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Bass numbers of local cohomology of cover ideals of graphs

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Abstract
We develop splitting techniques to study the Lyubeznik numbers of cover ideals of graphs which allow us to describe them for large families of graphs including forests, cycles, wheels and cactus graphs. More generally, we are able to compute all the Bass numbers and the shape of the injective resolution of local cohomology modules by considering the connected components of all the induced subgraphs. Indeed, our method gives us a very simple criterion for the vanishing of these local cohomology modules in terms of the number of connected components of the induced subgraphs.

Keywords Local cohomology · Injective resolution · Graphs · Connectivity

Mathematics Subject Classification Primary 13D45 · 13C11 · 05C38 · 05C40

1 Introduction

Let $G = (V_G, E_G)$ be a finite graph in the set of vertices $V_G = \{x_1, \ldots, x_n\}$ and the set of edges E_G. We will assume that the graph is simple so no multiple edges between vertices or loops are allowed. In order to study such a graph from an algebraic point of view, one may associate a monomial ideal in the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$,

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where $\mathbb{K}$ is a field. There are several ways to do so, but the most used in the literature are:

- **Edge ideal:** $I(G) = (x_i x_j \mid \{x_i, x_j\} \in E_G)$.
- **Cover ideal:** $J(G) = \bigcap_{\{x_i, x_j\} \in E_G} (x_i, x_j)$.

That is, the edges of the graph describe the generators of the edge ideal and the primary components of the cover ideal. Another approach would be through the Stanley–Reisner theory that associates a squarefree monomial ideal to a simplicial complex (see [11,32]). In our case, $I(G)$ is the Stanley–Reisner ideal associated with the simplicial complex

$$\Delta(I(G)) := \{F \subseteq V_G \mid e \nsubseteq F, \forall e \in E_G\}$$

of independent sets of vertices. On the other hand, $J(G)$ is the Stanley–Reisner ideal associated with the simplicial complex $\Delta(J(G))$ whose facets are complements of edges and we have that $\Delta(I(G))$ is the Alexander dual of $\Delta(J(G))$. Although we are just going to use the terminology from cover and edge ideals, we encourage the interested reader to follow their own preferences.

A common theme in combinatorial commutative algebra has been to understand graph theoretic properties of G from the algebraic properties of the associated ideal and vice versa. For instance, a lot of attention has been paid to the study of free resolutions of edge ideals and its associated invariants such as Betti numbers, projective dimension or Castelnuovo–Mumford regularity. Notice that edge ideals are a very particular class of squarefree monomial ideals so one can use some of the techniques available in this context such as Hochster's formula [20], splitting techniques [14,15] or discrete Morse theory [10]. Despite these efforts, a full description of these algebraic invariants is only known for a few families of graphs.

The aim of this paper is to study Bass numbers of local cohomology modules supported on cover ideals of graphs. The choice of cover ideals instead of edge ideals is because free resolutions and local cohomology modules of any squarefree monomial ideal are related via Alexander duality (see [8,28,29,31]) and, in the particular case we are considering, edge and cover ideals are Alexander dual to each other. Moreover, it seems more natural to use the primary decomposition of an ideal if we want to use the Mayer–Vietoris sequence to study local cohomology modules. We remark that the study of the Lyubeznik numbers of edge ideals of graphs should be of independent interest because no relations are known between the Lyubeznik numbers of a squarefree monomial ideal and its Alexander dual.

In order to compute the Bass numbers of local cohomology modules of any squarefree monomial ideal, we may refer to the work of Yanagawa [34] or the work of the first author with his collaborators in [1,2,8]. Indeed, one can use the computational algebra system `Macaulay 2` [18] to compute them as it is shown in [6]. We point out that, using the restriction functor, we may just reduce to the case of studying Bass numbers with respect to the homogeneous maximal ideal, which are also known as the Lyubeznik numbers [27].

The methods presented in [8] may seem quite appropriate for the case of cover ideals of graphs. Namely, in order to compute the Lyubeznik numbers, one has to describe the linear strands of the free resolution of the corresponding edge ideal and compute the homology groups of a complex of $\mathbb{K}$-vector spaces associated with these linear strands. However, even though one may find some explicit free resolution of edge ideals of graphs in the literature, it seems quite complicated to give closed formulas for the Lyubeznik numbers even for simple families of graphs.

In this paper, we shift gears and we present some splitting techniques that would allow us to compute the Lyubeznik numbers of large families of graphs without any previous description of its local cohomology modules or equivalently, the free resolution of the corresponding edge ideals. The idea behind these splitting techniques is to relate the Lyubeznik numbers of our initial graph to the Lyubeznik numbers of the subgraph obtained by removing a vertex. Indeed, the Lyubeznik table remains invariant when we remove a whisker or even a 3− or 4-cycle. Moreover, we can control the Lyubeznik table when we remove degree two vertices or a dominating vertex. To compute all the Lyubeznik numbers of any given graph in a fixed number of vertices is out of the scope of this work, but we can reduce enormously the number of cases that we have to consider by a simple inspection of the shape of the graph. More generally, we can compute all the Bass numbers of local cohomology modules just considering subgraphs of our initial graph. In particular, we can describe the linear strands of the injective resolution of these modules. The structure of these injective resolutions depend on the number of connected components of the corresponding subgraphs. Quite nicely, we deduce a vanishing criterion for local cohomology modules depending of these connected components of the subgraphs.

We should mention that, in general, the Lyubeznik numbers depend on the characteristic of the base field. However, all the methods we develop here are independent of the characteristic, meaning that the Lyubeznik numbers of a graph will depend on the characteristic if and only if the Lyubeznik numbers of the graph obtained after removing a vertex also depend on the characteristic.

The organization of this paper is as follows. In Sect. 2, we introduce all the basics on local cohomology supported on squarefree monomial ideals and its injective resolution. In particular we introduce Bass numbers and how to describe them using the graded pieces of the composition of local cohomology modules. Since we can always reduce to the case of the Lyubeznik numbers, we briefly recall in Sect. 2.1 its definition and the main properties we are going to use throughout this work. In Sect. 2.2 we review the relation between the Lyubeznik numbers and linear strands of the Alexander dual ideal. Finally, in Sect. 2.3 we propose the notion of MV-splitting (see Definition 2.8) together with an application of the long exact sequence of local cohomology modules (see Discussion 2.12) that will be crucial later on. The reason of working in the general framework of squarefree monomial ideals is that, even though we want to study cover ideals of graphs, we will have to leave this context when applying these techniques. Moreover, all these splitting methods could be also applied for any squarefree monomial ideal.

In Sect. 3 we focus on the study of the Lyubeznik numbers of cover ideals of graphs. Our first result is Theorem 3.3 in which we describe the Lyubeznik table associated with the cover ideal of a simple connected graph for which the MV-splitting satisfies some extra conditions. These conditions are naturally satisfied when we consider splitting vertices, and thus, we specialize to this case. In Proposition 3.5, we prove that the Lyubeznik table remains invariant after removing vertices of degree one. In Proposition 3.6, we prove that the Lyubeznik table is also invariant if we remove a handle, which is a 3- or 4-cycle having a degree two vertex. More generally, we describe in Proposition 3.7 and Corollary 3.8 the Lyubeznik table of any graph as long as we find degree two splitting vertices. In this way, we can apply recursion to reduce the computation to the case of a smaller graph.

In Sect. 3.2, we apply these splitting techniques to compute the Lyubeznik table of some families of examples. We prove that trees have trivial Lyubeznik table and we deduce a formula for the case of forests. Any cone of a graph, for example a wheel, also has trivial Lyubeznik table. The results on degree two vertices allow us to compute the case of cycles and, more generally, the family of graphs obtained by joining cycles in such a way that we can still find degree two vertices that we can remove in order to simplify the graph. Indeed, after removing whiskers and handles we may consider cycles joined by paths or sharing edges. Such an example would be the case of cactus graphs or cycles with chords.

In Sect. 4, we study all the Bass numbers of the cover ideal of a graph by considering the Lyubeznik numbers of the corresponding subgraphs. In Sect. 4.1, we pay attention to a class of graphs (that include forests and Cohen–Macaulay graphs) whose local cohomology modules have a linear injective resolution. In particular, we give a closed formula for these Bass numbers in Theorem 4.4. Quite surprisingly, we provide in Proposition 4.6 a vanishing criterion for the local cohomology modules in terms of the number of connected components of the subgraphs. Using Alexander duality, it also gives a formula for the projective dimension of the edge ideal of such a graph. We also study the injective resolution of local cohomology modules of graphs obtained by joining cycles in Sect. 4.2. Finally, we also provide a vanishing criterion for the local cohomology modules associated with the corresponding subgraphs in Propositions 4.12 and 4.13.

2 Bass numbers of local cohomology modules

Throughout this section, we will assume the general framework of a squarefree monomial ideals in a polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$ with coefficients over a field $\mathbb{K}$. Namely, a squarefree monomial ideal $J \subseteq R$ is generated by monomials of the form $\mathbf{x} := x_1^{a_1} \dots x_n^{a_n}$, where $\boldsymbol{\alpha} = (a_1, \dots, a_n) \in \{0, 1\}^n$. Its minimal primary decomposition is given in terms of face ideals $\mathfrak{p}_\alpha := \langle x_i \mid a_i \neq 0 \rangle$, $\boldsymbol{\alpha} \in \{0, 1\}^n$. For simplicity, we will denote the homogeneous maximal ideal $\mathfrak{m} := \mathfrak{p}_{\mathbf{1}} = (x_1, \dots, x_n)$, where $\mathbf{1} = (1, \dots, 1)$. As usual, we denote $|\alpha| = a_1 + \cdots + a_n$ and $\varepsilon_1, \dots, \varepsilon_n$ will be the standard basis of $\mathbb{Z}^n$. The **Alexander dual** of the ideal J is the squarefree monomial ideal $J^\vee \subseteq R$ defined as $J^\vee = (\mathbf{x}^{\mathbf{1}}, \dots, \mathbf{x}^{\mathbf{s}})$ associated with the minimal primary decomposition $J = \mathfrak{p}_{\alpha_1} \cap \cdots \cap \mathfrak{p}_{\alpha_s}$.

Let $\mathbb{Z}^\alpha \subseteq \mathbb{Z}^n$ be the coordinate space spanned by $\{\varepsilon_i \mid a_i = 1\}, \alpha \in \{0, 1\}^n$. The **restriction** of R to the face ideal $\mathfrak{p}_\alpha \subseteq R$ is the $\mathbb{Z}^\alpha$-graded $\mathbb{K}$-subalgebra of R

$$R_{\mathfrak{p}_\alpha} := \mathbb{K}[x_i \mid a_i = 1].$$

Let $J = \mathfrak{p}_{\alpha_1} \cap \cdots \cap \mathfrak{p}_{\alpha_s}$ be the minimal primary decomposition of a squarefree monomial ideal $J \subseteq R$. Then, the restriction of J to the face ideal $\mathfrak{p}_\alpha$ is the squarefree monomial ideal

$$J_{\mathfrak{p}_\alpha} = \bigcap_{\alpha_j \leq \alpha} \mathfrak{p}_{\alpha_j} \subseteq R_{\mathfrak{p}_\alpha}.$$

Moreover, the restriction of a local cohomology module is

$$[H_J^r(R)]_{\mathfrak{p}_\alpha} = H^r_{J_{\mathfrak{p}_\alpha}}(R_{\mathfrak{p}_\alpha})$$

Roughly speaking, restriction gives us a functor that plays the role of the localization functor. For details and further considerations we refer to [29].

A key fact in its study is that local cohomology modules $H_J^r(R)$ supported on monomial ideals are $\mathbb{Z}^n$-graded modules. Indeed, these modules satisfy some nice properties since they fit, modulo a shifting by **1**, into the category of **straight** (resp. **1**-determined) modules introduced by Yanagawa [34] (resp. Miller [29]). In what follows, we are going to introduce the basic notions that we are going to use in this work. Most of them can be found in textbooks such as [11] and [30] or the lecture notes [5].

In order to give a module structure to the straight module $H_J^r(R)$ we have to describe:

- The graded pieces $[H_J^r(R)]_{-\alpha}$ for all $\alpha \in \{0, 1\}^n$.
- The multiplication morphisms: $\cdot x_i : [H_J^r(R)]_{-\alpha} \longrightarrow [H_J^r(R)]_{-(\alpha-\varepsilon_i)}$.

This structure has been described by Terai [33] and Mustaţă [31] in terms of some simplicial complexes associated with the monomial ideal J. The approach considered in [7] gives an interpretation in terms of the components appearing in the minimal primary decomposition of J which will be more convenient for our purposes.

Let $\mathcal{P}_J$ be the partially ordered set consisting of the sums of ideals in the minimal primary decomposition of J ordered by reverse inclusion. Namely, if $J = \mathfrak{p}_{\alpha_1} \cap \cdots \cap \mathfrak{p}_{\alpha_s}$ is the minimal primary decomposition we have that any ideal $J_p \in \mathcal{P}_J$ is a certain sum $J_p = \mathfrak{p}_{\alpha_{i_1}} + \cdots + \mathfrak{p}_{\alpha_{i_j}}$ and, since the sum of face ideals is a face ideal we have that $J_p = \mathfrak{p}_\alpha$ for some $\alpha \in \{0, 1\}^n$. In what follows, we will just denote by $\mathfrak{p}_\alpha$, or simply α, the elements of $\mathcal{P}_J$.

Let $1_{\mathcal{P}_J}$ be a terminal element that we add to the poset. To any $\alpha \in \mathcal{P}_J$, we may consider the **order complex** associated with the subposet $(\alpha, 1_{\mathcal{P}_J}) := \{z \in \mathcal{P}_J \mid \alpha < z < 1_{\mathcal{P}}\}$ and the dimensions of the reduced simplicial homology groups

$$m_{r,\alpha} := \dim_{\mathbb{K}} \widetilde{H}_{|\alpha|-r-1}((\alpha, 1_{\mathcal{P}_J}); \mathbb{K}).$$

Then, the graded pieces of the local cohomology modules of J can be described as follows:

$$[H_J^r(R)]_{-\alpha} = \bigoplus_{\alpha \in \mathcal{P}_J} [H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)^{m_{r,\alpha}}]_{-\alpha} \tag{2.1}$$

The category of straight modules is a category with enough injective modules. Indeed, the indecomposable injective objects are the shifted injective envelopes

$E_\alpha := {}^*\mathrm{E}_R(R/\mathfrak{p}_\alpha)(\mathbf{1})$, $\alpha \in \{0,1\}^n$, and every graded injective module is isomorphic to a unique (up to order) direct sum of indecomposable injectives. It follows that the **minimal $\mathbb{Z}^n$-graded injective resolution** of a local cohomology module $H^r_J(R)$ is an exact sequence:

$$\mathbb{I}_\bullet(H^r_J(R)): \quad 0 \longrightarrow H^r_J(R) \longrightarrow I_0 \xrightarrow{d^0} I_1 \xrightarrow{d^1} \cdots \longrightarrow I_m \xrightarrow{d^m} 0\,,$$

where the pth term is

$$I_p = \bigoplus_{\alpha\in\mathbb{Z}^n} E_\alpha^{\mu_p(\mathfrak{p}_\alpha, H^r_J(R))}$$

and the invariants defined by $\mu_p(\mathfrak{p}_\alpha, H^r_J(R))$ are the **Bass numbers** of $H^r_J(R)$. Given an integer ℓ, the **ℓ-linear strand** of $\mathbb{I}_\bullet(H^r_J(R))$ is the complex:

$$\mathbb{I}_\bullet^{<\ell>}(H^r_J(R)): \quad 0 \longrightarrow I_0^{<\ell>} \longrightarrow I_1^{<\ell>} \longrightarrow \cdots \longrightarrow I_m^{<\ell>} \longrightarrow 0\,,$$

where

$$I_p^{<\ell>} = \bigoplus_{|\alpha|=p+\ell} E_\alpha^{\mu_p(\mathfrak{p}_\alpha, H^r_J(R))},$$

Remark 2.1 The ($\mathbb{Z}^n$-graded) Bass numbers coincide with the usual Bass numbers in the minimal injective resolution of $H^r_J(R)$ as it was proved by Goto and Watanabe in [17]. Indeed, they provided a method to compute the Bass numbers with respect to any prime ideal. Namely, given any prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, let $\mathfrak{p}_\alpha$ be the largest face ideal contained in $\mathfrak{p}$. If $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}_\alpha) = s$, then $\mu_p(\mathfrak{p}_\alpha, H^r_J(R)) = \mu_{p+s}(\mathfrak{p}, H^r_J(R))$.

The Bass numbers of straight modules, and local cohomology modules in particular, were already studied and described in [34]. The approach that we will use in this work is using the graded pieces of the composition of local cohomology modules. Namely, using [8, Corollary 3.6] (see also [4]), we have:

Proposition 2.2 *Let $J \subseteq R$ be a squarefree monomial ideal and $\mathfrak{p}_\alpha \subseteq R$ be a face ideal, $\alpha \in \{0,1\}^n$. Then, the Bass numbers of the local cohomology module $H^r_J(R)$ with respect to $\mathfrak{p}_\alpha$ are*

$$\mu_p(\mathfrak{p}_\alpha, H^r_J(R)) = \dim_{\mathbb{K}}[H^p_{\mathfrak{p}_\alpha}(H^r_J(R))]_{-\alpha}.$$

In particular, the Bass numbers with respect to the homogeneous maximal ideal $\mathfrak{m}$ are

$$\mu_p(\mathfrak{m}_\alpha, H^r_J(R)) = \dim_{\mathbb{K}}[H^p_{\mathfrak{m}}(H^r_J(R))]_{-\mathbf{1}}.$$

Remark 2.3 Bass numbers behave well with respect to the restriction functor so we may always assume that the face ideal $\mathfrak{p}_\alpha$ is the maximal ideal. Namely, we have

$$\mu_p(\mathfrak{p}_\alpha, H^r_J(R)) = \mu_p(\mathfrak{p}_\alpha R_{\mathfrak{p}_\alpha}, [H^r_J(R)]_{\mathfrak{p}_\alpha}).$$

2.1 Lyubeznik numbers

In the seminal works of Huneke and Sharp [22] and Lyubeznik [27], it is proven that the Bass numbers of local cohomology modules are all finite. This prompted Lyubeznik to introduce a new set of invariants defined as follows:

Let A be a Noetherian local ring that admits a surjection from an n-dimensional regular local ring $(R, \mathfrak{m})$ containing its residue field $\mathbb{K}$, and $J \subseteq R$ be the kernel of the surjection. Then, the Bass numbers

$$\lambda_{p,i}(A) := \mu_p(\mathfrak{m}, H^{n-i}_J(R))$$

depend only on A, i and p, but not on the choice of R or the surjection $R \longrightarrow A$. More generally, all the Bass numbers $\mu_p(\mathfrak{p}, H^{n-i}_J(R))$ are invariants of the local ring A as it was proved later on in [3]. Bass numbers behave well with respect to completion so we may always assume that A is a quotient of a formal power series ring R. Considering a squarefree monomial ideal as an ideal in the polynomial or the formal power series ring makes no difference since the Bass numbers of the corresponding local cohomology modules coincide. It is for this reason that we will keep considering, for simplicity, just the case of R being a polynomial ring.

The Lyubeznik numbers satisfy $\lambda_{d,d}(A) \neq 0$ and $\lambda_{p,i}(A) \neq 0$ implies $0 \leq p \leq i \leq d$, where $d = \dim A$. A way to collect these invariants is by means of the so-called **Lyubeznik table**:

$$\Lambda(A) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix}$$

and we say that the Lyubeznik table is **trivial** if $\lambda_{d,d} = 1$ and the rest of these invariants vanish.

The highest Lyubeznik number $\lambda_{d,d}(A)$ has an interesting interpretation in terms of the so-called **Hochster-Huneke graph** defined in [21], which is the graph whose vertices are the top-dimensional minimal primes of A and we have an edge between two vertices $\mathfrak{p}$ and $\mathfrak{q}$ if and only if $\mathrm{ht}(\mathfrak{p} + \mathfrak{q}) = 1$. The following result was proved by Lyubeznik [26] when $\mathbb{K}$ is a positive characteristic field, and a characteristic-free proof was given by Zhang [35]. To avoid technicalities in the statement of the result, we will restrict ourselves to the case of squarefree monomial ideals in a polynomial ring.

Theorem 2.4 *Let $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ be a squarefree monomial ideal and $A = R/I$. The highest Lyubeznik number $\lambda_{d,d}(A)$ equals the number of connected components of the Hochster–Huneke graph of A.*

Another property that we are going to use in this work is the following Thom–Sebastiani type formula for the case of squarefree monomial ideals that was proved in [9].

Proposition 2.5 *Let $I \subseteq R = \mathbb{K}[x_1, \ldots, x_m]$ and $J \subseteq S = \mathbb{K}[y_1, \ldots, y_n]$ be squarefree monomial ideals in two disjoint sets of variables and set $T = \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Then, the Lyubeznik numbers of $T/IT \cap JT$ have the following form:*

i) If either the height of I or the height of J is 1, then $T/IT \cap JT$ has trivial Lyubeznik table.
ii) If both the height of I and the height of J are ≥ 2, then we have:

$$\begin{aligned}\lambda_{p,i}(T/IT \cap JT) &= \lambda_{p,i}(T/IT) + \lambda_{p,i}(T/JT) \\ &\quad + \sum_{\substack{q+r=p+\dim T \\ j+k=i+\dim T-1}} \lambda_{q,j}(T/IT)\lambda_{r,k}(T/JT) \\ &= \lambda_{p-n,i-n}(R/I) + \lambda_{p-m,i-m}(S/J) \\ &\quad + \sum_{\substack{q+r=p \\ j+k=i-1}} \lambda_{q,j}(R/I)\lambda_{r,k}(S/J).\end{aligned}$$

The following particular case will be very useful later on.

Corollary 2.6 *Let $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ be a squarefree monomial ideal admitting a decomposition $I = I_1 \cap \cdots \cap I_c$ in disjoint sets of variables such that* $\dim R/I_j = d$ *and $\Lambda(R/I_j)$ are trivial for $j = 1, \ldots, c$. Then,*

$$\lambda_{d-2k,d-k}(R/I) = \binom{c}{k+1} \quad \text{for} \quad k = 0, \ldots, c-1$$

and the rest of the Lyubeznik numbers are zero.

Proof First, we notice that the matrices $\Lambda(R/I_i)$ have the same size for all i. In the case that $c = 2$ we have, using Proposition 2.5, $\lambda_{d,d} = 2$ and $\lambda_{d-2,d-1} = 1$. Then, we proceed using induction on the number of components. □

A general formula for the case of c disjoint sets of variables could be worked out, but we will just focus on finding the smallest integer i for which there exist p such that $\lambda_{p,i}(R/I) \neq 0$.

Corollary 2.7 *Let $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ be a squarefree monomial ideal admitting a decomposition $I = I_1 \cap \cdots \cap I_c$ in disjoint sets of variables. Let i_j be the smallest integer for which there exist p such that $\lambda_{p,i_j}(R/I_j) \neq 0$ for $j = 1, \ldots, c$. Then, the smallest i for which there exist p such that $\lambda_{p,i}(R/I) \neq 0$ is $i = (i_1 + \cdots + i_c) + (c-1)$.*

Proof In the case that $c = 2$, let q and r be integers such that $\lambda_{q,i_1}(R/I_1) \neq 0$, $\lambda_{r,i_2}(S/I_2) \neq 0$. Then, $\lambda_{q+r,i_1+i_2+1}(T/I_1T + I_2T) \neq 0$ using Proposition 2.5 and it gives the smallest integer i satisfying this property. Then, we proceed using induction on the number of components. □

2.2 Local cohomology modules and free resolutions

A way to interpret the Lyubeznik numbers for the case of squarefree monomial ideals is in terms of the linear strands of the free resolution of the Alexander dual of the ideal. This approach was given in [8] and further developed in [9], and we will briefly recall it here.

Let $J^\vee$ be the Alexander dual of a squarefree monomial ideal $J \subseteq R$. Its minimal **$\mathbb{Z}$-graded free resolution** is an exact sequence of free $\mathbb{Z}$-graded R-modules:

$$\mathbb{L}_\bullet(J^\vee): \quad 0 \longrightarrow L_m \xrightarrow{d_m} \cdots \longrightarrow L_1 \xrightarrow{d_1} L_0 \longrightarrow J^\vee \longrightarrow 0$$

where the jth term is of the form

$$L_j = \bigoplus_{\ell \in \mathbb{Z}} R(-\ell)^{\beta_{j,\ell}(J^\vee)},$$

and the matrices of the morphisms $d_j : L_j \longrightarrow L_{j-1}$ do not contain invertible elements. The $\mathbb{Z}$-graded **Betti numbers** of $J^\vee$ are the invariants $\beta_{j,\ell}(J^\vee)$. Given an integer r, the r**-linear strand** of $\mathbb{L}_\bullet(J^\vee)$ is the complex:

$$\mathbb{L}_\bullet^{<r>}(J^\vee): \quad 0 \longrightarrow L_{n-r}^{<r>} \xrightarrow{d_{n-r}^{<r>}} \cdots \longrightarrow L_1^{<r>} \xrightarrow{d_1^{<r>}} L_0^{<r>} \longrightarrow 0\ ,$$

where

$$L_j^{<r>} = R(-j-r)^{\beta_{j,j+r}(J^\vee)},$$

and the differentials $d_j^{<r>} : L_j^{<r>} \longrightarrow L_{j-1}^{<r>}$ are the corresponding components of d_j.

We point out that these differentials can be described using the so-called **monomial matrices** introduced by Miller in [29] (see also [30]). These are matrices with scalar entries that keep track of the degrees of the generators of the summands in the source and the target. Now we construct a complex of $\mathbb{K}$-vector spaces

$$\mathbb{F}_\bullet^{<r>}(J^\vee)^*: \quad 0 \longleftarrow \underbrace{\mathbb{K}^{\beta_{n-r,n}(J^\vee)}}_{\deg 0} \longleftarrow \cdots \longleftarrow \underbrace{\mathbb{K}^{\beta_{1,1+r}(J^\vee)}}_{\deg n-r-1} \longleftarrow \underbrace{\mathbb{K}^{\beta_{0,r}(J^\vee)}}_{\deg n-r} \longleftarrow 0$$

where the morphisms are given by the transpose of the corresponding monomial matrices, and thus, we reverse the indices of the complex. Then, the Lyubeznik numbers are described by means of the homology groups of these complexes. Namely, the result given in [8, Corollary 4.2] is the following characterization

$$\lambda_{p,n-r}(R/J) = \dim_{\mathbb{K}} H_p(\mathbb{F}_\bullet^{<r>}(J^\vee)^*). \tag{2.2}$$

2.3 Mayer–Vietoris splitting

A successful technique used in the study of free resolutions of monomial ideals was developed by Eliahou and Kervaire in [14] and refined by Francisco, Hà and Van Tuyl in [15] under the terminology of **splittings** of monomial ideals and **Betti splittings**, respectively.

An analogous technique can be used to study local cohomology modules.

Definition 2.8 Let $J \subseteq R$ be a squarefree monomial ideal. We say that the decomposition $J = L \cap K$ is a MV-splitting if the Mayer–Vietoris sequence

$$\cdots \longrightarrow H^r_{L+K}(R) \longrightarrow H^r_L(R) \oplus H^r_K(R) \longrightarrow H^r_J(R) \longrightarrow H^{r+1}_{L+K}(R) \longrightarrow \cdots$$

splits into short exact sequences

$$0 \longrightarrow H^r_L(R) \oplus H^r_K(R) \longrightarrow H^r_J(R) \longrightarrow H^{r+1}_{L+K}(R) \longrightarrow 0$$

for all r.

Remark 2.9 Using Alexander duality, we have that this notion is equivalent to the fact that $J^\vee$ admits a Betti splitting $J^\vee = L^\vee + K^\vee$ in the sense of [15], which means that the $\mathbb{Z}^n$-graded Betti numbers satisfy

$$\beta_{i,\alpha}(J^\vee) = \beta_{i,\alpha}(L^\vee) + \beta_{i,\alpha}(K^\vee) + \beta_{i-1,\alpha}(L^\vee \cap K^\vee)$$

Certainly, we have a MV-splitting if the $\mathbb{Z}^n$-graded morphisms $H^r_{L+K}(R) \longrightarrow H^r_L(R) \oplus H^r_K(R)$ are zero for all r. Sufficient conditions for this vanishing can be given in terms of the posets of sums of ideals associated with L, K and $L + K$. The following result, which uses the terminology of Eq. (2.1), can be understood as a reinterpretation of [15, Theorem 2.3].

Proposition 2.10 *Let $J = L \cap K$ be a decomposition of a squarefree monomial ideal $J \subseteq R$. Consider the posets $\mathcal{P}_L, \mathcal{P}_K$ and $\mathcal{P}_{L+K}$ associated with the primary decompositions of the ideals L, K and $L + K$, respectively. Assume that $m_{r,\alpha}(L + K) \neq 0$ implies $m_{r,\alpha}(L) = m_{r,\alpha}(K) = 0$ for any r and any $\alpha \in \{0, 1\}^n$. Then, the decomposition $J = L \cap K$ is a MV-splitting.*

Proof The assumptions we are considering are telling us that $[H^r_{L+K}(R)]_{-\alpha} \neq 0$ implies $[H^r_L(R)]_{-\alpha} = [H^r_K(R)]_{-\alpha} = 0$ by means of Eq. (2.1), and thus, the $\mathbb{Z}^n$-graded morphisms $H^r_{L+K}(R) \longrightarrow H^r_L(R) \oplus H^r_K(R)$ are zero for all r. □

Corollary 2.11 *Let $J = L \cap K$ be a decomposition of a squarefree monomial ideal $J \subseteq R$. Assume that the posets $\mathcal{P}_L, \mathcal{P}_K$ and $\mathcal{P}_{L+K}$ associated with the primary decompositions of the ideals L, K and $L + K$ have no face ideal in common. Then, the decomposition $J = L \cap K$ is a MV-splitting.*

We want to apply these splitting techniques to the study of the composition of local cohomology modules. The following discussion will be crucial in the rest of this work.

Discussion 2.12 The degree $-\mathbf{1}$ part of the long exact sequence of local cohomology associated with the short exact sequences

$$0 \longrightarrow H^r_L(R) \oplus H^r_K(R) \longrightarrow H^r_J(R) \longrightarrow H^{r+1}_{L+K}(R) \longrightarrow 0 \tag{2.3}$$

obtained in a MV-splitting is

$$\begin{aligned}\cdots \longrightarrow [H^{p-1}_{\mathfrak{m}}(H^{r+1}_{L+K}(R))]_{-\mathbf{1}} &\xrightarrow{\partial^r_{p-1}} [H^p_{\mathfrak{m}}(H^r_L(R))]_{-\mathbf{1}} \oplus [H^p_{\mathfrak{m}}(H^r_K(R))]_{-\mathbf{1}} \\ &\longrightarrow [H^p_{\mathfrak{m}}(H^r_J(R))]_{-\mathbf{1}} \longrightarrow [H^p_{\mathfrak{m}}(H^{r+1}_{L+K}(R))]_{-\mathbf{1}} \\ &\xrightarrow{\partial^r_p} [H^{p+1}_{\mathfrak{m}}(H^r_L(R))]_{-\mathbf{1}} \oplus [H^{p+1}_{\mathfrak{m}}(H^r_K(R))]_{-\mathbf{1}} \longrightarrow \cdots\end{aligned} \tag{2.4}$$

Equivalently, it is the long exact sequence of $\mathbb{K}$-vector spaces whose dimensions are the corresponding Lyubeznik numbers. Namely,

$$\begin{aligned}\cdots \longrightarrow \mathbb{K}^{\lambda_{p-1,n-r-1}(R/L+K)} &\xrightarrow{\partial^r_{p-1}} \mathbb{K}^{\lambda_{p,n-r}(R/L)} \oplus \mathbb{K}^{\lambda_{p-1,n-r}(R/K)} \\ &\longrightarrow \mathbb{K}^{\lambda_{p-1,n-r}(R/J)} \longrightarrow \mathbb{K}^{\lambda_{p,n-r-1}(R/L+K)} \\ &\xrightarrow{\partial^r_p} \mathbb{K}^{\lambda_{p+1,n-r}(R/L)} \oplus \mathbb{K}^{\lambda_{p+1,n-r}(R/K)} \longrightarrow \cdots\end{aligned} \tag{2.5}$$

Therefore, if we want to compute the Lyubeznik numbers of R/J in terms of the Lyubeznik numbers of R/L, R/K and $R/L+K$, we need to control the connecting morphisms ∂^r_p's.

Using the methods considered in [8], we may give an interpretation of these differentials in terms of linear strands. First, the short exact sequence (2.3) corresponds to the short exact sequence of complexes of $\mathbb{K}$-vector spaces

$$0 \longleftarrow \mathbb{F}^{<r>}_\bullet(L^\vee)^* \oplus \mathbb{F}^{<r>}_\bullet(K^\vee)^* \longleftarrow \mathbb{F}^{<r>}_\bullet(J^\vee)^* \longleftarrow \mathbb{F}^{<r+1>}_\bullet((L+K)^\vee)^* \longleftarrow 0$$

and the long exact sequence (2.4) corresponds to

$$\begin{aligned}\cdots \longleftarrow H_{p-1}(\mathbb{F}^{<r+1>}_\bullet((L+K)^\vee)^*) &\xleftarrow{\partial^r_{p-1}} H_p(\mathbb{F}^{<r>}_\bullet(L^\vee)^*) \oplus H_p(\mathbb{F}^{<r>}_\bullet(K^\vee)^*) \\ &\longleftarrow H_p(\mathbb{F}^{<r>}_\bullet(J^\vee)^*) \longleftarrow H_p(\mathbb{F}^{<r+1>}_\bullet((L+K)^\vee)^*) \\ &\xleftarrow{\partial^r_p} H_{p+1}(\mathbb{F}^{<r>}_\bullet(L^\vee)^*) \oplus H_{p+1}(\mathbb{F}^{<r>}_\bullet(K^\vee)^*) \longleftarrow \cdots\end{aligned}$$

3 Lyubeznik tables of cover ideals of graphs

Let $G = (V_G, E_G)$ be a simple finite graph in the set of vertices $V_G = \{x_1, \ldots, x_n\}$ and the set of edges E_G. For simplicity, we will also assume that G is connected. For a vertex x_i, we consider its **neighbour set** $N(x_i) = \{x_j \in G \mid \{x_i, x_j\} \in E_G\}$. The **degree** of a vertex is the cardinal of its neighbour set.

Let $J(G) \subseteq R$ be the cover ideal of G where $R = \mathbb{K}[x_1, \ldots, x_n]$ is a polynomial ring with coefficients in a field $\mathbb{K}$. In this section, we will develop MV-splitting techniques to study the Lyubeznik numbers of $R/J(G)$. To start with, we recall that since $J(G)$ is a pure height two ideal, the dimension of the support of $H^r_{J(G)}(R)$ is strictly smaller than $n-r$ for $r > 2$ (see [1, Theorem 3.8]). Then, it follows from [1, Theorem 4.4] that all the entries in the main diagonal of the Lyubeznik table are zero except for the highest Lyubeznik number.

Lemma 3.1 *Let $J(G)$ be the cover ideal of a simple connected graph G. Then, the highest Lyubeznik number is $\lambda_{d,d}(R/J(G)) = 1$*

Proof The vertices of the Hochster–Huneke graph of $J(G)$ correspond to the edges of G, and the edges of the Hochster–Huneke graph correspond to adjacent edges of G. Therefore, the Hochster–Huneke graph has just one connected component since the graph G is connected. □

Under these restrictions, the shape of the Lyubeznik table is

$$\Lambda(R/J(G)) = \begin{pmatrix} 0 & \lambda_{0,1} & \cdots & \lambda_{0,d-1} & \lambda_{0,d} \\ & 0 & \cdots & \lambda_{1,d-1} & \lambda_{1,d} \\ & & \ddots & \vdots & \vdots \\ & & & 0 & \lambda_{d-1,d} \\ & & & & 1 \end{pmatrix}$$

In the case that $R/J(G)$ is Cohen–Macaulay, we have that the Lyubeznik table is trivial (see [1, Remark 4.2]). Recall that, combining the results in [13] with [16], we have the following characterization of this property.

Proposition 3.2 *Let G be a simple graph. Then, the following are equivalent:*

i) The cover ideal $J(G)$ is Cohen–Macaulay.
ii) The edge ideal $I(G)$ has a linear resolution.
iii) The complement graph G^c is chordal.

Free resolutions of edge ideals have been extensively studied over the last years, and we may find in the literature several families of Cohen–Macaulay graphs. For example,

- **Complete graphs** K_n [23].
- **Complete bipartite graphs** $K_{n,m}$ and in particular **star graphs** $K_{1,m}$ [23].
- **Ferrers graphs** [12].

The simplest examples of ideals with non-trivial Lyubeznik table are minimal non-Cohen–Macaulay squarefree monomial ideals (see [25]). The unique minimal non-Cohen–Macaulay squarefree monomial ideal of pure height two in $R = \mathbb{K}[x_1, \ldots, x_n]$ is the cover ideal of the **complement of a cycle**:

$$J(C_n^c) = (x_1, x_3) \cap \cdots \cap (x_1, x_{n-1}) \cap (x_2, x_4) \cap \cdots \cap (x_2, x_n) \cap (x_3, x_5) \cap \cdots \cap (x_{n-2}, x_n).$$

Its Lyubeznik table is of the form (see [8])

$$\Lambda(R/J(C_n^c)) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & 0 & & 0 & 0 & 1 \\ & & & \ddots & & 0 & 0 \\ & & & & & \vdots & \vdots \\ & & & & & 0 & 0 \\ & & & & & & 1 \end{pmatrix}$$

To provide a full description of all the possible Lyubeznik tables of cover ideals of graphs is completely out of the scope of this work. Our aim is to introduce some Mayer–Vietoris splitting techniques that will allow us to compute large families of examples. To such purpose, we will follow the ideas considered in Discussion 2.12. To start with, we consider the case where $J(G) = L \cap K$ is a MV-splitting with the extra assumption that the Lyubeznik table of R/K is trivial.

Theorem 3.3 *Let $J(G) \subseteq R$ be the cover ideal of a simple connected graph G. Let $J(G) = L \cap K$ be a MV-splitting such that $\Lambda(R/K)$ is trivial. Then,*

i) If $\Lambda(R/L)$ and $\Lambda(R/L+K)$ are trivial, then $\Lambda(R/J(G))$ is trivial.
ii) If $\Lambda(R/L+K)$ is trivial, then $\Lambda(R/J(G)) = \Lambda(R/L)$.
iii) If $\Lambda(R/L)$ is trivial and

$$\Lambda(R/L+K) = \begin{pmatrix} \lambda'_{0,0} & \cdots & \lambda'_{0,d-1} \\ & \ddots & \vdots \\ & & \lambda'_{d-1,d-1} \end{pmatrix},$$

then the Lyubeznik table of $R/J(G)$ is

$$\Lambda(R/J(G)) = \begin{pmatrix} 0 & \lambda'_{0,0} & \cdots & \lambda'_{0,d-2} & \lambda'_{0,d-1} \\ & 0 & \cdots & \lambda'_{1,d-2} & \lambda'_{1,d-1} \\ & & \ddots & \vdots & \vdots \\ & & & 0 & \lambda'_{d-1,d-1} - 1 \\ & & & & 1 \end{pmatrix}.$$

Proof Assume that $\Lambda(R/K)$ is trivial and recall that, using Lemma 3.1, the highest Lyubeznik number of the cover ideal of a graph is one. In particular, we have $[H_{\mathfrak{m}}^{n-2}(H_K^2(R))]_{-\mathbf{1}} = \mathbb{K}$ and $[H_{\mathfrak{m}}^{n-2}(H_L^2(R))]_{-\mathbf{1}} = \mathbb{K}$. Then, for $r = 2$, the long exact sequence 2.4 considered in Discussion 2.12

$$\cdots \longrightarrow [H_{\mathfrak{m}}^{n-3}(H_{L+K}^{3}(R))]_{-\mathbf{1}} \xrightarrow{\partial_{n-3}^{2}} [H_{\mathfrak{m}}^{n-2}(H_{L}^{2}(R))]_{-\mathbf{1}} \oplus [H_{\mathfrak{m}}^{n-2}(H_{K}^{2}(R))]_{-\mathbf{1}} \longrightarrow [H_{\mathfrak{m}}^{n-2}(H_{J(G)}^{2}(R))]_{-\mathbf{1}} \longrightarrow 0$$

turns out to be

$$\cdots \longrightarrow \mathbb{K}^{\lambda_{n-3,n-2}(R/L)} \longrightarrow \mathbb{K}^{\lambda_{n-3,n-2}(R/J(G))} \longrightarrow \mathbb{K}^{\lambda_{n-3,n-3}(R/L+K)} \xrightarrow{\partial_{n-3}^{2}} \mathbb{K} \oplus \mathbb{K} \longrightarrow \mathbb{K} \longrightarrow 0$$

Moreover, for $r > 2$ and any p, the long exact sequence becomes

$$\cdots \longrightarrow \mathbb{K}^{\lambda_{p-1,n-(r+1)}(R/L+K)} \xrightarrow{\partial_{p-1}^{r}} \mathbb{K}^{\lambda_{p,n-r}(R/L)} \longrightarrow \mathbb{K}^{\lambda_{p,n-r}(R/J(G))} \longrightarrow \mathbb{K}^{\lambda_{p,n-(r+1)}(R/L+K)} \xrightarrow{\partial_{p}^{r}} \cdots$$

Now we are ready to consider all the cases:

i) If $\Lambda(R/L)$ and $\Lambda(R/L+K)$ are trivial, then the Lyubeznik table of $R/J(G)$ is trivial as well. Notice that for $r = 2$ we have

$$\cdots \longrightarrow \mathbb{K}^{\lambda_{n-3,n-2}(R/J(G))} \longrightarrow \mathbb{K} \xrightarrow{\partial_{n-3}^{2}} \mathbb{K} \oplus \mathbb{K} \longrightarrow \mathbb{K} \longrightarrow 0,$$

and thus, $\lambda_{n-3,n-2}(R/J(G)) = 0$ and the vanishing of the rest of the Lyubeznik numbers follow immediately.

ii) If $\Lambda(R/L+K)$ is trivial, then we have

$$0 \longrightarrow \mathbb{K}^{\lambda_{n-3,n-2}(R/L)} \longrightarrow \mathbb{K}^{\lambda_{n-3,n-2}(R/J(G))} \longrightarrow \mathbb{K} \xrightarrow{\partial_{n-3}^{2}} \mathbb{K} \oplus \mathbb{K} \longrightarrow \mathbb{K} \longrightarrow 0,$$

and thus, $\lambda_{n-3,n-2}(R/L) = \lambda_{n-3,n-2}(R/J(G))$. The rest of the Lyubeznik numbers also coincide so we get $\Lambda(R/J(G)) = \Lambda(R/L)$.

iii) If $\Lambda(R/L)$ is trivial, then we have

$$0 \longrightarrow \mathbb{K}^{\lambda_{n-3,n-2}(R/J(G))} \longrightarrow \mathbb{K}^{\lambda_{n-3,n-3}(R/L+K)} \xrightarrow{\partial_{n-3}^{2}} \mathbb{K} \oplus \mathbb{K} \longrightarrow \mathbb{K} \longrightarrow 0,$$

and thus, $\lambda_{n-3,n-2}(R/J(G)) = \mathbb{K}^{\lambda_{n-3,n-3}(R/L+K)} - 1$. The rest of the Lyubeznik numbers satisfy $\lambda_{p,n-r}(R/J(G)) = \lambda_{p,n-(r+1)}(R/L+K)$, and the result follows.

□

3.1 Splitting vertices

Let $J(G) \subseteq R$ be the cover ideal of a simple connected graph G. The easiest way to provide a MV-splitting $J(G) = L \cap K$ satisfying that the Lyubeznik table of R/K

is trivial is by means of a **splitting vertex**. Namely, we fix a vertex, say x_n, and we decompose the ideal $J(G)$ depending on the edges that contain this vertex.

$$J(G) = \underbrace{\left(\bigcap_{x_i, x_j \notin N_G(x_n)} (x_i, x_j) \right)}_{L} \cap \underbrace{\left(\bigcap_{x_k \in N_G(x_n)} (x_k, x_n) \right)}_{K}$$

Observe that $J(G) = K$ when G is a star graph with x_n being the internal vertex so in this case, we will pick up a different vertex. In general, we have:

- $L = J(G \setminus \{x_n\})$ is the cover ideal of the subgraph obtained removing the vertex x_n.
- $K = J(K_{1,g})$ is the cover ideal of a star graph with $g = \deg(x_n)$.
- $L + K$ is a height 3 monomial ideal which admits a (non-necessarily minimal) primary decomposition of the form:

$$L + K = \bigcap_{x_k \in N_G(x_n)} \left[\left(\bigcap_{x_i, x_j \notin N_G(x_n), N_G(x_k)} (x_i, x_j, x_k, x_n) \right) \cap \left(\bigcap_{x_l \in N_G(x_k)} (x_l, x_k, x_n) \right) \right].$$

Of course we can make it minimal removing conveniently the extra components. Notice that $\Lambda(R/K)$ is trivial. In order to check that this decomposition indeed provides a MV-splitting, we only need to invoke [19, Theorem 4.2] where it is proved that every vertex is a splitting vertex except for some limit cases where the vertex is isolated or its complement consists of isolated vertices.

Remark 3.4 The operations of sum and intersection distribute over each other in the case of squarefree monomial ideals. This gives us a very simple method to find (non-necessarily minimal) primary decompositions of sums $L + K$ of squarefree monomial ideals in terms of the primary decompositions of L and K, respectively. Namely,

$$\underbrace{(I_1 \cap \cdots \cap I_c)}_{L} + \underbrace{(J_1 \cap \cdots \cap J_d)}_{K} = (I_1 + J_1) \cap \cdots \cap (I_c + J_d).$$

In order to make it minimal, we just remove the extra components. Throughout the rest of this work, we will use this fact without further reference.

3.1.1 Splitting vertices of degree one

Let x_n be a splitting vertex of a graph G. Assume that its degree is one and, for simplicity, we will take x_{n-1} as the unique vertex in its neighbourhood. We can rephrase it by saying that we are adding a **whisker** to the vertex x_{n-1} of the graph $G \setminus \{x_n\}$.

Proposition 3.5 *Let $J(G) \subseteq R$ be the cover ideal of a simple connected graph G. Let $x_n \in V_G$ be a vertex of degree one. Then, $\Lambda(R/J(G)) = \Lambda(R/J(G \setminus \{x_n\}))$.*

Proof Let x_{n-1} be the unique vertex in the neighbourhood of x_n. Then, the MV-splitting $J_G = L \cap K$ given by x_n has $K = (x_{n-1}, x_n)$ and

$$\begin{aligned} L + K &= \left(\bigcap_{x_i, x_j \notin N_G(x_{n-1})} (x_i, x_j, x_{n-1}, x_n) \right) \cap \left(\bigcap_{x_l \in N_G(x_{n-1})} (x_l, x_{n-1}, x_n) \right) \\ &= \underbrace{\left[\left(\bigcap_{x_i, x_j \notin N_G(x_{n-1})} (x_i, x_j) \right) \cap \left(\bigcap_{x_l \in N_G(x_{n-1})} (x_l) \right) \right]}_{M} + (x_{n-1}, x_n) \end{aligned}$$

The ideal M is a height one ideal in two sets of disjoint variables. Therefore, its Lyubeznik table is trivial because of Proposition 2.5. Given the isomorphism

$$\mathbb{K}[x_1, \ldots, x_n]/L + K \cong \mathbb{K}[x_1, \ldots, x_{n-2}]/M,$$

we get that $\Lambda(R/L + K)$ is trivial as well. Then, the result follows using Theorem 3.3 and the fact that $L = J(G \setminus \{x_n\})$. □

3.1.2 Splitting vertices of degree two

Let x_n be a splitting vertex of a graph G. Assume that its degree is two and the vertices in its neighbourhood are x_{n-1} and x_{n-2}. In this case, we also have the invariance of the Lyubeznik table after removing the splitting vertex under certain extra conditions.

Proposition 3.6 *Let $J(G) \subseteq R$ be the cover ideal of a simple connected graph G. Let $x_n \in V_G$ be a vertex of degree 2 with $N_G(x_n) = \{x_{n-2}, x_{n-1}\}$. If any of the following conditions hold:*

i) $\{x_{n-1}, x_{n-2}\} \in E_G$,
ii) there exists $x_c \in N_G(x_{n-1}) \cap N_G(x_{n-2})$,

then $\Lambda(R/J(G)) = \Lambda(R/J(G \setminus \{x_n\}))$.

Proof We have a MV-splitting $J(G) = L \cap K$ where $K = (x_{n-1}, x_n) \cap (x_{n-2}, x_n)$ and

$$\begin{aligned} L = &\left(\bigcap_{x_i, x_j \notin N_G(x_n)} (x_i, x_j) \right) \cap \left(\bigcap_{\substack{x_a \in N_G(x_{n-1}) \\ x_a \notin N_G(x_{n-2})}} (x_a, x_{n-1}) \right) \cap \left(\bigcap_{\substack{x_b \in N_G(x_{n-2}) \\ x_b \notin N_G(x_{n-1})}} (x_b, x_{n-2}) \right) \\ &\cap \left(\bigcap_{\substack{x_c \in N_G(x_{n-1}) \\ x_c \in N_G(x_{n-2})}} (x_c, x_{n-2}) \cap (x_c, x_{n-1}) \right) \cap (x_{n-2}, x_{n-1}) \end{aligned}$$

Under the assumptions we are considering, at least one of the last components in this decomposition must appear. Therefore,

$$L + K = \underbrace{\left(\bigcap_{\substack{x_i, x_j \notin N_G(x_n) \\ x_i, x_j \notin N_G(x_{n-1})}} (x_i, x_j, x_{n-1}, x_n) \right) \cap \left(\bigcap_{\substack{x_a \in N_G(x_{n-1}) \\ x_a \notin N_G(x_{n-2})}} (x_a, x_{n-1}, x_n) \right)}_{M}$$

$$\underbrace{\cap \left(\bigcap_{\substack{x_c \in N_G(x_{n-1}) \\ x_c \in N_G(x_{n-2})}} (x_c, x_{n-1}, x_n) \right) \cap (x_{n-2}, x_{n-1}, x_n)}_{M}$$

$$\underbrace{\cap \left(\bigcap_{\substack{x_i, x_j \notin N_G(x_n) \\ x_i, x_j \notin N_G(x_{n-2})}} (x_i, x_j, x_{n-2}, x_n) \right) \cap \left(\bigcap_{x_b \in N_G(x_{n-2})} (x_b, x_{n-2}, x_n) \right)}_{N}$$

Notice that we can rephrase the ideals M and N as

$$M = \left[\left(\bigcap_{\substack{x_i, x_j \notin N_G(x_n) \\ x_i, x_j \notin N_G(x_{n-1})}} (x_i, x_j) \right) \cap \left(\bigcap_{\substack{x_a \in N_G(x_{n-1}) \\ x_a \notin N_G(x_{n-2})}} (x_a) \right) \cap \left(\bigcap_{\substack{x_c \in N_G(x_{n-1}) \\ x_c \in N_G(x_{n-2})}} (x_c) \right) \cap (x_{n-2}) \right] + (x_{n-1}, x_n)$$

$$N = \left[\left(\bigcap_{\substack{x_i, x_j \notin N_G(x_n) \\ x_i, x_j \notin N_G(x_{n-2})}} (x_i, x_j) \right) \cap \left(\bigcap_{x_b \in N_G(x_{n-2})} (x_b) \right) \right] + (x_{n-2}, x_n)$$

so both $\Lambda(R/M)$ and $\Lambda(R/N)$ are trivial using Proposition 2.5. Moreover, in the case that condition $i)$ is satisfied, we have

$$M + N = \left[\left(\bigcap_{\substack{x_i, x_j \notin N_G(x_n) \\ x_i, x_j \notin N_G(x_{n-1}) \\ x_i, x_j \notin N_G(x_{n-2})}} (x_i, x_j) \right) \cap \left(\bigcap_{\substack{x_b \in N_G(x_{n-2}) \\ x_b \notin N_G(x_{n-1})}} (x_b) \right) \cap \left(\bigcap_{\substack{x_c \in N_G(x_{n-1}) \\ x_c \in N_G(x_{n-2})}} (x_c) \right) \right] + (x_{n-2}, x_{n-1}, x_n)$$

Under condition $ii)$ we have

$$M+N=\left[\left(\bigcap_{\substack{x_i,x_j\notin N_G(x_n)\\ x_i,x_j\notin N_G(x_{n-1})\\ x_i,x_j\notin N_G(x_{n-2})}}(x_i,x_j)\right)\cap\left(\bigcap_{\substack{x_a\in N_G(x_{n-1})\\ x_a\notin N_G(x_{n-2})\\ x_b\in N_G(x_{n-2})\\ x_b\notin N_G(x_{n-1})}}(x_a,x_b)\right)\cap\left(\bigcap_{\substack{x_c\in N_G(x_{n-1})\\ x_c\in N_G(x_{n-2})}}(x_c)\right)\right]+(x_{n-2},x_{n-1},x_n)$$

In any case the Lyubeznik table of $R/M+N$ is trivial since we are dealing with a height one ideal in a disjoint set of variables, we can apply Proposition 2.5 once again. As a consequence of Theorem 3.3, the Lyubeznik table $\Lambda(R/L+K)$ is trivial and the result follows applying Theorem 3.3 once more. To finish the proof, we need to check that the decomposition $L+K=M\cap N$ is, indeed, a MV-splitting.

If $\{x_{n-2},x_{n-1}\}$ is not an edge of G, we have that the variable x_{n-1} appears in all the components of the primary decomposition of M but not in N. We also have that x_{n-2} appears in all the components of N but not in M and both x_{n-2},x_{n-1} appear in $M+N$. In particular, the posets associated with M, N and $M+N$ do not have common ideals. Then, the result follows from Corollary 2.11.

When $\{x_{n-2},x_{n-1}\}$ is an edge of G, we have to be more careful. The variable x_{n-1} appears in all the components of $M+N$ but not in N, and thus, its corresponding posets do not have common ideals so we only have to compare $M+N$ with M. Indeed, since the variables x_a's and x_b's do not belong to both ideals and x_{n-1},x_n do so we may just assume that the ideals are

$$M=\underbrace{\left(\bigcap_{\substack{x_i,x_j\notin N_G(x_n)\\ x_i,x_j\notin N_G(x_{n-1})}}(x_i,x_j)\right)\cap\left(\bigcap_{\substack{x_c\in N_G(x_{n-1})\\ x_c\in N_G(x_{n-2})}}(x_c)\right)}_{Q}\cap(x_{n-2})$$

$$M+N=\left[\left(\bigcap_{\substack{x_i,x_j\notin N_G(x_n)\\ x_i,x_j\notin N_G(x_{n-1})}}(x_i,x_j)\right)\cap\left(\bigcap_{\substack{x_c\in N_G(x_{n-1})\\ x_c\in N_G(x_{n-2})}}(x_c)\right)\right]+(x_{n-2}),$$

and thus, we have a MV-splitting

$$0\longrightarrow H^r_Q(R)\oplus H^r_{(x_{n-2})}(R)\longrightarrow H^r_M(R)\longrightarrow H^{r+1}_{M+N}(R)\longrightarrow 0$$

so the graded pieces of $H^{r+1}_{M+N}(R)$ are related to the graded pieces of $H^r_M(R)$ instead of those of $H^{r+1}_M(R)$. □

We can still say something about the Lyubeznik numbers of $J(G)$ in the event that the hypothesis of the previous proposition does not hold. In this case, the Lyubeznik table of $R/L+K$ is not trivial so we need to control the connecting morphisms

$$\mathbb{K}^{\lambda_{p-1,n-(r+1)}(R/L+K)} \xrightarrow{\partial^r_{p-1}} \mathbb{K}^{\lambda_{p,n-r}(R/L)}$$

considered in Discussion 2.12. If the Lyubeznik table of $L = J(G \setminus \{x_n\})$ is trivial or, more generally, the connecting morphisms are zero, we can give a formula for the Lyubeznik numbers of $J(G)$ in terms of those of $J(G \setminus \{x_n\})$ and the Lyubeznik numbers associated with the graph

$$H := (G \setminus \{x_n\}) \cup \bigcup_{\substack{x_a \in N_G(x_{n-1}) \\ x_b \in N_G(x_{n-2})}} \{x_a, x_b\}$$

whose cover ideal is

$$J(H) = \left(\bigcap_{\substack{x_i, x_j \notin N_G(x_{n-2}) \\ x_i, x_j \notin N_G(x_{n-1})}} (x_i, x_j) \right) \cap \left(\bigcap_{\substack{x_a \in N_G(x_{n-2}) \\ x_b \in N_G(x_{n-1})}} (x_a, x_b) \right)$$

In other words, the graph H is obtained by adjoining to $G \setminus \{x_n\}$, a complete bipartite graph in the set of vertices $N_G(x_{n-2})$ and $N_G(x_{n-1})$.

Proposition 3.7 *Let $J(G) \subseteq R$ be the cover ideal of a simple connected graph G. Let $x_n \in V_G$ be a vertex of degree 2 with $N_G(x_n) = \{x_{n-1}, x_{n-2}\}$. Assume that conditions $i)$ and $ii)$ of Proposition 3.6 do not hold and that the connecting morphisms ∂^r_p are zero for all $r > 2$ and for all p. Then, we have*

$$\lambda_{d,d}(R/J(G)) = 1, \quad \lambda_{d-1,d}(R/J(G)) = \lambda_{d-1,d}(R/J(G \setminus \{x_n\})) + 1,$$
$$\lambda_{p,r}(R/J(G)) = \lambda_{p,r}(R/J(G \setminus \{x_n\})) + \lambda_{p,r-2}(R/J(H))$$

for $r = 2, \ldots, d-1$ and $p = 0, \ldots, r-2$ and the rest of the Lyubeznik numbers satisfy

$$\lambda_{p,r}(R/J(G)) = \lambda_{p,r}(R/J(G \setminus \{x_n\})).$$

That is,

$$\Lambda(R/J(G)) = \begin{pmatrix} \lambda_{0,0} & \lambda_{0,1} & \lambda_{0,2}+\lambda'_{0,0} & \lambda_{0,3}+\lambda'_{0,1} & \cdots & \lambda_{0,d-1}+\lambda'_{0,d-3} & \lambda_{0,d} \\ & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3}+\lambda'_{1,1} & \cdots & \lambda_{1,d-1}+\lambda'_{1,d-3} & \lambda_{1,d} \\ & & \lambda_{2,2} & \lambda_{2,3} & \cdots & \lambda_{2,d-1}+\lambda'_{2,d-3} & \lambda_{2,d} \\ & & & & \ddots & \vdots & \vdots \\ & & & & & \lambda_{d-3,d-1}+\lambda'_{d-3,d-3} & \lambda_{d-3,d} \\ & & & & & \lambda_{d-2,d-1} & \lambda_{d-2,d} \\ & & & & & \lambda_{d-1,d-1} & \lambda_{d-1,d}+1 \\ & & & & & & 1 \end{pmatrix}$$

where

$$\Lambda(R/J(G\setminus\{x_n\})) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix},$$

$$\Lambda(\mathbb{K}[x_1,\ldots,x_{n-3}]/J(H)) = \begin{pmatrix} \lambda'_{0,0} & \cdots & \lambda'_{0,d-3} \\ & \ddots & \vdots \\ & & \lambda'_{d-3,d-3} \end{pmatrix}$$

Proof Following the same approach as in the proof of Proposition 3.6, we have a MV-splitting $J(G) = L \cap K$ with $K = (x_{n-1}, x_n) \cap (x_{n-2}, x_n)$, but in this case we also have

$$L = \left(\bigcap_{x_i,x_j \notin N_G(x_n)} (x_i, x_j)\right) \cap \left(\bigcap_{\substack{x_a \in N_G(x_{n-1}) \\ x_a \notin N_G(x_{n-2})}} (x_a, x_{n-1})\right) \cap \left(\bigcap_{\substack{x_b \in N_G(x_{n-2}) \\ x_b \notin N_G(x_{n-1})}} (x_b, x_{n-2})\right)$$

$$L + K = \underbrace{\left(\bigcap_{\substack{x_i,x_j \notin N_G(x_n) \\ x_i,x_j \notin N_G(x_{n-1})}} (x_i, x_j, x_{n-1}, x_n)\right) \cap \left(\bigcap_{\substack{x_a \in N_G(x_{n-1}) \\ x_a \notin N_G(x_{n-2})}} (x_a, x_{n-1}, x_n)\right)}_{M}$$
$$\cap \underbrace{\left(\bigcap_{\substack{x_i,x_j \notin N_G(x_n) \\ x_i,x_j \notin N_G(x_{n-1})}} (x_i, x_j, x_{n-2}, x_n)\right) \cap \left(\bigcap_{\substack{x_b \in N_G(x_{n-2}) \\ x_b \notin N_G(x_{n-1})}} (x_b, x_{n-2}, x_n)\right)}_{N}$$

Once again we rewrite the ideals M and N as

$$M = \left[\left(\bigcap_{\substack{x_i,x_j \notin N_G(x_n)\\ x_i,x_j \notin N_G(x_{n-1})}} (x_i, x_j)\right) \cap \left(\bigcap_{\substack{x_a \in N_G(x_{n-1})\\ x_a \notin N_G(x_{n-2})}} (x_a)\right)\right] + (x_{n-1}, x_n)$$

$$N = \left[\left(\bigcap_{\substack{x_i,x_j \notin N_G(x_n)\\ x_i,x_j \notin N_G(x_{n-2})}} (x_i, x_j)\right) \cap \left(\bigcap_{\substack{x_b \in N_G(x_{n-2})\\ x_b \notin N_G(x_{n-1})}} (x_b)\right)\right] + (x_{n-2}, x_n)$$

so both $\Lambda(R/M)$ and $\Lambda(R/N)$ are trivial. Moreover,

$$M + N = \left[\underbrace{\left(\bigcap_{\substack{x_i,x_j \notin N_G(x_{n-2})\\ x_i,x_j \notin N_G(x_{n-1})}} (x_i, x_j)\right) \cap \left(\bigcap_{\substack{x_a \in N_G(x_{n-1})\\ x_b \in N_G(x_{n-2})}} (x_a, x_b)\right)}_{J(H)}\right] + (x_{n-2}, x_{n-1}, x_n).$$

We have that the variable x_{n-1} appears in all the components of the primary decomposition of M but not in N. We also have that x_{n-2} appears in all the components of N but not in M and both x_{n-2}, x_{n-1} appear in $M + N$. In particular, the posets associated with M, N and $M + N$ do not have common ideals. Therefore, $L + K = M \cap N$ is a MV-splitting by using Corollary 2.11.

Applying the long exact sequence of local cohomology modules to the short exact sequence

$$0 \longrightarrow H^r_M(R) \oplus H^r_N(R) \longrightarrow H^r_{L+K}(R) \longrightarrow H^{r+1}_{M+N}(R) \longrightarrow 0$$

we get, for $r = 3$,

$$\cdots \longrightarrow H^{n-4}_{\mathfrak{m}}(H^4_{M+N}(R)) \xrightarrow{\partial^3_{n-4}} H^{n-3}_{\mathfrak{m}}(H^3_M(R)) \oplus H^{n-3}_{\mathfrak{m}}(H^3_N(R)) \longrightarrow H^{n-3}_{\mathfrak{m}}(H^3_{L+K}(R)) \longrightarrow 0$$

Since $\Lambda(R/M)$ and $\Lambda(R/N)$ are trivial and taking into account that $\operatorname{ht} M = \operatorname{ht} N = 3$ and $\operatorname{ht} M + N = 5$, we get

$[H^{n-3}_{\mathfrak{m}}(H^3_{L+K}(R))]_{-\mathbf{1}} \cong \mathbb{K}^2$

$[H^{p}_{\mathfrak{m}}(H^3_{L+K}(R))]_{-\mathbf{1}} \cong 0 \qquad \forall p < n - 3$

$[H^{p}_{\mathfrak{m}}(H^r_{L+K}(R))]_{-\mathbf{1}} \cong [H^{p}_{\mathfrak{m}}(H^{r+1}_{M+N}(R))]_{-\mathbf{1}} \qquad \forall p, \forall r > 3.$

Now, if we go back to the long exact sequence 2.4 considered in Discussion 2.12 to compute the Lyubeznik table of $R/J(G)$, we get

$$0 \longrightarrow \mathbb{K}^{\lambda_{n-3,n-2}(R/L)} \longrightarrow \mathbb{K}^{\lambda_{n-3,n-2}(R/J(G))} \longrightarrow \mathbb{K}^{\lambda_{n-3,n-3}(R/L+K)} \xrightarrow{\partial^3_{n-3}} \mathbb{K} \oplus \mathbb{K} \longrightarrow \mathbb{K} \longrightarrow 0$$

and $\lambda_{p,n-2}(R/L) = \lambda_{p,n-2}(R/J(G))$ for all $p < n-3$. Moreover, for $r > 2$ and any p, the long exact sequence becomes

$$\cdots \longrightarrow \mathbb{K}^{\lambda_{p-1,n-(r+1)}(R/L+K)} \xrightarrow{\partial^r_{p-1}} \mathbb{K}^{\lambda^r_{p,n-r}(R/L)} \longrightarrow \mathbb{K}^{\lambda_{p,n-r}(R/J(G))} \longrightarrow \mathbb{K}^{\lambda_{p,n-(r+1)}(R/L+K)} \xrightarrow{\partial^r_p} \cdots$$

Therefore, we have:
$\lambda_{d,d}(R/J(G)) = 1,$
$\lambda_{d-1,d}(R/J(G)) = \lambda_{d-1,d}(R/L) + 1,$
$\lambda_{p,d}(R/J(G)) = \lambda_{p,d}(R/L) \quad \forall p < d-1$
and the rest of the Lyubeznik numbers depend on the connecting morphisms

$$\mathbb{K}^{\lambda_{p-1,n-(r+2)}(R/M+N)} = \mathbb{K}^{\lambda_{p-1,n-(r+1)}(R/L+K)} \xrightarrow{\partial^r_{p-1}} \mathbb{K}^{\lambda_{p,n-r}(R/L)}.$$

Assuming that the connecting morphisms are zero, we get

$$\lambda_{p,n-r}(R/J(G)) = \lambda_{p,n-r}(R/L) + \lambda_{p,n-(r+1)}(R/L+K)$$

and the result follows since we have $L = J(G \setminus \{x_n\})$ and an isomorphism

$$\mathbb{K}[x_1, \ldots, x_n]/M + N \cong \mathbb{K}[x_1, \ldots, x_{n-3}]/J(H),$$

and thus, $\Lambda(R/M+N) = \Lambda(\mathbb{K}[x_1, \ldots, x_{n-3}]/J(H))$. $\square$

We highlight the following particular case.

Corollary 3.8 *Under the assumptions of Proposition 3.7, if $\Lambda(R/J(G \setminus \{x_n\}))$ is trivial, then the Lyubeznik table of $R/J(G)$ is*

$$\Lambda(R/J(G)) = \begin{pmatrix} 0 & 0 & \lambda'_{0,0} & \lambda'_{0,1} & \cdots & \lambda'_{0,d-3} & 0 \\ & 0 & 0 & \lambda'_{1,1} & \cdots & \lambda'_{1,d-3} & 0 \\ & & 0 & 0 & \cdots & \lambda'_{2,d-3} & 0 \\ & & & & \ddots & \vdots & \vdots \\ & & & & & \lambda'_{d-3,d-3} & 0 \\ & & & & & 0 & 0 \\ & & & & & 0 & 1 \\ & & & & & & 1 \end{pmatrix}$$

where

$$\Lambda(\mathbb{K}[x_1,\ldots,x_{n-3}]/J(H)) = \begin{pmatrix} \lambda'_{0,0} & \cdots & \lambda'_{0,d-3} \\ & \ddots & \vdots \\ & & \lambda'_{d-3,d-3} \end{pmatrix}$$

3.1.3 Splitting vertices of maximal degree

We turn our attention to the case of a splitting vertex x_n of degree $n-1$. That is, $\{x_i, x_n\}$ is an edge of G for $i = 1, \ldots, n-1$.

Proposition 3.9 *Let $J(G) \subseteq R$ be the cover ideal of a simple connected graph G. Let $x_n \in V_G$ be a vertex of degree $n-1$. Then, the Lyubeznik table of $R/J(G)$ is trivial.*

Proof We have a MV-splitting $J_G = L \cap K$ where $K = (x_1, x_n) \cap \cdots \cap (x_{n-2}, x_n)$ and, given the fact that all the vertices are in the neighbourhood of x_n, we have

$$L + K = \bigcap_{\substack{i,j \neq n \\ \{x_i,x_j\} \in E_G}} (x_i, x_j, x_n) = L + (x_n)$$

Recall that $\Lambda(R/K)$ is trivial so the long exact sequence associated with the MV-splitting reduces to

$$\cdots \longrightarrow H^p_{\mathfrak{m}}(H^r_{J(G)}(R)) \longrightarrow H^p_{\mathfrak{m}}(H^{r+1}_{L+(x_n)}(R)) \xrightarrow{\partial^r_p} H^{p+1}_{\mathfrak{m}}(H^r_L(R)) \longrightarrow \cdots$$

Using the interpretation of the connecting morphisms ∂^r_p's in terms of the corresponding linear strands given at the end of Sect. 2, we observe that we are comparing the linear strands $\mathbb{F}^{<r+1>}_\bullet((L + (x_n))^\vee)^*$ and $\mathbb{F}^{<r>}_\bullet(L^\vee)^*$ which are essentially the same (modulo a shifting), so the induced morphisms in homology are isomorphisms and the result follows. □

3.2 Examples

The MV-splitting techniques developed in the previous section allow us to compute the Lyubeznik table of many families of graphs directly from the combinatorics of the graph without an explicit computation of the corresponding local cohomology modules. The idea is to choose a convenient splitting vertex and reduce the computation to the case of a graph in a smaller number of vertices.

An interpretation of Proposition 3.5 is that the Lyubeznik table remains invariant under the operation of removing whiskers. In this way, we may simplify our original graph and, in the case of acyclic graphs, we can deduce the triviality of the Lyubeznik table by reducing the computation to the case of a single edge. In particular, we get:

Corollary 3.10 *The Lyubeznik table of the cover ideal of a* **path** *is trivial.*

Corollary 3.11 *The Lyubeznik table of the cover ideal of a* **tree** *is trivial.*

Using Corollary 2.6, we can also consider the case of forests.

Corollary 3.12 *The Lyubeznik numbers of the cover ideal of a* **forest** *with c connected components are*

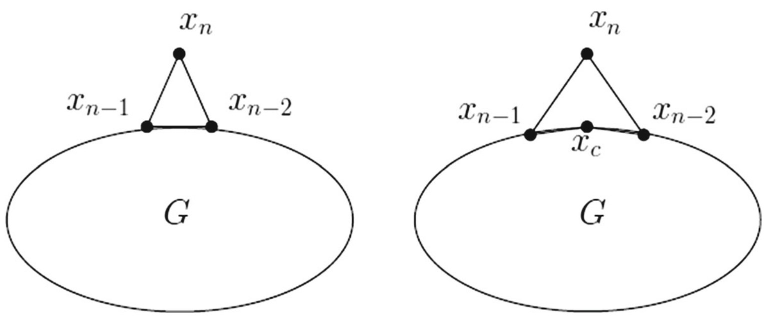
$$\lambda_{d-2k,d-k}(R/J(G)) = \binom{c}{k+1} \quad \text{for} \quad k = 0, \ldots, c-1$$

and the rest of Lyubeznik numbers are zero.

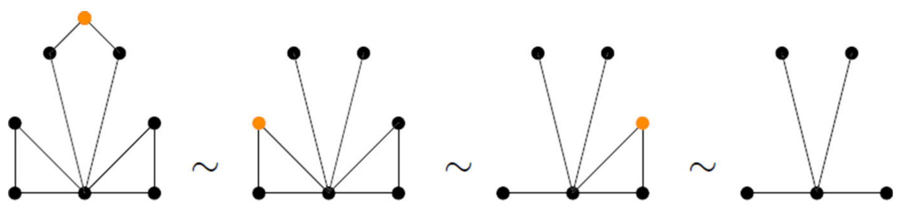
Another source of examples of trivial Lyubeznik tables is using Proposition 3.9. It says that the **cone** of any graph G has a trivial Lyubeznik table. In particular,

Corollary 3.13 *The Lyubeznik table of the cover ideal of a* **wheel** *is trivial.*

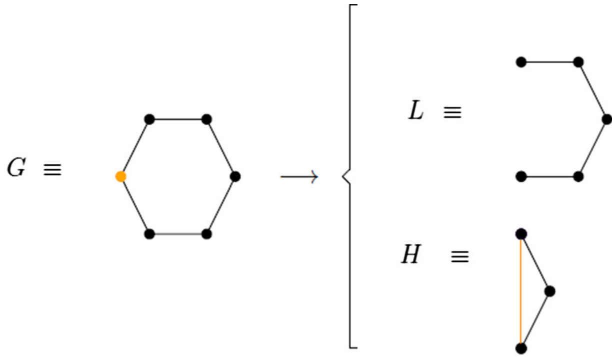
Another way of simplifying our original graph is by means of Proposition 3.6, which says that we can remove what we call **handles** (or equivalently 3− and 4-cycles) having the following shape:



This gives us a very visual method to reduce the computation of Lyubeznik tables of graphs. For example, removing the yellow vertices indicated below does not modify the Lyubeznik table, and thus, in the end, we see that the Lyubeznik table of the following graph is trivial.



Now we turn our attention to the case of cycles. Applying iteratively Corollary 3.8, we will obtain a closed formula for the Lyubeznik numbers. To illustrate our methods, we present the case of a 6-cycle. The corresponding graphs L and H are represented as follows:



Notice that L is a path so it has trivial Lyubeznik table, and thus, we can use Corollary 3.8. Moreover, we have that H is a 3-cycle.

Proposition 3.14 *The Lyubeznik numbers of the cover ideal of a n-**cycle** with $n = 3k + \ell$, $\ell \in \{-1, 0, 1\}$ are*

$$\lambda_{d-3i,d-i}(R/J(G)) = 1, \quad \lambda_{d-3i-1,d-i}(R/J(G)) = 1 \quad \text{for} \quad i = 0, \dots, k-2.$$
$$\lambda_{d-3i,d-i}(R/J(G)) = 1 \quad \text{for} \quad i = k-1.$$

and the rest of Lyubeznik numbers are zero.

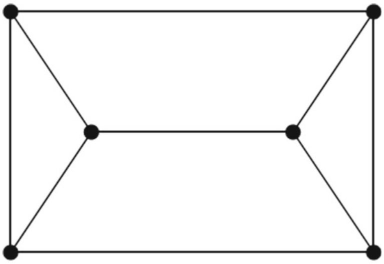
Proof Notice that, if we denote G the n-cycle, then $G \setminus \{x_n\}$ is a path so its Lyubeznik table is trivial. On the other hand, the graph H obtained from G is a $(n-3)$-cycle so we can use induction on k and Corollary 3.8 to produce the formula for the Lyubeznik numbers. □

Example 3.15 The Lyubeznik table of a cycle C_n in n vertices for $n = 5, \dots, 11$ is:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 1 \\ & & & & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 1 \\ & & & & & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 1 \\ & & & & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ &0&0&0&0&1&0&0\\ &&0&0&0&0&0&0\\ &&&0&0&0&1&0\\ &&&&0&0&1&0\\ &&&&&0&0&0\\ &&&&&&0&1\\ &&&&&&&1 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ &0&0&0&0&0&0&0&0\\ &&0&0&0&0&1&0&0\\ &&&0&0&0&0&0&0\\ &&&&0&0&0&1&0\\ &&&&&0&0&1&0\\ &&&&&&0&0&0\\ &&&&&&&0&1\\ &&&&&&&&1 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&1&0&0&0\\ &0&0&0&0&0&0&0&0&0\\ &&0&0&0&0&0&1&0&0\\ &&&0&0&0&0&1&0&0\\ &&&&0&0&0&0&0&0\\ &&&&&0&0&0&1&0\\ &&&&&&0&0&1&0\\ &&&&&&&0&0&0\\ &&&&&&&&0&1\\ &&&&&&&&&1 \end{pmatrix}$$

Certainly, a new source for finding non-trivial Lyubeznik tables is to consider graphs obtained by joining cycles. After removing whiskers and handles, we reduce to the case of cycles joined by paths or sharing edges in such a way that we can still find degree two vertices that we can remove in order to simplify the graph. For example, the complement of a 6-cycle



can be interpreted as joining two 3-cycles and two 4-cycles. However, all the vertices have degree three so we cannot apply our methods.

To start our study, we will consider the following joining operation.

Definition 3.16 Let C_m and C_n be two cycles in m and n vertices, respectively. We say that they are L-joined and we will denote by $C_m \circledast C_n$ the corresponding graph if they share at most one edge or if they are joined by a path.

When we remove a vertex from $C_m \circledast C_n$, we do not obtain a subgraph L with trivial Lyubeznik table. However, we still can use Proposition 3.7 to obtain the following formula for the Lyubeznik numbers.

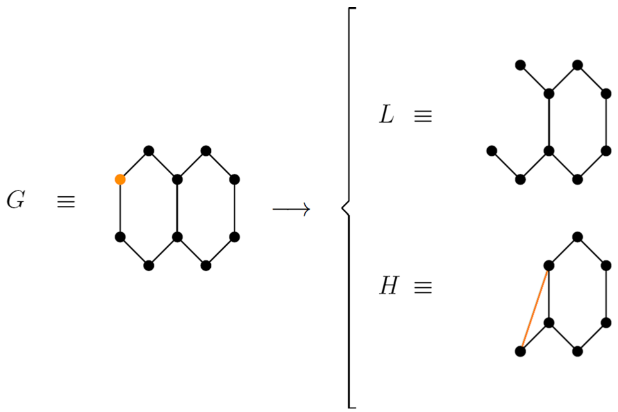
Proposition 3.17 *Let C_m and C_n be two cycles with $m \leq n$ and $m = 3k_1 + \ell_1$, $n = 3k_2 + \ell_2$ and $\ell_1, \ell_2 \in \{-1, 0, 1\}$. Then the Lyubeznik numbers of the cover ideal of $G = C_m \circledast C_n$ are:*

- *For $i = 0, \dots, k_1 - 2$*
  $\lambda_{d-3i,d-i}(R/J(G)) = i + 1, \qquad \lambda_{d-3i-1,di}(R/J(G)) = i + 2$
- *For $i = k_1 - 1, \dots, k_2 - 2$*
  $\lambda_{d-3i,d-i}(R/J(G)) = k_1, \qquad \lambda_{d-3i-1,di}(R/J(G)) = k_1$
- *For $i = k_2 - 1, \dots, k_1 + k_2 - 2$*
  $\lambda_{d-3i,d-i}(R/J(G)) = k_1 + k_2 - i - 1, \qquad \lambda_{d-3i-1,d-i}(R/J(G)) = k_1 + k_2 - i - 2$

and the rest of Lyubeznik numbers are zero. Here we follow the convention that in the case where $k_1 = k_2$ the set of indices $i = k_1 - 1, \ldots, k_2 - 2$ is empty.

Proof We are going to use Proposition 3.7 so we will follow the same terminology used in its proof. Pick a splitting vertex x_m of degree two in C_m. Notice that $G \setminus \{x_m\}$ has the same Lyubeznik table as C_n and the graph H associated with G is $C_{m-3} \circledast C_n$ so we can proceed by induction on k_1. The fact that C_m and C_n share at most one edge implies that the induction step will end up in a graph H having the Lyubeznik table of C_n. □

We illustrate the case $G = C_6 \circledast C_6$ as follows:



More generally, we can consider the family of graphs obtained using the $\circledast$ operation, which is a family that includes **cycles with chords** or **cactus graphs**. We can iterate the methods used in Proposition 3.17 in order to compute the Lyubeznik table of any L-joined graph of the form $C_{n_1} \circledast \cdots \circledast C_{n_r}$. We are not going to give a closed formula for its Lyubeznik table since it depends on the indices $n_i = 3k_i + \ell_i$, with $\ell_i \in \{-1, 0, 1\}$, for $i = 1, \ldots, r$ and one should distinguish too many cases which makes it very tedious and not very illustrative.

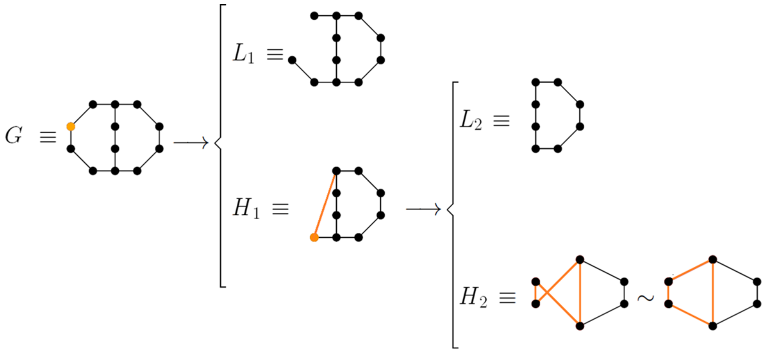
We will just point out that the non-vanishing Lyubeznik numbers are

$$\lambda_{d-3i,d-i}(R/J(G)), \qquad \lambda_{d-3i-1,d-i}(R/J(G)) \qquad \text{for } i = 0, \ldots, k_1 + \cdots + k_r - r$$

Moreover, we have $\lambda_{d,d} = 1$, $\lambda_{d-1,d} = r$ and $\lambda_{d-3(k_1+\cdots+k_r-r),d-(k_1+\cdots+k_r-r)} = 1$.

In the case that we have two cycles C_m and C_n sharing more than one edge, we can still use iteratively Proposition 3.7 to compute its Lyubeznik table. However, in this case we will not end up with a graph H having the Lyubeznik table of C_n as the following example shows.

Example 3.18 Let G be two 8-cycles sharing four consecutive vertices. The Lyubeznik table of G can be obtained, using Proposition 3.7, from the Lyubeznik tables of two graphs L_1 and H_1. Notice that L_1 has the same Lyubeznik type as an 8-cycle C_8, but in order to get the Lyubeznik table of H_1 we have to apply Proposition 3.7 once again. We illustrate the procedure as follows:



We have that L_2 is an 8-cycle so we know its Lyubeznik table by Proposition 3.14. On the other hand, H_2 is a graph in 6 vertices whose Lyubeznik table is trivial by using Proposition 3.6. Therefore, we can use Proposition 3.7 to compute the Lyubeznik table of H_1

$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ &0&0&0&0&1&0&0\\ &&0&0&0&0&0&0\\ &&&0&0&0&1&0\\ &&&&0&0&1&0\\ &&&&&0&0&0\\ &&&&&&0&1\\ &&&&&&&1 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&0\\ &0&0&0&0\\ &&0&0&0\\ &&&0&0\\ &&&&1 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} 0&0&0&0&0&0&0&0\\ &0&0&0&0&1&0&0\\ &&0&0&0&0&0&0\\ &&&0&0&0&1&0\\ &&&&0&0&2&0\\ &&&&&0&0&0\\ &&&&&&0&2\\ &&&&&&&1 \end{pmatrix}$$

From the Lyubeznik table of H_1 given above and the one for L_1, we deduce the Lyubeznik table of G.

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0\\ &0&0&0&0&0&0&0&0&0&0\\ &&0&0&0&0&0&0&0&0&0\\ &&&0&0&0&0&0&0&0&0\\ &&&&0&0&0&0&1&0&0\\ &&&&&0&0&0&0&0&0\\ &&&&&&0&0&0&1&0\\ &&&&&&&0&0&1&0\\ &&&&&&&&0&0&0\\ &&&&&&&&&0&1\\ &&&&&&&&&&1 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0\\ &0&0&0&0&0&0&1&0&0&0\\ &&0&0&0&0&0&0&0&0&0\\ &&&0&0&0&0&0&1&0&0\\ &&&&0&0&0&0&3&0&0\\ &&&&&0&0&0&0&0&0\\ &&&&&&0&0&0&3&0\\ &&&&&&&0&0&2&0\\ &&&&&&&&0&0&0\\ &&&&&&&&&0&2\\ &&&&&&&&&&1 \end{pmatrix}$$

Remark 3.19 The number of connected graphs in a given number of vertices is very large so it seems infeasible to find all the possible configurations of Lyubeznik tables. If the number of vertices is small, we can use our methods to reduce the number of examples we have to consider and compute the Lyubeznik numbers of the remaining graphs using the algorithm presented in [6].

Table 1 Number of graphs in n vertices with given Lyubeznik type

n	Trivial	Cycle	Complement cycle	Total
4	6	–	–	6
5	20	1	–	21
6	106	5	1	112

We made the computations in the case we have either 4, 5 or 6 vertices, and the possible Lyubeznik types are reflected in the following table. Not surprisingly, having trivial Lyubeznik table is the most common situation (Table 1).

In order to expand this list, it would be desirable to develop techniques to deal with splitting vertices of degree bigger than two and identify graphs with non-trivial Lyubeznik tables other than cycles or complement of cycles.

4 Bass numbers of local cohomology modules of cover ideals of graphs

Let G be a simple graph and, given $\alpha \in \{0, 1\}^n$, we denote by G_α the subgraph of G obtained by removing the vertices x_i such that $\alpha_i = 0$. Indeed, we will only consider those α's for which G_α is not a set of isolated vertices, which means that $E_{G_\alpha} \neq \emptyset$. In the case that G_α has an isolated vertex, we can assume that we are working in a smaller set of vertices by removing the isolated one. Notice that the cover ideal of G_α is an ideal in the polynomial ring $R_{\mathfrak{p}_\alpha} = \mathbb{K}[x_i \mid \alpha_i = 1]$. As we mentioned in Remark 2.3, Bass numbers behave well with respect to restriction, and thus, the Bass numbers of the local cohomology module $H^{n-i}_{J(G)}(R)$ with respect to the face ideal $\mathfrak{p}_\alpha$ are nothing but the Lyubeznik numbers corresponding to the subgraph G_α. More precisely,

$$\mu_p(\mathfrak{p}_\alpha, H^{n-i}_{J(G)}(R)) = \lambda_{p,i}(R_{\mathfrak{p}_\alpha}/J(G_\alpha))$$

Certainly, G_α are not necessarily connected graphs, and thus, one needs to use Proposition 2.5 in order to compute these Lyubeznik numbers. In what follows, we will denote c_α as the number of connected components of G_α and $c_{max} = \max\{c_\alpha \mid \alpha \in \{0, 1\}^n\}$.

Example 4.1 The maximal number of connected components of a path or a cycle is achieved when we remove every third vertex from the graph. For an n-path, we have $c_{max} = \lceil \frac{n+1}{3} \rceil$. However, for n-cycle we have to be a little more careful and we have $c_{max} = \lfloor \frac{n}{3} \rfloor$. In particular, for $n = 3k - 1$ we have that the n-path has $c_{max} = k$ and the n-cycle has $c_{max} = k - 1$. For the rest of the cases, they coincide.

4.1 Linear injective resolutions

Let G be a graph that $R/J(G)$ is Cohen–Macaulay. The Bass numbers of this class of graphs are completely determined as it has been shown in [8] using a simple spectral sequence argument. For completeness, we include the result here but giving an equivalent description in terms of the connectivity of the subgraphs.

Proposition 4.2 *Let $J(G) \subseteq R$ be the cover ideal of a simple connected graph G. Then, the following are equivalent*

i) $R/J(G)$ is Cohen–Macaulay.
ii) $\mu_p(\mathfrak{p}_\alpha, H^2_{J(G)}(R)) = \delta_{p,|\alpha|-2}$
iii) G_α is connected and $\Lambda(R_{\mathfrak{p}_\alpha}/J(G_\alpha))$ is trivial for all $\alpha \in \{0,1\}^n$.

In this case, the $\mathbb{Z}^n$-graded injective resolution $\mathbb{I}_\bullet(H^2_{J(G)}(R))$ has a very rigid structure which resembles the injective resolution of Gorenstein rings. Namely, we have:

$$0 \longrightarrow H^2_{J(G)}(R) \longrightarrow \bigoplus_{|\alpha|=2} E_\alpha \longrightarrow \bigoplus_{|\alpha|=3} E_\alpha \longrightarrow \cdots \longrightarrow$$

$$\longrightarrow \bigoplus_{|\alpha|=n-1} E_\alpha \longrightarrow E_{\mathbf{1}} \longrightarrow 0,$$

where at each component of the resolution we are only considering those α's such that $\mathfrak{p}_\alpha \in \operatorname{Supp} R/J(G)$. Notice that this resolution is linear, it only has one linear strand.

We will show next that we may find non-Cohen–Macaulay graphs still having a rigid injective resolution. To such purpose let us consider the following family.

Assumptions 4.3 Let $J(G) \subseteq R$ be the cover ideal of a simple (not necessarily connected) graph G such that, for any $\alpha \in \{0,1\}^n$, all the connected components of G_α have trivial Lyubeznik table.

As shown in Sect. 3, the condition of having trivial Lyubeznik table is very common and is not difficult to find families of graphs satisfying Assumptions 4.3. An interesting example would be the case of **forests**, but we may also include Cohen–Macaulay graphs. As a direct consequence of Corollary 2.6, we get:

Theorem 4.4 *Let $J(G) \subseteq R$ be an ideal satisfying Assumptions 4.3. Then, the Bass numbers of the corresponding local cohomology modules are*

$$\mu_p(\mathfrak{p}_\alpha, H^{k+1}_{J(G)}(R)) = \delta_{p,|\alpha|-2k} \cdot \binom{c_\alpha}{k} \quad \text{for} \quad k = 1, \ldots, c_\alpha,$$

where $\delta_{i,j}$ is Kronecker's delta function.

In this case, we also have a rigid injective resolution in the sense that they only have one linear strand.

Example 4.5 Let G be a 5-path. We have that all the subgraphs G_α without isolated vertices are connected except for the case $\alpha = (1,1,0,1,1)$ in which G_α has two connected components. Applying the above result, we get the following linear injective resolutions

$$0 \longrightarrow H^2_{J(G)}(R) \longrightarrow \bigoplus_{|\alpha|=2} E_\alpha \longrightarrow \bigoplus_{|\alpha|=3} E_\alpha \longrightarrow$$

$$\longrightarrow E^2_{(1,1,0,1,1)} \oplus \left(\bigoplus_{\substack{|\alpha|=4 \\ \alpha\neq(1,1,0,1,1)}} E_\alpha \right) \longrightarrow E_{\mathbf{1}} \longrightarrow 0,$$

$$0 \longrightarrow H^3_{J(G)}(R) \longrightarrow E_{(1,1,0,1,1)} \longrightarrow 0,$$

where we are only considering those α's such that $\mathfrak{p}_\alpha \in \mathrm{Supp} R/J(G)$. Notice that in this case, we have two local cohomology modules different from zero.

Quite surprisingly, we may provide a vanishing criterion for the local cohomology modules in terms of the connected components of the subgraphs. Recall that the **cohomological dimension** of an ideal J is the maximum r for which $H^r_J(R) \neq 0$.

Proposition 4.6 *Let $J(G) \subseteq R$ be an ideal satisfying Assumption 4.3. Then,*

$$H^{k+1}_{J(G)}(R)) \neq 0 \quad \text{for} \quad k = 1, \ldots, c_{max}.$$

In particular, the cohomological dimension of $J(G)$ is

$$\mathrm{cd}(J(G), R) = c_{max} + 1.$$

Proof A local cohomology module is different from zero if it has a non-vanishing Bass number. Then, the result follows from Theorem 4.4. □

Using the relation between the cohomological dimension and the projective dimension of the Alexander dual ideal given in [13], we deduce the following result which, in particular, gives a very simple description of the projective dimension of edge ideals of forests (compare with the results in [24]).

Proposition 4.7 *Let $J(G) \subseteq R$ be an ideal satisfying Assumption 4.3. Then, the projective dimension of the corresponding edge ideal $I(G)$ is*

$$\mathrm{pd}(R/I(G)) = c_{max} + 1.$$

4.2 Nonlinear injective resolutions

As we have seen through the examples in Sect. 3, the easiest way to find nonlinear injective resolutions is to consider the case of cycles. Let us illustrate this fact with the following examples.

Example 4.8 Let G be a 6-cycle. We have that all the subgraphs G_α are connected paths except for the cases $\alpha = (1, 1, 0, 1, 1, 0), (1, 0, 1, 1, 0, 1), (0, 1, 1, 0, 1, 1)$ in which G_α has two connected components. Therefore, we have the injective resolutions

$$0 \longrightarrow H^2_{J(G)}(R) \longrightarrow \bigoplus_{|\alpha|=2} E_\alpha \longrightarrow \bigoplus_{|\alpha|=3} E_\alpha \longrightarrow$$

$$\longrightarrow E^2_{(1,1,0,1,1,0)} \oplus E^2_{(1,0,1,1,0,1)} \oplus E^2_{(0,1,1,0,1,1)} \oplus \left(\bigoplus_{\substack{|\alpha|=4 \\ \text{rest of } \alpha' s}} E_\alpha \right) \longrightarrow$$

$$\longrightarrow \bigoplus_{|\alpha|=5} E_\alpha \oplus E_{\mathbf{1}} \longrightarrow E_{\mathbf{1}} \longrightarrow 0,$$

$$0 \longrightarrow H^3_{J(G)}(R) \longrightarrow E_{(1,1,0,1,1,0)} \oplus E_{(1,0,1,1,0,1)} \oplus E_{(0,1,1,0,1,1)} \longrightarrow E_{\mathbf{1}} \longrightarrow 0,$$

where we are only considering those α's such that $\mathfrak{p}_\alpha \in \operatorname{Supp} R/J(G)$. In this case, we have that both injective resolutions have two linear strands.

Example 4.9 Let G be a 6-wheel where x_6 is the dominating vertex. We have that all the subgraphs G_α are connected except for the case $\alpha = (1, 1, 1, 1, 1, 0)$ in which G_α is a 5-cycle. Therefore, we have the injective resolutions

$$0 \longrightarrow H^2_{J(G)}(R) \longrightarrow \bigoplus_{|\alpha|=2} E_\alpha \longrightarrow \bigoplus_{|\alpha|=3} E_\alpha \longrightarrow \begin{matrix} E_{(1,1,1,1,1,0)} \\ \oplus \\ \left(\bigoplus_{|\alpha|=4} E_\alpha \right) \end{matrix} \longrightarrow$$

$$\longrightarrow E_{(1,1,1,1,1,0)} \oplus \left(\bigoplus_{\substack{|\alpha|=5 \\ a_6=1}} E_\alpha \right) \longrightarrow E_1 \longrightarrow 0,$$

$$0 \longrightarrow H^3_{J(G)}(R) \longrightarrow E_{(1,1,1,1,1,0)} \longrightarrow 0,$$

where we are only considering those α's such that $\mathfrak{p}_\alpha \in \operatorname{Supp} R/J(G)$. The injective resolution of $H^2_{J(G)}$ has two linear strands.

More generally, we can consider the following family of examples.

Assumptions 4.10 Let $J(G) \subseteq R$ be the cover ideal of a simple (not necessarily connected) graph G that is obtained by joining cycles and paths in such a way that we can still find degree two vertices that we can remove in order to simplify the graph.

Even though it is not possible to give a closed formula for the Bass numbers $\mu_p(\mathfrak{p}_\alpha, H^{k+1}_{J(G)}(R))$ as the one given in Theorem 4.4, the methods developed in this work allow us to compute them. To do so, we must perform the following steps:

- Describe the connected components of G_α.
- Compute the Lyubeznik numbers of each component as in Sect. 3.
- Apply Proposition 2.5.

We can also discuss the vanishing of local cohomology modules depending on the connected components of the corresponding subgraphs, but the results are not going to be as clean as in Proposition 4.6. Indeed, using Proposition 3.14 and Example 4.1, we deduce the following formula for the case of cycles.

Proposition 4.11 *Let G be a cycle of the form C_{3k-1}. Then,*

$$H^{k+1}_{J(G)}(R)) \neq 0 \quad \text{for} \quad k = 1, \dots, c_{max} + 1.$$

On the other hand, if G is a cycle of the form C_{3k} or C_{3k+1} we have

$$H^{k+1}_{J(G)}(R)) \neq 0 \quad \text{for} \quad k = 1, \dots, c_{max}.$$

Some partial results that we can provide are the following.

Proposition 4.12 *Let G be a graph in n vertices and assume that, given $\alpha \in \{0, 1\}^n$, the subgraph G_α has r connected components having the Lyubeznik type of cycles $C_{n_1}, \dots, C_{n_r}$ and s connected components having trivial Lyubeznik table. Assume that $n_j = 3k_j + \ell_j$ with $\ell_j \in \{-1, 0, 1\}$. Then, if we denote $d_\alpha := |\alpha| - 2 = \dim(R_{\mathfrak{p}_\alpha}/J(G_\alpha))$, we have*

$$H^{d_\alpha - i}_{J(G_\alpha)}(R_{\mathfrak{p}_\alpha})) \neq 0 \quad \text{for} \quad i = 0, \dots, (k_1 + \dots + k_r + s) - 1.$$

In particular, the cohomological dimension of $J(G_\alpha)$ is

$$\mathrm{cd}(J(G_\alpha), R_{\mathfrak{p}_\alpha}) = |\alpha| - (k_1 + \dots + k_r + s) - 1.$$

Proof Consider the decomposition $G_\alpha = G_1 \cup G_2$ where $\alpha = \alpha_1 + \alpha_2$ and G_1 is the subgraph in $|\alpha_1|$ vertices containing the components corresponding to cycles and G_2 is the subgraph in $|\alpha_2|$ vertices containing the rest. For G_2, as a consequence of Corollary 2.6, we have

$$H^{d_2 - i}_{J(G_2)}(R_{\mathfrak{p}_{\alpha_2}})) \neq 0 \quad \text{for} \quad i = 0, \dots, s - 1,$$

where $d_2 = |\alpha_2| - 2$.

On the other hand, using Corollaries 3.14 and 2.7, we have that the smallest i for which the Lyubeznik number $\lambda_{p,i}$ corresponding to G_1 is nonzero is

$$i = (n_1 - 2) - (k_1 - 1) + \dots + (n_r - 2) - (k_r - 1) + (r - 1)$$

$$= (n_1 + \cdots + n_r) - 2 - 2(r-1) - (k_1 + \cdots + k_r) + r + (r-1)$$
$$= |\alpha_1| - 2 - (k_1 + \cdots + k_r) + 1$$

Moreover, since the non-vanishing local cohomology modules of a cycle are consecutive, that is $H^k_{J(C_n)}(R) \neq 0$ where k runs from two to the cohomological dimension, it follows from Proposition 2.5 that the same consecutiveness property holds for G_1. Namely, we have

$$H^{d_1-i}_{J(G_1)}(R_{\mathfrak{p}_{\alpha_1}}) \neq 0 \quad \text{for} \quad i = 0, \ldots, (k_1 + \cdots + k_r) - 1,$$

where $d_1 = |\alpha_1| - 2$.

Finally, applying Corollary 2.7 for G_1 and G_2, we have that the smallest i for which $\lambda_{p,i}(R_{\mathfrak{p}_\alpha}/J(G_\alpha)) \neq 0$ is

$$\begin{aligned} i &= (|\alpha_1| - 2 - (k_1 + \cdots + k_r) + 1) + (|\alpha_2| - 2 - (s-1)) + 1 \\ &= |\alpha_1| + |\alpha_2| - 2 - (k_1 + \cdots + k_r + s) + 1 \\ &= |\alpha| - 2 - (k_1 + \cdots + k_r + s) + 1 \\ &= d_\alpha - (k_1 + \cdots + k_r + s) + 1 \end{aligned}$$

Once again Proposition 2.5 gives the consecutiveness of the non-vanishing local cohomology modules, and the result follows. □

More generally, and using the same type of arguments as above, we have.

Proposition 4.13 *Let G be a graph in n vertices and assume that, given $\alpha \in \{0,1\}^n$, the subgraph G_α has r connected components having the Lyubeznik type of cycles $C_{n_{1,i}} \circledast \cdots \circledast C_{n_{t_i,i}}$, for $i = 1, \ldots, r$ and s connected components having trivial Lyubeznik table. Assume that $n_{j,i} = 3k_{j,i} + \ell_{j,i}$ with $\ell_{j,i} \in \{-1, 0, 1\}$. Then, if we denote $d_\alpha := |\alpha| - 2 = \dim(R_{\mathfrak{p}_\alpha}/J(G_\alpha))$, we have*

$$H^{d_\alpha - i}_{J(G_\alpha)}(R_{\mathfrak{p}_\alpha})) \neq 0$$

for $i = 0, \ldots, (k_{1,1} + \cdots + k_{t_1,1} + \cdots + k_{1,r} + \cdots + k_{t_r,r} + s) - (t_1 + \cdots + t_r) + (r-1)$.

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