{T} \\ * & * & 2\tilde{T}X^{-1} + \tilde{T}D + D^T\tilde{T} \end{bmatrix}.$$

For the slope restriction:

$$\begin{aligned} \phi \in \Phi_{sr}^{[0,\mu]} &\iff 0 \leq \frac{\phi_{k+1,i} - \phi_{k,i}}{q_{k+1,i} - q_{k,i}} \leq \mu_i, \quad i = 1, \ldots, n_q \\ &\iff (\phi_{k+1,i} - \phi_{k,i}) \left[\mu_i^{-1}(\phi_{k+1,i} - \phi_{k,i}) - (q_{k+1,i} - q_{k,i})\right] \leq 0. \end{aligned} \tag{22}$$

A useful notation for using the S-procedure with condition (22) is

$$\sum_{i=1}^{n_q} 2\nu_i (\phi_{k+1,i} - \phi_{k,i}) \left[\mu_i^{-1}(\phi_{k+1,i} - \phi_{k,i}) - (q_{k+1,i} - q_{k,i})\right] = \zeta_k^T S_3 \zeta_k \leq 0,$$

where $N := \operatorname{diag}\{\nu_1, \ldots, \nu_{n_q}\} \succcurlyeq 0$, and $S_3 = S_3^T$ is defined as

$$S_3 := \begin{bmatrix} 0 & -(CA - C)^T N & (CA - C)^T N \\ * & 2NM^{-1} - N(CB - D) - (CB - D)^T N & (CB - D)^T N - N(2M^{-1} + D) \\ * & * & 2NM^{-1} + ND + D^T N \end{bmatrix}.$$

Finally, applying the S-procedure gives that, if the LMI $G := A_a^T P A_a - E_a^T P E_a + U_1 + U_2 - S_1 - S_2 - S_3 \prec 0$ is feasible, then $\Delta V(x_k) < 0$ is satisfied $\forall k \geq 0$, and the system is globally asymptotically stable.

□

The sufficient condition for the stability of system (13) given by Theorem 1 can be used to find a lower bound for the stability/robustness margin for each nonlinear input p_i, which is the maximum value of ξ_i such that the sufficient condition for stability is satisfied.

Since stability depends on the n_q components of the nonlinear input vector p, a simplified case for which to find the robustness margin is that in which all components ξ_i of the vector ξ are equal, thus finding the maximum sector boundedness restriction imposed on all nonlinear input components such that the sufficient condition for stability is satisfied.

Thus, the lower bound on the robustness margin is the value of ξ that solves the optimization:

$$\begin{array}{rl} \max & \xi \\ \text{subject to} & G \prec 0 \\ & Q, \tilde{Q}, T, \tilde{T}, N \succcurlyeq 0 \\ & P = P^T \succcurlyeq 0, \;\; P_{11} \succ 0 \end{array}$$

with all matrices as defined in Theorem 1.

Because the condition in Theorem 1 is not a necessary condition, this margin bound can be conservative with respect to the real robustness margin.

2.2 Numerical Results for Stability Analysis

The sufficient condition for global asymptotic stability imposed by Theorem 1 is used to find lower bounds on the robustness margin of several examples of systems of the form (13). All examples consider sector bounds and slope restrictions with ξ_i and μ_i are taken to be the same for all nonlinearities ϕ_i. In addition, μ is taken to be linearly dependent on ξ.

Theorem 1 deals with systems of the form (13) that can either have $D = 0$ or $D \neq 0$. All examples in this section have $D = 0$ to allow comparison with results from other criteria in the literature.

Example 1:

$$G(z) = \frac{-0.5z + 0.1}{(z^2 - z + 0.89)\,(z + 0.1)}, \;\; \mu = 2\xi.$$

Example 2:

$$A = \begin{bmatrix} 0.2948 & 0 & 0 & 0 & 0 \\ 0 & 0.4568 & 0 & 0 & 0 \\ 0 & 0 & 0.0226 & 0 & 0 \\ 0 & 0 & 0 & 0.3801 & 0 \\ 0 & 0 & 0 & 0 & -0.3270 \end{bmatrix}, \;\; B = \begin{bmatrix} -1.1878 & 0.2341 \\ -2.2023 & 0.0215 \\ 0.9863 & -1.0039 \\ -0.5186 & -0.9471 \\ 0.3274 & -0.3744 \end{bmatrix},$$

$$C = \begin{bmatrix} -1.1859 & 1.4725 & -1.2173 & -1.1283 & -0.2611 \\ -1.0559 & 0.0557 & -0.0412 & -1.3493 & 0.9535 \end{bmatrix}, \;\; D = 0_{2\times 2} \;\; \mu = \xi.$$

Example 3:

$$A = \begin{bmatrix} 0.0469 & -0.3992 & -0.0835 \\ 0.3902 & -0.5363 & -0.2744 \\ 0.4378 & -1.3576 & 0.4651 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5673 & -0.2785 \\ 0.1155 & -0.0649 \\ -2.1849 & -0.5976 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.3587 & -1.0802 & -0.6802 \\ -1.3833 & -1.0677 & 1.1497 \end{bmatrix}, \quad D = 0_{2\times 2}, \quad \mu = \xi.$$

Example 4:

$$A = \begin{bmatrix} 0.4030 & 0 & 0 \\ 0 & -0.1502 & 0 \\ 0 & 0 & -0.1502 \end{bmatrix}, \quad B = \begin{bmatrix} -0.2494 \\ 0.2542 \\ -0.2036 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.9894 & 0.6649 & 0.4339 \end{bmatrix}, \quad D = 0, \quad \mu = 2\xi.$$

Example 5:

$$A = \begin{bmatrix} 0.4783 & 0 & 0 & 0 \\ 0 & 0.7871 & 0 & 0 \\ 0 & 0 & 0.7871 & 1 \\ 0 & 0 & 0 & 0.7871 \end{bmatrix}, \quad B = \begin{bmatrix} -1.5174 \\ 1.2181 \\ 0.2496 \\ -0.5181 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.8457 & -2.0885 & 1.2190 & 0.1683 \end{bmatrix}, \quad D = 0, \quad \mu = 2\xi.$$

Example 6:

$$A = \begin{bmatrix} 0.5359 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9417 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9802 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5777 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1227 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0034 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5721 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2870 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3599 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad D = 0_{4\times 4}, \quad \mu = \xi.$$

For each example, the lower bound on the robustness margin is calculated as the upper bound for ξ: the system will be globally asymptotically stable for any value of ξ smaller than the upper bound, provided that the system is nominally stable, i.e., that the system without the nonlinearities is stable.

The results from the most relevant criteria in the literature are obtained from [15] and shown in Table 1. For Theorem 1, LMI feasibility results are obtained using the LMI solver tools in MATLAB's Robust Control Toolbox, and the upper bound for ξ is found by the bisection method, as explained in Section 1.3.2. These results are also shown in Table 1.

Table 1: Lower bounds on the robustness margin for ξ obtained by different criteria. The first comparison is with respect to the latest criterion for $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$, while the second comparison is with respect to the previous least conservative criterion.

	Ex 1	Ex 2	Ex 3	Ex 4	Ex 5	Ex 6
Circle $\left(\phi \in \Phi_{sb}^{[0,\xi]}\right)$	1.0273	0.18358	0.21792	2.91387	0.03660	0.03716
Tsypkin $\left(\phi \in \Phi_{sb}^{[0,\xi]}\right)$	1.0273	0.18358	0.21792	2.91387	0.03660	0.03716
Haddad et al. $\left(\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}\right)$	1.0273	0.18358	0.21792	2.91387	0.03660	0.03716
Kapila et al. $\left(\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\infty]}\right)$	1.0273	0.18358	0.21792	2.91387	0.03660	0.03716
Park et al. $\left(\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\infty]}\right)$	1.7252	0.18358	0.21792	2.91387	0.03660	0.03716
Theorem 1 $\left(\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}\right)$	2.4475	0.18362	0.46429	2.91387	0.03660	0.13088
Improvement of Theorem 1 over Haddad et al.	138%	0.02%	113%	0%	0%	252%
Improvement of Theorem 1 over Park et al.	42%	0.02%	113%	0%	0%	252%

Theorem 1 is less conservative with respect to previous literature results form some examples, and the same value for others. While the lower bound on the margin remains the same in Examples 4 and 5 and practically unchanged for Example 2, the margin is significantly increased for Examples 1, 3, and 6. Theorem 1 gives tighter lower bounds on the real robustness margin and is a clear and substantial improvement with respect to previously published stability criteria.

The main consideration leading to an improved criterion is finding bounds on the variation of the Lyapunov function that are based on the slope restriction instead of the sector boundedness constraint. By being a local rather than a global condition, the slope restriction provides more information about the behavior of the nonlinearities and thus leads to tighter bounds in Lemma 1.

3 Gain Analysis

Gain analysis is also important for performance analysis of dynamical systems. Gain analysis can quantify about how much an input noise signal is amplified at the output. This chapter provides a method to determine an upper bound on the value of the gain, based on the bounds for the variation of the Lyapunov function (14) obtained in the stability analysis. Thus, the reduction in conservatism is equally present in this result.

3.1 Analytical LMI Derivation for Gain Analysis

For the gain analysis, the system (13) is expressed in the input-output formulation:

$$\begin{cases} x_{k+1} = Ax_k + B_p p_k + B_w w_k \\ q_k = C_q x_k + D_{qp} p_k + D_{qw} w_k \\ z_k = C_z x_k + D_{zp} p_k + D_{zw} w_k \\ p_k = -\phi(q_k) \end{cases} \tag{23}$$

where $w \in \mathbb{R}^{n_w}$ and $z \in \mathbb{R}^{n_z}$ are the input and output vectors, respectively, $B_w \in \mathbb{R}^{n \times n_w}$, $D_{qw} \in \mathbb{R}^{n_q \times n_w}$, $D_{zw} \in \mathbb{R}^{n_z \times n_w}$, and the rest of variables and matrices are the same as defined in (13).

The ℓ_2-gain of the system (23) is defined by

$$\sup_{\|w\|_2 \neq 0} \frac{\|z\|_2}{\|w\|_2},$$

where the ℓ_2-norm of the input w is defined by

$$\|w\|_2 \equiv \sqrt{\sum_{k=0}^{\infty} w_k^T w_k},$$

and analogously for the output z.

Theorem 2: An upper bound on the ℓ_2-gain of system (23) can be obtained as the value of γ that solves the optimization:

$$\begin{aligned} \min \quad & \gamma^2 \\ \text{subject to} \quad & H \preccurlyeq 0 \\ & Q, \tilde{Q}, T, \tilde{T}, N \succcurlyeq 0 \\ & P = P^T \succcurlyeq 0, \;\; P_{11} \succ 0 \end{aligned}$$

where $Q, \tilde{Q}, T, \tilde{T}, N \in \mathbb{R}^{n_q \times n_q}$ are diagonal matrices, $P \in \mathbb{R}^{(n+n_p+n_q+n_w)\times(n+n_p+n_q+n_w)}$, and $H = H^T$ is defined by the entries

$$\begin{aligned}H_{11} = {} & A^T(P_{11} + C_q^T P_{13}^T + P_{13}C_q + C_q^T P_{33} C_q)A - P_{11} - C_q^T P_{13}^T - P_{13}C_q - C_q^T P_{33} C_q \\ & + A^T C_q^T \tilde{Q} X C_q A - C_q^T \tilde{Q} X C_q + C_z^T C_z,\end{aligned}$$

$$\begin{aligned}H_{12} = {} & A^T(P_{11} + C_q^T P_{13}^T + P_{13}C_q + C_q^T P_{33} C_q)B_p - P_{12} - C_q^T P_{23}^T - P_{13} D_{qp} - C_q^T P_{33} D_{qp} \\ & + A^T C_q^T \tilde{Q} X C_q B_p - C_q^T \tilde{Q} X D_{qp} + (C_q A - C_q)^T \tilde{Q} + C_z^T D_{zp} - C_q^T T + (C_q A - C_q)^T N,\end{aligned}$$

$$\begin{aligned}H_{13} = {} & A^T(P_{11} + C_q^T P_{13}^T + P_{13}C_q + C_q^T P_{33} C_q)B_w - P_{13} D_{qw} - C_q^T P_{33} D_{qw} - P_{14} - C_q^T P_{34} \\ & + A^T C_q^T \tilde{Q} X C_q B_w - C_q^T \tilde{Q} X D_{qw} + C_z^T D_{zw},\end{aligned}$$

$$\begin{aligned}H_{14} = {} & A^T P_{12} + A^T C_q^T P_{23}^T + A^T P_{13} D_{qp} + A^T C_q^T P_{33} D_{qp} - (C_q A - C_q)^T Q + A^T C_q^T \tilde{Q} X D_{qp} \\ & - A^T C_q^T \tilde{T} - (C_q A - C_q)^T N,\end{aligned}$$

$$H_{15} = A^T P_{13} D_{qw} + A^T C_q^T P_{33} D_{qw} + A^T P_{14} + A^T C_q^T P_{34} + A^T C_q^T \tilde{Q} X D_{qw},$$

$$\begin{aligned}H_{22} = {} & B_p^T \left(P_{11} + C_q^T P_{13}^T + P_{13}C_q + C_q^T P_{33} C_q\right) B_p - P_{22} - D_{qp}^T P_{23}^T - P_{23} D_{qp} - D_{qp}^T P_{33} D_{qp} \\ & - Q M^{-1} + B_p^T C_q^T \tilde{Q} X C_q B_p - D_{qp}^T \tilde{Q} X D_{qp} + \tilde{Q}(C_q B_p - D_{qp}) + (C_q B_p - D_{qp})^T \tilde{Q} \\ & - \tilde{Q} M^{-1} + D_{zp}^T D_{zp} - 2 T X^{-1} - T D_{qp} - D_{qp}^T T - 2 N M^{-1} + N(C_q B_p - D_{qp}) \\ & + (C_q B_p - D_{qp})^T N,\end{aligned}$$

$$\begin{aligned}H_{23} = {} & B_p^T \left(P_{11} + C_q^T P_{13}^T + P_{13}C_q + C_q^T P_{33} C_q\right) B_w - P_{23} D_{qw} - D_{qp}^T P_{33} D_{qw} - P_{24} - D_{qp}^T P_{34} \\ & + B_p^T C_q^T \tilde{Q} X C_q B_w - D_{qp}^T \tilde{Q} X D_{qw} + \tilde{Q}(C_q B_w - D_{qw}) + D_{zp}^T D_{zw} - T D_{qw} \\ & + N(C_q B_w - D_{qw}),\end{aligned}$$

$$\begin{aligned}H_{24} = {} & B_p^T P_{12} + B_p^T C_q^T P_{23}^T + B_p^T P_{13} D_{qp} + B_p^T C_q^T P_{33} D_{qp} + M^{-1} Q - (C_q B_p - D_{qp})^T Q \\ & + B_p^T C_q^T \tilde{Q} X D_{qp} + \tilde{Q} D_{qp} + M^{-1} \tilde{Q} - B_p^T C_q^T \tilde{T} + 2 M^{-1} N - (C_q B_p - D_{qp})^T N + N D_{qp},\end{aligned}$$

$$H_{25} = B_p^T P_{13} D_{qw} + B_p^T C_q^T P_{33} D_{qw} + B_p^T P_{14} + B_p^T C_q^T P_{34} + B_p^T C_q^T \tilde{Q} X D_{qw} + \tilde{Q} D_{qw} + N D_{qw},$$

$$\begin{aligned}H_{33} = {} & B_w^T \left(P_{11} + C_q^T P_{13}^T + P_{13}C_q + C_q^T P_{33} C_q\right) B_w - D_{qw}^T P_{33} D_{qw} - P_{34}^T D_{qw} - D_{qw}^T P_{34} - P_{44} \\ & + B_w^T C_q^T \tilde{Q} X C_q B_w - D_{qw}^T \tilde{Q} X D_{qw} + D_{zw}^T D_{zw} - \gamma^2 I,\end{aligned}$$

$$H_{34} = B_w^T P_{12} + B_w^T C_q^T P_{23}^T + B_w^T P_{13} D_{qp} + B_w^T C_q^T P_{33} D_{qp} - (C_q B_w - D_{qw})^T Q + B_w^T C_q^T \tilde{Q} X D_{qp}$$
$$- B_w^T C_q^T \tilde{T} - (C_q B_w - D_{qw})^T N,$$

$$H_{35} = B_w^T P_{13} D_{qw} + B_w^T C_q^T P_{33} D_{qw} + B_w^T P_{14} + B_w^T C_q^T P_{34} + B_w^T C_q^T \tilde{Q} X D_{qw},$$

$$H_{44} = P_{22} + D_{qp}^T P_{23}^T + P_{23} D_{qp} + D_{qp}^T P_{33} D_{qp} - Q D_{qp} - D_{qp}^T Q - Q M^{-1} + D_{qp}^T \tilde{Q} X D_{qp} - \tilde{Q} M^{-1}$$
$$- 2\tilde{T} X^{-1} - \tilde{T} D_{qp} - D_{qp}^T \tilde{T} - 2N M^{-1} - N D_{qp} - D_{qp}^T N,$$

$$H_{45} = P_{23} D_{qw} + D_{qp}^T P_{33} D_{qw} + P_{24} + D_{qp}^T P_{34} - Q D_{qw} + D_{qp}^T \tilde{Q} X D_{qw} - \tilde{T} D_{qw} - N D_{qw},$$

$$H_{55} = D_{qw}^T P_{33} D_{qw} + P_{34}^T D_{qw} + D_{qw}^T P_{34} + P_{44} + D_{qw}^T \tilde{Q} X D_{qw},$$

and where $M := \operatorname{diag}\{\mu_1, \ldots, \mu_{n_q}\}$, $X := \operatorname{diag}\{\xi_1, \ldots, \xi_{n_q}\}$.

Proof: If there exists a Lyapunov function $V(x_k)$ such that

$$\Delta V(x_k) + z_k^T z_k - \gamma^2 w_k^T w_k \leq 0, \quad \forall k \geq 0, \tag{24}$$

then the ℓ_2-gain of the system is less than or equal to γ, which follows since the inequality (24) implies

$$\sum_{k=0}^{K} \left(\Delta V(x_k) + z_k^T z_k - \gamma^2 w_k^T w_k \right) \leq 0 \quad \forall K \geq 0 \iff$$
$$\iff V(x_{K+1}) + \sum_{k=0}^{K} \left(z_k^T z_k - \gamma^2 w_k^T w_k \right) \leq 0 \quad \forall K \geq 0 \implies$$
$$\implies \sum_{k=0}^{K} \left(z_k^T z_k - \gamma^2 w_k^T w_k \right) \leq 0 \ \forall K \geq 0 \ \text{(since } V \geq 0) \iff$$
$$\iff \sum_{k=0}^{K} z_k^T z_k \leq \gamma^2 \sum_{k=0}^{K} w_k^T w_k \ \forall K \geq 0 \implies$$
$$\implies \|z\|_2^2 \leq \gamma^2 \|w\|_2^2 \implies \frac{\|z\|_2}{\|w\|_2} \leq \gamma \quad \forall \|w\|_2 \neq 0.$$

The Lyapunov function used for the system (23) is again function (14). For the input-output formulation, the $\bar{x}_k$ and P are expanded to

$$\bar{x}_k := \begin{bmatrix} x_k \\ p_k \\ q_k \\ w_k \end{bmatrix}, \quad P^T = P := \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{12}^T & P_{22} & P_{23} & P_{24} \\ P_{13}^T & P_{23}^T & P_{33} & P_{34} \\ P_{14}^T & P_{24}^T & P_{34}^T & P_{44} \end{bmatrix} \succcurlyeq 0,$$

and, once again,

$$P_{11} \succ 0, \;\; Q := \mathrm{diag}\{Q_{ii}\} \succcurlyeq 0, \;\; \tilde{Q} := \mathrm{diag}\{\tilde{Q}_{ii}\} \succcurlyeq 0.$$

The variation of the Lyapunov function is, again, defined as in (19), although with ζ_k, A_a, and E_a now being defined as

$$\zeta_k := \begin{bmatrix} x_k \\ p_k \\ w_k \\ p_{k+1} \\ w_{k+1} \end{bmatrix}, \; A_a := \begin{bmatrix} A & B_p & B_w & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ C_qA & C_qB_p & C_qB_w & D_{qp} & D_{qw} \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \; E_a := \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ C_q & D_{qp} & D_{qw} & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix}.$$

Using, again, the bounds (15) for the integral $\int_{q_{k,i}}^{q_{k+1,i}} \phi_i(\sigma)d\sigma$, the integral terms of the Lyapunov function variation can be upper bounded by the linear matrix forms

$$2\sum_{i=1}^{n_q} Q_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \phi_i(\sigma)d\sigma \leq \zeta_k^T L_1 \zeta_k,$$

where $L_1 = L_1^T$ is defined as

$$L_1 := \begin{bmatrix} 0 & 0 & 0 & -(C_qA - C_q)^T Q & 0 \\ * & -QM^{-1} & 0 & M^{-1}Q - (C_qB_p - D_{qp})^T Q & 0 \\ * & * & 0 & -(C_qB_w - D_{qw})^T Q & 0 \\ * & * & * & -QD_{qp} - D_{qp}^T Q - QM^{-1} & -QD_{qw} \\ * & * & * & * & 0 \end{bmatrix}$$

and

$$2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \left[\xi_i\sigma - \phi_i(\sigma)\right] d\sigma \leq \zeta_k^T L_2 \zeta_k,$$

where $L_2 = L_2^T$ is defined by the block entries

$$\begin{aligned}
L_{2,11} &= A^T C_q^T \tilde{Q} X C_q A - C_q^T \tilde{Q} X C_q, \\
L_{2,12} &= A^T C_q^T \tilde{Q} X C_q B_p - C_q^T \tilde{Q} X D_{qp} + (C_qA - C_q)^T \tilde{Q}, \\
L_{2,13} &= A^T C_q^T \tilde{Q} X C_q B_w - C_q^T \tilde{Q} X D_{qw}, \\
L_{2,14} &= A^T C_q^T \tilde{Q} X D_{qp}, \\
L_{2,15} &= A^T C_q^T \tilde{Q} X D_{qw}, \\
L_{2,22} &= B_p^T C_q^T \tilde{Q} X C_q B_p - D_{qp}^T \tilde{Q} X D_{qp} + \tilde{Q}(C_qB_p - D_{qp}) + (C_qB_p - D_{qp})^T \tilde{Q} - \tilde{Q}M^{-1}, \\
L_{2,23} &= B_p^T C_q^T \tilde{Q} X C_q B_w - D_{qp}^T \tilde{Q} X D_{qw} + \tilde{Q}(C_qB_w - D_{qw}),
\end{aligned}$$

$$L_{2,24} = B_p^T C_q^T \tilde{Q} X D_{qp} + \tilde{Q} D_{qp} + M^{-1}\tilde{Q},$$
$$L_{2,25} = B_p^T C_q^T \tilde{Q} X D_{qw} + \tilde{Q} D_{qw},$$
$$L_{2,33} = B_w^T C_q^T \tilde{Q} X C_q B_w - D_{qw}^T \tilde{Q} X D_{qw},$$
$$L_{2,34} = B_w^T C_q^T \tilde{Q} X D_{qp},$$
$$L_{2,35} = B_w^T C_q^T \tilde{Q} X D_{qw},$$
$$L_{2,44} = D_{qp}^T \tilde{Q} X D_{qp} - \tilde{Q} M^{-1},$$
$$L_{2,45} = D_{qp}^T \tilde{Q} X D_{qw},$$
$$L_{2,55} = D_{qw}^T \tilde{Q} X D_{qw}.$$

The terms $z_k^T z_k$ and $w_k^T w_k$ from the inequality (24) can also be expressed in matrix form as

$$z_k^T z_k = \zeta_k^T Z \zeta_k, \quad w_k^T w_k = \zeta_k^T W \zeta_k,$$

where $Z = Z^T$ and $W = W^T$ are defined as

$$Z := \begin{bmatrix} C_z^T C_z & C_z^T D_{zp} & C_z^T D_{zw} & 0 & 0 \\ * & D_{zp}^T D_{zp} & D_{zp}^T D_{zw} & 0 & 0 \\ * & * & D_{zw}^T D_{zw} & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}, \quad W := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & I & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}.$$

Since

$$\Delta V(x_k) + z_k^T z_k - \gamma^2 w_k^T w_k \leq \zeta_k^T (A_a^T P A_a - E_a^T P E_a + L_1 + L_2 + Z - \gamma^2 W)\zeta_k, \quad \forall k \geq 0,$$

then

$$\zeta_k^T (A_a^T P A_a - E_a^T P E_a + L_1 + L_2 + Z - \gamma^2 W)\zeta_k \leq 0, \quad \forall k \geq 0 \implies \\ \implies \Delta V(x_k) + z_k^T z_k - \gamma^2 w_k^T w_k \leq 0, \quad \forall k \geq 0. \tag{25}$$

Thus, the upper bound on the ℓ_2-gain of the system is found as the minimum value of γ such that the left-hand side of (25) holds, given that ζ_k safisfies the sector-boundedness and slope-restriction conditions on ϕ. These conditions on ζ_k will be introduced again via the S-procedure. The matrix-form of the conditions is

$$\zeta_k^T S_1 \zeta_k \leq 0 \text{ and } \zeta_k^T S_2 \zeta_k \leq 0$$

for the sector boundedness at the sampling instances k and $k+1$, respectively, and

$$\zeta_k^T S_3 \zeta_k \leq 0$$

for the slope restriction. The matrices $S_1 = S_1^T$, $S_2 = S_2^T$ and $S_3 = S_3^T$ are obtained analogously to the proof of Theorem 1 and, for the input-output formulation system (23), are defined as

$$S_1 := \begin{bmatrix} 0 & C_q^T T & 0 & 0 & 0 \\ * & \begin{pmatrix} 2TX^{-1} + \\ +TD_{qp} + D_{qp}^T T \end{pmatrix} & TD_{qw} & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}, \quad S_2 := \begin{bmatrix} 0 & 0 & 0 & A^T C_q^T \tilde{T} & 0 \\ * & 0 & 0 & B_p^T C_q^T \tilde{T} & 0 \\ * & * & 0 & B_w^T C_q^T \tilde{T} & 0 \\ * & * & * & \begin{pmatrix} 2\tilde{T}X^{-1} + \\ +\tilde{T}D_{qp} + D_{qp}^T \tilde{T} \end{pmatrix} & \tilde{T}D_{qw} \\ * & * & * & * & 0 \end{bmatrix},$$

and

$$S_3 := \begin{bmatrix} 0 & -(C_qA - C_q)^T N & 0 & (C_qA - C_q)^T N & 0 \\ * & \begin{pmatrix} 2NM^{-1} - \\ -N(C_qB_p - D_{qp}) - \\ -(C_qB_p - D_{qp})^T N \end{pmatrix} & -N(C_qB_w - D_{qw}) & \begin{pmatrix} -2M^{-1}N + \\ +(C_qB_p - D_{qp})^T N - \\ -ND_{qp} \end{pmatrix} & -ND_{qw} \\ * & * & 0 & (C_qB_w - D_{qw})^T N & 0 \\ * & * & * & \begin{pmatrix} 2NM^{-1} + \\ +ND_{qp} + D_{qp}^T N \end{pmatrix} & ND_{qw} \\ * & * & * & * & 0 \end{bmatrix},$$

where, again, $T, \tilde{T}, N \in \mathbb{R}^{n_q \times n_q}$ are diagonal positive semidefinite matrices.

Finally, applying the S-procedure gives that an upper bound on the ℓ_2-gain of the system is the minimum value of γ such that the LMI $H := A_a^T P A_a - E_a^T P E_a + L_1 + L_2 + Z - \gamma^2 W - S_1 - S_2 - S_3 \preccurlyeq 0$ holds.

$\square$

The RMS-gain of the system (23) is defined by

$$\sup_{\|w\|_{RMS} \neq 0} \frac{\|z\|_{RMS}}{\|w\|_{RMS}},$$

where the RMS-norm of the input w is defined by

$$\|w\|_{RMS} \equiv \sqrt{\limsup_{K \to \infty} \sum_{k=0}^{K} w_k^T w_k}$$

and analogously for the output z.

The same optimization in Theorem 2 can be used to find the RMS-gain of the system, which follows since inequality (24) also implies that

$$\sum_{k=0}^{K} z_k^T z_k \leq \gamma^2 \sum_{k=0}^{K} w_k^T w_k \ \ \forall K \geq 0 \ \ (\text{since } V \geq 0) \iff$$

$$\iff \frac{1}{K}\sum_{k=0}^{K} z_k^T z_k \leq \gamma^2 \frac{1}{K}\sum_{k=0}^{K} w_k^T w_k \ \ \forall K \geq 0 \implies$$

$$\implies \limsup_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K} z_k^T z_k \leq \gamma^2 \limsup_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K} w_k^T w_k \iff$$

$$\iff \|z\|_{RMS}^2 \leq \gamma^2 \|w\|_{RMS}^2 \implies \frac{\|z\|_{RMS}}{\|w\|_{RMS}} \leq \gamma \ \ \forall \|w\|_{RMS} \neq 0.$$

Thus, the upper bound on the ℓ_2-gain and RMS-gain obtained through Theorem 2 are the same.

As with the robustness margin bound derived from Theorem 1, since condition (24) is sufficient but not necessary for the ℓ_2-gain and the RMS-gain to be smaller or equal to γ, the results might also present conservatism with respect to the true value of the gain.

3.2 Numerical Results for Gain Analysis

The optimization in Theorem 2 is solved for an extended version of Example 2 from Section 2.2, i.e.,

$$A = \begin{bmatrix} 0.2948 & 0 & 0 & 0 & 0 \\ 0 & 0.4568 & 0 & 0 & 0 \\ 0 & 0 & 0.0226 & 0 & 0 \\ 0 & 0 & 0 & 0.3801 & 0 \\ 0 & 0 & 0 & 0 & -0.3270 \end{bmatrix}, \ B_p = \begin{bmatrix} -1.1878 & 0.2341 \\ -2.2023 & 0.0215 \\ 0.9863 & -1.0039 \\ -0.5186 & -0.9471 \\ 0.3274 & -0.3744 \end{bmatrix}, \ B_w = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$C_q = \begin{bmatrix} -1.1859 & 1.4725 & -1.2173 & -1.1283 & -0.2611 \\ -1.0559 & 0.0557 & -0.0412 & -1.3493 & 0.9535 \end{bmatrix}, \ D_{qp} = 0_{2\times 2}, \ D_{qw} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \ D_{zp} = 0, \ D_{zw} = 1, \ \mu = \xi,$$

by using the LMI solver tools from the MATLAB Robust Control Toolbox and by implementing the bisection method to find the lowest value of γ^2 for which the LMI constraints are feasible, as explained in Section 1.3.2. This procedure gives that the ℓ_2-gain, and equivalently an RMS-gain, for the system is less than or equal to 2.42751 for $\xi = 0.01$ and 2.50171 for $\xi = 0.1$.

The same method was applied to an extended version of Example 3 from Section 2.2, i.e.,

$$A=\begin{bmatrix} 0.0469 & -0.3992 & -0.0835 \\ 0.3902 & -0.5363 & -0.2744 \\ 0.4378 & -1.3576 & 0.4651 \end{bmatrix},\ B_p=\begin{bmatrix} -0.5673 & -0.2785 \\ 0.1155 & -0.0649 \\ -2.1849 & -0.5976 \end{bmatrix},\ B_w=\begin{bmatrix} 0.0962 & -0.5829 \\ -0.0482 & 0.4739 \\ -1.1274 & 1.1238 \end{bmatrix},$$

$$C_q=\begin{bmatrix} 0.3587 & -1.0802 & -0.6802 \\ -1.3833 & -1.0677 & 1.1497 \end{bmatrix},\ D_{qp}=0_{2\times 2},\ D_{qw}=\begin{bmatrix} 0.1562 & 0.4342 \\ 0.5472 & 0.0356 \end{bmatrix},$$

$$C_z=\begin{bmatrix} 0.9792 & 0.1112 & -0.8091 \\ 0.6970 & 1.3471 & -0.0023 \end{bmatrix},\ D_{zp}=\begin{bmatrix} 0.0010 & -0.7238 \\ 1.2356 & 0.2360 \end{bmatrix},\ D_{zw}=\begin{bmatrix} 0.5474 & 0.0242 \\ 0.2762 & 0.0486 \end{bmatrix},$$

$$\mu=\xi.$$

The ℓ_2-gain, and RMS-gain, is obtained to be less than or equal to 4.76056 for $\xi = 0.01$ and 5.41419 for $\xi = 0.1$.

4 State Feedback Controller Design

This chapter derives sufficient BMI conditions for a proportional state feedback controller to globally asymptotically stabilize a nominally unstable system of the form (13). The conditions are based on the bounds from the stability analysis and, thus, conservatism is correspondingly reduced.

4.1 Analytical BMI Derivation for State Feedback Controller Design

The system controlled by state feedback has the form

$$\begin{cases} x_{k+1} = Ax_k + Bp_k + B_u u_k = (A + B_u K)x_k + Bp_k \\ q_k = Cx_k + Dp_k \\ u_k = Kx_k \\ p_k = -\phi(q_k) \end{cases} \tag{26}$$

where $u \in \mathbb{R}^{n_u}$ is the input control variable, $B_u \in \mathbb{R}^{n \times n_u}$, $K \in \mathbb{R}^{n_u \times n}$ is the state feedback controller matrix, and the rest of the variables and matrices are as defined for the system (13).

Theorem 3: A sufficient condition for the controller matrix K to globally asymptotically stabilize the system (26) is the existence of a positive definite matrix $P = P^T \in \mathbb{R}^{(n+n_p+n_q)\times(n+n_p+n_q)}$, with a positive definite submatrix $P_{11} = P_{11}^T \in \mathbb{R}^{n\times n}$, a diagonal positive definite matrix $\tilde{Q} \in \mathbb{R}^{n_q \times n_q}$, and diagonal positive semidefinite matrices $Q, T, \tilde{T}, N \in \mathbb{R}^{n_q \times n_q}$ that satisfy the bilinear matrix inequality

$$J^T = J := \left[\begin{array}{c|c|c} J_1 & J_2 & J_3 \\ \hline J_2^T & -P & 0 \\ \hline J_3^T & 0 & J_4 \end{array}\right] \prec 0, \tag{27}$$

where

$$J_1 := \begin{bmatrix} \begin{pmatrix} -P_{11} - C^T P_{13}^T - \\ -P_{13}C - C^T P_{33} C - \\ -C^T \tilde{Q} X C \end{pmatrix} & \begin{pmatrix} -P_{12} - C^T P_{23}^T - P_{13} D - \\ -C^T P_{33} D + \\ +A^T C^T \tilde{Q} X C B + \\ +K^T B_u^T C^T \tilde{Q} X C B - \\ -C^T \tilde{Q} X D + A^T C^T \tilde{Q} + \\ +K^T B_u^T C^T \tilde{Q} - C^T \tilde{Q} - \\ -C^T T + A^T C^T N + \\ +K^T B_u^T C^T N - C^T N \end{pmatrix} & \begin{pmatrix} -A^T C^T Q - K^T B_u^T C^T Q + \\ +C^T Q + A^T C^T \tilde{Q} X D + \\ +K^T B_u^T C^T \tilde{Q} X D - \\ -A^T C^T \tilde{T} - K^T B_u^T C^T \tilde{T} - \\ -A^T C^T N - K^T B_u^T C^T N + \\ +C^T N \end{pmatrix} \\ * & \begin{pmatrix} -P_{22} - D^T P_{23}^T - P_{23} D - \\ -D^T P_{33} D - Q M^{-1} + \\ +B^T C^T \tilde{Q} X C B - \\ -D^T \tilde{Q} X D + \tilde{Q}(CB - D) + \\ +(CB - D)^T \tilde{Q} - \tilde{Q} M^{-1} - \\ -2T X^{-1} - T D - D^T T - \\ -2N M^{-1} + N(CB - D) + \\ +(CB - D)^T N \end{pmatrix} & \begin{pmatrix} M^{-1} Q - (CB - D)^T Q + \\ +B^T C^T \tilde{Q} X D + \\ +\tilde{Q} D + \tilde{Q} M^{-1} - B^T C^T \tilde{T} - \\ -(CB - D)^T N + \\ +2N M^{-1} + N D \end{pmatrix} \\ * & * & \begin{pmatrix} -Q D - D^T Q - Q M^{-1} + \\ +D^T \tilde{Q} X D - \tilde{Q} M^{-1} - \\ -2\tilde{T} X^{-1} - \tilde{T} D - D^T \tilde{T} - \\ -2N M^{-1} - N D - D^T N \end{pmatrix} \end{bmatrix},$$

$$J_2 := \begin{bmatrix} \begin{pmatrix} A^T P_{11} + K^T B_u^T P_{11} + \\ +A^T C^T P_{13}^T + \\ +K^T B_u^T C^T P_{13}^T \end{pmatrix} & \begin{pmatrix} A^T P_{12} + K^T B_u^T P_{12} + \\ +A^T C^T P_{23}^T + \\ +K^T B_u^T C^T P_{23}^T \end{pmatrix} & \begin{pmatrix} A^T P_{13} + K^T B_u^T P_{13} + \\ +A^T C^T P_{33} + \\ +K^T B_u^T C^T P_{33} \end{pmatrix} \\ B^T P_{11} + B^T C^T P_{13}^T & B^T P_{12} + B^T C^T P_{23}^T & B^T P_{13} + B^T C^T P_{33} \\ P_{12}^T + D^T P_{13}^T & P_{22} + D^T P_{23}^T & P_{23} + D^T P_{33} \end{bmatrix},$$

$$J_3 := \begin{bmatrix} A^T C^T \tilde{Q} X + K^T B_u^T C^T \tilde{Q} X \\ 0 \\ 0 \end{bmatrix},$$

$$J_4 := -\tilde{Q} X.$$

Proof: Let $\tilde{A} := A + B_u K$. Then the stability of system (26) is equivalent to the stability of the system (13) by substituting the original matrix A by $\tilde{A}$. Thus, the controller matrix K stabilizes the system if $\tilde{G} := \tilde{A}_a^T P \tilde{A}_a - E_a^T P E_a + \tilde{U}_1 + \tilde{U}_2 - S_1 - \tilde{S}_2 - \tilde{S}_3 \prec 0$, where all matrices are defined in Theorem 1, and tildes have been used to indicate matrices where matrix A is present and therefore replaced by $\tilde{A}$. A first glance at this inequality shows that the term

$\tilde{A}_a^T P \tilde{A}_a$ will give rise to the submatrix variables in P being pre- & post- multiplied by matrix variable K. To address these higher order terms, the Schur complement lemma is applied as

$$\left.\begin{array}{c} \tilde{G} \prec 0 \\ P \succ 0 \end{array}\right\} \iff \left[\begin{array}{cc} -E_a^T P E_a + \tilde{U}_1 + \tilde{U}_2 - S_1 - \tilde{S}_2 - \tilde{S}_3 & \tilde{A}_a^T P \\ P\tilde{A}_a & -P \end{array}\right] \prec 0.$$

In order for the Schur complement lemma to hold, matrix P must be positive definite rather than merely positive semidefinite. This matrix inequality is not yet bilinear, given that the trilinear term $\tilde{A}^T C^T \tilde{Q} X C \tilde{A}$ is in the submatrix $\tilde{U}_{2,11}$. Let

$$V := \left[\begin{array}{ccc} \tilde{A}^T C^T \tilde{Q} X C \tilde{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and let $\tilde{\tilde{U}}_2 := \tilde{U}_2 - V$. The Schur complement lemma is then applied once more to give

$$\left.\begin{array}{r} \left[\begin{array}{cc} -E_a^T P E_a + \tilde{U}_1 + \tilde{\tilde{U}}_2 - S_1 - \tilde{S}_2 - \tilde{S}_3 & \tilde{A}_a^T P \\ P\tilde{A}_a & -P \end{array}\right] - \left[\begin{array}{cc} -V & 0 \\ 0 & 0 \end{array}\right] \prec 0 \\ \tilde{Q}X \succ 0 \end{array}\right\} \iff J \prec 0.$$

Note that here, again, using the Schur complement lemma imposes that $\tilde{Q}$ and X be positive definite. □

4.2 Illustrative Numerical Example of a State Feedback Controller

The BMI obtained in Theorem 3 has the advantage that it is an LMI for fixed K. This property is used for obtaining numerical results showing the stabilization of a system through a matrix K. In particular, the numerical results illustate how Theorem 3 holds for an example for which K is known, which is implemented in LMI software as explained in Section 1.3.2.

The system with the form (26) defined by the matrices

$$A = \left[\begin{array}{cccc} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 \\ 0 & 0 & 1 & 0 \end{array}\right], \ B = \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array}\right], \ B_u = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right], \ C_q = \left[\begin{array}{cccc} 0.8 & -0.5 & 0 & 1 \end{array}\right], \qquad (28)$$

with $K = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \end{array}\right]$ and with $\xi = \mu = 2$ is not globally asymptotically stable. Unstable behavior of the system can be seen in Figure 1.

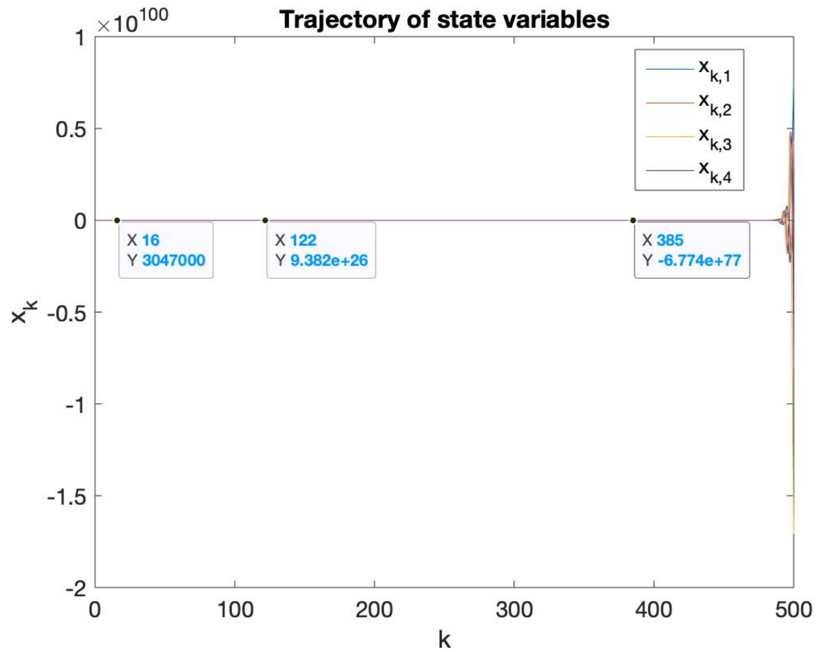
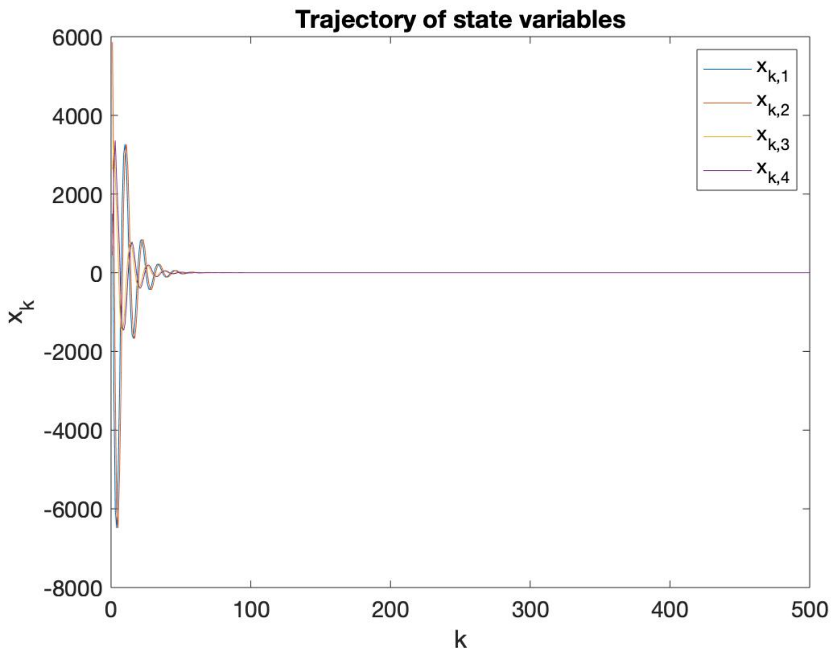


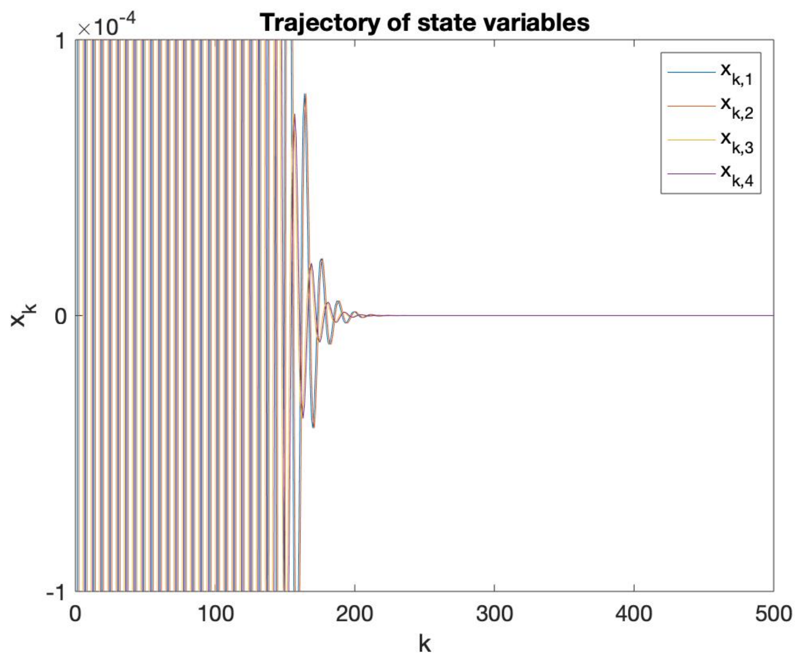
Figure 1: First 500 iterations of the system (28) without control. The initial values of the four state variables are set to random values between 0 and 10^4. Some points are labeled to show that the absolute values of the state variables are many (and increasing) orders of magnitude larger than their initial values.

Theorem 3 holds for this example since the BMI (27) is infeasible for system (28) with $K = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$.

The BMI is feasible with the state feedback controller $K = \begin{bmatrix} 0 & 0 & -1 & -1 \end{bmatrix}$, implying that the controlled system is globally asymptotically stable. This stable behavior is shown in Figure 2, again as predicted by Theorem 3.



(a)



(b)

Figure 2: (a) First 500 iterations of the system (28) with the state feedback controller $K = \left[\begin{array}{cccc} 0 & 0 & -1 & -1 \end{array}\right]$. The initial values of the four state variables are set to random values between 0 and 10^4. (b) Close-up of Figure (a), which confirms that the values of the state variables converge to zero.

5 State Estimator Design

In many real systems for which the dynamics are known, their variables cannot be measured. It can be useful in practice to be able to generate a state estimate $\hat{x}_k$, and the output of a state estimator can be fed to a state feedback controller to collectively produce an output feedback controller.

Obtaining an estimator system that converges to the true values of the variables involves analyzing the dynamics of the estimation error variables, to ensure that they absolutely converge to zero. The nonlinearities in the error dynamics system do not have a local slope restriction and, thus, the tight integral bounds obtained in the stability analysis are not applicable to this chapter. New bounds for the Lyapunov function variation integrals are developed based on the sector boundedness of the nonlinearities.

5.1 Analytical BMI Derivation for State Estimator Design

The design problem is to find an estimator matrix $L \in \mathbb{R}^{n \times n_y}$ to estimate the state of the system

$$\begin{cases} x_{k+1} = Ax_k + B_p p_k + B_u u_k \\ q_k = C_q x_k + D_{qp} p_k + D_{qu} u_k \\ y_k = C_y x_k + D_{yu} u_k \\ p_k = -\phi(q_k) \end{cases} \tag{29}$$

through the state estimator system

$$\begin{cases} \hat{x}_{k+1} = A\hat{x}_k + B_p \hat{p}_k + B_u u_k + L(\hat{y}_k - y_k) \\ \hat{q}_k = C_q \hat{x}_k + D_{qp} \hat{p}_k + D_{qu} u_k \\ \hat{y}_k = C_y \hat{x}_k + D_{yu} u_k \\ \hat{p}_k = -\phi(\hat{q}_k) \end{cases} \tag{30}$$

with estimation error dynamics

$$\begin{cases} \hat{x}_{k+1} - x_{k+1} = (A + LC_y)(\hat{x}_k - x_k) + B_p(\hat{p}_k - p_k) \\ \hat{q}_k - q_k = C_q(\hat{x}_k - x_k) + D_{qp}(\hat{p}_k - p_k) \\ \hat{y}_k - y_k = C_y(\hat{x}_k - x_k) \\ \hat{p}_k - p_k = -\left(\phi(\hat{q}_k) - \phi(q_k)\right) \equiv -f(\hat{q}_k - q_k; q_k) \end{cases} \tag{31}$$

where all variables and matrices in the system (29) are as defined in the system (23) with the control variable $u \in \mathbb{R}^{n_u}$ being analogous to the input vector w, and all variables with hats in the system (30) are the estimates of the corresponding variables in the system (29).

Considering the dynamics of the error (31) in place of the dynamics of the original system (13) leads to another Lur'e problem where now the nonlinearity $f \notin \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$. In fact,

$\phi \in \Phi_{sr}^{[0,\mu]} \iff f(\,\cdot\,;q) \in \Phi_{sb}^{[0,\mu]}$, the sector-boundedness restriction applying to the variable $\hat{q}-q$. This relationship follows from

$$\phi \in \Phi_{sr}^{[0,\mu]} \iff 0 \le \frac{\phi_i(\hat{\sigma})-\phi_i(\sigma)}{\hat{\sigma}-\sigma} \le \mu_i \iff 0 \le \frac{f_i(\nu;\sigma)}{\nu} \le \mu_i \iff$$
$$\iff f_i(\nu;\sigma)[\mu_i^{-1} f_i(\nu;\sigma)-\nu] \le 0, \quad \forall \sigma \neq \hat{\sigma} \in \mathbb{R},\ \nu \equiv \hat{\sigma}-\sigma,\ i=1,\dots,n_q.$$

Lemma 2: Let $\hat{q}_k - q_k$, $\hat{q}_{k+1}-q_{k+1}$ be any two consecutive sampling instances of $\hat{q}-q$ and $f(\,\cdot\,;q) \in \Phi_{sb}^{[0,\mu]}$ as defined in system (31). Then, for each $i=1,\dots,n_q$,

$$-\frac{\mu_i}{2}\left(\hat{q}_{k,i}-q_{k,i}\right)^2 \le \int_0^{\hat{q}_{k+1,i}-q_{k+1,i}} f_i(\sigma;q_{k+1})d\sigma - \int_0^{\hat{q}_{k,i}-q_{k,i}} f_i(\sigma;q_k)d\sigma \le \frac{\mu_i}{2}(\hat{q}_{k+1,i}-q_{k+1,i})^2. \quad (32)$$

Proof:

$$f(\,\cdot\,;q) \in \Phi_{sb}^{[0,\mu]} \iff \begin{cases} 0 \le f_i(\sigma;q) \le \mu_i\sigma, & \forall \sigma \ge 0, \\ \mu_i\sigma \le f_i(\sigma;q) \le 0, & \forall \sigma < 0, \end{cases}$$

which directly implies that, for the two sampling instances $k, k+1$, for $\hat{q}_{k,i}-q_{k,i} \ge 0$ and $\hat{q}_{k+1,i}-q_{k+1,i} \ge 0$,

$$0 \le \int_0^{\hat{q}_{k,i}-q_{k,i}} f_i(\sigma;q_k)d\sigma \le \frac{\mu_i}{2}\left(\hat{q}_{k,i}-q_{k,i}\right)^2,$$
$$0 \le \int_0^{\hat{q}_{k+1,i}-q_{k+1,i}} f_i(\sigma;q_{k+1})d\sigma \le \frac{\mu_i}{2}\left(\hat{q}_{k+1,i}-q_{k+1,i}\right)^2.$$

These bounds are directly implied as well for $\hat{q}_{k,i}-q_{k,i} < 0$ and for $\hat{q}_{k+1,i}-q_{k+1,i} < 0$, by using that

$$\int_0^{\hat{q}_{k,i}-q_{k,i}} f_i(\sigma;q_k)d\sigma = -\int_{\hat{q}_{k,i}-q_{k,i}}^0 f_i(\sigma;q_k)d\sigma,$$
$$\int_0^{\hat{q}_{k+1,i}-q_{k+1,i}} f_i(\sigma;q_{k+1})d\sigma = -\int_{\hat{q}_{k+1,i}-q_{k+1,i}}^0 f_i(\sigma;q_{k+1})d\sigma.$$

Thus,

$$\begin{aligned} -\frac{\mu_i}{2}\left(\hat{q}_{k,i}-q_{k,i}\right)^2 &\le -\int_0^{\hat{q}_{k,i}-q_{k,i}} f_i(\sigma;q_k)d\sigma \\ &\le \int_0^{\hat{q}_{k+1,i}-q_{k+1,i}} f_i(\sigma;q_{k+1})d\sigma - \int_0^{\hat{q}_{k,i}-q_{k,i}} f_i(\sigma;q_k)d\sigma \\ &\le \int_0^{\hat{q}_{k+1,i}-q_{k+1,i}} f_i(\sigma;q_{k+1})d\sigma \le \frac{\mu_i}{2}\left(\hat{q}_{k+1,i}-q_{k+1,i}\right)^2. \end{aligned}$$

$\square$

The Lyapunov function (14) for system (31) with the nonlinearity in $\Phi_{sb}^{[0,\mu]}$ yields

$$V\left(\hat{x}_k - x_k\right) = \bar{x}_k^T P \bar{x}_k + 2\sum_{i=1}^{n_q} Q_{ii} \int_0^{\hat{q}_{k,i}-q_{k,i}} f_i(\sigma; q_k) d\sigma + 2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_0^{\hat{q}_{k,i}-q_{k,i}} [\mu_i \sigma - f_i(\sigma; q_k)] d\sigma, \tag{33}$$

where

$$\bar{x}_k := \begin{bmatrix} \hat{x}_k - x_k \\ \hat{p}_k - p_k \\ \hat{q}_k - q_k \end{bmatrix}, \quad P^T = P := \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix} \succcurlyeq 0, \quad P_{11} \succ 0$$

$$\text{and } Q_{ii} \geq 0, \ \tilde{Q}_{ii} \geq 0, \quad \forall i = 1, \ldots, n_q.$$

The variation in the Lyapunov function (33) between the two sampling instances $k, k+1$ is expressed as

$$\begin{aligned} \Delta V\left(\hat{x}_k - x_k\right) &= \zeta_k^T (A_a^T P A_a - E_a^T P E_a) \zeta_k \\ &+ 2\sum_{i=1}^{n_q} Q_{ii} \left(\int_0^{\hat{q}_{k+1,i}-q_{k+1,i}} f_i(\sigma; q_{k+1}) d\sigma - \int_0^{\hat{q}_{k,i}-q_{k,i}} f_i(\sigma; q_k) d\sigma \right) \\ &+ 2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \left\{ \int_{\hat{q}_{k,i}-q_{k,i}}^{\hat{q}_{k+1,i}-q_{k+1,i}} \mu_i \sigma d\sigma - \left(\int_0^{\hat{q}_{k+1,i}-q_{k+1,i}} f_i(\sigma; q_{k+1}) d\sigma - \int_0^{\hat{q}_{k,i}-q_{k,i}} f_i(\sigma; q_k) d\sigma \right) \right\}, \end{aligned} \tag{34}$$

where

$$\zeta_k := \begin{bmatrix} \hat{x}_k - x_k \\ \hat{p}_k - p_k \\ \hat{p}_{k+1} - p_{k+1} \end{bmatrix}, \ A_a := \begin{bmatrix} A + LC_y & B_p & 0 \\ 0 & 0 & I \\ C_q(A + LC_y) & C_q B_p & D_{qp} \end{bmatrix}, \ E_a := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ C_q & D_{qp} & 0 \end{bmatrix}.$$

In order to find an LMI-form condition that implies condition (38), Lemma 2 is used to find quadratic upper bounds on the two integral terms of $\Delta V(\hat{x}_k - x_k)$ by

$$\begin{aligned} &2\sum_{i=1}^{n_q} Q_{ii} \bigg(\int_0^{\hat{q}_{k+1,i}-q_{k+1,i}} f_i(\sigma; q_{k+1}) d\sigma - \int_0^{\hat{q}_{k,i}-q_{k,i}} f_i(\sigma; q_k) d\sigma \bigg) \\ &\qquad \leq 2\sum_{i=1}^{n_q} Q_{ii} \Big(\frac{\mu_i}{2} \left(\hat{q}_{k+1,i} - q_{k+1,i}\right)^2 \Big) = \zeta_k^T E_1 \zeta_k, \end{aligned}$$

where $M := \text{diag}\{\mu_1, \ldots, \mu_{n_q}\} \succcurlyeq 0$, and $E_1 = E_1^T$ is defined as

$$E_1 := \begin{bmatrix} (A + LC_y)^T C_q^T Q M C_q (A + LC_y) & (A + LC_y)^T C_q^T Q M C_q B_p & (A + LC_y)^T C_q^T Q M D_{qp} \\ * & B_p^T C_q^T Q M C_q B_p & B_p^T C_q^T Q M D_{qp} \\ * & * & D_{qp}^T Q M D_{qp} \end{bmatrix},$$

and, for the second integral term,

$$
\begin{aligned}
2\sum_{i=1}^{n_q}\tilde{Q}_{ii}\Bigg\{\int_{\hat{q}_{k,i}-q_{k,i}}^{\hat{q}_{k+1,i}-q_{k+1,i}}\mu_i\sigma d\sigma-\Bigg(\int_0^{\hat{q}_{k+1,i}-q_{k+1,i}}f_i(\sigma;q_{k+1})d\sigma-\int_0^{\hat{q}_{k,i}-q_{k,i}}f_i(\sigma;q_k)d\sigma\Bigg)\Bigg\}& \\
\leq 2\sum_{i=1}^{n_q}\tilde{Q}_{ii}\Big\{\frac{\mu_i}{2}\left((\hat{q}_{k+1,i}-q_{k+1,i})^2-(\hat{q}_{k,i}-q_{k,i})^2\right)+\frac{\mu_i}{2}\left(\hat{q}_{k,i}-q_{k,i}\right)^2\Big\}& \\
=2\sum_{i=1}^{n_q}\tilde{Q}_{ii}\Big(\frac{\mu_i}{2}(\hat{q}_{k+1,i}-q_{k+1,i})^2\Big).&
\end{aligned}
$$

It can be easily seen that the upper bound of the second integral term is equivalent to that of the first integral term. Because the problem to be solved will be a feasibility problem and the existence of a diagonal matrix $Q \succcurlyeq 0$ that solves it is equivalent to the existence of diagonal $(Q+\tilde{Q}) \succcurlyeq 0$ that solves it, the second nonlinear term can be omitted. Thus, the Lyapunov function to be used for the error dynamics system (31) is

$$
V(\hat{x}_k-x_k)=\bar{x}_k^T P\bar{x}_k+2\sum_{i=1}^{n_q}Q_{ii}\int_0^{\hat{q}_{k,i}-q_{k,i}}f_i(\sigma;q_k)d\sigma, \tag{35}
$$

with $\bar{x}_k, P, Q_{ii}$ as in (33), and with variation between the consecutive sampling instances $k, k+1$,

$$
\begin{aligned}
\Delta V(\hat{x}_k-x_k)=\;&\zeta_k^T(A_a^TPA_a-E_a^TPE_a)\zeta_k\\
&+2\sum_{i=1}^{n_q}Q_{ii}\Bigg(\int_0^{\hat{q}_{k+1,i}-q_{k+1,i}}f_i(\sigma;q_{k+1})d\sigma-\int_0^{\hat{q}_{k,i}-q_{k,i}}f_i(\sigma;q_k)d\sigma\Bigg)\\
\leq\;&\zeta_k^T(A_a^TPA_a-E_a^TPE_a+E_1)\zeta_k,
\end{aligned} \tag{36}
$$

with ζ_k, A_a, E_a as in (34).

Theorem 4: A sufficient condition for system (31) to be globally asymptotically stable is the existence of a positive definite matrix $P=P^T\in\mathbb{R}^{(n+n_p+n_q)\times(n+n_p+n_q)}$, with a positive definite submatrix $P_{11}=P_{11}^T\in\mathbb{R}^{n\times n}$, a diagonal positive definite matrix $Q\in\mathbb{R}^{n_q\times n_q}$ and diagonal positive semidefinite matrices $T,\tilde{T}\in\mathbb{R}^{n_q\times n_q}$ that satisfy the bilinear matrix inequality,

$$
R^T=R:=\left[\begin{array}{c|c|c} R_1 & R_2 & R_3\\ \hline R_2^T & -P & 0\\ \hline R_3^T & 0 & R_4\end{array}\right]\prec 0, \tag{37}
$$

where

$$R_1 := \begin{bmatrix} \begin{pmatrix} -P_{11} - C_q^T P_{13}^T - \\ -P_{13} C_q - C_q^T P_{33} C_q \end{pmatrix} & \begin{pmatrix} -P_{12} - C_q^T P_{23}^T - \\ -P_{13} D_{qp} - C_q^T P_{33} D_{qp} + \\ +A^T C_q^T Q M C_q B_p + \\ +C_y^T L^T C_q^T Q M C_q B_p - C_q^T T \end{pmatrix} & \begin{pmatrix} A^T C_q^T Q M D_{qp} + \\ +C_y^T L^T C_q^T Q M D_{qp} - \\ -A^T C_q^T \tilde{T} - C_y^T L^T C_q^T \tilde{T} \end{pmatrix} \\ * & \begin{pmatrix} -P_{22} - D_{qp}^T P_{23}^T - \\ -P_{23} D_{qp} - D_{qp}^T P_{33} D_{qp} + \\ +B_p^T C_q^T Q M C_q B_p - \\ -2TM^{-1} - T D_{qp} - D_{qp}^T T \end{pmatrix} & \begin{pmatrix} B_p^T C_q^T Q M D_{qp} - B_p^T C_q^T \tilde{T} \end{pmatrix} \\ * & * & \begin{pmatrix} D_{qp}^T Q M D_{qp} - 2\tilde{T} M^{-1} - \\ -\tilde{T} D_{qp} - D_{qp}^T \tilde{T} \end{pmatrix} \end{bmatrix},$$

$$R_2 := \begin{bmatrix} \begin{pmatrix} A^T P_{11} + C_y^T L^T P_{11} + \\ +A^T C_q^T P_{13}^T + \\ +C_y^T L^T C_q^T P_{13}^T \end{pmatrix} & \begin{pmatrix} A^T P_{12} + C_y^T L^T P_{12} + \\ +A^T C_q^T P_{23}^T + \\ +C_y^T L^T C_q^T P_{23}^T \end{pmatrix} & \begin{pmatrix} A^T P_{13} + C_y^T L^T P_{13} + \\ +A^T C_q^T P_{33} + \\ +C_y^T L^T C_q^T P_{33} \end{pmatrix} \\ B_p^T P_{11} + B_p^T C_q^T P_{13}^T & B_p^T P_{12} + B_p^T C_q^T P_{23}^T & B_p^T P_{13} + B_p^T C_q^T P_{33} \\ P_{12}^T + D_{qp}^T P_{13}^T & P_{22} + D_{qp}^T P_{23}^T & P_{23} + D_{qp}^T P_{33} \end{bmatrix},$$

$$R_3 := \begin{bmatrix} A^T C_q^T Q M + C_y^T L^T C_q^T Q M \\ 0 \\ 0 \end{bmatrix},$$

$$R_4 := -QM.$$

Proof: Given the Lyapunov function (35), a sufficient condition for the global asymptotic stability of the system is for the condition

$$\Delta V(\hat{x}_k - x_k) < 0, \;\; \forall k \geq 0 \tag{38}$$

to be satisfied. Given the upper bound for $\Delta V(\hat{x}_k - x_k)$ in (36),

$$\zeta_k^T (A_a^T P A_a - E_a^T P E_a + E_1) \zeta_k < 0 \implies \Delta V(\hat{x}_k - x_k) < 0. \tag{39}$$

Therefore, a sufficient condition for the global asymptotic stability of the system is for the left-hand side of (39) to be satisfied for all ζ_k that satisfy the sector-boundedness condition on $f(\,\cdot\,; q)$. This condition on ζ_k will be introduced via the S-procedure. First, the LMI-form of the condition is found. Let $f_k \equiv f(\hat{q}_k - q_k; q_k)$ and $f_{k+1} \equiv f(\hat{q}_{k+1} - q_{k+1}; q_{k+1})$. Then

$$f(\,\cdot\,; q) \in \Phi_{sb}^{[0,\mu]} \iff f_{k,i}[\mu_i^{-1} f_{k,i} - (\hat{q}_{k,i} - q_{k,i})] \leq 0, \;\; i = 1, \ldots, n_q, \;\; \forall k. \tag{40}$$

A useful notation for using the S-procedure with condition (40) is

$$\sum_{i=1}^{n_q} 2\tau_i f_{k,i}[\mu_i^{-1} f_{k,i} - (\hat{q}_{k,i} - q_{k,i})] = \zeta_k^T S_1 \zeta_k \leq 0,$$

where $T := \text{diag}\{\tau_1, \ldots, \tau_{n_q}\} \succcurlyeq 0$, and $S_1 = S_1^T$ is defined as

$$S_1 := \begin{bmatrix} 0 & C_q^T T & 0 \\ * & 2TM^{-1} + TD_{qp} + D_{qp}^T T & 0 \\ * & * & 0 \end{bmatrix}.$$

Similarly, for the next sampling instance,

$$\sum_{i=1}^{n_q} 2\tilde{\tau}_i f_{k+1,i}[\mu_i^{-1} f_{k+1,i} - (\hat{q}_{k+1,i} - q_{k+1,i})] = \zeta_k^T S_2 \zeta_k \leq 0,$$

where $\tilde{T} := \text{diag}\{\tilde{\tau}_1, \ldots, \tilde{\tau}_{n_q}\} \succcurlyeq 0$, and $S_2 = S_2^T$ is defined as

$$S_2 := \begin{bmatrix} 0 & 0 & (A + LC_y)^T C_q^T \tilde{T} \\ * & 0 & B_p^T C_q^T \tilde{T} \\ * & * & 2\tilde{T}M^{-1} + \tilde{T}D_{qp} + D_{qp}^T \tilde{T} \end{bmatrix}.$$

Finally, applying the S-procedure gives that, if $F := A_a^T P A_a - E_a^T P E_a + E_1 - S_1 - S_2 \prec 0$ is feasible, then $\Delta V\,(\hat{x}_k - x_k) < 0$ is satisfied $\forall k \geq 0$, and the system is globally asymptotically stable.

At first glance, this inequality for F shows that the term $A_a^T P A_a$ will give rise to the submatrix variables in P being pre- and post- multiplied by matrix variable L. To address these higher order terms, the Schur complement lemma is applied as

$$\left.\begin{array}{l} F \prec 0 \\ P \succ 0 \end{array}\right\} \iff \begin{bmatrix} -E_a^T P E_a + E_1 - S_1 - S_2 & A_a^T P \\ P A_a & -P \end{bmatrix} \prec 0.$$

As in the case of the state feedback controller design, in order for the Schur complement lemma to hold, matrix P must be positive definite rather than positive semidefinite.

This matrix inequality is not yet bilinear, given that the term $(A+LC_y)^T C_q^T QMC_q(A+LC_y)$ is in the submatrix $E_{1,11}$. Let

$$W := \begin{bmatrix} (A + LC_y)^T C_q^T QMC_q(A + LC_y) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and let $\tilde{E}_1 := E_1 - W$. The Schur complement lemma is then applied once more, yielding

$$\left.\begin{array}{r}\begin{bmatrix} -E_a^T P E_a + \tilde{E}_1 - S_1 - S_2 & \tilde{A}_a^T P \\ P\tilde{A}_a & -P \end{bmatrix} - \begin{bmatrix} -W & 0 \\ 0 & 0 \end{bmatrix} \prec 0 \\ QM \succ 0 \end{array}\right\} \iff R \prec 0.$$

Again, using the Schur complement lemma imposes that Q and M be positive definite.

□

5.2 Illustrative Numerical Example of a State Estimator

Analogously to Theorem 3, the BMI obtained in Theorem 4 has the advantage that it is an LMI for fixed L. This property is used for obtaining numerical results showing the stabilization of an error dynamics system through a matrix L. In particular, the numerical results illustate how Theorem 4 holds for an example for which L is known, which is implemented in LMI software as explained in Section 1.3.2.

The system (31) defined by the matrices

$$A = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ C_q = \begin{bmatrix} 0.8 & -0.5 & 0 & 1 \end{bmatrix},\ C_y = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \tag{41}$$

with $L = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ and $\mu = 2$ is not globally asymptotically stable. The unstable behavior of the system can be seen in Figure 3.

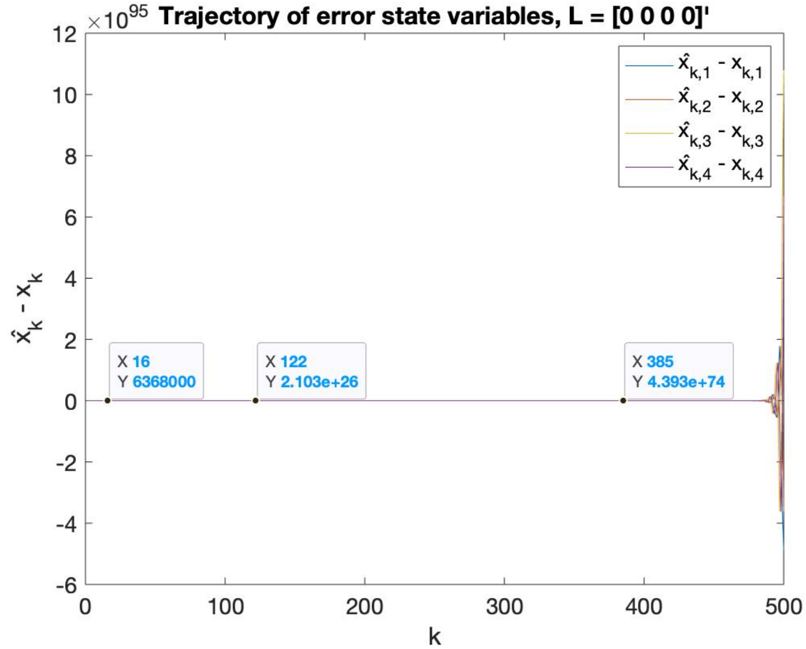
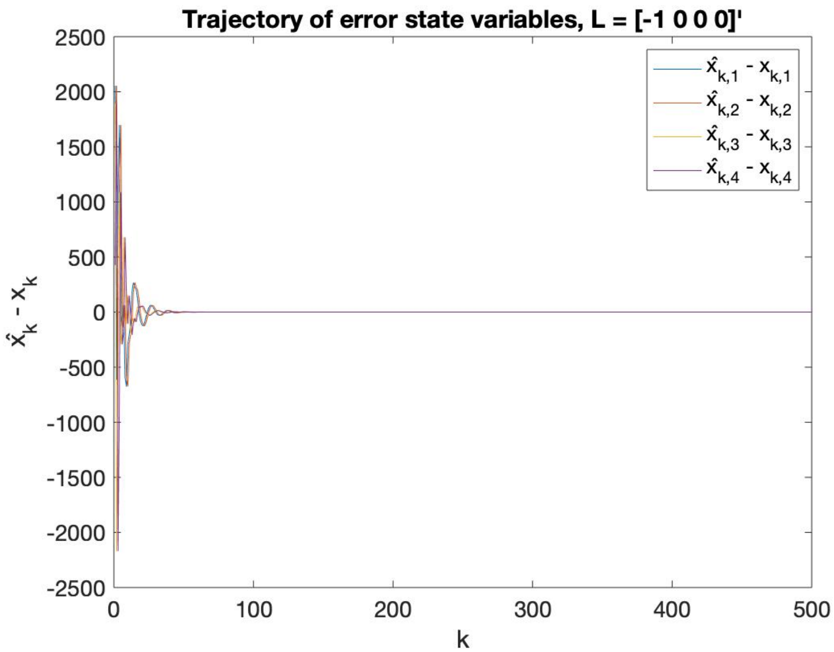


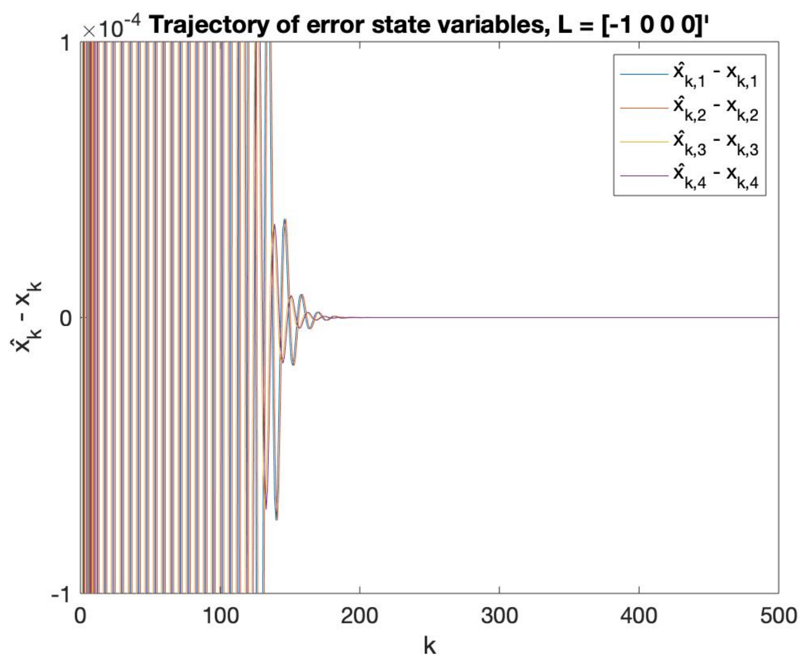
Figure 3: First 500 iterations of the error dynamics system (41) with zero estimator matrix L. The initial values of the four state variables are set to random values between 0 and 10^4. Some points are labeled to show that the absolute values of the state variables are many (and increasing) orders of magnitude higher than their initial values.

Theorem 4 holds for this example since the BMI (37) is infeasible for system (41) with $L = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$.

The BMI is found to be feasible for the estimator matrix $L = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}^T$, implying that the error dynamics system is globally asymptotically stable. This stable behavior is shown in Figure 4, again as predicted by Theorem 4.



(a)



(b)

Figure 4: (a) First 500 iterations of the system (41) with the estimator matrix $L = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}^T$. The initial values of the four error state variables are set to random values between 0 and 10^4. (b) Close-up of (a), which confirms that the values of the error state variables converge to zero.

6 Conclusions

6.1 Contributions

The main contributions of this thesis are:

- Rigorous mathematical formulations are developed for new criteria for the absolute stability, ℓ_2- and RMS-gain, state feedback controller design, and state estimator design of a discrete-time Lur'e system with sector-bounded, slope-restricted nonlinearities.

- Tight bounds are found for the variation in the integrals in the used modified Lur'e-Postnikov Lyapunov function, based on the local slope restriction on the system's nonlinearities. These bounds lead to a significant decrease in the conservatism of the conditions for Lyapunov stability of the system. The reduced conservatism is illustrated by obtaining the results for the lower bound on the robustness margin of six numerical examples and comparing them to the most relevant criteria in the literature.

- The same bounds are applied in the ℓ_2- and RMS-gain analysis and for state feedback controller design, and therefore conservatism is significantly decreased for those problems as well. The upper bound on the gain is obtained for illustrative numerical examples and the behavior of a nominally unstable system stabilized by state feedback is shown.

- The state estimator error dynamics are analyzed and the resulting nonlinearities are characterized, leading to a modification of the previously used Lyapunov function. Bounds on the variation of the integrals in the Lyapunov function are derived based on the sector boundedness of the nonlinearities. A sufficient condition for the stability of the error dynamics system is provided and illustrated through a numerical example.

- The criteria provided for the Lyapunov stability and gain of the system are based on LMIs and therefore lead to convex optimizations that are highly computationally efficient. The criteria provided for the state feedback controller design and state estimator design are based on BMIs, which can be solved with software that is readily available. Although BMIs are not as efficient for problems of large dimensions, the resulting problems are LMIs for fixed K or for fixed L.

- A software implementation in MATLAB is provided for the set of numerical examples of the four LMI criteria developed in the thesis. The provided scripts can be easily adapted to any other example of a Lur'e system with sector-bounded, slope-restricted nonlinearities. The bisection method scripts used to solve the feasibility optimization problems in the absolute stability and gain analysis can also be easily adapted to different desired accuracies in the results.

6.2 Further work

Natural continuations of this work might include:

- Studying the state feedback controller design for the case with $D_{qu} \neq 0$, and attempting to reduce the resulting matrix inequalities to BMIs. Although the addition of this term makes the reduction more complicated, the term appears in some systems including for some types of dynamic neural networks.
- Designing an estimator-based output feedback controller for the studied system. This approach would enable the control of systems with unmeasured variables.
- Implementing the optimizations for the design of a state feedback controller and a state estimator using a BMI solver and applying to several examples.
- Exploring how close iteratively solving the LMI optimizations of each of the variables in the BMI formulations converges to the global optimum, for a range of numerical examples.

7 References

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Appendices

A MATLAB Scripts for Chapter 2

To use these scripts, for each example i, run the code `findmargin_example`i`.m`, which calls the scripts `lmis_stability.m` and `bisection.m`.

findmargin_example1.m

```
% G(z) = (-0.5z + 0.1) / (z^3 - z^2 + 0.89z + 0.1z^2 - 0.1z + 0.089) =
%      = (-0.5z + 0.1) / (z^3 - 0.9z^2 + 0.79z + 0.089)
%
% ==>  b0 = 0, b1 = 0, b2 = -0.5, b3 = 0.1
%      (a0 = 1), a1 = -0.9, a2 = 0.79, a3 = 0.089
%
% Using the Observable Canonical Form

clear variables

% Previous results for bisection:
% xi = 2 feasible, xi = 3 infeasible
xi_a = 2; xi_b = 3;

% Definition of example
global A B C D n nq

A = [ 0.9    1  0;
     -0.79   0  1;
     -0.089  0  0 ];
B = [ 0  ;
     -0.5;
      0.1 ];
C = [1 0 0];
D = 0;

% Dimensions of the system
n = 3;
nq = 1;

mu_ct = 2;        % Linear dependence of mu on xi
nq_vec = [1 0];   % Parameter for matrix variables

%%

```

```
margin_example1 = bisection(@lmis_stability, xi_a, xi_b, mu_ct, nq_vec)
```

findmargin_example2.m

```
clear variables

% Previous results for bisection:
% xi = 0.1 feasible, xi = 0.5 infeasible
xi_a = 0.1; xi_b = 0.5;

% Definition of example
global A B C D n nq

A = diag([0.2948 0.4568 0.0226 0.3801 -0.3270]);
B = [-1.1878  0.2341;
     -2.2023  0.0215;
      0.9863 -1.0039;
     -0.5186 -0.9471;
      0.3274 -0.3744 ];
C = [-1.1859  1.4725 -1.2173 -1.1283 -0.2611;
     -1.0559  0.0557 -0.0412 -1.3493  0.9535 ];
D = zeros(2);

% Dimensions of the system
n = 5;
nq = 2;

mu_ct = 1;              % Linear dependence of mu on xi
nq_vec = [1 0; 1 0];    % Parameter for matrix variables

%%

margin_example2 = bisection(@lmis_stability, xi_a, xi_b, mu_ct, nq_vec)
```

findmargin_example3.m

```
%clear variables

% Previous results for bisection:
% xi = 0.1 feasible, xi = 0.5 infeasible
xi_a = 0.1; xi_b = 1;

% Definition of example
```

```
global A B C D n nq

A = [ 0.0469 -0.3992 -0.0835;
      0.3902 -0.5363 -0.2744;
      0.4378 -1.3576  0.4651 ];
B = [-0.5673 -0.2785;
      0.1155 -0.0649;
     -2.1849 -0.5976 ];
C = [ 0.3587 -1.0802 -0.6802;
     -1.3833 -1.0677  1.1497 ];
D = zeros(2);

% Dimensions of the system
n = 3;
nq = 2;

mu_ct = 1;              % Linear dependence of mu on xi
nq_vec = [1 0; 1 0];    % Parameter for matrix variables

%%

margin_example3 = bisection(@lmis_stability, xi_a, xi_b, mu_ct, nq_vec)
```

findmargin_example4.m

```
clear variables

% Previous results for bisection:
% xi = 1 feasible, xi = 3 infeasible
xi_a = 1; xi_b = 3;

% Definition of example
global A B C D n nq

A = diag([0.4030 -0.1502 -0.1502]);
B = [-0.2494;
      0.2542;
     -0.2036 ];
C = [0.9894 0.6649 0.4339];
D = 0;

% Dimensions of the system
n = 3;
nq = 1;

```

```
mu_ct = 2;          % Linear dependence of mu on xi
nq_vec = [1 0];     % Parameter for matrix variables

%%

margin_example4 = bisection(@lmis_stability, xi_a, xi_b, mu_ct, nq_vec)
```

findmargin_example5.m

```
clear variables

% Previous results for bisection:
% xi = 0.01 feasible, xi = 0.1 infeasible
xi_a = 0.01; xi_b = 0.1;

% Definition of example
global A B C D n nq

A = diag([0.4783 0.7871 0.7871 0.7871]); A(3,4) = 1;
B = [-1.5174;
      1.2181;
      0.2496;
     -0.5181 ];
C = [0.8457 -2.0885 1.2190 0.1683];
D = 0;

% Dimensions of the system
n = 4;
nq = 1;

mu_ct = 2;          % Linear dependence of mu on xi
nq_vec = [1 0];     % Parameter for matrix variables

%%

margin_example5 = bisection(@lmis_stability, xi_a, xi_b, mu_ct, nq_vec)
```

findmargin_example6.m

```
clear variables

% Previous results for bisection:
% xi = 0.1 feasible, xi = 1 infeasible
```

```
xi_a = 0.1; xi_b = 1;

% Definition of example
global A B C D n nq

A = diag([0.5359 0.9417 0.9802 0.5777 -0.1227 -0.0034 -0.5721 0.2870 ...
    -0.3599]);
B = [1 0 0 0;
     0 1 0 0;
     0 0 1 0;
     0 0 0 1;
     1 0 0 0;
     0 1 0 0;
     0 0 1 0;
     0 0 0 1;
     1 0 0 0 ];
C = [1 1 0 0 0 0 0 0 0;
     0 0 1 1 1 0 0 0 0;
     0 0 0 0 0 1 1 0 0;
     0 0 0 0 0 0 0 1 1 ];
D = zeros(4);

% Dimensions of the system
n = 9;
nq = 4;

mu_ct = 1;                          % Linear dependence of mu on xi
nq_vec = [1 0; 1 0; 1 0; 1 0];    % Parameter for matrix variables

%%

margin_example6 = bisection(@lmis_stability, xi_a, xi_b, mu_ct, nq_vec)
```

lmis_stability.m

```
function tmin = lmis_stability(xi, mu_ct, nq_vec)

poszero = 1e-12;      % Numerical zero for nonstrict LMIs
negzero = -poszero;   % For positive semidefinite LMIs

global A B C D n nq

% Definition of M and X: change xi according to example
mu = mu_ct * xi;

```

```
X = xi * eye(nq);
M = mu * eye(nq);
Xi = inv(X);
Mi = inv(M);

%%
setlmis([]);
P11 = lmivar(1, [n 1]);
P12 = lmivar(2, [n nq]);
P13 = lmivar(2, [n nq]);
P22 = lmivar(1, [nq 1]);
P23 = lmivar(2, [nq nq]);
P33 = lmivar(1, [nq 1]);

Q = lmivar(1, nq_vec);
tQ = lmivar(1, nq_vec);
T = lmivar(1, nq_vec);
tT = lmivar(1, nq_vec);
N = lmivar(1, nq_vec);


G = newlmi;

lmiterm([G 1 1 P11], A', A);
lmiterm([G 1 1 P13], A', C*A, 's');
lmiterm([G 1 1 P33], A'*C', C*A);
lmiterm([G 1 1 P11], -1, 1);
lmiterm([G 1 1 P13], -1, C, 's');
lmiterm([G 1 1 P33], -C', C);
lmiterm([G 1 1 tQ], A'*C', X*C*A);
lmiterm([G 1 1 tQ], -C', X*C);

lmiterm([G 1 2 P11], A', B);
lmiterm([G 1 2 -P13], A'*C', B);
lmiterm([G 1 2 P13], A', C*B);
lmiterm([G 1 2 P33], A'*C', C*B);
lmiterm([G 1 2 P12], -1, 1);
lmiterm([G 1 2 -P23], -C', 1);
lmiterm([G 1 2 P13], -1, D);
lmiterm([G 1 2 P33], -C', D);
lmiterm([G 1 2 tQ], A'*C', X*C*B);
lmiterm([G 1 2 tQ], -C', X*D);
lmiterm([G 1 2 tQ], (C*A - C)', 1);
lmiterm([G 1 2 T], -C', 1);
lmiterm([G 1 2 N], (C*A - C)', 1);

```

```
lmiterm([G 1 3 P12], A', 1);
lmiterm([G 1 3 -P23], A'*C', 1);
lmiterm([G 1 3 P13], A', D);
lmiterm([G 1 3 P33], A'*C', D);
lmiterm([G 1 3 Q], -(C*A - C)', 1);
lmiterm([G 1 3 tQ], A'*C', X*D);
lmiterm([G 1 3 tT], -A'*C', 1);
lmiterm([G 1 3 N], -(C*A - C)', 1);

lmiterm([G 2 2 P11], B', B);
lmiterm([G 2 2 P13], B', C*B, 's');
lmiterm([G 2 2 P33], B'*C', C*B);
lmiterm([G 2 2 P22], -1, 1);
lmiterm([G 2 2 P23], -1, D, 's');
lmiterm([G 2 2 P33], -D', D);
lmiterm([G 2 2 Q], -1, Mi);
lmiterm([G 2 2 tQ], B'*C', X*C*B);
lmiterm([G 2 2 tQ], -D', X*D);
lmiterm([G 2 2 tQ], 1, C*B - D, 's');
lmiterm([G 2 2 tQ], -1, Mi);
lmiterm([G 2 2 T], -2, Xi);
lmiterm([G 2 2 T], -1, D, 's');
lmiterm([G 2 2 N], -2, Mi);
lmiterm([G 2 2 N], 1, (C*B - D), 's');

lmiterm([G 2 3 P12], B', 1);
lmiterm([G 2 3 -P23], B'*C', 1);
lmiterm([G 2 3 P13], B', D);
lmiterm([G 2 3 P33], B'*C', D);
lmiterm([G 2 3 Q], (Mi - (C*B - D))', 1);
lmiterm([G 2 3 tQ], B'*C', X*D);
lmiterm([G 2 3 tQ], 1, D + Mi);
lmiterm([G 2 3 tT], -B'*C', 1);
lmiterm([G 2 3 N], -(C*B - D)', 1);
lmiterm([G 2 3 N], 1, 2*Mi + D);

lmiterm([G 3 3 P22], 1, 1);
lmiterm([G 3 3 P23], 1, D, 's');
lmiterm([G 3 3 P33], D', D);
lmiterm([G 3 3 Q], -1, Mi);
lmiterm([G 3 3 Q], -1, D, 's');
lmiterm([G 3 3 tQ], D', X*D);
lmiterm([G 3 3 tQ], -1, Mi);
lmiterm([G 3 3 tT], -2, Xi);
lmiterm([G 3 3 tT], -1, D, 's');
lmiterm([G 3 3 N], -2, Mi);
```

```
lmiterm([G 3 3 N], -1, D, 's');

P11l = newlmi;
lmiterm([-P11l 1 1 P11], 1, 1);

Pl = newlmi;
lmiterm([-Pl 1 1 P11], 1, 1);
lmiterm([-Pl 1 2 P12], 1, 1);
lmiterm([-Pl 1 3 P13], 1, 1);
lmiterm([-Pl 2 2 P22], 1, 1);
lmiterm([-Pl 2 3 P23], 1, 1);
lmiterm([-Pl 3 3 P33], 1, 1);
% semidefiniteness
lmiterm([Pl 1 1 0], negzero);
lmiterm([Pl 2 2 0], negzero);
lmiterm([Pl 3 3 0], negzero);

Ql = newlmi;
lmiterm([-Ql 1 1 Q], 1, 1);
% semidefiniteness
lmiterm([Ql 1 1 0], negzero);
tQl = newlmi;
lmiterm([-tQl 1 1 tQ], 1, 1);
% semidefiniteness
lmiterm([tQl 1 1 0], negzero);
Tl = newlmi;
lmiterm([-Tl 1 1 T], 1, 1);
% semidefiniteness
lmiterm([Tl 1 1 0], negzero);
tTl = newlmi;
lmiterm([-tTl 1 1 tT], 1, 1);
% semidefiniteness
lmiterm([tTl 1 1 0], negzero);
Nl = newlmi;
lmiterm([-Nl 1 1 N], 1, 1);
% semidefiniteness
lmiterm([Nl 1 1 0], negzero);

lmisys = getlmis;

%%
target = []; options=zeros(1,5);
options(2) = 300;
options(3) = 1;
[tmin,xfeas] = feasp(lmisys, options, target);
end
```

bisection.m

```
function p = bisection(f, a, b, mu_ct, nq_vec)

dif_tol = 1e-7;  % tolerance for iteration difference
max_it = 200;    % max number of iterations

if f(a, mu_ct, nq_vec)*f(b, mu_ct, nq_vec) > 0
    disp('Wrong choice')
else
    it = 1;
    dif_p = 1;
    p = (a + b)/2;
    fp = f(p, mu_ct, nq_vec);
    while(it < max_it && dif_p >= dif_tol)
        it = it + 1;
        if f(a, mu_ct, nq_vec)*fp < 0
            b = p;
        else
            a = p;
        end
        p_old = p;
        p = (a + b)/2;
        p_new = p;
        dif_p = abs(p_new - p_old);
        fp = f(p, mu_ct, nq_vec);
        if it == 100; ...
                disp('Stopped because bisection it = max_it'); end
        if dif_p < dif_tol; ...
                disp('Stopped because bisection dif_p < dif_tol'); end
    end
    if fp < 0; p = b; end
    % Note: this gives an infeasible bound
end
end
```

B MATLAB Scripts for Chapter 3

To use these scripts, for each example i, run the code `findgain_example`i`.m`, which calls the scripts `lmis_gain.m` and `bisection_semidef.m`.

findgain_example2.m

```
clear variables

% Previous results for bisection:
% gamma = 0.1 infeasible, gamma = 100 feasible
gamma_a = 0.1; gamma_b = 100;

% Definition of example
global A Bp Bw Cq Cz Dqp Dqw Dzp Dzw n nq nw

A = diag([0.2948 0.4568 0.0226 0.3801 -0.3270]);
Bp = [-1.1878  0.2341;
       -2.2023  0.0215;
        0.9863 -1.0039;
       -0.5186 -0.9471;
        0.3274 -0.3744 ];
Bw = [ 1;
        1;
        1;
        1;
        1];
Cq = [-1.1859  1.4725 -1.2173 -1.1283 -0.2611;
       -1.0559  0.0557 -0.0412 -1.3493  0.9535 ];
Dqw = zeros(2,1);
Dqp = zeros(2);
Cz = [1 0 0 0 0];
Dzp = 0;
Dzw = 1;

% Dimensions of the system
n = 5;
nq = 2;
nw = 1;

mu_ct = 1;              % Linear dependence of mu on xi
nq_vec = [1 0; 1 0];    % Parameter for matrix variables

%%

```

```
gain_example2 = bisection_semidef(@lmis_gain, gamma_a, gamma_b, ...
    mu_ct, nq_vec)
```

findgain_example3.m

```
clear variables

% Previous results for bisection:
% gamma = 0.1 infeasible, gamma = 100 feasible
gamma_a = 0.1; gamma_b = 100;

% Definition of example
global A Bp Bw Cq Cz Dqp Dqw Dzp Dzw n nq nw

A = [ 0.0469 -0.3992 -0.0835;
      0.3902 -0.5363 -0.2744;
      0.4378 -1.3576  0.4651 ];
Bp = [-0.5673 -0.2785;
       0.1155 -0.0649;
      -2.1849 -0.5976 ];
Bw = [ 0.0962 -0.5829;
      -0.0482  0.4739;
      -1.1274  1.1238 ];
Cq = [ 0.3587 -1.0802 -0.6802;
      -1.3833 -1.0677  1.1497 ];
Dqw = [ 0.1562  0.4342;
        0.5472  0.0356 ];
Dqp = zeros(2);
Cz = [ 0.9792 0.1112 -0.8091;
       0.6970 1.3471 -0.0023 ];
Dzp = [ 0.0010 -0.7238;
        1.2356 0.2360 ];
Dzw = [ 0.5474  0.0242;
        0.2762  0.0486 ];


% Dimensions of the system
n = 3;
nq = 2;
nw = 2;

mu_ct = 1;              % Linear dependence of mu on xi
nq_vec = [1 0; 1 0];    % Parameter for matrix variables

%%
```

```
gain_example3 = bisection_semidef(@lmis_gain, gamma_a, gamma_b, ...
    mu_ct, nq_vec)
```

lmis_gain.m

```
function tmin = lmis_gain(gamma, mu_ct, nq_vec)

poszero = 1e-12;      % Numerical zero for nonstrict LMIs
negzero = -poszero;   % For positive semidefinite LMIs

global A Bp Bw Cq Cz Dqp Dqw Dzp Dzw n nq nw

% Definition of M and X: change xi according to example
xi = 0.01;
% xi = 0.1;
mu = mu_ct * xi;

X = xi * eye(nq);
M = mu * eye(nq);
Xi = inv(X);
Mi = inv(M);

%%
setlmis([]);
P11 = lmivar(1, [n 1]);
P12 = lmivar(2, [n nq]);
P13 = lmivar(2, [n nq]);
P14 = lmivar(2, [n nw]);
P22 = lmivar(1, [nq 1]);
P23 = lmivar(2, [nq nq]);
P24 = lmivar(2, [nq nw]);
P33 = lmivar(1, [nq 1]);
P34 = lmivar(2, [nq nw]);
P44 = lmivar(1, [nw 1]);

Q = lmivar(1, nq_vec);
tQ = lmivar(1, nq_vec);
T = lmivar(1, nq_vec);
tT = lmivar(1, nq_vec);
N = lmivar(1, nq_vec);


H = newlmi;

```

```
lmiterm([H 1 1 P11], A', A);
lmiterm([H 1 1 P13], A', Cq*A, 's');
lmiterm([H 1 1 P33], A'*Cq', Cq*A);
lmiterm([H 1 1 P11], -1, 1);
lmiterm([H 1 1 P13], -1, Cq, 's');
lmiterm([H 1 1 P33], -Cq', Cq);
lmiterm([H 1 1 tQ], A'*Cq', X*Cq*A);
lmiterm([H 1 1 tQ], -Cq', X*Cq);
lmiterm([H 1 1 0], Cz'*Cz);

lmiterm([H 1 2 P11], A', Bp);
lmiterm([H 1 2 P13], A', Cq*Bp);
lmiterm([H 1 2 -P13], A'*Cq', Bp);
lmiterm([H 1 2 P33], A'*Cq', Cq*Bp);
lmiterm([H 1 2 P12], -1, 1);
lmiterm([H 1 2 -P23], -Cq', 1);
lmiterm([H 1 2 P13], -1, Dqp);
lmiterm([H 1 2 P33], -Cq', Dqp);
lmiterm([H 1 2 tQ], A'*Cq', X*Cq*Bp);
lmiterm([H 1 2 tQ], Cq', X*Dqp);
lmiterm([H 1 2 tQ], (Cq*A - Cq)', 1);
lmiterm([H 1 2 0], Cz'*Dzp);
lmiterm([H 1 2 T], -Cq', 1);
lmiterm([H 1 2 N], (Cq*A - Cq)', 1);

lmiterm([H 1 3 P11], A', Bw);
lmiterm([H 1 3 P13], A', Cq*Bw);
lmiterm([H 1 3 -P13], A'*Cq', Bw);
lmiterm([H 1 3 P33], A'*Cq', Cq*Bw);
lmiterm([H 1 3 P13], -1, Dqw);
lmiterm([H 1 3 P33], -Cq', Dqw);
lmiterm([H 1 3 P14], -1, 1);
lmiterm([H 1 3 P34], -Cq', 1);
lmiterm([H 1 3 tQ], A'*Cq', X*Cq*Bw);
lmiterm([H 1 3 tQ], -Cq', X*Dqw);
lmiterm([H 1 3 0], Cz'*Dzw);

lmiterm([H 1 4 P12], A', 1);
lmiterm([H 1 4 -P23], A'*Cq', 1);
lmiterm([H 1 4 P13], A', Dqp);
lmiterm([H 1 4 P33], A'*Cq', Dqp);
lmiterm([H 1 4 Q], -(Cq*A - Cq)', 1);
lmiterm([H 1 4 tQ], A'*Cq', X*Dqw);
lmiterm([H 1 4 tT], -A'*Cq', 1);
lmiterm([H 1 4 N], -(Cq*A - Cq)', 1);
```

```
lmiterm([H 1 5 P13], A', Dqw);
lmiterm([H 1 5 P33], A'*Cq', Dqw);
lmiterm([H 1 5 P14], A', 1);
lmiterm([H 1 5 P34], A'*Cq', 1);
lmiterm([H 1 5 tQ], A'*Cq', X*Dqw);

lmiterm([H 2 2 P11], Bp', Bp);
lmiterm([H 2 2 P13], Bp', Cq*Bp, 's');
lmiterm([H 2 2 P33], Bp'*Cq', Cq*Bp);
lmiterm([H 2 2 P22], -1, 1);
lmiterm([H 2 2 P23], -1, Dqp, 's');
lmiterm([H 2 2 P33], -Dqp', Dqp);
lmiterm([H 2 2 Q], -1, Mi);
lmiterm([H 2 2 tQ], Bp'*Cq', X*Cq*Bp);
lmiterm([H 2 2 tQ], -Dqp', X*Dqp);
lmiterm([H 2 2 tQ], 1, Cq*Bp - Dqp, 's');
lmiterm([H 2 2 tQ], -1, Mi);
lmiterm([H 2 2 0], Dzp'*Dzp);
lmiterm([H 2 2 T], -2, Xi);
lmiterm([H 2 2 T], -1, Dqp, 's');
lmiterm([H 2 2 N], -2, Mi);
lmiterm([H 2 2 N], 1, Cq*Bp - Dqp, 's');


lmiterm([H 2 3 P11], Bp', Bw);
lmiterm([H 2 3 -P13], Bp'*Cq', Bw);
lmiterm([H 2 3 P13], Bp', Cq*Bw);
lmiterm([H 2 3 P33], Bp'*Cq', Cq*Bw);
lmiterm([H 2 3 P23], -1, Dqw);
lmiterm([H 2 3 P33], -Dqp', Dqw);
lmiterm([H 2 3 P24], -1, 1);
lmiterm([H 2 3 P34], -Dqp', 1);
lmiterm([H 2 3 tQ], Bp'*Cq', X*Cq*Bw);
lmiterm([H 2 3 tQ], -Dqp', X*Dqw);
lmiterm([H 2 3 tQ], 1, Cq*Bw - Dqw);
lmiterm([H 2 3 0], Dzp'*Dzw);
lmiterm([H 2 3 T], -1, Dqw);
lmiterm([H 2 3 N], 1, Cq*Bw - Dqw);


lmiterm([H 2 4 P12], Bp', 1);
lmiterm([H 2 4 -P23], Bp'*Cq', 1);
lmiterm([H 2 4 P13], Bp', Dqp);
lmiterm([H 2 4 P33], Bp'*Cq', Dqp);
lmiterm([H 2 4 Q], Mi, 1);
lmiterm([H 2 4 Q], -(Cq*Bp - Dqp)', 1);
```

```
lmiterm([H 2 4 tQ], Bp'*Cq', X*Dqp);
lmiterm([H 2 4 tQ], 1, Dqp);
lmiterm([H 2 4 tQ], Mi, 1);
lmiterm([H 2 4 tT], -Bp'*Cq', 1);
lmiterm([H 2 4 N], 2*Mi, 1);
lmiterm([H 2 4 N], -(Cq*Bp - Dqp)', 1);
lmiterm([H 2 4 N], 1, Dqp);

lmiterm([H 2 5 P13], Bp', Dqw);
lmiterm([H 2 5 P33], Bp'*Cq', Dqw);
lmiterm([H 2 5 P14], Bp', 1);
lmiterm([H 2 5 P34], Bp'*Cq', 1);
lmiterm([H 2 5 tQ], Bp'*Cq', X*Dqw);
lmiterm([H 2 5 tQ], 1, Dqw);
lmiterm([H 2 5 N], 1, Dqw);

lmiterm([H 3 3 P11], Bw', Bw);
lmiterm([H 3 3 P13], Bw', Cq*Bw, 's');
lmiterm([H 3 3 P33], Bw'*Cq', Cq*Bw);
lmiterm([H 3 3 P33], -Dqw', Dqw);
lmiterm([H 3 3 P34], -Dqw', 1, 's');
lmiterm([H 3 3 P44], -1, 1);
lmiterm([H 3 3 tQ], Bw'*Cq', X*Cq*Bw);
lmiterm([H 3 3 tQ], -Dqw', X*Dqw);
lmiterm([H 3 3 0], Dzw'*Dzw);
lmiterm([H 3 3 0], -gamma^2);

lmiterm([H 3 4 P12], Bw', 1);
lmiterm([H 3 4 -P23], Bw'*Cq', 1);
lmiterm([H 3 4 P13], Bw', Dqp);
lmiterm([H 3 4 P33], Bw'*Cq', Dqp);
lmiterm([H 3 4 Q], -(Cq*Bw - Dqw)', 1);
lmiterm([H 3 4 tQ], Bw'*Cq', X*Dqp);
lmiterm([H 3 4 tT], -Bw'*Cq', 1);
lmiterm([H 3 4 N], -(Cq*Bw - Dqw)', 1);

lmiterm([H 3 5 P13], Bw', Dqw);
lmiterm([H 3 5 P33], Bw'*Cq', Dqw);
lmiterm([H 3 5 P14], Bw', 1);
lmiterm([H 3 5 P34], Bw'*Cq', 1);
lmiterm([H 3 5 tQ], Bw'*Cq', X*Dqw);

lmiterm([H 4 4 P22], 1, 1);
lmiterm([H 4 4 P23], 1, Dqp, 's');
lmiterm([H 4 4 P33], Dqp', Dqp);
lmiterm([H 4 4 Q], -1, Dqp, 's');
```

```
lmiterm([H 4 4 Q], -1, Mi);
lmiterm([H 4 4 tQ], Dqp', X*Dqp);
lmiterm([H 4 4 tQ], -1, Mi);
lmiterm([H 4 4 tT], -1, Xi);
lmiterm([H 4 4 tT], -1, Dqp, 's');
lmiterm([H 4 4 N], -2, Mi);
lmiterm([H 4 4 N], -1, Dqp, 's');

lmiterm([H 4 5 P23], 1, Dqw);
lmiterm([H 4 5 P33], Dqp', Dqw);
lmiterm([H 4 5 P24], 1, 1);
lmiterm([H 4 5 P34], Dqp', 1);
lmiterm([H 4 5 Q], -1, Dqw);
lmiterm([H 4 5 tQ], Dqp', X*Dqw);
lmiterm([H 4 5 tT], -1, Dqw);
lmiterm([H 4 5 N], -1, Dqw);

lmiterm([H 5 5 P33], Dqw', Dqw);
lmiterm([H 5 5 P34], Dqw', 1, 's');
lmiterm([H 5 5 P44], 1, 1);
lmiterm([H 5 5 tQ], Dqw', X*Dqw);

% semidefiniteness
lmiterm([-H 1 1 0], poszero);
lmiterm([-H 2 2 0], poszero);
lmiterm([-H 3 3 0], poszero);
lmiterm([-H 4 4 0], poszero);
lmiterm([-H 5 5 0], poszero);


P11l = newlmi;
lmiterm([-P11l 1 1 P11], 1, 1);

Pl = newlmi;
lmiterm([-Pl 1 1 P11], 1, 1);
lmiterm([-Pl 1 2 P12], 1, 1);
lmiterm([-Pl 1 3 P13], 1, 1);
lmiterm([-Pl 1 4 P14], 1, 1);
lmiterm([-Pl 2 2 P22], 1, 1);
lmiterm([-Pl 2 3 P23], 1, 1);
lmiterm([-Pl 2 4 P24], 1, 1);
lmiterm([-Pl 3 3 P33], 1, 1);
lmiterm([-Pl 3 4 P34], 1, 1);
lmiterm([-Pl 4 4 P44], 1, 1);
% semidefiniteness
lmiterm([Pl 1 1 0], negzero);
```

```
lmiterm([Pl 2 2 0], negzero);
lmiterm([Pl 3 3 0], negzero);
lmiterm([Pl 4 4 0], negzero);

Ql = newlmi;
lmiterm([-Ql 1 1 Q], 1, 1);
% semidefiniteness
lmiterm([Ql 1 1 0], negzero);
tQl = newlmi;
lmiterm([-tQl 1 1 tQ], 1, 1);
% semidefiniteness
lmiterm([tQl 1 1 0], negzero);
Tl = newlmi;
lmiterm([-Tl 1 1 T], 1, 1);
% semidefiniteness
lmiterm([Tl 1 1 0], negzero);
tTl = newlmi;
lmiterm([-tTl 1 1 tT], 1, 1);
% semidefiniteness
lmiterm([tTl 1 1 0], negzero);
Nl = newlmi;
lmiterm([-Nl 1 1 N], 1, 1);
% semidefiniteness
lmiterm([Nl 1 1 0], negzero);

lmisys = getlmis;

%%
target = []; options=zeros(1,5);
options(2) = 300;
options(3) = 10;
[tmin,xfeas] = feasp(lmisys, options, target);
end
```

bisection_semidef.m

```
function p = bisection_semidef(f, a, b, mu_ct, nq_vec)

% The goal is not to find a zero for tmin,
% but a value at which it is smaller than TOL.
TOL = 1e-10;
% The result of feasp then yields "may be feasible
% but not strictly feasible".

dif_tol = 1e-6;  % tolerance for iteration difference
```

```matlab
max_it = 100;      % max number of iterations

fa = f(a, mu_ct, nq_vec);
fb = f(b, mu_ct, nq_vec);

if (fa >= TOL && fb >= TOL) || (fa < TOL && fb < TOL)
    disp('Wrong choice')
else
    it = 1;
    dif_p = 1;
    p = (a + b)/2;
    fp = f(p, mu_ct, nq_vec);
    while (it < max_it && dif_p >= dif_tol)
        it = it + 1;
        fa = f(a, mu_ct, nq_vec);
        if (fa < TOL && fp >= TOL) || (fa >= TOL && fp < TOL)
            b = p;
        else
            a = p;
        end
        p_old = p;
        p = (a + b)/2;
        p_new = p;
        dif_p = abs(p_new - p_old);
        fp = f(p, mu_ct, nq_vec);
        if it == max_it; ...
                disp('Stopped because bisection it = max_it'); end
        if dif_p < dif_tol; ...
                disp('Stopped because bisection dif_p < dif_tol'); end
    end
    if fp > TOL; p = b; end
    % Note: this gives a feasible bound
end
end
```

C MATLAB Scripts for Chapter 4

The code `controller_feasibility.m` solves the feasibility problem for the given example. The code `controller_simulation.m` plots the behavior of the example. The controller matrix K has to be changed within the definition of the example in both scripts.

controller_feasibility.m

```
clear variables

poszero = 1e-12;      % Numerical zero for nonstrict LMIs
negzero = -poszero;   % For positive semidefinite LMIs

% Dimensions of the system
n = 4;
nq = 1;
nu = 1;

% Definition of example
A = [0.8  -0.25 0    1;
     1     0    0    0;
     0     0    0.2  0.3;
     0     0    1    0   ];
B = [0;
     0;
     1;
     0 ];
Bu = [1;
      0;
      0;
      0];
C = [0.8  -0.5  0  1];
D = 0;
K = zeros(1, n);
% K = [0 0 -1 -1]; % Select K for example

mu_ct = 1;           % Linear dependence of mu on xi
nq_vec = [1 0];      % Parameter for matrix variables

% Definition of M and X: change xi according to example
xi = 2;
mu = mu_ct * xi;

X = xi * eye(nq);
M = mu * eye(nq);
```

```
Xi = inv(X);
Mi = inv(M);

%%
setlmis([]);
P11 = lmivar(1, [n 1]);
P12 = lmivar(2, [n nq]);
P13 = lmivar(2, [n nq]);
P22 = lmivar(1, [nq 1]);
P23 = lmivar(2, [nq nq]);
P33 = lmivar(1, [nq 1]);

Q = lmivar(1, nq_vec);
tQ = lmivar(1, nq_vec);
T = lmivar(1, nq_vec);
tT = lmivar(1, nq_vec);
N = lmivar(1, nq_vec);


J = newlmi;

lmiterm([J 1 1 P11], -1, 1);
lmiterm([J 1 1 P13], -1, C, 's');
lmiterm([J 1 1 P33], -C', C);
lmiterm([J 1 1 tQ], -C', X*C);

lmiterm([J 1 2 P12], -1, 1);
lmiterm([J 1 2 -P23], -C', 1);
lmiterm([J 1 2 P13], -1, D);
lmiterm([J 1 2 P33], -C', D);
lmiterm([J 1 2 tQ], A'*C', X*C*B);
lmiterm([J 1 2 tQ], K'*Bu'*C', X*C*B);
lmiterm([J 1 2 tQ], -C', X*D);
lmiterm([J 1 2 tQ], A'*C', 1);
lmiterm([J 1 2 tQ], K'*Bu'*C', 1);
lmiterm([J 1 2 tQ], -C', 1);
lmiterm([J 1 2 T], -C', 1);
lmiterm([J 1 2 N], A'*C', 1);
lmiterm([J 1 2 N], K'*Bu'*C', 1);
lmiterm([J 1 2 N], -C', 1);

lmiterm([J 1 3 Q], -A'*C', 1);
lmiterm([J 1 3 Q], -K'*Bu'*C', 1);
lmiterm([J 1 3 Q], C', 1);
lmiterm([J 1 3 tQ], A'*C', X*D);
lmiterm([J 1 3 tQ], K'*Bu'*C', X*D);
```

```
lmiterm([J 1 3 tT], -A'*C', 1);
lmiterm([J 1 3 tT], -K'*Bu'*C', 1);
lmiterm([J 1 3 N], -A'*C', 1);
lmiterm([J 1 3 N], -K'*Bu'*C', 1);
lmiterm([J 1 3 N], C', 1);

lmiterm([J 1 4 P11], A', 1);
lmiterm([J 1 4 P11], K'*Bu', 1);
lmiterm([J 1 4 -P13], A'*C', 1);
lmiterm([J 1 4 -P13], K'*Bu'*C', 1);

lmiterm([J 1 5 P12], A', 1);
lmiterm([J 1 5 P12], K'*Bu', 1);
lmiterm([J 1 5 -P23], A'*C', 1);
lmiterm([J 1 5 -P23], K'*Bu'*C', 1);

lmiterm([J 1 6 P13], A', 1);
lmiterm([J 1 6 P13], K'*Bu', 1);
lmiterm([J 1 6 P33], A'*C', 1);
lmiterm([J 1 6 P33], K'*Bu'*C', 1);

lmiterm([J 1 7 tQ], A'*C', X);
lmiterm([J 1 7 tQ], K'*Bu'*C', X);

lmiterm([J 2 2 P22], -1, 1);
lmiterm([J 2 2 P23], -1, D, 's');
lmiterm([J 2 2 P33], -D', D);
lmiterm([J 2 2 Q], -1, Mi);
lmiterm([J 2 2 tQ], B'*C', X*C*B);
lmiterm([J 2 2 tQ], -D', X*D);
lmiterm([J 2 2 tQ], 1, C*B - D, 's');
lmiterm([J 2 2 tQ], -1, Mi);
lmiterm([J 2 2 T], -2, Xi);
lmiterm([J 2 2 T], -1, D, 's');
lmiterm([J 2 2 N], -2, Mi);
lmiterm([J 2 2 N], 1, C*B - D, 's');

lmiterm([J 2 3 Q], Mi, 1);
lmiterm([J 2 3 Q], -(C*B - D)', 1);
lmiterm([J 2 3 tQ], B'*C', X*D);
lmiterm([J 2 3 tQ], 1, D);
lmiterm([J 2 3 tQ], 1, Mi);
lmiterm([J 2 3 tT], -B'*C', 1);
lmiterm([J 2 3 N], -(C*B - D)', 1);
lmiterm([J 2 3 N], 2, Mi);
lmiterm([J 2 3 N], 1, D);
```

```
lmiterm([J 2 4 P11], B', 1);
lmiterm([J 2 4 -P13], B'*C', 1);

lmiterm([J 2 5 P12], B', 1);
lmiterm([J 2 5 -P23], B'*C', 1);

lmiterm([J 2 6 P13], B', 1);
lmiterm([J 2 6 P33], B'*C', 1);

lmiterm([J 3 3 Q], -1, D, 's');
lmiterm([J 3 3 Q], -1, Mi);
lmiterm([J 3 3 tQ], D', X*D);
lmiterm([J 3 3 tQ], -1, Mi);
lmiterm([J 3 3 tT], -2, Xi);
lmiterm([J 3 3 tT], -1, D, 's');
lmiterm([J 3 3 N], -2, Mi);
lmiterm([J 3 3 N], -1, D, 's');

lmiterm([J 3 4 -P12], 1, 1);
lmiterm([J 3 4 -P13], D', 1);

lmiterm([J 3 5 P22], 1, 1);
lmiterm([J 3 5 -P23], D', 1);

lmiterm([J 3 6 P23], 1, 1);
lmiterm([J 3 6 P33], D', 1);

lmiterm([J 4 4 P11], -1, 1);

lmiterm([J 4 5 P12], -1, 1);

lmiterm([J 4 6 P13], -1, 1);

lmiterm([J 5 5 P22], -1, 1);

lmiterm([J 5 6 P23], -1, 1);

lmiterm([J 6 6 P33], -1, 1);

lmiterm([J 7 7 tQ], -1, X);


P11l = newlmi;
lmiterm([-P11l 1 1 P11], 1, 1);

```

```
Pl = newlmi;
lmiterm([-Pl 1 1 P11], 1, 1);
lmiterm([-Pl 1 2 P12], 1, 1);
lmiterm([-Pl 1 3 P13], 1, 1);
lmiterm([-Pl 2 2 P22], 1, 1);
lmiterm([-Pl 2 3 P23], 1, 1);
lmiterm([-Pl 3 3 P33], 1, 1);

Ql = newlmi;
lmiterm([-Ql 1 1 Q], 1, 1);
% semidefiniteness
lmiterm([Ql 1 1 0], negzero);
tQl = newlmi;
lmiterm([-tQl 1 1 tQ], 1, 1);
Tl = newlmi;
lmiterm([-Tl 1 1 T], 1, 1);
% semidefiniteness
lmiterm([Tl 1 1 0], negzero);
tTl = newlmi;
lmiterm([-tTl 1 1 tT], 1, 1);
% semidefiniteness
lmiterm([tTl 1 1 0], negzero);
Nl = newlmi;
lmiterm([-Nl 1 1 N], 1, 1);
% semidefiniteness
lmiterm([Nl 1 1 0], negzero);

lmisys = getlmis;

%%
target = []; options=zeros(1,5);
options(2) = 300;
options(3) = 10;
feasp(lmisys, options, target)
```

controller_simulation.m

```
clear variables

% Dimensions of the system
n = 4;
nq = 1;
nu = 1;
ny = 1;

```

```
% Assign value of xi
constant = 2;

% Initial values of state variables
x = 1e4*rand(n,1);

% Number of simulation iterations
n_it = 499;

% Definition of example
A = [0.8  -0.25 0    1;
     1     0    0    0;
     0     0    0.2  0.3;
     0     0    1    0   ];
B = [0;
     0;
     1;
     0 ];
Bu = [1;
      0;
      0;
      0];
C = [0.8 -0.5 0 1];
D = 0;
K = zeros(1, n);
% K = [0 0 -1 -1]; % Select K for example

q = C*x;
p = - constant * q;

for i = 1:n_it
    x_next = (A + Bu*K)*x(:, end) + B*p(:, end);
    q_next = C*x_next;
    p_next = -constant*q_next;

    x = [x x_next];
    q = [q q_next];
    p = [p p_next];
end
%%

figure(1);
plot(x(1,:)); hold on;
plot(x(2,:)); plot(x(3,:)); plot(x(4,:)); hold off;
set(gca,'fontsize', 14);
legend('x_{k,1}', 'x_{k,2}', 'x_{k,3}', 'x_{k,4}');
```

```
title('Trajectory of state variables');
xlabel('k'); ylabel('x_k');

figure(2);
plot(x(1,:)); hold on;
plot(x(2,:)); plot(x(3,:)); plot(x(4,:)); hold off;
set(gca,'fontsize', 14);
legend('x_{k,1}', 'x_{k,2}', 'x_{k,3}', 'x_{k,4}');
ylim([-1 1]);
title('Trajectory of state variables');
xlabel('k'); ylabel('x_k');
```

D MATLAB Scripts for Chapter 5

The code `estimator_feasibility.m` solves the feasibility problem for the given example. The code `estimator_simulation.m` plots the behavior of the example. The estimator matrix L has to be changed within the definition of the example in both scripts.

estimator_feasibility.m

```
clear variables

poszero = 1e-12;      % Numerical zero for nonstrict LMIs
negzero = -poszero;   % For positive semidefinite LMIs

% Dimensions of the system
n = 4;
nq = 1;
ny = 1;

% Definition of example
A = [0.8  -0.25 0    1;
     1     0    0    0;
     0     0    0.2  0.3;
     0     0    1    0   ];
Bp = [0;
      0;
      1;
      0 ];
Cy = [0 0 1 1];
Cq = [0.8 -0.5 0 1];
Dqp = zeros(1);
L = zeros(1, n)';
% L = [-1 0 0 0]'; % Select L for example

mu_ct = 1;          % Linear dependence of mu on xi
nq_vec = [1 0];     % Parameter for matrix variables

% Definition of M and X: change according to example
xi = 2;
mu = mu_ct * xi;

X = xi * eye(nq);
M = mu * eye(nq);
Xi = inv(X);
Mi = inv(M);

```

```
%%
setlmis([]);
P11 = lmivar(1, [n 1]);
P12 = lmivar(2, [n nq]);
P13 = lmivar(2, [n nq]);
P22 = lmivar(1, [nq 1]);
P23 = lmivar(2, [nq nq]);
P33 = lmivar(1, [nq 1]);

Q = lmivar(1, nq_vec);
T = lmivar(1, nq_vec);
tT = lmivar(1, nq_vec);


R = newlmi;

lmiterm([R 1 1 P11], -1, 1);
lmiterm([R 1 1 P13], -1, Cq, 's');
lmiterm([R 1 1 P33], -Cq', Cq);

lmiterm([R 1 2 P12], -1, 1);
lmiterm([R 1 2 -P23], -Cq', 1);
lmiterm([R 1 2 P13], -1, Dqp);
lmiterm([R 1 2 P33], -Cq', Dqp);
lmiterm([R 1 2 Q], A'*Cq', M*Cq*Bp);
lmiterm([R 1 2 Q], Cy'*L'*Cq', M*Cq*Bp);
lmiterm([R 1 2 T], -Cq', 1);

lmiterm([R 1 3 Q], A'*Cq', M*Dqp);
lmiterm([R 1 3 Q], Cy'*L'*Cq', M*Dqp);
lmiterm([R 1 3 tT], -A'*Cq', 1);
lmiterm([R 1 3 tT], -Cy'*L'*Cq', 1);

lmiterm([R 1 4 P11], A', 1);
lmiterm([R 1 4 P11], Cy'*L', 1);
lmiterm([R 1 4 -P13], A'*Cq', 1);
lmiterm([R 1 4 -P13], Cy'*L'*Cq', 1);

lmiterm([R 1 5 P12], A', 1);
lmiterm([R 1 5 P12], Cy'*L', 1);
lmiterm([R 1 5 -P23], A'*Cq', 1);
lmiterm([R 1 5 -P23], Cy'*L'*Cq', 1);

lmiterm([R 1 6 P13], A', 1);
lmiterm([R 1 6 P13], Cy'*L', 1);
lmiterm([R 1 6 P33], A'*Cq', 1);
```

```
lmiterm([R 1 6 P33], Cy'*L'*Cq', 1);

lmiterm([R 1 7 Q], A'*Cq', M);
lmiterm([R 1 7 Q], Cy'*L'*Cq', M);

lmiterm([R 2 2 P22], -1, 1);
lmiterm([R 2 2 P23], -1, Dqp, 's');
lmiterm([R 2 2 P33], -Cq', Dqp);
lmiterm([R 2 2 Q], Bp'*Cq', M*Cq*Bp);
lmiterm([R 2 2 T], -2, Mi);
lmiterm([R 2 2 T], -1, Dqp, 's');

lmiterm([R 2 3 Q], Bp'*Cq', M*Dqp);
lmiterm([R 2 3 tT], -Bp'*Cq', 1);

lmiterm([R 2 4 P11], Bp', 1);
lmiterm([R 2 4 -P13], Bp'*Cq', 1);

lmiterm([R 2 5 P12], Bp', 1);
lmiterm([R 2 5 -P23], Bp'*Cq', 1);

lmiterm([R 2 6 P13], Bp', 1);
lmiterm([R 2 6 P33], Bp'*Cq', 1);

lmiterm([R 3 3 Q], Dqp', M*Dqp);
lmiterm([R 3 3 tT], -2, Mi);
lmiterm([R 3 3 tT], -1, Dqp, 's');

lmiterm([R 3 4 -P12], 1, 1);
lmiterm([R 3 4 -P13], Dqp', 1);

lmiterm([R 3 5 P22], 1, 1);
lmiterm([R 3 5 -P23], Dqp', 1);

lmiterm([R 3 6 P23], 1, 1);
lmiterm([R 3 6 P33], Dqp', 1);

lmiterm([R 4 4 P11], -1, 1);

lmiterm([R 4 5 P12], -1, 1);

lmiterm([R 4 6 P13], -1, 1);

lmiterm([R 5 5 P22], -1, 1);

lmiterm([R 5 6 P23], -1, 1);
```

```
lmiterm([R 6 6 P33], -1, 1);

lmiterm([R 7 7 Q], -1, M);


P11l = newlmi;
lmiterm([-P11l 1 1 P11], 1, 1);

Pl = newlmi;
lmiterm([-Pl 1 1 P11], 1, 1);
lmiterm([-Pl 1 2 P12], 1, 1);
lmiterm([-Pl 1 3 P13], 1, 1);
lmiterm([-Pl 2 2 P22], 1, 1);
lmiterm([-Pl 2 3 P23], 1, 1);
lmiterm([-Pl 3 3 P33], 1, 1);

Ql = newlmi;
lmiterm([-Ql 1 1 Q], 1, 1);
Tl = newlmi;
lmiterm([-Tl 1 1 T], 1, 1);
% semidefiniteness
lmiterm([Tl 1 1 0], negzero);
tTl = newlmi;
lmiterm([-tTl 1 1 tT], 1, 1);
% semidefiniteness
lmiterm([tTl 1 1 0], negzero);

lmisys = getlmis;

%%
target = []; options=zeros(1,5);
options(2) = 300;
options(3) = 10;
feasp(lmisys, options, target)
```

estimator_simulation.m

```
clear variables

% Dimensions of the system
n = 4;
nq = 1;
nu = 1;
ny = 1;
```

```
% Assign value of xi
constant = 2;

% Initial values of state variables
x = 1e4*rand(n,1);

% Number of simulation iterations
n_it = 499;

% Definition of example
A = [0.8  -0.25 0    1;
     1     0     0    0;
     0     0     0.2 0.3;
     0     0     1    0   ];
Bp = [0;
      0;
      1;
      0 ];
Cy = [0 0 1 1];
Cq = [0.8 -0.5 0 1];
Dqp = zeros(1);
L = zeros(1, n)';
% L = [-1 0 0 0]'; % Select L for example

q = Cq*x;
p = - constant * q;

for i = 1:n_it
    x_next = (A + L*Cy)*x(:, end) + Bp*p(:, end);
    q_next = Cq*x_next;
    p_next = -constant*q_next;

    x = [x x_next];
    q = [q q_next];
    p = [p p_next];
end
%%
figure(1);
plot(x(1,:)); hold on;
plot(x(2,:)); plot(x(3,:)); plot(x(4,:)); hold off;
set(gca,'fontsize', 14);
legend('x _{k,1} - x_{k,1}', 'x _{k,2} - x_{k,2}', ...
    'x _{k,3} - x_{k,3}', 'x _{k,4} - x_{k,4}');
title('Trajectory of error state variables');
xlabel('k'); ylabel('x _k - x_k');
```

```
figure(2);
plot(x(1,:)); hold on;
plot(x(2,:)); plot(x(3,:)); plot(x(4,:)); hold off;
set(gca,'fontsize', 14);
legend('x _{k,1} - x_{k,1}', 'x _{k,2} - x_{k,2}', ...
    'x _{k,3} - x_{k,3}', 'x _{k,4} - x_{k,4}');
ylim([-1e-4 1e-4]);
title('Trajectory of error state variables');
xlabel('k'); ylabel('x _k - x_k');
```