Global Hamiltonian dynamics on singular symplectic manifolds

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Summary

In this thesis, we study the Reeb and Hamiltonian dynamics on singular symplectic and contact manifolds. Those structures are motivated by singularities coming from classical mechanics and fluid dynamics.

We start by studying generalized contact structures where the non-integrability condition fails on a hypersurface, the critical hypersurface. Those structures, called b-contact structures, arise from hypersurfaces in b-symplectic manifolds that have been previously studied extensively in the past. Formerly, this odd-dimensional counterpart to b-symplectic geometry has been neglected in the existing vast literature. Examples are given and local normal forms are proved. The local geometry of those manifolds is examined using the language of Jacobi manifolds, which provides an adequate set-up and leads to understanding the geometric structure on the critical hypersurface. We further consider other types of singularities in contact geometry, as for instance higher order singularities, called b^m-contact forms, or singularities of folded type.

Obstructions to the existence of those structures are studied and the topology of b^m-contact manifolds is related to the existence of convex contact hypersurfaces and further relations to smooth contact structures are described using the desingularization technique.

We continue examining the dynamical properties of the Reeb vector field associated to a given b^m-contact form. The relation of those structures to celestial mechanics underlines the relevance for existence results of periodic orbits of the Hamiltonian vector field in the b^m-symplectic setting and Reeb vector fields for b^m-contact manifolds. In this light, we prove that in dimension 3, there are always infinitely many periodic Reeb orbits on the critical surface, but describe exam-

ples without periodic orbits away from it in any dimension. We prove that there are traps for this vector field and discuss possible extensions to prove the existence of plugs. We will see that in the case of overtwisted disks away from the critical hypersurface and some additional conditions, Weinstein conjecture holds: more precisely there exists either a periodic Reeb orbit away from the critical hypersurface or a 1-parametric family in the neighbourhood of it. The mentioned results shed new light towards a singular version for this conjecture.

The obtained results are applied to the particular case of the restricted planar circular three body problem, where we prove that after the McGehee change, there are infinitely many non-trivial periodic orbits at the manifold at infinity for positive energy values.

Resumen

En esta tesis, estudiamos la dinámica de Reeb y Hamiltoniana en variedades simplécticas y de contacto con singularidades. El estudio de estas variedades está motivado por singularidades que tienen su origen en la mecánica clásica y la dinámica de fluidos.

Empezamos estudiando una generalización de las estructuras de contacto, en la cual la condición de no integrabilidad falla en una hipersuperficie, llamada la hipersuperficie crítica. Estas estructuras geométricas, llamadas estructuras de b-contacto, surgen de hipersuperficies en variedades b-simplécticas, estudiadas en el pasado. Hasta el momento, este equivalente de dimensión impar de la geometría b-simpléctica ha sido desatendido en la literatura existente. Después de los primeros ejemplos, probamos la existencia de formas locales. Estudiamos la geometría local de estas variedades usando el lenguaje de variedades de Jacobi, que resultan ser técnicas adecuadas para entender la estructura geométrica en la hipersuperficie crítica. Consideramos también singularidades de orden superior, formas de b^m-contacto, y singularidades de tipo *folded*.

Continuamos con el estudio de las obstrucciones a la existencia de estas estructuras y relacionamos la topología de variedades de b^m-contacto con la existencia de hipersuperficies convexas. Describimos relaciones entre formas de b^m-contacto y formas de contacto diferenciables usando técnicas de desingularización.

Examinamos las propiedades del campo de Reeb asociado a una forma de b^m-contacto dada. La relación de estas estructuras con la mecánica celeste pone en relieve la importancia del estudio de órbitas periódicas de este campo vectorial. Comprobamos que, en dimensión 3, el campo de Reeb en la hipersuperficie crítica admite infinitas

órbitas periódicas. Sin embargo, describimos ejemplos sin órbitas periódicas fuera de la hipersuperficie crítica en cualquier dimensión. Comprobamos la existencia de *traps* y discutimos la posible existencia de *plugs*. En el caso de un disco *overtwisted* fuera de la hipersuperficie se satisface la conjetura de Weinstein: en concreto, o bien existe una órbita periódica de Reeb fuera de la hipersuperficie de contacto o bien existe una familia de órbitas periódicas en un entorno de la hipersuperficie. Estos resultados sugieren una versión singular de dicha conjetura.

Aplicamos los resultados obtenidos al caso del problema de los tres cuerpos restringido circular: comprobamos que después del cambio de coordenadas de McGehee, existen infinitas órbitas periódicas en la variedad en el infinito para valores positivos de la energía.

Chapter 1

Introduction

Ce qui nous rend ces solutions périodiques si précieuses, c'est quelles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable.
Henri Poincaré, *Les méthodes nouvelles de la mécanique céleste*

The branch of symplectic and contact geometry emerged as a set-up for the study of classical Hamiltonian systems, as for instance celestial mechanics. The equations of motion of the Hamiltonian system can in this light be geometrically interpreted as the flow of the Hamiltonian vector field associated to the smooth energy function H, called Hamiltonian, on a symplectic manifold (W, ω). In view of the epigraph due to Henri Poincaré, the holy grail in Hamiltonian systems is the existence of periodic orbits.

Symplectic and contact geometry flourished to an independent field of research over the last decades. The quest for periodic orbits motivated several important conjectures that gave rise to spectacular development. One such example is the Weinstein conjecture asserting that for a compact contact manifold, there always exists a periodic orbit of an intrinsically defined vector field, called the Reeb vector field. For Hamiltonian systems, this is saying that the Hamiltonian vector field always admits a periodic orbit on a special kind of level-sets. In the view of the chronological order of appearance of the theories, it is natural to apply the known cases of this conjecture and the developed techniques to the initial context, as for instance to examples of celestial mechanics.

The three body problem consists in studying the dynamics of three massive bodies in space whose motion is determined by Newton's law. A simplification of this problem is obtained assuming that one of the bodies is massless and therefore does not alter the motion of the two others. This configuration models the movement of the earth, sun and a satellite, which corresponds to the massless body and whose periodic orbits one would like to compute. For simplicity, we further assume that the motion of the earth and the moon is described by circular motions around their center of gravity and that the movement of the satellite is bound to happen in the same plane. This problem is classically known as the restricted planar circular three body problem.

In [AFKP12], the authors study the level-set of the Hamiltonian for low energy values and use classical regularization methods to regularize collision of the satellite with the earth. They show that a connected component of the regularized level-set of H is a compact contact manifold whose associated Reeb vector field describes the Hamiltonian dynamics. As a consequence of the positive answers to the Weinstein conjecture, there exist periodic orbits of the Hamiltonian vector field on that regularized level-set.

Due to collision, the existence of the periodic orbit is guaranteed

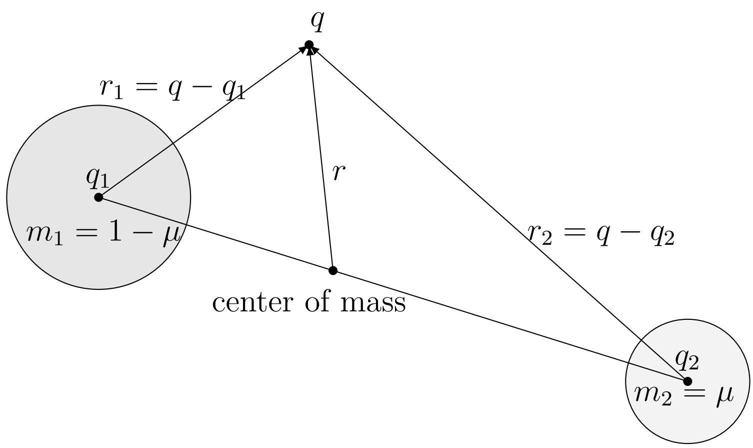


Figure 1.1: The three-body problem: Earth-Moon-Satellite model.

only on the regularized level-set. Ignoring those classical regularization techniques yield singularities in the symplectic form and therefore classical results from symplectic geometry do not apply. More precisely, while the Weinstein conjecture on contact manifolds is well-studied and fairly well understood (see [Hof93, Tau07]) there is an important gap in the current state of art, namely the consideration of singularities in those theories.

We now describe the singularities occurring in the planar restricted three body problem as in [BDM+19]. The Hamiltonian of the satellite is given by

$$H(q,p,t) = \frac{|p|^2}{2} - U(q,t), (q,p) \in \mathbb{R}^2 \times \mathbb{R}^2,$$

where the potential is given by

$$U(q,t) = \frac{1-\mu}{|q-q_1|} + \frac{\mu}{|q-q_2|},$$

where $q_1 = q_1(t)$ the position of the planet with mass $1-\mu$ at time t and $q_2 = q_2(t)$ the position of the one with mass μ. Passing to polar coordinates through a symplectic coordinate change (that is changing the momenta accordingly) given by

$$(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$$

followed by the classical McGehee blow-up (x, α, y, G), where $r = \frac{2}{x^2}, x \in \mathbb{R}^+$, the standard symplectic form on the cotangent bundle of $\mathbb{R}^2$ gives then rise to a geometric structure which is singular. It is given by

$$-\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG.$$

This non-smooth differential form is symplectic away from the line at infinity but is singular on the hypersurface given by $\{x = 0\}$.

Those geometric structures are called b^3-symplectic structures and play the central theme in this thesis. Over the last decade, the geometry and topology of b^m-symplectic manifolds has developed into an active field of research. However, the study of the odd-dimensional counterpart has been neglected. As motivated by this example, the understanding of the global Hamiltonian dynamics on b^m-symplectic manifolds are of fundamental importance, and in particular the one on level-sets of b^m-symplectic manifolds.

This thesis main goal is therefore to consider hypersurfaces of contact type in b^m-symplectic manifolds. The so obtained geometric structures are called b^m-contact structures and this thesis is a first step towards filling the gap in the existent literature of understanding the dynamics on symplectic and contact manifold in the presence of singularities.

We study the local properties of those structures, as well as their topology. Last but not least, we investigate the dynamical properties of the associated Reeb vector field of those and apply the results to the motivating examples of the planar restricted circular three body problem described above. We give a more detailed outline of the results below.

1.1 Structure and results of this thesis

We outline the content of each of the chapters of this thesis and underline the main results.

1.1.1 Chapter 2: Preliminaries

We introduce the geometric structures studied in this thesis, that is symplectic, contact, Poisson and Jacobi manifolds. We give a biased review on the most important results on contact topology and Reeb and Hamiltonian dynamics. We end this chapter with an introduction to b^m-symplectic geometry and more general singular symplectic structures, called E-symplectic structures.

1.1.2 Chapter 3: b^m-Contact geometry

This chapter is devoted to the study of the geometry of contact structures that cease to be contact along a hypersurface, but where some degeneracy conditions are met. This constitutes the odd-dimensional analogue to b^m-symplectic manifold. Geometrically, the hyperplane distribution becomes tangent to the given hypersurface.

Definition 1.1.1 (Definition 3.0.1). Let $(M$ be a (2n+1)-dimensional manifold and $Z \subset M$ a given hypersurface. A b-contact structure is the distribution given by the kernel of a one b-form $\xi = \ker \alpha \subset {}^bTM$, $\alpha \in {}^b\Omega^1(M)$, that satisfies $\alpha \wedge (d\alpha)^n \neq 0$ as a section of $\Lambda^{2n+1}({}^bT^*M)$.

We say that α is a b-contact form and the pair (M, ξ) a b-contact manifold.

Associated to a b-contact form, there exists a unique vector field, called the Reeb vector field, defined by the following equations:

$$\begin{cases} \iota_{R_\alpha} d\alpha = 0 \\ \alpha(R_\alpha) = 1. \end{cases}$$

We give first examples and prove a local normal theorem for those structures.

Theorem A (Theorem 3.1.1)**.** *Let α be a b-contact form inducing a b-contact structure ξ on a b-manifold (M, Z) of dimension $(2n+1)$ and $p \in Z$. There exists a local chart $(\mathcal{U}, z, x_1, y_1, \dots, x_n, y_n)$ centred at p such that on $\mathcal{U}$ the hypersurface Z is locally defined by $z = 0$ and*

1. *if $R_p \neq 0$*

   (a) *ξ_p is singular, then*

   $$\alpha|_{\mathcal{U}} = dx_1 + y_1 \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

   (b) *ξ_p is regular, then*

   $$\alpha|_{\mathcal{U}} = dx_1 + y_1 \frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

2. *if $R_p = 0$, then $\tilde{\alpha} = f\alpha$ for $f(p) \neq 0$, where*

   $$\tilde{\alpha}_p = \frac{dz}{z} + \sum_{i=1}^{n} x_i dy_i.$$

We study b^m-contact manifolds in the language of Jacobi manifolds and prove that they are dual to Jacobi manifolds that satisfy certain transversality conditions.

Theorem B (Theorem 3.2.5). *Let $(M^{2n+1}, \ker \alpha)$ be a b-contact manifold. Then (M, Λ, R) is a Jacobi manifold, where Λ is the bi-vector field computed as in Equation 2.3, satisfying the following transversality conditions $\Lambda^n \wedge R \pitchfork 0$. Conversely, any odd-dimensional Jacobi manifold that satisfies that transversality condition defines a b-contact structure.*

Viewing b-contact manifolds as a particular case of Jacobi manifolds opens the door to study the induced structure of the b-contact structure on the critical set. Indeed, the Jacobi structure induces a characteristic foliation on the manifold. Depending on the parity of the dimension of the leaf, the induced structure is either contact in the odd-dimensional case or locally conformally symplectic in the even-dimensional case. In the case of a b-contact manifold, the induced structure on the critical set is as follows.

Theorem C (Theorem 3.3.1). *Let $(M^{2n+1}, \xi = \ker \alpha)$ be a b-contact manifold and $p \in Z$. We denote $\mathcal{F}_p$ the leaf of the singular foliation $\mathcal{F}$ going through p. Then*

1. *if ξ_p is regular, that is $\mathcal{F}_p$ of dimension $2n$, then the induced structure on $\mathcal{F}_p$ is locally conformally symplectic;*
2. *if ξ_p is singular, that is $\mathcal{F}_p$ of dimension $2n-1$, then the induced structure on $\mathcal{F}_p$ is contact.*

We finish this chapter by describing other singularities in contact geometry, as for instance folded contact structures.

1.1.3 Chapter 4: Obstructions and constructions on b^m-symplectic and b^m-contact manifolds

This chapter is devoted to topological constructions and the study of obstruction theory of b^m-contact and b^m-symplectic manifolds.

We open the chapter by studying the cohomological obstructions to the existence of b^m-symplectic manifolds.

We related the study of b^m-contact manifolds to the study of smooth contact geometry. In the case of even powers, we prove the following.

Theorem D (Theorem 4.2.4). *Let $(M^{2n+1}, \ker \alpha)$ a b^{2k}-contact structure with critical hypersurface Z. Assume that α is almost convex. Then there exists $\epsilon_0 > 0$ and a family of usual contact forms α_ϵ for all $\epsilon \in (0, \epsilon_0)$ which coincides with the b^{2k}-contact form α outside of an ϵ-neighbourhood of Z. The family of bi-vector fields $\Lambda_{\alpha_\epsilon}$ and the family of vector fields R_{α_ϵ} associated to the Jacobi structure of the contact form α_ϵ converges to the bivector field Λ_α and to the vector field R_α in the C^{2k-1}-topology as $\epsilon \to 0$.*

A similar result is proved for b^{2k+1}-contact structures, where the desingularization yields folded contact structures, see Theorem 4.2.5.

We prove that convex hypersurfaces in contact manifolds can be realized as critical sets of b^m-contact manifolds. In the case of b^{2k}-contact structures, we prove the following theorem.

Theorem E (Theorem 4.3.1). *Let (M, ξ) be a contact manifold and let Z be a convex hypersurface in M. Then M admits a b^{2k}-contact structure for all $k \in \mathbb{N}^*$ that has Z as critical set. The dividing set of the convex hypersurface $\Sigma \subset Z$, a codimension 2 submanifold, cor-*

responds to the set where the rank associated b^{2k}-contact distribution drops and the induced structure is contact.

As before, a similar result holds for b^{2k+1}-contact structures, where we consider two connected components of the hypersurface to overcome orientation issues, see Theorem 4.3.5.

In particular, in the 3-dimensional case, every generic surface is the critical surface of a b^m-contact manifold (Corollary 4.3.2). A combination of the before mentioned results yield the generic existence of folded contact forms in dimension 3 (Corollary 4.3.8).

Similar techniques yield the realization of b^m-symplectic manifolds for cosymplectic hypersurfaces in symplectic manifolds.

Theorem F (Theorem 4.4.1)**.** *Let $Z \subset (W^{2n}, \omega)$ be a hypersurface in a symplectic manifold and assume that there exists a symplectic vector field X that is transverse everywhere to Z. Then Z can be realized as the critical hypersurface of a family of b^{2k}-symplectic structures.*

Once more, a similar result is proven for b^{2k+1}-symplectic structures.

1.1.4 Chapter 5: b^m-Reeb dynamics and the singular Weinstein conjecture

We study the dynamics of the Reeb vector field on b^m-contact manifolds.

We show that in dimension 3, there are always infinitely many periodic Reeb orbits on the critical set.

Theorem G (Proposition 5.1.1)**.** *Let (M, α) be a 3-dimensional b^m-contact manifold and assume the critical hypersurface Z to be closed.*

Then there exists infinitely many periodic Reeb orbits on Z.

In contrast to this, we show that there are examples in any dimension without periodic orbits away from Z.

In order to produce counter-examples to existence results of periodic orbits, a common strategy is to introduce local modifications to the flow of the given vector field to destroy a finite number of periodic orbits. This construction is known in the literature as trap construction. A plug additionally demands that no new periodic orbits are created. See Subsection 2.4.3 for precise definitions.

Based on the existence result proved in Chapter 4, we prove that there are traps for the flow of the Reeb vector field in any dimension.

Theorem H (Theorem 5.2.1)**.** *There exist b^m-contact traps in any dimension.*

We continue by discussing possible approaches to strengthen this result that potentially could lead to the existence of plugs for the b^m-Reeb flow.

We further apply Hofer's methods to open overtwisted contact manifold that are $\mathbb{R}^+$-invariant in the open ends. More precisely, the $\mathbb{R}^+$-action originates from a strict contact vector field that exists in the open ends, that it is transverse to the boundary of the open ends and complete for positive time, see Definition 5.5.4. We prove that in the presence of an overtwisted disk there is either a periodic orbit in away from the $\mathbb{R}^+$-invariant part or a 1-parametric family of periodic Reeb orbits see Theorem 5.5.8. As a consequence of this more general theorem, we obtain the following result.

Theorem I (Theorem 5.5.2)**.** *Let (M, α) be a compact b^m-contact* *manifold with critical set Z. Assume there exists an overtwisted disk in $M \setminus Z$ and assume that α is $\mathbb{R}^+$-invariant in a tubular neighbourhood around Z. Then at least one of the following statement holds:*

1. *There exists a periodic Reeb orbit in $M \setminus Z$.*
2. *There exists a family of periodic Reeb orbits approaching the critical set Z.*

Furthermore, the periodic orbits are contractible loops in the symplectization.

We examine possible applications of our techniques to a conjecture on the existence on infinitely many periodic Reeb orbits on overtwisted contact manifolds.

By the aforementioned results, we are lead to formulate a generalization of Weinstein conjecture for b^m-contact manifolds, that conjectures the existence of singular periodic orbits away from the critical hypersurface.

1.1.5 Chapter 6: Applications of b^m-contact geometry: the three body problem and fluid dynamics

We end this thesis with applications of the results mentioned in the previous chapters to a particular case of physical systems: celestial mechanics and fluid dynamics.

In the case of the restricted planar circular three body problem, we will prove that the dynamics are described by a b^3-Reeb vector field on positive energy level-sets and that there are infinitely many periodic orbits on the critical set.

Theorem J (Theorem 6.2.2)**.** *After the McGehee blow-up, the Liouville vector field Y in the fibres of the phase-space is a b^3-vector field that is everywhere transverse to energy level-set $\Sigma_c = H^{-1}(c)$ for $c > 0$, where H denotes the Hamiltonian associated to the RPC3BP. The level-sets $(\Sigma_c, \iota_Y \omega)$ for $c > 0$ are b^3-contact manifolds. Topologically, the critical set is a cylinder and the Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.*

Due to non-compactness of the critical set Z, the proof does not use directly Theorem G, but does use that the Reeb vector field on the critical set is a Hamiltonian vector field (Corollary 3.1.5).

In the case of fluid dynamics, we show that so-called rotational b-Beltrami vector fields exhibit different dynamical behavior than rotational Beltrami vector field by means of studying the dynamics of the latter ones on an invariant hypersurface. We end this thesis with the description of possible further applications of the theory of b^m-contact forms to fluid dynamics.

1.2 Publications resulting from this thesis

The results mentioned above can be found in the following articles:

- *An invitation to singular symplectic geometry*, joint with Roisin Braddell, Amadeu Delshams, Eva Miranda, and Arnau Planas. International Journal of Geometric Methods in Modern Physics 16, no. supp01 (2019): 1940008.

- *The geometry and topology of contact structures with singularities*, joint with Eva Miranda, arXiv 1806.05638, (submitted for publication).

- *The singular Weinstein conjecture*, joint with Eva Miranda, arXiv 2005.09568.

and the following conference proceedings:

- *Do overtwisted contact manifolds admit infinitely many periodic Reeb orbits?*, to appear in Extended Abstract, Birkhäuser.

Chapter 2

Preliminaries

"El mundo es eso – reveló –. Un montón de gente, un mar de fueguitos."
Eduardo Galeano, *El libro de los abrazos.*

In this section we introduce the main objects of this thesis. We will first briefly introduce two well-known geometric structures: symplectic and contact structures. We assume the reader familiar with the basic notions and only include the necessary material to make this thesis somewhat self-contained. We include a section on contact topology, especially focusing on the theory of convex contact geometry.

We include a section on dynamical results concerning the Reeb and Hamiltonian vector field.

We then introduce two generalizations of those geometric structures, namely Poisson and Jacobi manifolds. We analyse the interplay between those geometric structures.

Motivated by the examples in the introduction, we introduce certain kinds of mild singularities of symplectic structures, called b^m-symplectic structures and will see that they correspond to a particular case of Poisson manifolds.

The next chapter then consists in introducing those mild singularities on contact manifolds. We remark that the dynamical understanding of Poisson manifolds, or even Jacobi manifolds, are out of reach, as the global picture of those is too complicated and far from being understood. This thesis can be summarized as a first step to understand the dynamics of those by considering the upcoming generalizations.

2.1 Symplectic manifolds

Definition 2.1.1. A symplectic manifold W is a manifold equipped with a non-degenerate closed 2-form ω. The form ω is called the symplectic form and (M, ω) symplectic manifold.

By non-degeneracy, symplectic manifolds are always even dimensional and we will always assume in what follows that the dimension is $2n$. Furthermore, as ω^n is a volume form, W is orientable.

Any orientable surface with the area form is an example of a symplectic manifold. An important example is given by the cotangent bundle.

An obstruction for a closed even dimensional manifold to admit a symplectic structure lies in the second cohomology class. Assume $H^2(M) = 0$, so $\omega = d\eta$. By Stokes theorem,

$$0 \neq \int_M \omega^n = \int_{\partial M} \eta \wedge \omega^{n-1} = 0.$$

This implies in particular that S^{2n} does not admit a symplectic structure for $n > 1$.

The importance of symplectic structures in classical mechanics originates in the following example.

Example 2.1.2. Let X be any manifold of dimension n and take its cotangent bundle T^*X. Taking coordinate charts $(U; x_1, \ldots, x_n)$ of X, the manifold T^*X of dimension $2n$ admits an atlas $(T^*U; x_1, \ldots, x_n, y_1, \ldots y_n)$, where y_i are the coefficients of $\lambda \in T^*X$ in the basis given by $(dx_1, \ldots, dx_n)$. Let us take the 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. This form is globally defined and it is symplectic.

The only local invariant of symplectic manifolds is the dimension, as every symplectic manifold is locally described by the last example.

Theorem 2.1.3 (Darboux theorem for symplectic manifolds)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$ and let $p \in M$. Then there exists an open neighbourhood $\mathcal{U} \ni p$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that $\omega|_{\mathcal{U}} = \sum_{i=1}^n dx_i \wedge dy_i$.*

The chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ is called *Darboux chart.* Loosely speaking, Darboux theorem implies that the deep results concerning symplectic manifolds are *global* result, hence the difference between symplectic geometry and topology.

Definition 2.1.4. Two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) are symplectomorphic if there exists a diffeomorphism $\varphi : M_1 \to M_2$ preserving the symplectic structure, i.e. $\varphi^*\omega_2 = \omega_1$.

In this thesis, we are mainly interested in the study of the Hamiltonian vector field. From the non-degeneracy of the symplectic form follows that for each smooth function $f : W \to \mathbb{R}$, we can associate a unique vector field X_f defined by the equation $i_{X_f}\omega = -df$, where i denotes the interior product. The Hamiltonian vector field is symplectic, that is it preserves the symplectic form: $\mathcal{L}_{X_f}\omega = 0$.

The Poisson bracket associated to a symplectic manifold is defined as follows:

$$\{f, g\} = \omega(X_f, X_g).$$

By closeness and non-degeneracy of the symplectic form, the associated Poisson bracket turns $C^\infty(W)$ in a Lie algebra and furthermore, the bracket satisfies Leibniz rule. Poisson brackets will play an important role in what follows and will be explained in see Section 2.5.

We conclude this section by motivating the name of Hamiltonian vector field as was promised already in the introduction. We point out that the flow of a Hamiltonian vector field in $\mathbb{R}^{2n}$ with the standard symplectic structure is described by the Hamilton equations. Indeed, a direct computation yields

$$X_H = \sum_{i=1}^{n} \Big(\frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} \Big)$$

and the trajectory $(x(t), y(t))$ of the vector field X_H satisfies the equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i},$$

which are exactly the Hamilton equations.

2.2 Contact Manifolds

Contact geometry can be considered as the odd-dimensional analogue of symplectic geometry with many links between those two theories that we will underline.

Definition 2.2.1. Let M be a manifold of dimension $(2n+1)$ and let ξ be a hyperplane distribution on M locally described by $\xi = \ker \alpha$ where $\alpha \in \Omega^1(M)$. We say that (M, ξ) is a contact manifold if $\alpha \wedge (d\alpha)^n \neq 0$. The hyperplane distribution ξ is called contact structure or also sometimes contact distribution.

The geometric meaning of being non-integrable is that the hyperplanes twist too much to be even locally tangent to a hypersurface. The contact condition $\alpha \wedge (d\alpha)^n \neq 0$ is at the opposite of integrability as stated in Frobenius theorem.

Locally, it is always true that the hyperplane distribution ξ can be written as the kernel of a 1-form α. In what follows, we always assume that ξ is cooriented, hence we can assume that $\xi = \ker \alpha$ where α is a globally defined 1-form. The form α is called *contact form*. We see that for each $p \in M$, $(\xi_p, d\alpha|_p)$ is a linear symplectic vector space of dimension $2n$.

Let us remark that if α is a contact form, then $f\alpha$, where $f : M \to \mathbb{R} \setminus \{0\}$ is a non-vanishing smooth function, is also a contact form, as can been seen from the computation

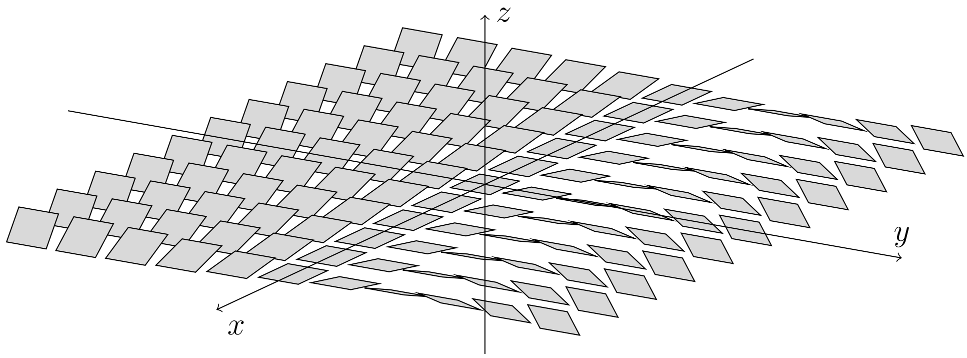
$$(f\alpha) \wedge \Big(d(f\alpha)\Big)^n = f\alpha \wedge (f d\alpha + df \wedge \alpha)^n = f^{n+1} \alpha \wedge d\alpha \neq 0.$$

Hence contact forms are defined only up to multiplication of a non-vanishing function.

As in the symplectic case, the manifold needs to be orientable as $\alpha \wedge (d\alpha)^n$ is non-vanishing and hence a contact form.

The standard example is the following.

Example 2.2.2. 1. Take $\mathbb{R}^{2n+1}$ with coordinates $x_1, \dots, x_n, y_1, \dots, y_n, z$ and consider the hyperplane distribution given by the kernel of the form $\alpha = dz + \sum_{i=1}^{n} x_i dy_i$. This 1-form satisfies the non-integrability condition.



As in symplectic geometry, there do not exist other local invariants than the dimension of the manifold.

Theorem 2.2.3 (Darboux theorem for contact manifolds)**.** *Let (M, α) be a contact manifold of dimension $2n+1$ and let $p \in M$. Then there exists an open neighbourhood $\mathcal{U} \ni p$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ such that $\alpha|_{\mathcal{U}} = dz + \sum_{i=1}^{n} x_i dy_i$.*

On contact manifolds, there exists one particular vector field: the *Reeb vector field.*

Definition 2.2.4. The Reeb vector field R_α of a contact manifold $(M, \ker \alpha)$ is defined by the equations

$$\begin{cases} i_{R_\alpha} \alpha = 1 \\ i_{R_\alpha} d\alpha = 0. \end{cases}$$

Those equations defined R_α uniquely: as $d\alpha$ is skew-symmetric form of maximal rank $2n$, so it has a one dimensional kernel. So the second equation defined a line field on M. The first condition then normalizes this line field: the contact condition implies that α is non-zero on that line field.

In the example of a Darboux chart, the Reeb vector field is given by $\frac{\partial}{\partial z}$. We will discuss the Reeb vector field, together with the Hamiltonian vector field in Section 2.4.

As in the symplectic case, it is possible to define *contact Hamiltonian vector fields.* Given a function $H \in C^\infty(M)$, the two following equations define a unique vector field by the same arguments as before:

$$\begin{cases} i_{X_H}\alpha = H \\ i_{X_f}d\alpha = -dH + R_\alpha(H)\alpha. \end{cases}$$

The following lemma is well-known:

Lemma 2.2.5. *Let (M, α) a contact manifold and let $H \in C^\infty(M)$ be a positive Hamiltonian, that is $H > 0$. Then the Hamiltonian vector field X_H is the Reeb vector field of the contact form $\frac{1}{H}\alpha$.*

Definition 2.2.6. Two contact structures $(M_1, \ker\alpha_1)$ and $(M_2, \ker\alpha_2)$ are called contactomorphic if there exists a diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^*\alpha_2 = f\alpha_1$, where f is a non-vanishing function.

2.2.1 Symplectization and hypersurfaces of contact type

Symplectic and contact are even respectively odd-dimensional cousins: a contact manifold times the real line always admits a symplectic structure, the *symplectization.*

Let $(M, \ker\alpha)$ be a contact manifold. Then $(\mathbb{R}\times M, d(e^t\alpha))$, where t is the $\mathbb{R}$-coordinate, is a symplectic manifold. The symplectic manifold $\mathbb{R} \times M$ is called the *symplectization* of M. Note that the symplectization is a non-compact manifold.

The other way around, hypersurfaces in symplectic manifolds admitting a Liouville vector field that is transverse to it are contact. Those hypersurfaces are *hypersurfaces of contact type.*

Definition 2.2.7. A vector field X on a symplectic manifold (W, ω) is called a *Liouville vector field* if $\mathcal{L}_X \omega = \omega$.

This vector field is defined up to addition of a symplectic vector field. In the case of the standard symplectic structure on $\mathbb{R}^{2n}$, this vector field is given by the radial vector field $X = \frac{1}{2}\sum_{i=1}^{n}(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i})$.

Let (W, ω) be a symplectic manifold of dimension $(2n+2)$, with Liouville vector field X. Let H be a hypersurface in W, transverse to the Liouville vector field. Then $(H, i_X\omega)$ is a contact manifold of dimension $(2n+1)$:

$$(i_X\omega) \wedge (d i_X \omega)^n = i_X\omega \wedge \omega^n = \frac{1}{n+1} i_X(\omega^{n+1}),$$

which is a volume on H provided that the Liouville vector field X is transverse to the hypersurface H.

This method gives rise to a lot of examples.

Example 2.2.8. Take $\mathbb{R}^{2n}$ with its standard symplectic structure. The Liouville vector field is given by the radial vector field $\frac{1}{2}\sum_{i=1}^{n}(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i})$, which is transverse to the $(2n+1)$-dimensional unit sphere S^{2n+1}. Hence $(S^{2n+1}, i_X\omega)$ is a contact manifold.

The method of going back and forwards between symplectic and contact manifolds has been fruitful in the past as we will see for example in Section 2.4.4.

In the case of level-sets for a Hamiltonian function that are of contact type, the Hamiltonian vector field is a reparametrization of the Reeb vector field.

Lemma 2.2.9 ([Gei08], Lemma 1.4.10)**.** *Let $H \in C^\infty(W,\omega)$ be a Hamiltonian function and assume that $H^{-1}(0)$ is a a regular level-set that is of contact type (with contact form $\alpha = \iota_Y\omega$ for Y some Liouville vector field). Then the Reeb flow of α is a reparametrization of the Hamiltonian vector field on $H^{-1}(0)$.*

2.3 Review on contact topology

We move on to results that have a more topological flavour. Those results will play an important role in Chapter 4.

2.3.1 Existence of contact structures

First of all, we mention that Gromov's h-principle proves that on open odd-dimensional manifolds [Gro69], there is no obstruction to the existence of contact structures.

Theorem 2.3.1. *Every open odd-dimensional manifold is contact.*

The case of closed manifolds is much more intricate: the 3-dimensional case was proved by Martinet–Lutz [Mar71, Lut77] and the higher dimensional case was eventually solved by Bormann-Eliashberg–Murphy [BEM+15].

Theorem 2.3.2. *[Eli89, BEM+15] Every almost contact structure on a closed manifold M is homotopic to a contact structure.*

A particular class of 3-dimensional contact manifolds plays an important role in this proof: that of the *overtwisted* ones. The notion exists also in higher dimensions, see also [BEM$^+$15] but its definition is more involved. Let $\Sigma \subset (M^3, \ker \alpha)$ be an embedded surface in a 3-dimensional contact manifold. The *characteristic distribution* is defined by the intersection of the tangent space of Σ with the contact distribution. The rank changes as the intersection can be transverse or the tangent planes coincides. It follows from the contact condition that the tangent planes can only coincide at isolated points. Therefore the characteristic distribution integrates to a singular foliation of Σ.

Definition 2.3.3. A 3-dimensional contact manifold $(M, \xi = \ker \alpha)$ is called overtwisted if there exists an embedded disk D^2 such that the boundary of $T\partial D \subset \xi|_{\partial D}$ and $TD \cap \xi$ defines a 1-dimensional foliation except on a unique elliptic[1] singular point $e \in \text{int}D$ with $T_eD = \xi_p$. The disk D is called overtwisted disk and we will denote it by D_{OT}. The point e is called the elliptic singularity.

A contact manifold that is not overtwisted is called *tight*.

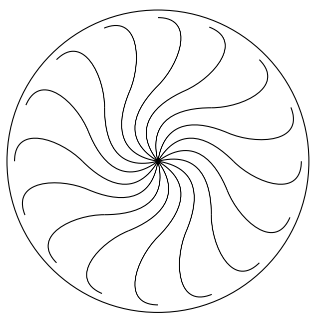


Figure 2.1: An overtwisted disk

[1]For a precise definition of elliptic singular point, we refer to Page 55 of [AH19].

The notion of overtwisted contact manifolds was generalized to higher dimensions in [Nie06] and is non-trivial. See also [BEM$^+$15], where another definition of overtwisted contact structures in higher dimension was introduced. Both have been proved to be equivalent in [CMP19].

2.3.2 Convex hypersurfaces

The theory of convex hypersurfaces was introduced by Giroux [Gir91, Gir99]. The main of convex theory in contact topology is to recover the data of a contact structure using an isotopy class of curves, called the dividing set.

Definition 2.3.4. A contact vector field on $(M, \ker\alpha)$ is a vector field X such that X preserves the contact structure, that is $\mathcal{L}_X\alpha = g\alpha$ for $g \in C^\infty(M)$. A hypersurface $Z \subset M$ is convex if there exists a contact vector field that is transverse to it. The *dividing set* of a convex hypersurface Z is given by the set of points $\Sigma = \{x \in Z | X_x \in \ker\alpha_x\}$.

Lemma 2.3.5 (Theorem 2.3.1 in [Gei08]). *Contact vector fields are in a bijective correspondence with contact Hamiltonian vector fields.*

It follows from the contact condition that the diving set is a codimension 2 submanifold without boundary and can be shown to be a hypersurface of Z. Convex hypersurfaces exist in abundance.

Theorem 2.3.6 ([Gir91]). *Let $(M, \ker\alpha)$ be a 3-dimensional contact manifold. Then any closed surface is C^∞-close to a convex surface.*

In higher dimensions, this result does not hold for generic hypersurfaces, see [Mor11]. However, even though genericity does not hold, examples are given by boundaries of Legendrian neighbourhoods. For a proof of this fact, see Page 35 of [AH19].

Using the flow of the transverse contact vector field, there exist coordinates in a semi-local neighbourhood around the convex hypersurface that we are going to use in later chapters.

Lemma 2.3.7. *Let $\Sigma \subset (M, \ker \alpha)$ be a convex hypersurface and let X be a contact vector field transverse to Z. Then there exist coordinates around Z such that $\alpha = f(udt + \beta)$ where $u \in C^\infty(Z)$, $\beta \in \Omega^1(Z)$ and $f \in C^\infty(M)$ is positive. In this coordinates $\frac{\partial}{\partial t} = X$.*

Proof. We take the flow of the contact vector field X. By definition, it satisfies $\mathcal{L}_X\alpha = g\alpha$. By Lemma 2.3.5, $X = X_H$ where $H = \alpha(X)$. The transversality condition implies that $H > 0$. A straightforward computation yields that $\mathcal{L}_X(\frac{1}{H}\alpha) = 0$, and we can hence assume without loss of generality that the contact form defining the distribution satisfies $\mathcal{L}_X\alpha = 0$. As X is transverse to Z, the flow defines a coordinate t around Z. The 1-form writes $\alpha = udt + \beta$ where u is smooth function and β a 1-form. We want to show that both u and β do not depend on the coordinate t. We compute the Lie derivative of α with respect to $\frac{\partial}{\partial t}$. By Cartans formula,

$$\begin{aligned}
\mathcal{L}_{\frac{\partial}{\partial t}}\alpha &= d\iota_{\frac{\partial}{\partial t}}\alpha + \iota_{\frac{\partial}{\partial t}}d\alpha \\
&= du + \iota_{\frac{\partial}{\partial t}}(du_t \wedge dt + d\beta_t + dt \wedge \frac{\partial \beta}{\partial t}) \\
&= \frac{\partial u_t}{\partial t}dt + du_t - du_t - \frac{\partial \beta}{\partial t} \\
&= \frac{\partial u_t}{\partial t}dt - \frac{\partial \beta}{\partial t},
\end{aligned}$$

and hence $\frac{\partial u_t}{\partial t} = \frac{\partial \beta}{\partial t} = 0$. □

Giroux proved a criteria, based on the connectedness of the dividing set, whether or not a neighbourhood of the convex hypersurface admits an overtwisted disk or not.

Theorem 2.3.8 (Giroux criterion [Gir99])**.** *Let* $(M^3, \ker\alpha)$ *be a contact manifold of dimension* 3 *and* $Z \subset M$ *a convex surface with dividing set given by* Σ*. The convex surface* Z *has a tight neighbourhood if and only if one of the following conditions is satisfied:*

1. *no component of* Σ *bounds a disk in* Z*,*

2. Z *is a sphere and* Σ *is connected.*

Note that in the case where $Z = S^2$ and $\sharp|\Sigma| = 2$, the contact manifold is overtwisted.

2.4 An excursion on the dynamics on contact and symplectic manifolds

In this section, we give a review on classical, but also some rather recent results on Hamiltonian and Reeb dynamics. As both symplectic and contact geometry historically arise from classical mechanics as pointed out in the direction, existence result of periodic orbit plays a central role in the theory.

2.4.1 On Reeb dynamics

In contact geometry, the main actor is the Reeb vector field associated to a contact form. In 1979, Weinstein [Wei79] formulated the following conjecture:

Conjecture 2.4.1 (Weinstein conjecture, [Wei79]). *Let (M, α) be a closed contact manifold. Then there exists at least one periodic Reeb orbit.*

At the time of stating the conjecture, several particular cases were already known, as for example the case of star-shaped level sets of a Hamiltonian function on $(\mathbb{R}^{2n}, \omega_{st})$, see [Rab78a]. The quest for positive answers of the conjecture has lead to spectacular development in contact topology. Most notably, Hofer proved in [Hof93] that overtwisted contact manifolds always admit periodic orbits. He proved that the existence of periodic orbits is equivalent to the existence of finite energy J-holomorphic planes in the symplectization. We outline the proof in a Subsection 2.4.4.

In dimension three, the conjecture is known to be true in full generality since the work of Taubes [Tau07].

A refinement of Weinstein conjecture is about the minimal number of periodic Reeb orbits. There are examples of compact contact manifolds exhibiting only finitely many periodic Reeb orbits, namely ellipsoids of irrational axis in the standard symplectic space.

Example 2.4.2. Consider the unit sphere S^3 in the symplectic manifold $(\mathbb{R}^4, \omega = dx_1 \wedge dy_1 + \epsilon dx_2 \wedge dy_2)$ where ϵ is fixed. The contact form is given by contracting the symplectic form with the radial vector field. When $\epsilon = 1$ all the orbits are periodic and they are in fact described by the Hopf fibration of S^3. When ϵ is irrational, the only two Reeb orbits that are preserved under the change of the contact form are the ones given in the (x_i, y_i)-plane.

The other example of contact manifold having finitely many periodic Reeb orbits is the quotient of the last example to Lens spaces.

The authors [CGH+16] proved the following dichotomy: in dimension 3, there are 2 or infinitely many. In the sequel [CGHP19], it is shown that there are always infinitely, under the condition that the contact form satisfies some non-degeneracy condition and being torsion-free.

It is well-known that the contact structure given in Example 2.4.2 is tight. This follows directly from symplectic fillability. We therefore conjecture

Conjecture 2.4.3. *Compact overtwisted contact manifolds always admit infinitely many periodic Reeb orbits.*

The novelty of this conjecture is that there is no condition of non-degeneracy or being torsion-free involved. In a later chapter, we will come back to this conjecture.

2.4.2 On Hamiltonian dynamics

As already mentioned, the Reeb flow can be viewed as a particular case of Hamiltonian flow on a fixed level-set. As hypersurfaces of contact type consists of special kinds of Hamiltonian level-sets with periodic orbit, it is natural to ask if in general the Hamiltonian vector field X_H admit periodic orbits in all level-sets of $H \in C^\infty(W, \omega)$. A good formulation of this questions is assuming that the level-sets are compact as for example the standard symplectic space with Hamiltonian function $\{x_1 = c \in \mathbb{R}\}$ answers this question negatively.

Conjecture 2.4.4 (Hamiltonian Seifert conjecture)**.** *Let $H \in C^\infty(\mathbb{R}^{2n}, \omega_{st})$ be a smooth Hamiltonian and assume that the level-set $H^{-1}(a)$ is proper. Then there exists a periodic orbit of X_H on $H^{-1}(a)$.*

This question has been open (and is still open in full-generality as we will see) for many years. A good reason to believe this conjecture to hold is the fact that for a large class of symplectic manifolds, the Hamiltonian flow admits periodic orbits on "almost all" level-sets. More precisely, let us mention the following almost-existence theorem:

Theorem 2.4.5 ([HZ11]). *Let (W, ω) be a symplectic manifold of finite Hofer–Zehnder capacity. Then for all $H : M \to \mathbb{R}$ such that $\{H \leq a\}$ is compact, almost all level-sets contain periodic orbits.*

This theorem is proved using powerful variational methods arising from the least action principle. Periodic orbits of X_H are in a one-to-one correspondence with the critical points of the action functional $\mathcal{A}_H$. The action of a contractible loop on a symplectic manifold (W, ω) is given by

$$\mathcal{A}_H(\gamma) = \int_{D^2} u^*\omega + \int_{S^1} H(\gamma(t))dt,$$

where $u : D^2 \to W$ is such that $u(\partial D^2) = \gamma$. Here γ is assumed to be periodic.

A value a of a Hamiltonian H is called *aperiodic* if the level $\{H = a\}$ carries no periodic orbits and we denote by $\mathcal{AP}_H$ the set of aperiodic values. Theorem 2.4.5 can be restated that $\mathcal{AP}_H$ is of measure zero for many symplectic manifolds and the Hamiltonian Seifert conjecture can be restated by saying that $\mathcal{AP}_H$ for H a proper, smooth function in $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ is empty.

Even though the almost existence theorem (Theorem 2.4.5) provides a good motivation for Hamiltonian Seifert conjecture, this conjecture turns out to be false.

Theorem 2.4.6 ([Gin97]). *Let $2n \geq 6$. There exists a proper smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$ such that the flow of X_H does not have any* *periodic orbits on the level set* $\{H = 1\}$.

The theorem was independently proved by Herman [Her95] for C^2-Hamiltonian functions. In dimension 4, a C^2-counterexample is proved in [GG03].

We conclude from the last two results that for many manifolds, $\mathcal{AP}_H$ is of measure zero but can be non-empty. In [Gin01], the following question is raised:

Question 2.4.7. *Let M be a symplectic manifold of bounded Hofer–Zehnder capacity and H a smooth proper function on M. How large can the set $\mathcal{AP}_H$ of regular aperiodic values be?*

For a review of the known results concerning this question, see [Gin01]. The proof of Theorem 2.4.6 is based on a plug construction.

2.4.3 Traps and Plugs

By the flow-box theorem, the flow of a non-singular vector field on a n-dimensional manifold locally looks like the linear flow, that is: on $D^{n-1} \times [0,1]$ the flow is given by $\Psi_t : (x,s) \to (x, s+t)$, where $t \in \mathbb{R}$ and D^{n-1} denotes a disk of dimension $n-1$.

Definition 2.4.8. A *trap* is a smooth vector field on the manifold $D^{n-1} \times [0,1]$ such that

1. the flow of the vector field is given by $\frac{\partial}{\partial t}$ near the boundary of $\partial D \times [0,1]$, where t is the coordinate on $[0,1]$;
2. there are no periodic orbits contained in $D \times [0,1]$;
3. the orbit entering at the origin of the disk $D \times \{0\}$ does not leave $D \times [0,1]$ again.

If the vector field additionally satisfies *entry-exit matching condition*, meaning that the orbit entering at $(x, 0)$ leaves at $(x, 1)$ for all $x \in D \setminus \{0\}$, then the trap is called a *plug*.

As a result of the flow-box theorem, traps can be introduced to change the local dynamics of a flow of a vector field and "trap" a given orbit. However, the introduction of a trap can change the global dynamical behaviour drastically. A plug additionally asks for matching condition at entry and exit in order not to change the global dynamics of the vector field. The vector field in question often satisfies some geometric properties (as for example volume-preserving, a Reeb vector field, a Hamiltonian vector field,. . . , etc). The crux in the construction of traps and plugs is to produce a vector field satisfying the given geometric constraint.

Traps and plug have been successfully used to construct counter-example in existence theorem for many geometric flows. For instance, Kuperberg constructed a plug in [Kup94] to find a smooth non-singular vector field without periodic orbits on any closed manifold of dimension 3. The special case of S^3 is known as counter-example to the Seifert conjecture. In the contact case, by the positive answers of Weinstein conjecture, there cannot exist plugs for the Reeb flow. Furthermore, it is a corollary of a theorem of Eliashberg and Hofer [EH$^+$94] that in dimension 3, Reeb traps do not exist. The same was conjectured in higher dimension, but Reeb traps were later proved to exist in dimension higher than 5, see [GRZ14].

Theorem 2.4.9 ([GRZ14])**.** *There is a contact form α on $\mathbb{R}^{2n+1}$ for $n \geq 2$, defining the standard contact structure, that is $\ker \alpha = \xi_{st}$ satisfying the following:*

1. *R_α has a compact invariant set (and hence orbits bounded in forward and backward time).*

2. *There are Reeb orbits which are bounded in forward time and whose z-component goes to $-\infty$ for $t \to -\infty$.*

3. *$\alpha = \alpha_{st}$ outside a compact set.*

4. *R_α does not have any periodic orbits.*

2.4.4 Hofer's proof for overtwisted contact manifold

Existence of periodic Reeb orbits where successfully proved by Hofer [Hof93] for overtwisted contact manifolds. The proof is thoroughly explained in the book [AH19].

We will review the proof which is based on J-holomorphic curve techniques applied in the symplectization of the contact manifold (M, α). The almost-complex structure J in the symplectization is compatible with the contact form.

More precisely, the almost-complex structure considered in the symplectization of the contact manifold (M, α) is constructed as follows. First fix a complex structure J_ξ on the plane-field $\xi = \ker \alpha$ that is compatible with α, i.e. $d\alpha(J_\xi \cdot, J_\xi \cdot) = d\alpha(\cdot, \cdot)$ and $d\alpha(\cdot, J_\xi \cdot) > 0$. We then extend the complex structure J_ξ on ξ to an almost complex structure J on $M \times \mathbb{R}$, compatible with $\omega = d(e^t \alpha)$ in the following way:

- $J|_\xi = J_\xi$,

- $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$,

- $\omega(\cdot, J\cdot) > 0$,
- $J(\frac{\partial}{\partial t}) = R_\alpha$.

A J-holomorphic curve is a map from a Riemann surface punctured in a finite set Γ to the symplectization $\widetilde{u} : (\Sigma \setminus \Gamma, j) \to (\mathbb{R} \times M, J)$ satisfying the non-linear Cauchy–Riemann equation $d\widetilde{u} \circ j = J \circ d\widetilde{u}$. Given a J-holomorphic curve, the energy is defined by

$$E(\widetilde{u}) = \sup_{\phi \in \mathcal{C}} \int_{\Sigma \setminus \Gamma} \widetilde{u}^* d(\phi\alpha)$$

where $\mathcal{C}$ is the set of all smooth maps $\phi : \mathbb{R} \to [0, 1]$ satisfying that $\phi' \geq 0$. In what follows, we will always denote $\widetilde{u} = (a, u)$, where $a : \Sigma \setminus \Gamma \to \mathbb{R}$ and $u : \Sigma \setminus \Gamma \to \mathbb{R}$. The *horizontal energy*, also called the $d\alpha$-energy, is defined to be

$$E^h(u) = \int_{\Sigma \setminus \Lambda} u^* d\alpha.$$

It is clear from the definitions that $E^h(u) \leq E(\widetilde{u})$. Hofer proved the following result, reducing the quest for periodic Reeb orbits to the existence of non-constant finite energy planes. Those finite energy planes arise from a careful bubbling-off analysis à la Uhlenbeck—Sachs.

Theorem 2.4.10. *Let $\widetilde{u} : \mathbb{C} \to \mathbb{R} \times M$ be a non-constant J-holomorphic plane such that $E(\widetilde{u}) < \infty$. Then there exists a periodic Reeb orbit in M.*

The J-holomorphic curve $\widetilde{u}$ as in Theorem 2.4.10 is called *finite energy plane.*

Out of the data of the overtwisted disk, Hofer proved the existence of family of J-holomorphic curves emanating from the elliptic point e of the overtwisted disk D_{OT} and that satisfy some additional properties which are key to study this family. We denote by $D^*_{OT} = D_{OT} \setminus \{e\}$.

Theorem 2.4.11. *Let D be the 2-disk. There is a continuous map*

$$\Psi : D \times [0, \epsilon) \to \mathbb{R} \times M$$

such that for each $\widetilde{u}_t(\cdot) = \Psi(\cdot, t)$

1. *$\widetilde{u}_t : D \to M \times \mathbb{R}$ is J-holomorphic,*
2. *$\widetilde{u}_t(\partial D) \subset D^*_{OT} \subset \{0\} \times M$ for $t \in (0, \epsilon)$,*
3. *$\widetilde{u}_t|_{\partial D} : \partial D \to D^*_{OT}$ has winding number 1 for $t \in (0, \epsilon)$,*
4. *$\Psi|_{D\times(0,\epsilon)}$ is a smooth map,*
5. *$\Psi(z, 0) = e$ for all $z \in D$,*
6. *$Ind(\widetilde{u}_t) = 2$.*

The family $\{\widetilde{u}_t\}_{t\in[0,\epsilon[}$ is called *Bishop family*. It is then studied whether or not the family can be extended. First, as an application of the maximum principle, the Bishop family restricted to the boundary is necessarily transverse to the characteristic foliation on the overtwisted disk and foliate a neighbourhood of the elliptic singularity e by circles.

Lemma 2.4.12. *The Bishop family is transverse to the characteristic foliation of the overtwisted disk $\xi|_{D^*_{OT}} \cap TD^*_{OT}$.*

It is shown that if the gradient of $\widetilde{u}_t$ is uniformly bounded in the interval $[0, T]$, then the family $\{u_t\}_{t\in[0,\epsilon[}$ can be maximally extended. However, this results leads to a contradiction with the transversality in Lemma 2.4.12. Hence the norm of the gradient blows up. There are basically two different possibilities for the gradient to blow up: it can blow up at the boundary of the J-holomorphic disk or in the interior.

A careful analysis then shows that in the case where the gradient blows up on the boundary, so called disk bubbling happens, which once more contradicts transversality of the Bishop family with the characteristic foliation. The only possibility is that the Bishop family blows up in the interior. The blow-up of the norm of the gradient in the interior of the disk is giving rise to bubbling phenomena. A carefully chosen reparametrization of the bubble converges uniformly to a non-constant finite energy plane. Hence by Theorem 2.4.10 there exists a periodic Reeb orbit.

2.5 Poisson manifolds

Poisson manifold come as a natural generalization of symplectic manifolds. First, by non-degeneracy of the symplectic form, to every smooth function is associated a Hamiltonian vector field. Poisson structure can be seen as a relaxation of the symplectic condition for which it still makes sense to talk about Hamiltonian vector fields.

Definition 2.5.1. A Poisson manifold M is a smooth manifold equipped with a skew-symmetric $\mathbb{R}$-bilinear map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

that satisfies

1. Jacobi identity, i.e. $\forall f, g, h \in C^\infty(M), \{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$ and
2. Leibniz's rule, i.e. $\forall f, g \in C^\infty(M), \{fg,h\} = f\{g,h\} + \{f,h\}g$.

This definition can be summarized by saying that a Poisson bracket on a manifold M is a Lie bracket on the space of smooth function

$C^\infty(M)$ that satisfies Leibniz property. By the skew-symmetry and Leibniz rule, the Poisson bracket is a bi-derivation and hence given by a bi-vector field Π, defined by

$$\Pi(df, dg) = \{f, g\}.$$

The Jacobi identity is translated by the fact that the Schouten–Nijenhuis bracket of Π with itself is zero, i.e. $[\Pi, \Pi] = 0$. We don't go into details about the Schouten–Nijenhuis bracket, but only say that it is a generalization of the Lie bracket to multi-vector fields, see [LGPV12] for a reference.

Hence an equivalent definition of Poisson manifold is the following:

Definition 2.5.2. A Poisson manifold is a manifold M equipped with a bi-vector field Π satisfying $[\Pi, \Pi] = 0$.

First note that there are no topological obstructions for the existence of a Poisson structure. Every manifold M admits the trivial Poisson structure $\Pi = 0$.

We continue by giving some examples of Poisson manifolds.

Example 2.5.3.

1. We already pointed out that symplectic manifolds (M, ω) admit a Poisson bracket as can be seen from the formula $\{f, g\} = \omega(X_f, X_g)$. We denote the associated Poisson bi-vector field by $\Pi = \omega^{-1}$. The Jacobi identity for this bracket is a consequence for the closeness of the symplectic form: expanding, one can check that $d\omega(X_f, X_g, X_h) = 2\Big(\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}\Big)$.
2. Consider $\mathbb{R}^{2n}$ equipped with the bi-vector field $\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$. This is a Poisson structure, but not a

symplectic structure due to the factor x_1. Taking the inverse of the Poisson structure, we get a 2-form $\frac{1}{x_1}dx_1 \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i$, which is not smooth at $x_1 = 0$, but symplectic outside of this critical locus. This example is important to keep in mind for the next chapter as we will rigorously give meaning to this kind symplectic structures with singularities.

Each Poisson structure admits a characteristic distribution, which is spanned by the Hamiltonian vector fields. This distribution is called *characteristic space* and its dimension is called the *rank* of the Poisson structure. The rank of the Poisson structure depends of course on the base point $x \in M$ and we say that a Poisson structure is *regular* if the rank is constant all over the manifold and it is said to be *singular* if the rank varies. It is immediate that the rank is an even integer.

The theorem of Frobenius says that a regular distribution is integrable if and only if it is involutive. As the rank in general is not constant, the arguments are a little bit more involved but the Stefan–Sussmann theorem [Sus73] asserts that a singular distribution integrates to a singular foliation if and only if the distribution is by a family of smooth vector field and invariant with respect to this family. The theorem of Stefan–Sussmann is satisfied by the characteristic space and we therefore speak about the *characteristic foliation* of the Poisson structure.

The behaviour of this foliation is explained by the Weinstein splitting theorem [Wei83], which studies the local normal form of Poisson structures:

Theorem 2.5.4 (Weinstein splitting theorem)**.** *Let (M, Π) be a Poisson manifold of dimension n and let the rank be $2k$ at the point $p \in M$.*

Then on a neighbourhood of p there exists a coordinate system

$$(x_1, y_1, \dots, x_k, y_k, z_1, \dots, z_{N-2k})$$

such that the Poisson structure can be written as

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i,j=1}^{n-2k} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{2.1}$$

where f_{ij} are functions which depend only on the variables $(z_1, \dots, z_{n-2k})$ and which vanish at the origin.

2.6 Jacobi manifolds

In this section, we introduce a further generalization of Poisson manifolds. The reason for this is that contact structures are not a particular case of Poisson structures. By replacing the Leibniz rule for Poisson manifolds by a weaker condition, namely that the bracket is a *local* derivation, we obtain what is called a *Jacobi structure* and covers also contact structures.

A bracket on $C^\infty(M)$ is said to have the property of being local if $\mathrm{supp}(\{f, g\}) \subset \mathrm{supp}(f) \cap \mathrm{supp}(g)$ for any $f, g \in C^\infty(M)$. We also say that the bracket gives rise to a *local Lie algebra structure*. Lichnérowicz proved in [Lic78] that this definition of Jacobi manifold is equivalent to the following definition. The fact that the bracket is a local derivation implies that the bracket is of the form

$$\{f, g\} = \Lambda(df, dg) + fR(g) - gR(f)$$

where Λ is a bi-vector field and R a vector field. This bracket satisfies Jacobi identity if and only if

$$[\Lambda, \Lambda] = 2R \wedge \Lambda, \quad [\Lambda, R] = 0 \tag{2.2}$$

where the bracket in the last equations is the Schouten-Nijenhuis bracket.

Definition 2.6.1. A Jacobi structure on a manifold M is a triplet (M, Λ, R) where Λ is a bi-vector field and R is a vector field satisfying the compatibility equations (2.2).

Example 2.6.2. Every Poisson manifold is a Jacobi manifold by taking $\Lambda = \Pi$ and $R = 0$. In particular, symplectic manifolds are Jacobi manifolds.

Another important example of Jacobi manifold is the one of contact manifold, see [Vai02]. Let (M, α) is a contact manifold, then (M, Λ, R) is a Jacobi structure, where R is the Reeb vector field and the bi-vector field Λ is defined by

$$\Lambda(df, dg) = d\alpha(X_f, X_g),$$

where X_f, X_g are the contact Hamiltonian vector fields of f and g. We give an alternative way to define the Jacobi structure associated to the contact structure and we will make use of this alternative description in later chapters.

Let us denote the bi-vector field dual to $d\alpha$, by Π. More precisely, as α is contact, $(d\alpha)|_\xi$ is symplectic and hence the dual is well-defined. Furthermore, we denote by X a Liouville vector field relatively to $d\alpha$, i.e. $\mathcal{L}_X d\alpha = d\alpha$. Eventually, we define the bi-vector field

$$\Lambda = \Pi + R \wedge X. \tag{2.3}$$

The vector and bi-vector fields satisfies the following equations:

- $\mathcal{L}_X \Pi = \Pi$,

- $\mathcal{L}_R\Pi = 0$,
- $[\Pi, \Pi] = 0$.

The following lemma characterizes the Jacobi structure.

Lemma 2.6.3. *The Jacobi structure associated to* (M, α) *is given by* Λ *and* R *if and only if* $R \wedge [X, R] \wedge X = 0$*.*

Proof. Let us check the two conditions of a Jacobi manifold, which are $[\Lambda, \Lambda] = 2R \wedge \Lambda$ and $[\Lambda, R] = 0$. The second equation writes

$$\begin{aligned}[\Lambda, R] &= [\Pi + R \wedge X, R] = [\Pi, R] + [\Pi, R \wedge X] \\ &= 0 + [\Pi, R] \wedge X - R \wedge [\Pi, X] = R \wedge \Pi = 0.\end{aligned}$$

As for the first one, we do the following computation:

$$[\Lambda, \Lambda] = [\Pi, \Pi] + 2[\Pi, R \wedge X] + [R \wedge X, R \wedge X].$$

Here, the first term is zero. The second term, using a well-known identity of the Schouten-bracket, gives us

$$2[\Pi, R \wedge X] = 2[\Pi, R] \wedge X - 2R \wedge [\Pi, X] = 0 + 2R \wedge \Pi = 0.$$

For the third term, using the same identity, we obtain

$$\begin{aligned}&[R \wedge X, R \wedge X] \\ &= R \wedge [X, R] \wedge X + [R, R] \wedge X \wedge X - R \wedge R \wedge [X, X] - R \wedge [R, X] \wedge X \\ &= 2R \wedge [X, R] \wedge X.\end{aligned}$$

□

Note that a sufficient condition for (M, Λ, R) to be a Jacobi manifold is to ask that X and R commute.

For each smooth function f, the Hamiltonian vector field on the Jacobi manifold is given by $X_f = \Lambda(df, \cdot) + fR$. Note that this definition is compatible with the one of Hamiltonian vector fields on Poisson manifolds (i.e. when $R = 0$). Furthermore, as in the Poisson case, the distribution spanned by those Hamiltonian vector fields can be shown to be integrable and integrates to a singular foliation. However, this foliation does not need to have only even dimensional leaves. One can show that the Jacobi structure induces two different structures on the leaves. If the leaf is of odd dimension, the induced structure is contact. In the even-dimensional case, the leaf is locally conformally symplectic, see [Vai02] for instance for the proof.

Definition 2.6.4. A locally conformally symplectic manifold is a manifold of even dimension W^{2n} equipped with a non-degenerate two form $\omega \in \Omega^2(W)$ that is locally conformally closed, that is for every $p \in W$, there exists function defined on an open neighbourhood $\mathcal{U} \ni p$, $\sigma \in C^\infty(\mathcal{U}, \mathbb{R})$ such that $e^\sigma \omega|_{\mathcal{U}}$ is closed.

The condition of the 2-form ω to be locally closed is equivalent to the existence of a closed 1-form $\alpha \in \Omega^1(W)$ such that $d\omega = \alpha \wedge \omega$. Locally conformally symplectic manifold regained recent attention, notably in the work of [CM16].

A theorem similar to Weinstein splitting theorem in the set-up of Jacobi manifold has been proved by [DLM91]. To state it, let us first introduce some notation.

We recall local structure theorems of Jacobi manifolds, proved in [DLM91]. To state it, we first introduce some notation. The subindex of the multi-vector fields here denotes the dimension of the Euclidean space that those multi vector fields are considered in.

- $\Lambda_{2q} = \sum_{i=1}^{q} \frac{\partial}{\partial x_{i+q}} \wedge \frac{\partial}{\partial x_i}$
- $Z_{2q} = \sum_{i=1}^{q} x_{i+q} \frac{\partial}{\partial x_{i+q}}$
- $R_{2q+1} = \frac{\partial}{\partial x_0}$
- $\Lambda_{2q+1} = \sum_{i=1}^{q} (x_{i+q} \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_i}) \wedge \frac{\partial}{\partial x_{i+q}}$

The local structure theorem states as follows.

Theorem 2.6.5 ([DLM91]). *Let (M^m, Λ, R) be a Jacobi manifold, x_0 a point of M and S be the leaf of the characteristic foliation going through x_0.*

If S is of dimension $2q$, then there exists a neighbourhood of x_0 that is diffeomorphic to $U_{2q} \times N$ where U_{2q} is an open neighbourhood containing the origin of $\mathbb{R}^{2q}$ and (N, Λ_N, R_N) is a Jacobi manifold of dimension $m-2q$. The diffeomorphism preserves the Jacobi structure, where the Jacobi structure on $U_{2q} \times N$ is given by

$$R_{U_{2q}\times N} = \Lambda_N, \qquad R_{U_{2q}\times N} = \Lambda_{2q} + \Lambda_N - Z_{2q} \wedge R_N.$$

If S is of dimension $2q+1$, then there exists a neighbourhood of x_0 that is diffeomorphic to $U_{2q+1} \times N$ where U_{2q+1} is an open neighbourhood containing the origin of $\mathbb{R}^{2q+1}$ and (N, Λ_N, R_N) is a homogeneous Poisson manifold of dimension $m-2q-1$. The diffeomorphism preserves the Jacobi structure, where the Jacobi structure on $U_{2q} \times N$ is given by

$$R_{U_{2q+1}\times N} = R_{2q+1}, \qquad \Lambda_{U_{2q+1}\times N} = \Lambda_{2q+1} + \Lambda_N + E_{2q+1} \wedge Z_N.$$

2.6.1 Geography of symplectic, contact, Poisson and Jacobi manifolds

We depicture the relation between the four geometric structures introduced. We schematically can visualize them as follows.

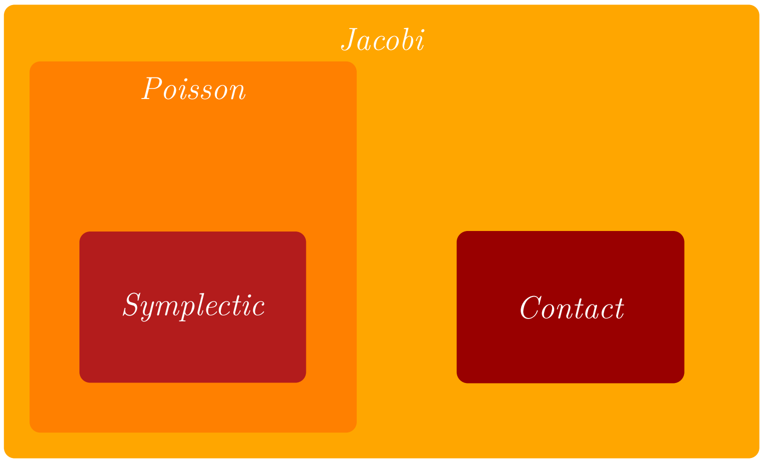


Figure 2.2: Geography of Jacobi, Poisson, Symplectic and Contact manifolds

We already mentioned that the symplectic and contact world are connected via the symplectization or taking hyperplanes transverse to the Liouville vector field.

Poisson and Jacobi manifold behave in the same way by taking the *Poissonization.*

Proposition 2.6.6. *Let* (M, Λ, R) *be a Jacobi manifold. Then* $(M \times \mathbb{R}, \Pi = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge R))$ *is a Poisson manifold. Furthermore, the vector field* $T = \frac{\partial}{\partial \tau}$ *satisfies*

$$L_T \Pi = -\Pi$$

.

The Poisson manifold $(M \times \mathbb{R}, \Pi)$ is the Poissonization of the Jacobi manifold.

Proof. $(M \times \mathbb{R}, \Pi := e^{-\tau}(\Lambda + \frac{\partial}{\partial \tau} \wedge R))$ is a Poisson manifold because

$$\begin{aligned}[\Pi, \Pi] &= [e^{-\tau}\Lambda, e^{-\tau}\Lambda] + 2[e^{-\tau}\Lambda, e^{-\tau}\frac{\partial}{\partial \tau} \wedge R] + [e^{-\tau}\frac{\partial}{\partial \tau} \wedge R, e^{-\tau}\frac{\partial}{\partial \tau} \wedge R] \\ &= e^{-2\tau}[\Lambda, \Lambda] - 2e^{-2\tau}\Lambda \wedge R = e^{-2\tau}(2R \wedge \Lambda - 2\Lambda \wedge R) = 0,\end{aligned}$$

where we used the compatibility condition (Equation 2.2). □

Furthermore, the later is said to be *homogeneous* because the vector field $T = \frac{\partial}{\partial \tau}$ satisfies

$$L_T \Pi = -\Pi.$$

2.7 Symplectic structures with singularities

In this section, we introduce geometric structures that are symplectic away from a given hypersurface. On the hypersurface, the symplectic form is singular. The singularities are of the simplest possible type. These geometric structures can be viewed as a particular case of Poisson manifolds, but are close enough to symplectic manifolds to import well-known techniques from symplectic geometry and topology, that generally fail to exist in Poisson geometry.

There is a one to one correspondence between symplectic forms and non-degenerate Poisson structures on manifolds. This section focuses on an exposition of structures which are the "next best" case, i.e. manifolds where these structures are non-degenerate away from a hypersurface of the manifold and behave well on the singular hypersurface.

Definition 2.7.1 (b-Poisson structure). Let (M^{2n}, Π) be an oriented Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM) \tag{2.4}$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface and we say that Π is a b-Poisson structure on (M^{2n}, Z) and (M^{2n}, Z, Π) is a b-Poisson manifold. The hypersurface Z is called singular hypersurface.

We already gave an example of a b-Poisson manifold, see Example 2.5.3, where both the non-degenerate and the b-Poisson case are given. In the non-degenerate case, the dual of the Poisson bi-vector field gives rise to the symplectic form. We cannot simply take the dual in the b-Poisson case, as we will obtain 2-form which is not smooth anymore. In what follows, we give a formal sense to this singularities.

Let us remark that the classification of b-Poisson structures on compact surfaces was initiated by Radko [Rad02]. The study of b-Poisson in higher dimension was then studied in [GMP14] by Guillemin, Miranda and Pires.

2.7.1 b-Geometry

The notion of b-manifold was initiated in the work of Melrose [Mel93], hence the letter "b", coming from *boundary*. In the scenario of b-Poisson manifold, we view a distinguished hypersurface in the manifold as boundary.

Definition 2.7.2. A b-manifold (M, Z) is an oriented manifold M together with an oriented hypersurface Z. A b-map is a map

$$f : (M_1, Z_1) \to (M_2, Z_2) \tag{2.5}$$

so that f is transverse to Z_2 and $f^{-1}(Z_2) = Z_1$. A b-vector field is a vector field on (M, Z) which is everywhere tangent to Z.

One easily sees that the set of b-vector fields is a Lie subalgebra of the Lie algebra of vector fields. The surprising result is that they are in fact sections of a vector bundle of M. Take U to be an open neighbourhood around a point $p \in Z$ and assume that Z is locally given by the level set of a locally defined function f. We call f the *defining function*. Then the vector field $f\frac{\partial}{\partial f}$ is tangent to Z. Take a coordinate chart on U of the form $(f, x_2, \dots, x_n)$ for which the b-vector fields restricted to U form a free C^∞-module with a finite basis given by

$$\left(f\frac{\partial}{\partial f}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right).$$

According to a theorem of Serre–Swan [Swa62], there exists a vector bundle having the b-vector fields as sections. This vector bundle is called b-tangent bundle and we denote it by ${}^bTM$. At points $p \in M \backslash Z$, the b-tangent space coincides with the usual one, i.e. ${}^bT_pM = T_pM$. On points $p \in Z$, the restriction of a b-vector field to Z yields a vector field on Z. The vector bundle morphism

$${}^bTM|_Z \to TM|_Z$$

is in fact surjective and the kernel is the line bundle generated by $f\frac{\partial}{\partial f}$, which is called the *normal b-bundle*.

From here on, we can imitate the construction of classical differential geometry to define the co-vectors associated and differential forms of this vector bundle. The b-cotangent bundle ${}^bT^*M$ of a b-manifold is defined as the dual of ${}^bTM$ and a local basis is given by

$$(\frac{df}{f}, dx_2, \ldots, dx_n)$$

where the form $\frac{df}{f}$ is well-defined on the b-tangent bundle. Differential forms for this vector bundle are called b-forms. A b-form of degree k is defined as section of the vector bundle ${}^b\Omega^k(M) = \Lambda^k({}^bT^*M)$. Fixing a defining function f, every b-form of degree k can be decomposed as follows:

$$\alpha \wedge \frac{df}{f} + \beta \quad \text{where } \alpha \in \Omega^{k-1}(M), \beta \in \Omega^k(M). \tag{2.6}$$

This decomposition makes it possible to define an exterior derivative by

$$d(\alpha \wedge \frac{df}{f} + \beta) = d\alpha \wedge \frac{df}{f} + d\beta, \tag{2.7}$$

the notion of *closed* and *exact* forms are naturally extended, $d^2 = 0$ and so de Rham cohomology can be extended to the b-setting. By a theorem of Mazzeo–Melrose, it can be computed in terms of the cohomology of M and Z, see [GMP14].

Theorem 2.7.3 (Mazzeo–Melrose for b-cohomology)**.** *The b-cohomology can be computed as follows:* ${}^bH^*(M) = H^*(M) \bigoplus H^{*-1}(Z)$.

Equipped with the theory of b-calculus, we are now ready to rigorously define the dual of b-Poisson structures.

2.7.2 b-Symplectic Manifolds

Definition 2.7.4. Let (W^{2n}, Z) be a b-manifold. Let $\omega \in {}^b\Omega^2(W)$ be a closed b-form. We say that ω is b-symplectic if ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*W)$ for all $p \in M$.

Let us give some examples of b-symplectic manifolds.

Example 2.7.5.
1. A compact example of a b-symplectic manifold is given by $(S^2, \frac{dh}{h} \wedge d\theta)$. The critical surface is given by $Z = h^{-1}(0)$.
2. Take $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$ and the b-form
$$\frac{dx_1}{x_1} \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$
This is a b-symplectic manifold and we will see that this constitutes the local model.

There exist two approaches to prove Darboux theorem for b-symplectic manifold. One is using Moser's trick and translating some classical results from symplectic geometry to the b-setting. Another is to first prove that there is a one to one correspondence between b-Poisson manifolds and b-symplectic manifolds and then to use Weinstein's splitting theorem. The transversality condition then guarantees the local normal form.

Theorem 2.7.6 (b-Darboux theorem for b-symplectic manifolds, [GMP14])**.** *Let ω be a b-symplectic form on (W^{2n}, Z). Let $p \in Z$. Then we can find a local coordinate chart $(x_1, y_1, \ldots, x_n, y_n)$ centred at p such that hypersurface Z is locally defined by $y_1 = 0$ and*
$$\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

The dual of the b-Darboux theorem gives a local normal form of type

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}. \tag{2.8}$$

Viewing b-symplectic manifolds from the Poisson point of view gives us a nice interpretation of the characteristic foliation: away from the critical set, the rank is the same as the dimension and therefore has one leave of maximal dimension. However as Π^n cuts the zero section transversally, the rank drops by 2, and we therefore have a codimension 1 foliation of symplectic leaves, which is given in the local Darboux coordinates by $\ker dx_1$.

There exists a intrinsically defined vector field, that can be shown to be transverse to the symplectic leaves: the *modular vector field.* Take a volume form Ω on W. The map

$$f \mapsto \frac{\mathcal{L}_{X_f}\Omega}{\Omega}$$

is a derivation, hence a vector field denoted by v_{mod}. It can be shown that it is independent on the choice of the volume form and that it is a Poisson vector field, i.e. $\mathcal{L}_{v_{\text{mod}}}\Pi = 0$. In the Darboux coordinates, it is given by $\frac{\partial}{\partial x_1}$. The existence of this transverse vector field implies that the critical set Z is in fact a cosymplectic manifold: the one form, dual to this vector field together with the symplectic form on the leaves of the symplectic foliation defines the cosymplectic structure.

Definition 2.7.7. A cosymplectic manifold is a manifold M^{2n+1} together with a closed one-form η and a closed two-form ω such that $\eta \wedge \omega^n$ is a volume form.

b^m-symplectic manifolds

In the same way we defined b-forms, we can define higher order singularities by taking the C^∞-module with a finite basis given by

$$\left(x_1^m \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right)$$

where the critical hypersurface is defined by x_1 and by the Serre–Swan theorem, consider the vector bundle admitting those vector fields as sections. Repeating the same construction, we obtain b^m-symplectic manifolds. Those higher order singularities were extensively studied in [Sco16]. We only mention here that a Darboux theorem for those structures has been proved in [GMW19], asserting that locally, b^m-structures are of the form

$$\omega = dx_1 \wedge \frac{dy_1}{y_1^m} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Desingularizing b^m-symplectic structures

We briefly describe the main theorem of [GMW19]. It relates the topology of b^m-symplectic manifolds to the one of symplectic and folded-symplectic manifolds. Folded-symplectic structures are in some sense dual to b-symplectic structures as will be clear from the definition, and their study has been initiated in [GSW00].

Definition 2.7.8. A 2-form $\omega \in \Omega^2(W)$ on a $2n$-dimensional manifold W is folded-symplectic if ω^n is transverse to the zero-section and $\omega^{n-1}|_Z \neq 0$, where Z denotes the hypersurface give by $Z := (\omega^n)^{-1}(0)$.

Similarly, by replacing the transversality by higher order singularities one obtains *m-folded symplectic forms*.

The relation between symplectic, b^m-symplectic and m-folded symplectic structures is being done by ingeniously *desingularizing* the b^m-symplectic structure, depending on the parity of the singularity.

Theorem 2.7.9 ([GMW19]). *Given a b^m-symplectic structure ω on a compact manifold (W^{2n}, Z):*

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If $m = 2k+1$, there exists a family of folded-symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.*

It follows immediately from this theorem that b^m-symplectic manifolds are symplectic for m even and folded-symplectic for m even and therefore the obstruction of existence of symplectic (respectively folded-symplectic) translate directly to obstructions for the existence of b^m-symplectic manifolds depending on the parity of m. Note that every four dimensional manifold admits a folded-symplectic structure [DS+10]. However, there are cohomological obstructions to the existence of b^m-symplectic structures as we will see in Section 4.1.

2.7.3 Quick guide through b^m-symplectic literature

As the central theme of this thesis are b^m-symplectic structures, we include here a guide through b^m-symplectic geometry. Beside the main sources already cited [Rad02, GMP14, GMPS15, GMPS17, GMW18, Sco16], we would like to mention that b^m-symplectic structures also appear under the name of *log-symplectic structures*, see [Cav17, GL14, GLPR17, CK16] to name a few.

2.7.4 E-symplectic structures

We briefly discuss a far-reaching generalization of b^m-symplectic structures, studied in [MS18]. Note that the b^m-(co)tangent bundle is defined by the Serre–Swan theorem. This construction, along with the extension of the exterior derivative, opens up the door to study symplectic structures over the b-tangent bundle. It is possible to generalize this by imposing the necessary condition to mimic this construction.

Let E be a locally free submodule of $\mathfrak{X}(M)$. Hence by Serre–Swan theorem, there exists a vector bundle, that we denote E-tangent bundle, ${}^ETM$ whose sections are given by E. The dual, respectively sections of k-wedge powers of the dual, are denoted by ${}^ET^*M$, respectively ${}^E\Omega(M)$.

To extend the exterior to ${}^E\Omega^p(M)$, we further ask that E is involutive, that is $[E,E]=E$ and then define the exterior derivative via Cartan formula. In what follows, we always assume that E is an involutive, locally free submodule of $\mathfrak{X}(M)$.

The bundle ${}^ETM \to M$ is a Lie algebroid, whose bracket is the standard bracket for vector fields and the anchor map is given by the inclusion $E \subset \mathfrak{X}(M)$.

Definition 2.7.10. A closed non-degenerate 2-form of ${}^E\Omega(M)$ is called an E-symplectic form. The manifold (M,E,ω) is an E-symplectic manifold.

It is clear that b^m-symplectic forms are a particular case of E-symplectic forms. We enumerate other examples.

1. C-symplectic: this is generalization of b-symplectic manifolds where the hypersurface is allowed to have self-intersections. Here C stands for *corner*.
2. Elliptic-symplectic: the C^∞module $E \subset \mathfrak{X}(\mathbb{R}^2)$ is generated by the two vector fields $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.
3. regular symplectic foliations: E is spanned by the vector fields tangent to the foliation.

In addition, many other examples can be constructed by describing explicitly the submodule E.

2.8 Contact structures with singularities

We briefly resume the preliminaries and also will give the main lines of the following chapters.

Besides the classical geometric structures, the preliminaries exclusively treats the appearance of singularities in symplectic geometry, where Poisson geometry provides the right viewpoint. Having introduced the odd-dimensional counterpart to symplectic manifolds, we investigate the analogue of those kind of singularities in contact manifolds. In view of the geography map shown in Figure 2.2, it is clear that Jacobi manifolds will provide the proper set-up. The following chapters are going to be dedicated to complete the geographical picture given by Figure 2.3.

We will analyse the interplay between contact geometry (to be precise, Moser's trick) to prove local normal forms and use the structural

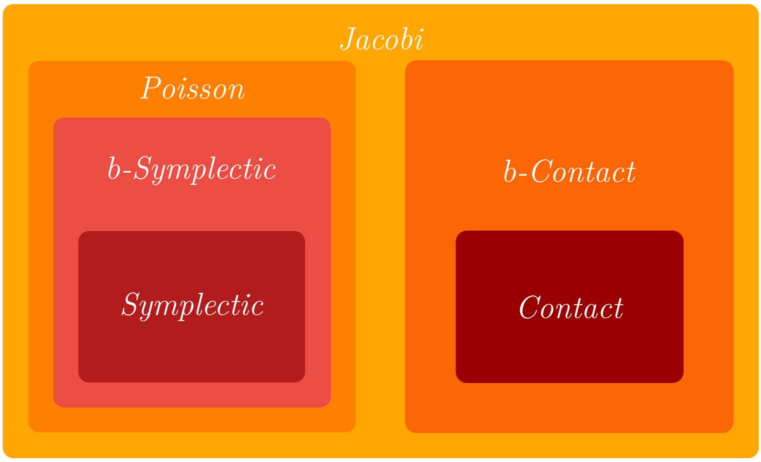


Figure 2.3: b-symplectic and b-contact geography

theorems of Jacobi manifolds (that is Theorem 2.6.5) to analyse the set where the contact form is singular. In the light of Section 2.3 and 2.4, we are going to study the topology of those manifolds, but also the dynamical behaviour of the Reeb vector field and the Hamiltonian vector field.

Chapter 3

b^m-Contact Geometry

Connaître ce n'est pas démontrer, ni expliquer. C'est accéder à la vision.
Antoine de Saint-Exupéry, *Le Petit Prince.*

In this chapter, we introduce the local geometry of the contact counterpart of b^m-symplectic manifolds. Inspired by the definition of b-symplectic manifolds, we define the contact case as follows:

Definition 3.0.1. Let (M, Z) be a (2n+1)-dimensional b-manifold. A b-contact structure is the distribution given by the kernel of a one b-form $\xi = \ker \alpha \subset {}^bTM$, $\alpha \in {}^b\Omega^1(M)$, that satisfies $\alpha \wedge (d\alpha)^n \neq 0$ as a section of $\Lambda^{2n+1}({}^bT^*M)$. We say that α is a b-contact form and the pair (M, ξ) a b-contact manifold.

Analogous to the symplectic case, the hypersurface Z is called *critical hypersurface*. In what follows, we always assume that Z is non-empty. Away from the critical set Z the b-contact structure is a smooth contact structure. The former definition fits well with what is standard in contact geometry where coorientable contact manifolds are considered (i.e. there exists a global defining contact form whose kernel defines the given contact structure). We only consider the case of coorientable b-contact structures in this thesis, that is we assume that $\xi \subset {}^bTM$ is given by the kernel of a globally defined b-form of degree 1.

Example 3.0.2. Let (M, Z) be a b-manifold of dimension n. Let $z, y_i, i = 2, \dots, n$ be the local coordinates for the manifold M on a neighbourhood of a point in Z, with Z defined locally by $z = 0$ and $x_i, i = 1, \dots, n$ be the fibre coordinates on ${}^bT^*M$, then the canonical Liouville one-form is given in these coordinates by

$$x_1 \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i.$$

The bundle $\mathbb{R} \times {}^bT^*M$ is a b-contact manifold with b-contact structure defined as the kernel of the one-form

$$\alpha = dt + x_1 \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

where t is the coordinate on $\mathbb{R}$. The critical set is given by $\widetilde{Z} = Z \times \mathbb{R}$. Using the definition of the extended de Rham derivative,

$$\alpha \wedge (d\alpha)^n = n! \frac{dz}{z} \wedge dt \wedge dx_1 \wedge dx_2 \wedge dy_2 \cdots \wedge dx_n \wedge dy_n \neq 0.$$

Away from $\widetilde{Z}$, $\xi = \ker \alpha$ is a non-integrable hyperplane field distribution, as in usual contact geometry. On the critical set however, ξ is tangent to $\tilde{Z}$. This comes from the definition of b-vector fields. Viewed as smooth distribution, we cannot say that ξ is a hyperplane field since the rank of ξ may drop by 1. In this example this is the case for the codimension 2 submanifold $x_1 = 0$.

As we will see in the next example, the rank does not necessarily drop.

Example 3.0.3. Let us take $\mathbb{R}^{2n+1}$ with coordinates $(z, x_1, \ldots, x_n, y_1, \ldots, y_n)$. We consider the distribution of the kernel of $\alpha = \frac{dz}{z} + \sum_{i=1}^n x_i dy_i$. The critical set is given by $z = 0$ and the rank when ξ is viewed as (that is through the inclusion of ${}^b\mathfrak{X}(M) \subset \mathfrak{X}(M)$) a smooth distribution does not drop on the critical set: on the critical set, the distribution is spanned by $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, i = 1, \ldots n\}$.

Using the two last examples and a generalization of Möbius transformations, we can construct b-contact structures on the unit ball with critical set given by the unit sphere.

Example 3.0.4. Let us denote the unit ball of dimension n by D^n and the half-space, that is $\mathbb{R}^n$ where the first coordinate is positive, by $\mathbb{R}^n_+$. The Möbius transformation maps the open half-space diffeomorphically to the closed 2-ball minus a point by the following map:

$$\begin{aligned} \Phi : \{z \in \mathbb{C} | \Re(z) > 0\} &\to D^2 \setminus \{(1,0)\} \\ z &\mapsto \frac{z-1}{z+1}. \end{aligned}$$

This map can easily be generalized to all dimension and the inverse is given by

$$\Psi : D^n \setminus \{(1,0,\dots,0)\} \to \mathbb{R}^n_+$$
$$(x_1,\dots,x_n) \mapsto \frac{1}{(x_1-1)^2+\sum_{i=2}^n x_i^2}\Big(1-\sum_{i=1}^n x_i^2, 2x_2,\dots,2x_n\Big).$$

We now provide $\mathbb{R}^{2n+1}_+$ with the b-contact structures described in Example 3.0.2 (respectively 3.0.3) and pull-back the b-contact form. We obtain hence two different b-contact structures on the unit ball minus a point and the critical set is given by the unit sphere S^{2n-2} minus the point $(1,0,\dots,0)$.

It is not possible to compactify this example by adding the point. This can be seen when computing the hyperplane distribution of the pushforward under Φ. Alternatively, this follows as we will see in one of the two cases from Theorem 3.1.5. However, we will see that the 3-sphere does admit a b-contact structure, induced by a b-symplectic structure, see Example 3.4.3.

Example 3.0.5. A compact example admitting a b-contact structure is given by $S^2 \times S^1$. Let us consider the 2-sphere S^2, with coordinates (θ, h) where $\theta \in (0,2\pi)$ is the angle and $h \in (0,1)$ is the height, and the 1-sphere S^1 with coordinate $\varphi \in (0,2\pi)$. Then $(S^2 \times S^1, \alpha = \sin\varphi d\theta + \cos\varphi \frac{dh}{h})$ is a b-contact manifold. Once more, the rank on the critical set changes when $\cos\varphi = 0$, where instead of a plane-distribution, we are dealing with a line distribution.

Example 3.0.6 (Non-orientable example). Coorientable contact manifolds are always orientable as $\alpha \wedge (d\alpha)^n$ is a volume form. There are b-contact forms on non-orientable manifolds. Consider the example of the b-contact form on the 3-torus given by $(\mathbb{T}^2 \times S^1, \alpha =$

$cos\theta\frac{dx}{\sin 2\pi x} + \sin\theta dy)$. Consider the group action $\mathbb{Z}/2\mathbb{Z}$ that acts on $(x,y) \in \mathbb{T}^2$ by $\mathrm{Id}\cdot(x,y) = (x,y)$ and $-\mathrm{Id}\cdot(x,y) = (1-x,y)$. The orbit space by this action is the Klein bottle. The b-contact form is invariant under the action of the group and therefore descends to $\mathbb{K}\times S^1$ where $\mathbb{K}$ is the Klein bottle. The manifold $\mathbb{K}\times S^1$ is of course non-orientable.

Example 3.0.7 (Product examples). Let (N^{2n+1},α) be a b-contact manifold and let $(M^{2m},d\lambda)$ be an exact symplectic manifold, then $(N\times M,\alpha+\lambda)$ is a b-contact manifold. It is easy to check that $\tilde{\alpha}=\alpha+\lambda$ satisfies $\tilde{\alpha}\wedge(d\tilde{\alpha})^{n+m}\neq 0$.

In the same way if (N^{2n+1},α) is a contact manifold and $(M^{2m},d\lambda)$ be an exact b-symplectic manifold (where exactness is understood in the b-complex), then $(N\times M,\alpha+\lambda)$ is a b-contact manifold.

3.1 The b-contact Darboux theorem

In usual contact geometry, the Reeb vector field R_α of a contact form α is given by the equations

$$\begin{cases}\iota_{R_\alpha}d\alpha = 0\\ \alpha(R_\alpha)=1.\end{cases}$$

In the case where we change the tangent bundle by ${}^bTM$, the existence is given by the same reasoning: $d\alpha$ is a bilinear, skewsymmetric 2-form on the space of b-vector fields ${}^bTM$, hence the rank is an even number. As $\alpha\wedge(d\alpha)^n$ is non-vanishing and of maximum degree, the rank of $d\alpha$ must be $2n$, its kernel is 1-dimensional and α is non-trivial on that line field. So a global vector field is defined by the normalization condition.

By the same reasoning, we can define the b-contact vector fields: for every function $H \in C^\infty(M)$, there exists a unique b-vector field X_H defined by the equations

$$\begin{cases} \iota_{X_H}\alpha = H \\ \iota_{X_H}d\alpha = -dH + R_\alpha(H)\alpha. \end{cases}$$

A direct computation yields that in Example 3.0.2, the Reeb vector field is given by $\frac{\partial}{\partial t}$. In Example 3.0.3, the Reeb vector field is given by $z\frac{\partial}{\partial z}$ and hence singular. We will see that, roughly speaking, the Reeb vector field locally classifies b-contact structures.

We now prove a Darboux theorem for b-contact manifolds. The proof follows the one of usual contact geometry as in [Gei08]. More precisely, it makes use of Moser's path method. There are two differences from the standard Darboux theorem: the first one is that there exist two local models, depending on whether or not the Reeb vector field is vanishing on the critical set Z. The second one is that in the case where the Reeb vector field is singular, the local expression of the contact form only holds pointwise, see for instance Example 3.1.3. Furthermore, in the case where the Reeb vector field is singular, this linearisation is done up to multiplication of a non-vanishing function. The proof is not following Moser's path method in this case as the flow of the Reeb vector field is stationary.

Theorem 3.1.1. *Let α be a b-contact form inducing a b-contact structure ξ on a b-manifold (M, Z) of dimension $(2n+1)$ and $p \in Z$. We can find a local chart $(\mathcal{U}, z, x_1, y_1, \dots, x_n, y_n)$ centred at p such that on $\mathcal{U}$ the hypersurface Z is locally defined by $z = 0$ and*

1. *if $R_p \neq 0$*

(a) ξ_p *is singular, then*

$$\alpha|_{\mathcal{U}} = dx_1 + y_1\frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

(b) ξ_p *is regular, then*

$$\alpha|_{\mathcal{U}} = dx_1 + y_1\frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

2. if $R_p = 0$*, then* $\tilde{\alpha} = f\alpha$ *for* $f(p) \neq 0$*, where*

$$\tilde{\alpha}_p = \frac{dz}{z} + \sum_{i=1}^{n} x_i dy_i.$$

Proof. We may assume without loss of generality that $M = \mathbb{R}^{2n+1}$ and that p is the origin of $\mathbb{R}^{2n+1}$. Let us choose linear coordinates on $T_p\mathbb{R}^{2n+1}$. By the non-integrability condition, $d\alpha$ has rank $2n$ and α is non-trivial on the kernel of $d\alpha$. We first choose the vector belonging to the kernel of $d\alpha$ and then complete a symplectic basis of $d\alpha$.

Let us first treat the case where $\ker d\alpha \subset T_pZ$: We choose x_1 such that $\frac{\partial}{\partial x_1} \in \ker d\alpha$ and $\alpha(\frac{\partial}{\partial x_1}) = 1$. Now let us take $V \in \ker\alpha$, but $V \notin T_pZ$ such that $\iota_V d\alpha \neq 0$. As $V \notin T_pZ$, V belongs to the kernel of the a vector bundle morphism

$${}^bTM|_Z \to TZ$$

as explained in [GMP14]. We take the coordinate z such that $V = z\frac{\partial}{\partial z}$. We then choose a coordinate y_1 such that $\frac{\partial}{\partial y_1} \in \ker\alpha$ and $d\alpha(z\frac{\partial}{\partial z}, \frac{\partial}{\partial y_1}) = 1$.

We complete a symplectic basis of $d\alpha$ and we can choose the remaining $2n-2$ coordinates x_i and y_i in both cases so that for all $i = 2, \dots, n$ that $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \in T_pZ$.

We now set

$$\alpha_0 = dx_1 + y_1 \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i \text{ when } \ker d\alpha \subset Z \tag{3.1}$$

when ξ_p is singular and when ξ_p is regular we set

$$\tilde{\alpha}_0 = dx_1 + y_1 \frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i \text{ when } \ker d\alpha \subset Z.$$

By the choice of the basis, it is clear that at the origin,

$$\begin{cases} \alpha = \alpha_0 \\ d\alpha = d\alpha_0 \end{cases}$$

when ξ_p is singular. We only work out the details in this case, as the case ξ_p regular works analogously.

Note that, until this stage, we only used linear algebra arguments, which are more involved due to the structure of the vector bundle ${}^bTM$. Let us now apply Moser's path method. In a neighbourhood of p, we consider the family of b-forms of degree 1

$$\alpha_t = (1-t)\alpha_0 + t\alpha \quad \text{for} \quad t \in [0,1].$$

By the choice of basis, it is clear that at the origin,

$$\begin{cases} \alpha_t = \alpha \\ d\alpha_t = d\alpha \end{cases}$$

and so α_t is a path of b-contact forms in a neighbourhood of the origin. We want to show that there exists an isotopy $\psi_t : \mathcal{U} \mapsto \mathbb{R}^{2n+1}$ satisfying

$$\begin{cases} \psi_t^* \alpha_t = \alpha_0 \\ \psi_t(p) = p \\ \psi_t|_Z \subset Z. \end{cases} \tag{3.2}$$

Differentiating the first equation, we obtain $\mathcal{L}_{X_t}\alpha_t + \dot{\alpha}_t = 0$, where $X_t(p) = \frac{d\psi_s}{ds}\left(\psi_t^{-1}(p)\right)\Big|_{s=t}$. Inserting the splitting $X_t = H_t R_{\alpha_t} + Y_t$, where $H_t \in C^\infty(M)$ and $Y_t \in \ker \alpha_t$ and applying Cartan's formula, we obtain

$$\iota_{Y_t} d\alpha_t + dH_t + \dot{\alpha}_t = 0. \tag{3.3}$$

Evaluating this differential equation in the Reeb vector field R_{α_t}, we obtain

$$dH_t(R_{\alpha_t}) + \dot{\alpha}_t(R_{\alpha_t}) = 0. \tag{3.4}$$

This equation can be solved locally around the point p, as we can assume without loss of generality that R_{α_t} does not have closed orbits around that point. This is due to the fact that $R_{\alpha_t} \neq 0$. In fact, by the construction of the coordinate system $R_\alpha = \frac{\partial}{\partial x_1}$. Furthermore, as $\dot{\alpha}_t(p) = 0$, $dH_t(p) = 0$, and we can choose the constant of integration such that $H_t(p) = 0$. Once H_t is chosen, let us take a look at Equation (3.3), given by

$$\iota_{Y_t} d\alpha_t = -(dH_t + \dot{\alpha}_t).$$

We want to solve this equation for Y_t. By the previous observation and the fact that $d\alpha_t$ is a b-symplectic form, we obtain that $Y_t(p) = 0$, so $X_t(p) = 0$. Furthermore, it is clear that Y_t is a b-vector field because $d\alpha$ is a b-form. Integrating the vector field X_t gives us the isotopy ψ_t, satisfying the conditions of (3.2). This proves the first part of the theorem.

Let us now consider the case where $\ker d\alpha \not\subseteq T_pZ$, which corresponds to the case where $R_p = 0$ and $d\alpha$ is a smooth de Rham form. A b-form decomposes as $f\frac{dz}{z} + \beta$, where z is a defining function. As $d\alpha$ is smooth, the function f can only depend on z on Z and hence, $f(p) \neq 0$ as we would be in the smooth case otherwise. We choose a neighbourhood $\mathcal{U}$ around the origin such that f is non-vanishing on that neighbourhood. By dividing by f, the b-form $\tilde{\alpha} = \frac{dz}{z} + \tilde{\beta}$ defines the same distribution. Now take a contractible $2n$-dimensional disk $D^{2n} \ni p$ in $\mathcal{U}$. As $(D, d\alpha)$ is symplectic, we know by applying Darboux theorem for symplectic forms (we assume the disk D small enough), that there exist $2n$ functions x_i, y_i such that locally $d\alpha = \sum_{i=1}^n dx_i \wedge dy_i$. Now consider the b-form $\alpha - \sum_{i=1}^n x_i dy_i - \frac{dz}{z}$. This form is closed and smooth. Hence by Poincaré lemma for smooth forms, there exists a smooth function g such that

$$\tilde{\alpha} = \frac{dz}{z} + dg + \sum_{i=1}^n x_i dy_i.$$

We can change the defining function by $\tilde{z} = e^{-g}z$, so that $\frac{d\tilde{z}}{\tilde{z}} = \frac{dz}{z} + dg$. Now

$$\tilde{\alpha} = \frac{d\tilde{z}}{\tilde{z}} + \sum_{i=1}^n x_i dy_i.$$

As $\tilde{\alpha} \wedge (d\tilde{\alpha})^n = n\frac{d\tilde{z}}{\tilde{z}} \wedge \sum_{i=1}^n dx_i \wedge dy_i \neq 0$, the functions $\tilde{z}, x_i, y_i$ form a basis. □

Remark 3.1.2. It follows from the b-Darboux theorem that if $(M, \ker\alpha)$ is a b-contact manifold and $\ker\alpha_p$ is regular for $p \in Z$, then there is an open neighbourhood around p where $\ker\alpha$ is regular.

The following example shows that it is possible to have both local models appearing on one connected component of the critical set. Furthermore, it shows in the case where the Reeb vector field is singular, we can only prove the normal form pointwise and does not hold in a local neighbourhood as when the Reeb vector field is regular.

Example 3.1.3. $(S^2 \times S^1, \alpha = \sin\varphi d\theta + \cos\varphi\frac{dh}{h})$ where (θ, h) are the polar coordinates on S^2 and φ the coordinate on S^1. The Reeb vector field is given by $R_\alpha = \sin\varphi\frac{\partial}{\partial\theta} + \cos\varphi h\frac{\partial}{\partial h}$.

We will prove that there are at least two points where the Reeb vector field is singular in the compact, 3-dimensional case. This will be a corollary of the following. By definition of the b-tangent bundle, the Reeb vector field is tangent to the critical set. We can prove that in dimension 3, the Reeb vector field is in fact Hamiltonian with respect to the induced area form from the contact condition. We will prove the following theorem:

Theorem 3.1.4. *Let $(M, \alpha = u\frac{dz}{z} + \beta)$ be a b-contact manifold of dimension 3, where $u \in C^\infty(M)$ and $\beta \in \Omega^1(M)$ as in Equation 2.6. Then the restriction on Z of the 2-form $\Theta = ud\beta + \beta \wedge du$ is symplectic and the Reeb vector field is Hamiltonian with respect to Θ with Hamiltonian function $-u$, i.e. $\iota_R\Theta = du$.*

Proof. In the decomposition, α is given by $\alpha = u\frac{dz}{z} + \beta$, where $u \in C^\infty(M)$ and $\beta \in \Omega^1(M)$. As α is a b-contact form, we compute

$$d\alpha = du \wedge \frac{dz}{z} + d\beta + dz \wedge \frac{\partial \beta}{\partial z},$$

where $d\beta$ is the exterior derivative on Z. Hence

$$\alpha \wedge d\alpha = \frac{dz}{z} \wedge (ud\beta + \beta \wedge du) + \beta \wedge dz \wedge \frac{\partial \beta}{\partial z}.$$

As α is b-contact, this last expression is non-zero as a b-form of degree 3. We claim that at a point $p \in Z$, $\Theta := ud\beta + \beta \wedge du$ is non-vanishing. Assume by contradiction that $(ud\beta + \beta \wedge du)_p = 0$, but then $\alpha \wedge d\alpha$ is a smooth contact form at p which is a contradiction because $p \in Z$.

Hence the b-contact condition implies that $\Theta := ud\beta + \beta \wedge du$ is an area form and $\dim Z = 2$, it is symplectic. In the same decomposition, let us write the Reeb vector field as $R_\alpha = g \cdot z\frac{\partial}{\partial z} + X$, where $g \in C^\infty(M)$ and $X \in \mathfrak{X}(Z)$. As R_α is the Reeb vector field, we obtain the following equations:

$$\begin{aligned} g \cdot u + \beta(X) &= 1, \\ -g du + \iota_X d\beta &= 0, \\ \iota_X du &= 0. \end{aligned}$$

A straightforward computation using those equations yield that $\iota_X \Theta = du$, hence the restriction of R_α to Z is the Hamiltonian vector field for the function $-u$. □

In the compact case, the function u attains a minimum and maximum and therefore we obtain:

Corollary 3.1.5. *Let (M, α) be a 3-dimensional closed b-contact manifold. Then there are at least two points where the local normal form of α is described by the singular model of the Darboux theorem.*

We furthermore notice that u is not constant on the critical set in the compact case.

Proposition 3.1.6. *Let $(M, \alpha = u\frac{dz}{z} + \beta)$ be a 3-dimensional closed b-contact manifold. Then the function $u|_Z$ is non-constant.*

Proof. Assume by contradiction $u|_Z$ to be constant. Then the area form on Z given by $\Theta = ud\beta$ is exact. As the dimension of M is 3, it is an exact symplectic form on the closed surface Z. This contradicts Stokes theorem. □

Remark 3.1.7. As shown in Example 3.0.4, there is a b-contact structure on the unit disk under the pull-back under the Möbius transformation of the regular local model. It follows from the last corollary, that this example can not be compactified.

Example 3.1.8. As before, consider $(S^2 \times S^1, \alpha = \sin\varphi d\theta + \cos\varphi \frac{dh}{h})$. The Reeb vector field on the critical set is given by Hamiltonian vector field of the function $-\cos\varphi$ with respect to the area form $d\varphi \wedge d\theta$. Hence, on the critical set, the Reeb vector field vanishes when $\cos\varphi = 0$ and there are no periodic orbits of the Reeb vector field on the critical set.

A well known result in contact geometry is Gray's stability theorem, asserting that on a closed manifold, smooth families of contact structures are isotopic. The proof uses Moser's path method that works well in b-geometry. One proves in the same line the following stability result for b-contact manifolds.

Theorem 3.1.9. *Let (M, Z) compact b-manifold and let (ξ_t), $t \in [0,1]$ be a smooth path of b-contact structures. Then there exists an isotopy ϕ_t preserving the critical set Z such that $(\phi_t)_*\xi_0 = \xi_t$, or equivalently, $\phi_t^*\alpha_t = \lambda_t\alpha_0$ for a non-vanishing function λ_t.*

Proof. Assume that ϕ_t is the flow of a time-dependent vector field X_t. Deriving the equation, we obtain

$$d\iota_{X_t}\alpha_t + \iota_{X_t}d\alpha_t + \dot{\alpha} = \mu_t\alpha_t$$

where $\mu_t = \frac{\dot{\lambda}_t}{\lambda_t} \circ \phi_t^{-1}$. If X_t belongs to ξ_t, the first term of the last equation vanishes and applying then the Reeb vector field yields

$$\dot{\alpha}_t(R_{\alpha_t}) = \mu_t.$$

The equation given by

$$\iota_{X_t}d\alpha_t = \mu_t\alpha_t - \dot{\alpha}_t$$

then defines X_t because $(\mu_t\alpha_t - \dot{\alpha}_t)(R_{\alpha_t})$. We integrate the vector field X_t to find ϕ_t and as X_t is a vector field, tangent to the critical set, the flow preserves it. □

The compactness condition is necessary as it is shown in the next example.

Example 3.1.10. Consider the path of b-contact structures on $\mathbb{R}^3$ given by $\ker \alpha_t$ where $\alpha_t = (\cos\frac{\pi}{2}t - y\sin\frac{\pi}{2}t)\frac{dz}{z} + (\sin\frac{\pi}{2}t + y\cos\frac{\pi}{2}t)dx$. As $\alpha_0 = \frac{dz}{z} + ydx$ and $\alpha_1 = dx - y\frac{dz}{z}$, the two b-contact structures cannot be isotopic.

In the same lines, we prove the following semi-local result.

Theorem 3.1.11. *Let (M, Z) be a b-manifold and assume Z compact. Let $\xi_0 = \ker \alpha_0$ and $\xi_1 = \ker \alpha_1$ be two b-contact structures such that $\alpha_0|_Z = \alpha_1|_Z$. Then there exists a local isotopy ψ_t, $t \in [0,1]$ in an open neighbourhood $\mathcal{U}$ around Z such that $\psi_t^*\alpha_t = \lambda_t\alpha_0$ and $\psi_t|_Z = \mathrm{Id}$ where λ_t is a family of non-vanishing smooth functions.*

Proof. The proof is done following Moser's path method. Put $\xi_t = (1-t)\xi_0 + t\xi_1$, $t \in [0,1]$. Because the non-integrability condition is an open condition and $\xi_t|_Z = \xi_0|_Z = \xi_1|_Z$, there exists an open neighbourhood $\mathcal{U}$ containing Z such that ξ_t is a family of b-contact structures. We will prove that there exists an isotopy $\psi_t : \mathcal{U} \mapsto M$ such that $\psi_t^*\alpha_t = \lambda_t\alpha_0$, where λ_t is a non-vanishing smooth function and $\lambda_t|_Z = \mathrm{Id}$. Assume that ψ_t is the flow of a vector field X_t and differentiating, we obtain the following equation:

$$d\iota_{X_t}\alpha_t + \iota_{X_t}d\alpha_t + \dot{\alpha}_t = \mu_t\alpha_t,$$

where $\mu_t = \frac{d}{dt}(\log|\lambda_t|) \circ \psi_t^{-1}$. Taking $X_t \in \xi_t$, this equation writes down

$$\dot{\alpha}_t + \iota_{X_t} d\alpha_t = \mu_t \alpha_t. \tag{3.5}$$

Applying the Reeb vector field to both sides, we obtain the equation that defines μ_t:

$$\mu_t = \dot{\alpha}_t(R_{\alpha_t}).$$

As $\dot{\alpha}_t|_Z = 0$, $\mu_t|_Z = 0$ and hence X_t is zero on Z. By non-degeneracy of $d\alpha_t$ on ξ_t there exists a unique $X_t \in \xi_t$ solving Equation 3.5. Integrating X_t yields the desired result. □

Note that this proof fails if one wants to prove stability of b-contact forms, that is we cannot assume that $\lambda_t = \text{Id}$ in a neighbourhood of Z.

3.2 b-Jacobi manifolds

We already mentioned that in the b-symplectic manifolds, it is often helpful to look at b-symplectic manifolds as being the dual of a particular case of Poisson manifold. In contact geometry, Jacobi manifolds play this role as explained by the geographical map, see Figure 2.2.

Recall that a Jacobi structure on a manifold M is a triplet (M, Λ, R) where Λ is a smooth bi-vector field and R a vector field satisfying the following compatibility conditions:

$$[\Lambda, \Lambda] = 2R \wedge \Lambda, \qquad [\Lambda, R] = 0. \tag{3.6}$$

Definition 3.2.1. Let (M, Λ, R) be a Jacobi manifold of dimension $2n+1$. We say that M is a b-Jacobi manifold if $\Lambda^n \wedge R$ cuts the zero section of $\Lambda^{2n+1}(TM)$ transversally.

Note that this definition is similar to the one of b-Poisson manifolds, in the sense that it also asks the top wedge power to be transverse to the zero section. We denote the hypersurface given by the zero section of $\Lambda^{2n+1}(TM)$ by Z and we call it the *critical set.*

We will prove that b-contact manifolds and b-Jacobi manifolds are dual in some sense, as will be explained in the next two propositions. Before doing so, let us note that in the case where the dimension of the Jacobi manifold is $\dim M = 2n$, we can given an similar definition to the one of Definition 3.2.1 by asking that Λ^{2n} cuts the zero-section of $\Lambda^{2n}(TM)$ transversally. It should be possible to prove in the same lines that this case corresponds to locally conformally b-symplectic manifold.

Proposition 3.2.2. *Let $(M, \ker \alpha)$ be a b-contact manifold. Let Λ be the bi-vector field computed as in Equation 2.3 in Section 2.6 and let R be the Reeb vector field. Then (M, Λ, R) is a b-Jacobi manifold.*

Proof. As being b-Jacobi is a local condition, we can work in a local coordinate chart. Outside of the critical set, α is a contact form. Hence we can compute Λ as in Equation 2.3 in both local models of the Darboux theorem and Λ can smoothly be extended to the critical set Z. A straightforward computation now yields that for both local models $\Lambda^n \wedge R \pitchfork 0$. □

Recall that to every Jacobi manifold (M, Λ, R), one can associate a homogeneous Poisson manifold obtained through Poissonization, see Proposition 2.6.6. The same stays true in the b-scenario, although we need to assume that the b-Jacobi manifold is of odd dimension, as b-Poisson manifold are defined only for even dimensions.

Lemma 3.2.3. *The Poissonization of a b-Jacobi manifold of odd di-*

mension is a homogeneous b-Poisson manifold.

Proof. The proof is a straightforward computation:

$$\Pi^{n+1} = -e^{-(n+1)\tau}\frac{\partial}{\partial\tau} \wedge \Lambda^n \wedge R.$$

It follows from the definition of b-Jacobi that Π is transverse to the zero-section. □

We now prove that there is a one-to-one correspondence between b-Jacobi and b-contact manifolds. The proof is based on the local normal form of Jacobi structures, recalled in Section 2.6 in Theorem 2.6.5.

Proposition 3.2.4. *Let (M^{2n+1}, Λ, R) be a b-Jacobi manifold. Then M is a b-contact manifold.*

Proof. Let (M, Λ, R) be the b-Jacobi structure, so that $\Lambda^n \wedge R \pitchfork 0$. As usual, denote the critical hypersurface by $Z = (\Lambda^n \wedge R)^{-1}(0)$. First note that outside of Z, the leaf of the characteristic foliation is maximal dimensional. This is saying that outside of Z, the Jacobi structure is equivalent to a contact structure.

Consider a point $p \in Z$ and denote the leaf of the characteristic foliation by L. By the transversality condition, the dimension of the leaf needs to be of dimension $2n$ or $2n-1$. Indeed, as $(M \times \mathbb{R}, e^{-\tau}(\frac{\partial}{\partial\tau} \wedge R + \Lambda))$ is b-Poisson, the critical set of $M \times \mathbb{R}$ is foliated by symplectic manifolds of codimension 2, that is of dimension $2n$. Hence the critical set restricted to the hypersurface $\{\tau = 0\}$, which is identified to be the critical set Z of the initial manifold M, is foliated by codimension 1 and codimension 2 leaves.

Let us first consider the case where at the point $x \in Z$, the leaf is of dimension $2n$. We will prove that this case corresponds to the case where the R is singular, vanishing linearly. Let us apply Theorem 2.6.5. Hence the Jacobi manifold (N, Λ_N, E_N) is of dimension 1, hence Λ_N is zero. Hence Λ is given by

$$\Lambda = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_{i+n}} - \sum_{i=1}^{n} x_{i+n} \frac{\partial}{\partial x_{i+n}} \wedge E_N.$$

We now use the transversality condition on $\Lambda^n \wedge E_N$ to conclude that $E_N = z\frac{\partial}{\partial z}$. which is the same expression for the b-Jacobi structure associated to the b-contact form $\alpha = \frac{dz}{z} + \sum_{i=1}^{n} x_i dx_{i+n}$.

Let us consider the case where the leaf is of dimension $2n-1$. We will see that this corresponds to the case where the Reeb vector field is regular. According to Theorem 2.6.5, the bi-vector field is given by

$$\Lambda = \Lambda_{2n-1} + \Lambda_N + E \wedge Z_N$$

where (N, Λ_N, Z_N) is a homogeneous 2-dimensional Poisson manifold and

$$\Lambda_{2n-1} = \sum_{i=1}^{n-1} (x_{i+n-1} \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_i}) \wedge \frac{\partial}{\partial x_{i+q}}$$

. The transversality condition implies that $\Lambda_{2n-1}^{n-1} \wedge \Lambda_N \wedge \frac{\partial}{\partial x_0} \pitchfork 0$, hence Λ_N is a b-Poisson manifold. By [GMPS15], $\Lambda_N = z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial y}$. The homogeneous vector field Z_N is determined by equation $\mathcal{L}_{Z_N}\Lambda_N = -\Lambda_N$. Hence $Z_N = y\frac{\partial}{\partial y}$. Hence the Jacobi structure is given by $E = \frac{\partial}{\partial x_0}$ and

$$\Lambda = \sum_{i=1}^{n-1} (x_{i+n-1} \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_i}) \wedge \frac{\partial}{\partial x_{i+q}} + z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x_0} \wedge y\frac{\partial}{\partial y},$$

which is the Jacobi structure associated to the contact form $\alpha = dx_0 + y\frac{dz}{z} + \sum_{i=1}^{n-1} x_i dx_{i+q}$. □

Combining Proposition 3.2.4 and 3.2.2, we obtain the following correspondence.

Corollary 3.2.5. *Let* $(M^{2n+1}, \ker\alpha)$ *be a b-contact manifold. Then* (M, Λ, R) *is a Jacobi manifold, where* Λ *is the bi-vector field computed as in Equation 2.3, satisfying the following transversality conditions* $\Lambda^n \wedge R \pitchfork 0$*. Conversely, any odd-dimensional Jacobi manifold that satisfies that transversality condition defines a b-contact structure.*

3.3 Geometric structure on the critical set

To determine the induced structure of the b-contact structure on the critical set, we compute the associated Jacobi structure. As already mentioned in Section 2.6, the Hamiltonian vector fields defined by $X_H = \Lambda^\sharp(dH) + HR$ integrate to a foliation, where the leaves are either of contact or locally conformally symplectic type, depending on the parity of the dimension of the leaf.

The computation of a Jacobi structure associated to a contact structure is explained in Section 2.6. As we have proved a local norm form theorem, we can use the two local models to compute the associate Jacobi structure and check in both cases if $R \in \Lambda^\sharp$. We will prove

Theorem 3.3.1. *Let* $(M^{2n+1}, \xi = \ker\alpha)$ *be a b-contact manifold and* $p \in Z$*. We denote* $\mathcal{F}_p$ *the leaf of the singular foliation* $\mathcal{F}$ *going through* p*. Then*

1. *if* ξ_p *is regular, that is* $\mathcal{F}_p$ *of dimension* $2n$*, then the induced structure on* $\mathcal{F}_p$ *is locally conformally symplectic;*
2. *if* ξ_p *is singular, that is* $\mathcal{F}_p$ *of dimension* $2n-1$*, then the induced structure on* $\mathcal{F}_p$ *is contact.*

Proof. By Theorem 3.1.1, if ξ_p is singular, the Reeb vector field is not singular and the contact form can be written locally as $\alpha = dx_1 + y_1\frac{dz}{z} + \sum_{i=2}^n x_i dy_i)$. The Reeb vector field is given by $R = \frac{\partial}{\partial x_1}$, the dual of $d\alpha$ by $\Pi = z\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial z} + \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$. As Liouville vector field with respect to $d\alpha$, we take $X = \sum_{i=1}^n y_i \frac{\partial}{\partial y_i}$. The Jacobi structure associated to this b-contact structure is given by $\Lambda = \Pi + R \wedge X$.

On the critical set, we have

$$\Lambda\Big|_Z = \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i=1}^n y_i \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_i}.$$

Let us check if we can find a one form α such that $\Lambda\Big|_Z^{\#}(\alpha) = \frac{\partial}{\partial x_1}$. For $y_1 = 0$, this cannot be solved, hence the set $\{z = 0, y_1 = 0\}$ is a leaf with an induced contact structure.

If ξ_p is not singular and the Reeb vector is regular, the contact form can be written locally as $\alpha = dx_1 + y_1\frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^n x_i dy_i)$. A direct computation implies that the Reeb vector field lies in the distribution spanned by the bi-vector field Λ, hence the b-contact structure induces a locally conformally symplectic structure on the set $\{z = 0, y_1 \neq 0\}$.

Last, if ξ_p is not singular and the Reeb vector is singular, Theorem 3.1.1 yields that the Reeb vector field can be written as $z\frac{\partial}{\partial z}$. As the Reeb vector field is vanishing, the critical set equals the $2n$-dimensional leaf spanned by $\mathrm{Im}\Lambda^\sharp$. The induced structure on $\mathcal{F}_p$ is locally conformally symplectic. □

Remark 3.3.2. Let us consider the case where $\dim M = 3$ and the distribution ξ is singular. Then the induced structure on the critical set is given by $\Lambda|_Z = y_1\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial x_1}$. As the critical set is a surface, it is clear that this is a Poisson structure and furthermore, that it is transverse to the zero section. Hence we obtain an induced b-symplectic structure on the critical set. Note that this is not true for higher dimensions.

3.4 Symplectization and contactization

We will see that the symplectization of a b-contact manifold is a b-symplectic manifold.

Similarly, particular types of hypersurfaces in b-symplectic manifolds are b-contact.

Example 3.4.1. Let $(\mathbb{R}^4, \omega = \frac{1}{z} dz \wedge dt + dx \wedge dy)$ be a b-symplectic manifold. A Liouville vector field is given by $X = \frac{1}{2}(z\frac{\partial}{\partial z} + 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$. Note that Liouville vector fields are defined up to addition of symplectic vector fields, that is a vector field Y satisfying $\mathcal{L}_Y \omega = 0$. Another Liouville vector field is for example given by $t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$.

Let us take a b-symplectic manifold (W, ω) of dimension $(2n+2)$ and a Liouville vector field X on W that is transverse to a hypersurface H of W. Then $(H, \iota_X \omega)$ is a b-contact manifold of dimension $(2n+1)$ as $\iota_X \omega \wedge (d\iota_X \omega)^n = \frac{1}{n+1} \iota_X(\omega^{n+1})$ is a volume form provided that X is transverse to H. If H does not intersect the critical set, one obtains of course a smooth contact form. Due to the b-Darboux theorem, there are two local models for b-contact manifolds and we will see that we can obtain both structures, depending on the relative position of the hypersurface with the Reeb vector field on it.

Example 3.4.2. Let us take $(W = \mathbb{R}^4, \omega = \frac{1}{z} dz \wedge dt + dx \wedge dy)$ and the Liouville vector field $X = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$. The contraction of X with the b-symplectic form yields $\iota_X \omega = -\frac{t}{z} dz + x dy$. Let us take different hypersurfaces transverse to X and compute the induced b-contact form.

- If we take as hypersurface the hyperplane

$$M_1 = \{(1, y, -t, z), y, t, z \in \mathbb{R}\},$$

which is transverse to X, we obtain $\alpha = dy + t\frac{dz}{z}$, which is the regular local model.

- If we take as hypersurface the hyperplane

$$M_2 = \{(x, y, -1, z), x, y, z \in \mathbb{R}\},$$

which is transverse to X, we obtain $\alpha = \frac{dz}{z} + xdy$, which is the singular local model.

Example 3.4.3. The three dimensional sphere admits a b-contact structure. Consider the $\mathbb{R}^4$ with the standard b-symplectic structure $\omega = \frac{dx_1}{x_1} \wedge dy_1 + dx_2 \wedge dy_2$ and denote by S^3 the unit sphere in $\mathbb{R}^4$. The Liouville vector field $X = \frac{1}{2}x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} + \frac{1}{2}(x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2})$ is transverse to the sphere and hence $\iota_X\omega$ defines a b-contact form on S^3. The critical set is a 2 dimensional sphere, S^2, given by the intersection of the sphere with the hyperplane $z = 0$.

Example 3.4.4. The unit cotangent bundle of a b-manifold has a natural b-contact structure. Let (M, Z) be a b-manifold of dimension n with coordinates $z, y_i, i = 2, \ldots, n$ as in Example 3.0.2. It is shown in [GMP14] that the cotangent bundle has a natural b-symplectic structure defined by the b-form given by the exterior derivative $d\lambda = d\Big(x_1\frac{dz}{z} + \sum_{i=2}^n x_i dy_i\Big)$. The unit b-cotangent bundle is given by ${}^bT_1^*M = \{(z, y_2, \ldots, y_n, x_1, \ldots, x_n) \in {}^bT^*M |\ \sum_{i=1}^n x_i^2 = 1\}$. The vector field $\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}$ defined on the b-cotangent bundle ${}^bT^*M$ is a Liouville vector field, and is transverse to the unit b-cotangent bundle, and hence induces a b-contact structure on it.

From a Riemannian point of view, the Reeb vector field describes the geodesic flow associated to a b-metric, that is a bundle metric on ${}^bTM$. In the coordinate chart (z, y_i), a b-metric is give by $g = \frac{1}{z^2}dz\otimes dz + \sum_{i=2}^n dy_i \otimes dy_i$ and induces a bundle metric g^* on ${}^bT^*M$. The unit cotangent bundle is alternatively described by ${}^bT_1^*M = \{X \in {}^bT^*M | g^*(X, X) = 1\}$ and the associated Reeb vector field to the associated contact form as described above, is the push-forward under the bundle isomorphism of the geodesic vector field on ${}^bTM$.

We will compute a particular case of the unit cotangent bundle of a b-manifold.

Example 3.4.5. Consider the torus $\mathbb{T}^2$ as a b-manifold where the boundary component is given by two disjoint copies of S^1. The unit cotangent bundle $S^*\mathbb{T}^2$, diffeomorphic to the 3-torus $\mathbb{T}^3$ is a b-contact manifold with b-contact form given by $\alpha = \sin\phi \frac{dx}{\sin(x)} + \cos\phi dy$, where ϕ is the coordinate on the fibre and (x, y) the coordinates on $\mathbb{T}^2$.

We saw that hypersurfaces of b-symplectic manifolds that are transverse to a Liouville vector field have an induced b-contact structure. The next lemma describes which model locally defines the b-contact structure.

Lemma 3.4.6. *Let (W, ω) be a b-symplectic manifold and X a Liouville vector field transverse to a hypersurface H. Let R be the Reeb vector field defined on H for the b-contact form $\alpha = i_X\omega$. Then $R \in H^\perp$, where $H^\perp$ is the symplectic orthogonal of H.*

Proof. The Reeb vector field defined on H satisfies

$$i_R(d\alpha)|_H = i_R(di_X\omega)|_H = i_R\omega|_H = 0.$$

□

Hence if $H^\perp$ is generated by a singular vector field, the contact manifold (H, α) is locally of the second type as in the b-Darboux theorem. In the other case, the local model is given by the first type.

We now come back to the contactization of a b-symplectic manifold.

Theorem 3.4.7. *Let (M, α) be a b-contact manifold. Then $(M \times \mathbb{R}, \omega = d(e^t\alpha))$ is a b-symplectic manifold.*

Proof. It is clear that ω is a closed b-form. Furthermore, a direct computation yields

$$\Big((e^t d\alpha)\Big)^{n+1} = e^{t(n+1)} dt \wedge \alpha \wedge (d\alpha)^n,$$

which is non-zero as a b-form by the non-integrability condition. □

It is easy to see that $\frac{\partial}{\partial t}$ is a Liouville vector field of the symplectization $(M \times \mathbb{R}, d(e^t\alpha))$, which is clearly transverse to the submanifold $M \times \{0\}$. Hence, we obtain the initial contact manifold (M, α). This gives us the following proposition.

Proposition 3.4.8. *Every b-contact manifold can be obtained as a hypersurface of a b-symplectic manifold.*

Remark 3.4.9. Another close relation between the symplectic and the contact world is the contactization: take an exact symplectic manifold, i.e. $(M, d\beta)$, then $(M \times \mathbb{R}, \beta + dt)$, where t is the coordinate on $\mathbb{R}$, is contact. This remains true in the b-case. Furthermore, it is clear that by this construction, we obtain b-contact forms of the first type, as the Reeb vector field is given by $\frac{\partial}{\partial t}$.

3.5 Other singularities

In this section, we consider other singularities in contact geometry. The first example is obtained by considering higher order singularities instead of the transversality condition, obtaining b^m-contact structures.

3.5.1 b^m-contact structures

In the light of b^m-symplectic structures, we consider contact structures with higher order singularities. As explained in Subsection 2.7.2, the b^m-tangent bundle ${}^{b^m}TM$ is the vector bundle whose sections are given by

$$\left\{ z^m \frac{\partial}{\partial z}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}} \right\}.$$

As usual, here $Z \subset M$ denotes the hypersurface given as the zero level-set of the function z. Similar to b^m-symplectic manifolds, we define b^m-contact forms as follows.

Definition 3.5.1. A b^m-contact structure is the distribution given by the kernel of a one b^m-form $\xi = \ker \alpha \subset {}^{b^m}TM$, $\alpha \in {}^{b^m}\Omega^1(M)$, that satisfies $\alpha \wedge (d\alpha)^n \neq 0$ as a section of $\Lambda^{2n+1}({}^{b^m}T^*M)$. We say that α is a b-contact form and the pair (M, ξ) a b^m-contact manifold.

The proofs of the theorems of the previous sections, in particular Theorem 3.1.1 and Proposition 3.2.4 and the construction carried out in Section 3.4, generalize directly to this setting. For the sake of a clear notation, we do not write down the statements of the generalization, but only assert that b can be replaced by b^m in the statements.

3.5.2 Folded contact

We briefly introduce another kind of singularities of contact forms. Those correspond to the odd-dimensional counterpart of folded symplectic structures, see Definition 2.7.8.

Definition 3.5.2. A form $\alpha \in \Omega^1(M^{2n+1})$ is a folded contact form if $\alpha \wedge (d\alpha)^n \pitchfork 0$. The hypersurface $Z := \{x \in M | \alpha \wedge (d\alpha)^n(p) = 0\}$ is called *folding hypersurface*. The distribution $\ker \alpha$ is called folded contact structure and we will see that the rank of this distribution can change (as in the b^m-contact case).

As in the b^m-contact case, away from the folding hypersurface, the kernel of folded contact forms define a contact structure. Folded contact forms are dual in some sense to b^m-contact forms and we will see that they are related through the desingularization technique (see Proposition 4.2.5).

Example 3.5.3. The following are examples of folded contact forms on $\mathbb{R}^3$ with folding surface Z:

1. $\alpha = dz + x^2 dy$ with $Z = \{x = 0\}$;

2. $\alpha = dz + xydy$ with $Z = \{y = 0\}$;

3. $\alpha = ydz + xdy$ with $Z = \{x = -y\}$;

4. $\alpha = zdz + xdy$ with $Z = \{z = 0\}$.

The two last examples are different from the first two: the kernel of the first two defines a hyperplane distribution that is integrable along the folding hypersurface. This is contrast to the last two examples because there are points on the folding hypersurface where $\alpha(p) = 0$. At those singular points, the kernel is no longer a hyperplane distribution but spans the whole space. In other words, the Reeb vector field defined on $M \setminus Z$ extends smoothly over Z in the two first examples. In the singular points of the last two examples, this vector field blows-up.

This behaviour is to be compared to the Darboux theorem of b^m-contact forms, where the Reeb vector field can vanish on the critical hypersurface.

We don't enter in a more detailed study of folded contact forms but only remark that folded contact structures appear in the work of [Mar70, JZ01] under the name of *singular contact structures* that are structurally smooth (this condition is just the transversality condition as in Definition 3.5.2). The folding hypersurface is called Martinet hypersurface.

We also remark that this is a particular case of confoliation, as studied in [YET⁺98].

We will relate the study of folded contact to the one of b^m-contact geometry in Section 4.2. We also point out that the results, as explained in Section 3.4 also hold in the folded case: indeed the symplectization of a folded contact structure is a folded-symplectic structure.

We will see in Chapter 4 that in dimension 3, a generic surface in a manifold can be seen as the folding surface of a folded contact form, see Corollary 4.3.8.

3.5.3 E-contact

In this subsection, we investigate examples of E-contact forms. Those appear in the literature as contact Lie algebroids, see [IP17]. As in Subsection 2.7.4, we assume that E is a locally finitely generated submodule of $\mathfrak{X}(M)$ and that it is involutive. It follows from these assumptions that there exists a vector bundle ${}^E TM$ whose sections are given by E and that there is a well-defined exterior derivative for E-forms. We call the set of points such that at the level of germs ${}^E TM$ does not coincide with TM the *singular locus* of M.

Definition 3.5.4. Let $\alpha \in {}^E\Omega(M)$ an E-form of degree 1 and assume the rank of ${}^E TM$ is given by $(2n+1)$. We say that α is an E-contact form if $\alpha \wedge (d\alpha)^n \neq 0$ as section of the bundle $\Lambda^{2n+1}({}^E T^*M)$.

It follows from the definition that away from the singular locus, E-contact forms are just smooth contact forms.

We will construct several examples.

Example 3.5.5 (C-contact). This is a generalization of b-contact, where self-intersection of the critical hypersurface are allowed.

Example 3.5.6 (Elliptic contact). Consider $E = \langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\rangle$ on $\mathbb{R}^3$. Away from the codimension 2-submanifold (in this case the line $x = y = 0$), those forms define a contact structure. We call the resulting E-contact forms *elliptic* contact forms.

Example 3.5.7 (Isolated singular contact). Consider $E = \langle x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}\rangle$ on $\mathbb{R}^3$. The singular locus is given by the origin of $\mathbb{R}^3$.

Example 3.5.8. Consider a regular foliation $\mathfrak{F}$ of rank $(2n+1)$ of the manifold M^{2n+1+m} and consider E the vector fields tangent to $\mathfrak{F}$. A E-contact form is a smooth differential 1-form such that $\ker \alpha \cap T\mathfrak{F}$ defines a contact distribution on the leaf $\mathfrak{F}$. We call those E-contact forms *foliated contact forms*.

Remark 3.5.9. As before in the case of folded contact and folded symplectic structures, the symplectization of a E-contact form yields a E-symplectic form.

As in the case of b^m-contact forms, every E-contact form α defines a unique Reeb vector field $R_\alpha \in E \subset \mathfrak{X}(M)$ that can be viewed as a smooth vector field by the usual inclusion.

Remark 3.5.10. In the case of foliated contact forms, the Weinstein conjecture has been proved in [dPP18]. In Chapter 5, we will prove the existence of periodic Reeb orbits of E-contact forms on compact manifolds under some supplementary assumptions in Theorem 5.5.10, but do not recover the result proved in [dPP18], see also Remark 5.5.13. In the hypotheses of the theorem that we will prove for compact E-contact manifold, we ask the overtwisted disk to be contained in the interior of the complement of the singular locus. However, the singular locus for foliated contact forms is, in the general case, empty and hence Theorem 5.5.10 does not apply to that case.

Chapter 4

Obstructions and constructions on b^m-symplectic and b^m-contact manifolds

Ja, man muss seinen Traum finden, dann wird der Weg leicht. Aber es gibt keinen immerwährenden Traum, jeden löst ein neuer ab, und keinen darf man festhalten wollen.
Hermann Hesse, *Demian.*

In this chapter, we analyse the obstructions to the existence of b^m-symplectic structures. We desingularize b^m-contact structures to contact and folded contact structures and prove existence of b^m-contact structures through a singularization process. The same techniques yield the singularization of cosymplectic hypersurfaces in symplectic manifold to b^m-symplectic manifolds.

4.1 Obstructions to the existence of b^m-symplectic manifolds

The obstruction for the existence of a symplectic structure on an even-dimensional orientable manifold lies in the second cohomology class $H^2(W)$. A similar statement holds for b^m-symplectic manifolds, where the standard cohomology is substituted by the b-cohomology.

This following theorem has first been proved in [MT+14], here we use a more direct version involving Proposition 10 of [GMP14]. We only consider the orientable case.

Theorem 4.1.1. *Let (W^{2n}, ω) be an orientable b^m-symplectic manifold with compact critical hypersurface Z. Then there exists an element $c \in H^2(W)$ such that $c^{n-1} \neq 0$.*

Proof. Let f be the defining function for the critical set Z. Then we can decompose the b^m-symplectic form as

$$\omega = \alpha \wedge \frac{df}{f^m} + \beta,$$

with $\alpha \in \Omega^1(W)$ and $\beta \in \Omega^2(W)$. By Proposition 10 in [GMP14], which was strictly speaking only proved for b-symplectic manifold, but the b^m-case is the same, the forms α and β can be chosen without loss of generality to be closed. Furthermore, denoting $\iota : Z \hookrightarrow W$ the inclusion, $\iota^*(\alpha \wedge \beta^{n-1})$ is nowhere vanishing, hence it is a volume form on Z. We claim that the class $[\beta]$ satisfies the assumptions of the theorem. Indeed, $[\beta]^{n-1}$ needs to be non-zero. Assuming it to be zero leads to a contradiction by Stokes theorem:

$$0 \neq \int_Z \iota^*(\alpha \wedge \beta^{n-1}) = \int_{\partial Z} \iota^*(\alpha \wedge d\eta) = 0,$$

where we used that Z is compact. $\square$

This theorem gives cohomological obstructions for the existence of b^m-symplectic manifolds. It follows for instance that S^n for $n \geq 0$ does not admit any b^m-symplectic structure.

We point out that in contrast to symplectic geometry, non-orientable manifolds do possibly admit b^{2k-1}-symplectic structures, as for example the projective plane. This can be described by the 2-colorability of the graph whose edges are given by the connected components of the $W \setminus Z$ and the nodes by the connected components of Z, see [MP18].

4.2 Desingularization of b^m-contact manifolds

The proof is based on the idea of [GMW19]. However, in contrast to the symplectic case, we need an additional assumption in order to desingularize the b^m-contact form.

Recall that from Equation 2.6, it follows that a b^m-form $\alpha \in {}^{b^m}\Omega^1(M)$ decomposes $\alpha = u\frac{dz}{z^m} + \beta$ where $u \in C^\infty(M)$ and $\beta \in \Omega^1(M)$. In order to desingularize the b^m-contact forms, we will assume that β is the pull-back under the projection of a one-form defined on Z.

Definition 4.2.1. We say that a b^m-contact structure $(M, \ker\alpha)$ is almost convex if $\beta = \pi^*\tilde{\beta}$, where $\pi : \mathcal{N}(Z) \to Z$ is the projection from a tubular neighbourhood of Z to the critical set and $\tilde{\beta} \in \Omega^1(Z)$. We will abuse notation and write $\beta \in \Omega^1(Z)$. We say that a b^m-contact structure is convex if $\beta \in \Omega^1(Z)$ and $u \in C^\infty(Z)$.

Note that the this notion is to be compared to the one of convex hypersurfaces, which we will recall in the next section. As we will see in the next lemma, almost convex b^m-contact structures are semi-locally isotopic to convex ones.

Lemma 4.2.2. *Let $(M, \ker\alpha)$ be an almost convex b^m-contact manifold and let the critical hypersurface Z be compact. Then there exists a neighbourhood around the critical set $\mathcal{U} \supset Z$, such that α is isotopic to a convex b^m-contact form relative to Z on $\mathcal{U}$.*

Proof. Let $\alpha = u\frac{dz}{z^m} + \beta$ where $u \in C^\infty(M)$ and $\beta \in \Omega^1(Z)$. Put $\tilde{\alpha} = u_0\frac{dz}{z^m} + \beta$, where $u_0 = u|_Z \in C^\infty(Z)$, which is convex. Take the linear path between the two b^m-contact structures, which is a path of b^m-contact structures because ξ and $\tilde{\xi}$ equal on Z. Applying Theorem 3.1.11, we obtain that there exist a local diffeomorphism f preserving Z and a non-vanishing function λ such that on a neighbourhood of Z, $f^*\alpha = \lambda\tilde{\alpha}$. □

The next lemma gives intuition on this definition and gives a geometric characterization of the almost-convexity in terms of the f_ϵ-desingularized symplectization.

Lemma 4.2.3. *A b^m-contact manifold $(M, \ker\alpha)$ is almost-convex if and only if the vector field $\frac{\partial}{\partial t}$ is a Liouville vector field in the desingularization of the b^m-symplectic manifold obtained by the symplectization of $(M, \ker\alpha)$. Here t denotes the coordinate of the symplectization.*

Proof. Let $(M, \ker\alpha)$ be a almost-convex b^m-contact manifold. The symplectization is given by $(M \times \mathbb{R}, \omega = d(e^t\alpha))$. The desingularization technique of Theorem 2.7.9 produces a family of symplectic forms $\omega_\epsilon = ue^t dt \wedge df_\epsilon + e^t dt \wedge \beta + e^t du \wedge df_\epsilon + e^t d\beta$. From almost-convexity it follows that $\frac{\partial}{\partial t}$ preserves ω_ϵ, so $\frac{\partial}{\partial t}$ is a Liouville vector field.

To prove the converse, assume that $\frac{\partial}{\partial t}$ is a Liouville vector field in (M, ω_ϵ). It follows from the fact that $\mathcal{L}_{\frac{\partial}{\partial t}}\omega_\epsilon = \omega_\epsilon$ that $\beta \in \Omega^1(Z)$. □

We will see that under almost-convexity, the b^{2k}-contact forms can be desingularized.

Theorem 4.2.4. *Let $(M^{2n+1}, \ker\alpha)$ a b^{2k}-contact structure with critical hypersurface Z. Assume that α is almost convex. Then there exists $\epsilon_0 > 0$ and a family of usual contact forms α_ϵ, $\epsilon \in (0, \epsilon_0)$ which coincides with the b^{2k}-contact form α outside of an ϵ-neighbourhood of Z. The family of bi-vector fields $\Lambda_{\alpha_\epsilon}$ and the family of vector fields R_{α_ϵ} associated to the Jacobi structure of the contact form α_ϵ converges to the bivector field Λ_α and to the vector field R_α in the C^{2k-1}-topology as $\epsilon \to 0$.*

We call α_ϵ the f_ϵ-desingularization of α.

A corollary of this is that almost-convex b^m-contact forms admit a family of contact structures if m is even, and a family of folded-type contact structures is m is odd.

The proof of this theorem follows from the definition of convexity and makes use of the family of functions introduced in [GMW19].

Proof. By the decomposition lemma, $\alpha = u\frac{dz}{z^m} + \beta$. As α is almost convex, the contact condition writes down as follows:

$$\alpha \wedge (d\alpha)^n = \frac{dz}{z^m} \wedge (u(d\beta)^n + n\beta \wedge du \wedge (d\beta)^{n-1}) \neq 0.$$

In an ϵ-neighbourhood, we replace $\frac{dz}{z^m}$ by a smooth form. The expression depends on the parity of m.

Following [GMW19] we consider an odd smooth function $f \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $f'(x) > 0$ for all $x \in [-1,1]$ and satisfying outside that

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for} \quad x < -1, \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for} \quad x > 1. \end{cases} \tag{4.1}$$

Let $f_\epsilon(x)$ be defined as $\epsilon^{-(2k-1)} f(x/\epsilon)$.

We obtain the family of globally defined 1-forms given by $\alpha_\epsilon = udf_\epsilon + \beta$ that agrees with α outside of the ϵ-neighbourhood. Let us check that α_ϵ is contact inside this neighbourhood. Using the almost-convexity condition, the non-integrability condition on the b^m-form α writes down as follows:

$$\alpha_\epsilon \wedge (d\alpha_\epsilon)^n = dz \wedge (f'_\epsilon(z)ud\beta + f'_\epsilon(z)\beta \wedge du - \beta \wedge \frac{\partial \beta}{\partial z}).$$

We see that $\alpha_\epsilon \wedge d\alpha_\epsilon = f'_\epsilon(z) z^m \alpha \wedge d\alpha$ and hence α_ϵ is contact.

We denote by Λ_α and R_α the bi-vector field and vector field of the b-contact form α. Now let us check that the bi-vector field $\Lambda_{\alpha_\epsilon}$ and the vector field of R_{α_ϵ} corresponding to the Jacobi structure of the desingularization converge to Λ_α and R_α respectively.

Let us write R_α and Λ_α in a neighbourhood of a point $p \in Z$.

$$R_\alpha = gz^{2k}\frac{\partial}{\partial z} + X, \quad \Lambda_\alpha = z^{2k}\frac{\partial}{\partial z} \wedge Y_1 + Y_2 \wedge Y_3$$

where $g \in C^\infty(M)$ and $X, Y_i \in \mathfrak{X}(Z)$ for $i = 1,2,3$. The Jacobi structure associated to the desingularization is given by

$$R_{\alpha_\epsilon} = g\frac{1}{f'_\epsilon(z)}\frac{\partial}{\partial z} + X, \quad \Lambda_{\alpha_\epsilon} = \frac{1}{f'_\epsilon(z)}\frac{\partial}{\partial z} \wedge Y_1 + Y_2 \wedge Y_3.$$

The C^{2k-1}-convergence follows from this formulas. □

It is possible to desingularize b^{2k+1}-contact structures following [GMW19]. The resulting one-form of this desingularization is of folded contact structure defined in Subsection 3.5.2.

Theorem 4.2.5. *Let $(M^{2n+1}, \ker\alpha)$ a b^{2k+1}-contact structure with critical hypersurface Z. Assume that α is almost convex. Then there exist $\epsilon > 0$ and a family of folded contact forms α_ϵ for $\epsilon \in (0, \epsilon_0)$ which coincides with the b^{2k+1}-contact form α outside of an ϵ-neighbourhood of Z.*

Proof. By the decomposition lemma, $\alpha = u\frac{dz}{z^{2k+1}} + \beta$. As α is almost convex, the contact condition writes down as follows:

$$\alpha \wedge (d\alpha)^n = \frac{dz}{z^{2k+1}} \wedge (u(d\beta)^n + n\beta \wedge du \wedge (d\beta)^{n-1}) \neq 0.$$

In an ϵ-neighbourhood, we replace $\frac{dz}{z^{2k+1}}$ by a smooth form.

Following [GMW19] we consider an even smooth function given by $f_\epsilon(x) := \frac{1}{\epsilon^{2k}} f(\frac{x}{\epsilon})$ where $f \in C^\infty(\mathbb{R})$ satisfies

- $f > 0$ and $f(x) = f(-x)$,
- $f'(x) > 0$ if $x < 0$,
- $f(x) = -x^2 + 2$ if $x \in [-1, 1]$,
- $f(x) = \log(|x|)$ if $k = 0, x \in \mathbb{R} \setminus [-2, 2]$.
- $f(x) = \frac{-1}{(2k+2)x^{2k+2}}$ if $k > 0$, $x \in \mathbb{R} \setminus [-2, 2]$.

We define $\alpha_\epsilon = udf_\epsilon + \beta$. We see that $\alpha_\epsilon \wedge d\alpha_\epsilon = f'_\epsilon(z)dz \wedge (u(d\beta)^n + n\beta \wedge du \wedge (d\beta)^{n-1})$ and hence α_ϵ is folded contact: indeed f'_ϵ vanishes transversally at zero, and away from zero, this last expression is non-zero.

□

An alternative proof of the last two theorems would be to use the symplectization as explained in Section 3.4 and to use immediately Theorem 2.7.9 in the symplectization. The almost convex condition makes sure that the vector field in the direction of the symplectization is Liouville in the desingularization, see Lemma 4.2.3. Hence the induced structure is contact. Without the almost-convexity, the induced structure of the desingularized symplectic form on the initial manifold is not necessarily contact. This is saying that almost-convexity is a sufficient condition, but not a necessary condition to apply the desingularization method.

4.3 Existence of singular contact structures on a prescribed submanifold

In the present section, we will see that in presence of convex hypersurface in contact manifolds, the inverse construction holds.

Existence of contact structures on odd dimensional manifolds has been one of the leading questions in the field and the necessary results are contained in Section 2.3.

We will prove that convex hypersurfaces can be realized as the critical set of b^{2k}-contact structures. A similar result holds for b^{2k+1}-contact structures. However, in this case the critical set has two connected components, which correspond to two convex hypersurfaces arbitrarily close to a connected component of the given convex hypersurface.

Theorem 4.3.1. *Let (M, ξ) be a contact manifold and let Z be a convex hypersurface in M with dividing set given by Σ. Then M admits a b^{2k}-contact structure for all $k \in \mathbb{N}^*$ that has Z as critical set. The* *codimension* 2 *submanifold* Σ *corresponds to the set where the rank of the distribution drops and the induced structure on* Z *is contact.*

Using Giroux's genericity result, we obtain the following corollary in dimension 3:

Corollary 4.3.2. *Let* M *be a* 3*-dimensional manifold. Then for a generic surface* Z*, there exists a* b^{2k}*-contact structure on* M *realising* Z *as the critical set.*

Proof of the Corollary. Using Gromov's result in the open case and Lutz–Martinet for M closed (Theorem 2.3.1), we can equip M with a contact form. By the abundance of convex surfaces in contact manifolds, that is Theorem 2.3.6, a generic surface Z is convex and the conclusion follows from Theorem 4.3.1. □

Proof of Theorem 4.3.1. Using the transverse contact vector field, we find a tubular neighbourhood of Z diffeomorphic to $Z \times \mathbb{R}$ such that the contact structure is defined by the contact form $\alpha = udt + \beta$, where t is the coordinate on $\mathbb{R}$, $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$, see Lemma 2.3.7. Note that in general, α is multiplied by a non-vanishing function g that is not vertically invariant. As g is non-vanishing, the form divided by the function gives a contact form defining the same contact structure. The non-integrability condition then is equivalent of saying that $u(d\beta)^n + n\beta \wedge du \wedge d\beta$ is a volume form on Z. We will change the contact form to a b^{2k}-contact form.

Take $\epsilon > 0$. Let us take a function s_ϵ (that is smooth outside of $x = 0$) such that

1. $s_\epsilon(x) = x$ for $x \in \mathbb{R} \setminus [-2\epsilon, 2\epsilon]$,

2. $s_\epsilon(x) = -\frac{1}{x^{2k-1}}$ for $x \in [-\epsilon, 0[\cup]0, \epsilon]$,

3. $s'_\epsilon(x) > 0$ for all $x \in \mathbb{R}$.

Now consider $\alpha_\epsilon = uds_\epsilon + \beta$. By construction, α_ϵ is a b^{2k}-form that coincides with α outside of $Z \times (\mathbb{R} \setminus [-2\epsilon, 2\epsilon])$. Furthermore, α_ϵ satisfies the non-integrability condition on $Z \times]-2\epsilon, 2\epsilon[$ because $s'_\epsilon > 0$.

The rest of the statement follows from the discussion of Section 3.3. □

Remark 4.3.3. Note that there are many different choices for the function s_ϵ yielding the same result: the function s_ϵ only needs to allow singularities of the right order and have positive derivative. We call (M, α_ϵ) the s_ϵ-singularization of the contact manifold (M, α).

This proof only works for b^m-contact forms where m is even because it is essential that $s'_\epsilon > 0$. In the case where the complimentary set of the convex hypersurface is connected, the contact condition obstructs the existence of b^{2k+1}-contact structures on M having Z as critical set. This is because the contact condition induces an orientation on the manifold, whereas in the b^{2k+1}-contact case, the orientation changes when crossing the critical set. The same holds for symplectic surfaces: see for example [MP18] where this orientability issues were formulated using colorable graphs.

Lemma 4.3.4. *Let M be an orientable manifold with Z a hypersurface such that $M \setminus Z$ is connected. Then there exist no b^{2k+1}-contact form with critical set Z.*

Proof. Assume by contradiction that there is a b^{2k+1}-contact form. Let z be a defining function for the critical set. Around the critical set, the contact condition writes down as $\frac{1}{z^{2k+1}}\nu$, where ν is volume form on M. This expression has opposite signs on either side of Z.

As $M \setminus Z$ is connected, $\alpha \wedge (d\alpha)^n$ must vanish in $M \setminus Z$, which is in contradiction with the contact condition. □

To overcome this orientability issue, we prove existence of b^{2k+1}-contact structures with two disjoint critical sets contained in a tubular neighbourhood of a given convex hypersurface.

Theorem 4.3.5. *Let (M, ξ) be a contact manifold and let Z be a convex hypersurface in M with dividing set Σ. Then M admits a b^{2k+1}-contact structure for all $k \in \mathbb{N}$ that has two diffeomorphic connected components Z_1 and Z_2 as critical set. The codimension 2 submanifold Σ corresponds to the set where the rank of the distribution drops and the induced structures on Z is contact. Additionally, one of the hypersurfaces can be chosen to be Z.*

Proof. The proof follows from the same considerations as before, but replacing the vertically invariant contact form α defining the contact ξ by $\alpha_\epsilon = uds_\epsilon + \beta$, where $s_\epsilon :]-\epsilon, \epsilon[\to \mathbb{R}$ is given by

- $s_\epsilon(t) = |t|$ for $|t| \in [3\epsilon/4, \epsilon]$,
- $s_\epsilon(t) = \log |t - 3\epsilon/8|$ for $|t| \in [\epsilon/4, \epsilon/2]$ if $m = 1$,
- $s_\epsilon(t) = \frac{1}{2k(x-3\epsilon/8)^{2k}}$ for $|t| \in [\epsilon/4, \epsilon/2]$ if $m = 2k+1 \neq 1$,
- s_ϵ is odd, i.e. $s_\epsilon(-t) = -s_\epsilon(t)$,
- $s'_\epsilon(t) \neq 0$.

As before, $s'_\epsilon \neq 0$ guarantees that α_ϵ is a b^{2k+1}-contact form. As any other function with non-vanishing derivative and the right order of singularities gives rise to a b^{2k+1}-contact form, one of the two hypersurfaces can be chosen to be the initial convex hypersurface. □

Remark 4.3.6. Given a contact manifold with a convex hypersurface such that the complementary set of the hypersurface is not connected, it may, in some particular cases, also be possible to construct a b^{2k+1}-contact form admitting a unique connected component as critical set. This is related to extending a given contact form in a neighbourhood of a contact manifold with boundary to a globally defined contact form. More precisely, let α be the contact form. In a tubular neighbourhood around the convex hypersurface, we replace as before $\alpha = udt + \beta$ by $\alpha_\epsilon = uds_\epsilon + \beta$ where s_ϵ is given by

- $s_\epsilon(t) = t$ for $t > 2\epsilon$,
- $s_\epsilon(t) = \log t$ for $0 < t < \epsilon$,
- $s'_\epsilon(t) > 0$ for $t > 0$,
- s_ϵ is even, i.e. $s_\epsilon(-t) = s_\epsilon(t)$.

The form α_ϵ is a b^{2k+1}-contact form that agrees with α for $t > 2\epsilon$. However, it does not agree with α for $t < -2\epsilon$ and in fact, it may not always be possible to extend α_ϵ.

Remark 4.3.7. Recall that a b^m-contact form $\alpha = u\frac{dz}{z^m} + \beta$ is convex if $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$, see Definition 4.2.1. Note that the s_ϵ-singularization of is by construction a convex b^m-contact manifold.

As a corollary of Theorem 4.3.5, we will prove that any contact manifold is folded contact.

Corollary 4.3.8. *Let Z be a convex hypersurface in a contact manifold $(M, \ker\alpha)$. Then M admits a folded-contact form that has two* *connected components Z_1 and Z_2 as folded hypersurface, diffeomorphic to Z. In particular in dimension 3, a generic surface $Z \subset M$ can be realized as a connected component of the folded hypersurface of a folded contact form.*

Proof. The proof is a direct application of the existence theorem for $k = 1$ and the desingularization theorem. First, by Theorem 4.3.5, the hypersurface Z can be realized as one of two connected components of the critical set of a b-contact structure. As the obtained b-contact form is convex, see Remark 4.3.7, we then use the desingularization theorem (Theorem 4.2.5) to obtain a folded contact structure.

The genericity statement in dimension 3 follows as in Corollary 4.3.2. □

Remark 4.3.9. It follows from [BEM+15] that a necessary and sufficient condition for a manifold to admit a contact structure is that it is almost contact. It would be interesting to ask whether the almost contact condition can be relaxed to prove the existence of b^m-contact structures on closed manifolds. For an example it is well known that $SU(3)/SO(3)$ does not admit a contact structure, see [Gei08]. Another indication for this is given by examples of cooriented b^m-contact structures on non-orientable manifolds (see Example 3.0.6).

4.4 Singularization of cosymplectic hypersurfaces in symplectic manifolds

We apply the singularization techniques as introduced in Section 4.3 to the existence of b^m-symplectic structures. The role of convex hypersurfaces in contact geometry are played here by hypersurfaces that admit a transverse symplectic vector field.

Theorem 4.4.1. *Let $Z \subset (W^{2n}, \omega)$ be a hypersurface in a symplectic manifold and assume that there exists a symplectic vector field X defined in a tubular neighbourhood around Z that is transverse everywhere to Z. Then Z can be realized as the critical hypersurface of a family of b^{2k}-symplectic structures on W, coinciding with the given symplectic structure away from a tubular neighbourhood of Z.*

Proof. We will first prove that ω is symplectomorphic in a tubular neighbourhood to $\alpha \wedge dt + \beta$ where $\alpha \in \Omega^1(Z)$ and $\beta \in \Omega^2(Z)$ defines a cosymplectic structure and t denotes the flow of X. This part of the proof is done in Section 2.2 in [FTM17], but we include this here for the sake of completeness. It can also be found for example in Proposition 12 in [Bra19]. Let us denote the inclusion $i : Z \hookrightarrow W$. The 1-form $\alpha = \iota_X \omega$ is non-vanishing due to the non-degeneracy of the symplectic form and is closed as

$$d\iota_X \omega = \mathcal{L}_X \omega = 0.$$

We now claim that $\widetilde{\alpha} := i^*\alpha$ and $\widetilde{\omega} := i^*\omega$ defines a cosymplectic structure on Z. The 2-form $\widetilde{\omega}$ is closed because $d\omega = 0$. Furthermore $\widetilde{\alpha} \wedge \widetilde{\omega}^{n-1}$ is a volume form on Z. Indeed, consider at $p \in Z$ a basis by $\{v_2, \dots, v_{2n}\}$, such that $\{X, v_2, \dots, v_{2n}\}$ is a symplectic basis for $\tilde{\omega}$. We obtain

$$\widetilde{\alpha} \wedge \widetilde{\omega}^{n-1}(v_2, \dots, v_{2n}) = i^*(\iota_X \omega \wedge \omega^{n-1}(v_2, \dots, v_{2n})) = i^*\omega^n(X, \dots, v_{2n}).$$

This last expression is non-vanishing by the non-degeneracy of the symplectic form. Hence Z admits a cosymplectic structure and we define $\widehat{\omega} := dt \wedge \widetilde{\alpha} + \widetilde{\omega}$ which is symplectic and furthermore $\widehat{\omega} = \omega$

on Z. By Moser's path method $\widehat{\omega}$ and ω are symplectomorphic in a tubular neighbourhood.

We now apply the same techniques as in Theorem 4.3.1. We define

$$\omega_\epsilon := df_\epsilon \wedge \widetilde{\alpha} + \widetilde{\beta}$$

where f_ϵ is as in the proof of Theorem 4.3.1. It follows from the definitions that ω_ϵ is a b^{2k}-symplectic form and that it agrees with ω away from the ϵ-neighbourhood. □

By the same consideration, we can consider two disjoint copies of the hypersurface to prove a similar result for b^{2k+1}-symplectic structures.

Theorem 4.4.2. *Let (W, ω) be a symplectic manifold and let X be a symplectic vector field defined in a tubular neighbourhood around Z that is transverse to a hypersurface Z. Then W admits a b^{2k+1}-symplectic structure for all k that has two diffeomorphic connected components Z_1 and Z_2 as critical set, coinciding with the given symplectic structure away from a tubular neighbourhood of Z.*

Note that in contrast to convex hypersurfaces in contact geometry, whose existence is generic in dimension 3, hypersurface in symplectic manifolds that admit a transverse symplectic vector field are much less common.

Chapter 5

b^m-Reeb dynamics and the singular Weinstein conjecture

It's a trap!

Admiral Ackbar, *Return of the Jedi.*

This chapter is devoted to the study of the dynamics of the Reeb vector field on b^m-contact manifolds. The necessary preliminaries about Reeb and Hamiltonian dynamics are contained in Subsection 2.4.1.

As pointed out in the introduction, existence results on periodic orbits are of great importance on b^m-contact manifolds for applications. Examples to be mentioned are for instance the restricted planar circular three body problem, but also fluid dynamics (see Chapter 6). On b^m-contact manifolds, the geometric structure on the critical hypersurface, and hence also the dynamics, is fundamentally different from the complement of the critical set, where the b^m-contact form is contact. At first glimpse, this observation indicates the need to distinguish between those two cases and therefore examine the existence of periodic orbits away from the critical set or on the critical set for compact manifolds separately. We will thereby tackle the Weinstein conjecture for b^m-contact manifolds. Whereas the existence of periodic Reeb orbits on the critical set for 3-dimensional compact manifolds will follow rather easily from earlier results, understanding the Reeb dynamics away from the critical set is much more subtle and leads ultimately to a reformulation of the Weinstein conjecture: the singular Weinstein conjecture.

5.1 About the Weinstein conjecture for b^m-contact manifolds

We will see that there are examples of compact b^m-contact manifolds without any periodic Reeb orbits away from Z. This implies that asking for the existence of periodic orbits away from Z is not a meaningful formulation for the Weinstein conjecture for b^m-contact manifolds.

An immediate implication of this is that in the presence of singularities in the symplectic structure, there are b^m-symplectic manifolds that do not admit any periodic orbits of the Hamiltonian vector field away from the critical set. In particular, we prove that taking the symplectization, there are proper Hamiltonian functions on b^m-symplectic manifolds having no periodic orbits for the Hamiltonian flow away from $Z \times \mathbb{R}$.

In contrast to those non-existence results, we prove that in dimension 3, there are always infinitely many periodic Reeb orbits on the critical set whenever it is compact.

5.1.1 On the existence of periodic orbits on Z

As was observed in Theorem 3.1.4, the Reeb vector field on the critical set is a Hamiltonian vector field in the 3-dimensional case. This is only true in dimension 3, which comes from the fact that area forms are symplectic forms on surfaces. This will imply that on the critical set of closed 3-dimensional b^m-contact manifolds, there are infinitely many periodic Reeb orbits[1].

Proposition 5.1.1. *Let (M, α) be a 3-dimensional b^m-contact manifold and assume the critical hypersurface Z to be closed. Then there exists infinitely many periodic Reeb orbits on Z.*

Note that the critical hypersurface Z is closed if there exists a global function defining Z and the ambient manifold M is compact.

Proof. Let us denote the usual decomposition by $\alpha = u\frac{dz}{z} + \beta$. By Proposition 3.1.6, the function u is non-constant on Z. Furthermore by Theorem 3.1.4, the Reeb vector field is Hamiltonian on Z for the function $-u$. Let $p \in Z$ be a point such that $du_p \neq 0$. As the preimage of a closed topological set is closed and a closed set of a compact manifold is compact, the level-sets $u^{-1}(p)$ are given by circles and the Reeb vector field, contained in the level-set, is non-vanishing in view of

$$\iota_{R_\alpha}(ud\beta + \beta \wedge du) = du$$

[1] The author would like to thank Robert Cardona for pointing this out.

as in the proof of Corollary 3.1.5. Hence the Reeb vector field is periodic on $u^{-1}(p)$. □

The condition of M to be closed is necessary, as is seen from the next example.

Example 5.1.2. Consider $S^2 \times S^1 \setminus \{(p_N, \phi), (p_S, \phi)\}$ where ϕ is the angular coordinate on S^1 and (h, θ) are polar coordinates on S^2 and p_N (respectively p_S) denotes the north pole (respectively south pole). The b-form $\alpha = \frac{d\phi}{\sin\phi} + hd\theta)$ is a b-contact form whose Reeb vector $R_\alpha = \sin\phi \frac{\partial}{\partial\varphi}$ vanishes everywhere on Z and does not admit any periodic Reeb orbits. However this is a non-compact example and it fact, the north and south pole cannot be added to compactify this example as this would yield a trivial Boothby–Wang fibration of S^2 over S^1.

We conjecture that in higher dimension, a similar result to the one in Proposition 5.1.1 holds. However, as noticed before, the dimension 3 is particular here because area forms on surfaces are symplectic.

Conjecture 5.1.3. *Let (M, α) be a closed $(2n+1)$-dimensional b^m-contact manifold, $n \geq 1$. Then there exists infinitely many periodic Reeb orbits on Z.*

Remark 5.1.4. A 1-dimensional closed b-contact manifold without any periodic Reeb orbits is given by $(S^1, \frac{d\phi}{\sin\phi})$. In this example the singularity is transferred from the contact form to the orbit as marked points on the circle are declared as zeros of the vector field. If we try to upgrade this example to higher dimensions using circle actions and transferring the singularity to the orbit of the action it turns out that it is topologically obstructed. This follows from the remark in the

Example 5.1.2: Higher dimensional examples without any non-trivial periodic Reeb orbits cannot come from a S^1-action, as this would yield a trivial Boothby-Wang fibration. We still believe that enlarging the class of contact to b^m-contact structures and thereby admitting possible vanishing of the Reeb vector field allows to construct such examples and thus a counterexample to the smooth Weinstein conjecture for b^m-contact structures.

As in the 3-dimensional compact case b^m-contact manifolds always admit infinitely many periodic Reeb orbits on the critical set, a first *naive* generalization of the Weinstein conjecture would be that there are always periodic orbits away from the critical set. However, this is not true, as we will see in the next subsection.

5.1.2 On the non-existence of periodic orbits away from Z

Whereas on the critical set, there are always infinitely many periodic Reeb orbits, this is not true away from Z.

Claim 5.1.5. *There are compact b^m-contact manifolds (M, Z) of any dimension for all $m \in \mathbb{N}$ without periodic Reeb orbits on $M \setminus Z$.*

In what follows several examples where the Weinstein conjecture is not satisfied are given thus proving the claim. The first example is given by Example 3.4.3.

Example 5.1.6. Consider the example of the 3-sphere in the standard b-symplectic Euclidean space $(\mathbb{R}^4, \omega)$ where $\omega = \frac{dx_1}{x_1} \wedge dy_1 + dx_2 \wedge dy_2$ as in Example 3.4.3. The b-contact form is given by $\alpha = \iota_X \omega$ where

X is the Liouville vector field transverse to S^3 given by $X = \frac{1}{2}x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} + \frac{1}{2}(x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2})$. The Reeb vector field is given by

$$R_\alpha = 2x_1^2\frac{\partial}{\partial y_1} - x_1y_1\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial y_2} - 2y_2\frac{\partial}{\partial x_2}.$$

On the critical set, given by S^2, this vector field is giving rise to rotation and thus infinitely many periodic Reeb orbits, as is proved in Proposition 5.1.1. Away from Z, the Reeb vector field does not admit any periodic orbits. Indeed, the vector field can be interpreted as two uncoupled systems in the (x_1, y_1), respectively (x_2, y_2)-plane. The flow in the (x_1, y_1)-plane is clearly not periodic.

This example can be generalized to b^{2k+1}-contact forms for any $k \geq 1$ by considering $(\mathbb{R}^4, \omega_{\text{st}} = \frac{dx_1}{x_1^{2k+1}} \wedge dy_1 + dx_2 \wedge dy_2)$ and the Liouville vector field given by $X = \frac{1}{2}x_1^{2k+1}\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} + \frac{1}{2}(x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2})$ that is transverse to S^3 and hence $\alpha = \iota_X\omega$ is a b^m-contact form. The associated Reeb vector field does not admit any periodic orbits. The restriction on the parity on the b^m-contact form comes from the fact that transversality of a similar Liouville vector field with respect to S^3 fails.

The next example is given by the 3-torus, as in Example 3.4.5.

Example 5.1.7. Consider $(\mathbb{T}^3, \sin\phi\frac{dx}{\sin x} + \cos\phi dy)$. The Reeb vector field is given by $R_\alpha = \sin\phi \sin x\frac{\partial}{\partial x} + \cos\phi\frac{\partial}{\partial y}$. The critical set is given by two disjoint copies of the 2-torus $\mathbb{T}^2$ and the Reeb flow restricted to it is given by $\cos\phi\frac{\partial}{\partial y}$. As in the last example, the critical set Z is given by periodic orbits (except when $\cos\phi = 0$, where the Reeb vector field is singular). However, away from Z there are no periodic orbits.

This example can be generalized to higher order singularities: $\alpha =$

$\sin\phi\frac{dx}{\sin^m(x)} + \cos\phi dy)$ is a b^m-contact form that does not admit any periodic orbits away from Z.

Armed with these examples we conclude that there are examples of b^m-symplectic manifolds without periodic orbits of the Hamiltonian flow away from the critical hypersurface.

Claim 5.1.8. *There are b^m-symplectic manifolds with proper smooth Hamiltonian whose Hamiltonian flow does not have any periodic orbits away from Z.*

To see this, let (M, α) be a compact b^m-contact manifold without any periodic Reeb orbits away from the critical set and consider in its symplectization the Hamiltonian function e^t (where t is the coordinate in the symplectization). The Hamiltonian vector field is a reparametrization of the Reeb vector field on the level-sets and therefore provides an example of a proper smooth Hamiltonian containing no periodic orbits in the level-sets away from the critical hypersurface.

We define the set-of aperiodic values for a b^m-symplectic manifold (W, ω) and a Hamiltonian $H \in C^\infty(W)$ as follows

$$\begin{aligned} {}^{b^m}\mathcal{AP}_H := \{a \in \mathbb{R} | X_H \text{ does not admit and} \\ \text{periodic orbits on } H^{-1}(a) \text{ away from } Z\}. \end{aligned}$$

The corollary can be reformulated: on b^m-symplectic manifolds there is a proper smooth Hamiltonian such that ${}^{b^m}\mathcal{AP}_H = \mathbb{R}$.

This is in stark contrast to the almost-existence theorem (Theorem 2.4.5) in symplectic geometry, but we also remark that there is no notion of Hofer–Zehnder capacity in this setting. To the author's knowledge, there are no known examples of Hamiltonian having no periodic orbits on all level-sets (or equivalently having $\mathcal{AP}_H = \mathbb{R}$).

As mentioned before, by the partial positive results on the Weinstein conjecture, Reeb plugs cannot exist. The compact counterexamples in any dimension in the b^m-contact case however raise the following question:

Question 5.1.9. *Are there plugs for b^m-Reeb flows?*

More precisely, given a contact manifold, can the singularization be used to change the contact structure by a b^m-contact structure and thereby controlling the changed Reeb dynamics to destroy a given periodic orbit for the initial flow? A guideline example is Example 5.1.6, where the critical set Z is given by a 2-sphere and there are no periodic orbits away from Z. In the following section, we give a first approach towards understanding the existence of plugs. We will see that spheres can be inserted as hypersurfaces in local Darboux charts. The dynamics on those spheres is given as in Example 5.1.6 and a given orbit entering the Darboux charge is being captured as it approaches one of the fixed points of the sphere. This will yield the existence of traps for b^m-Reeb fields.

5.2 On the existence of traps and plugs for b^m-contact manifolds

As explained in Section 2.4.1 before, traps for Reeb flows do not exist in dimension 3, see [EH+94], but do exist in higher dimensions [GRZ14]. We prove that in the b-category, there is no restriction on the dimension.

Theorem 5.2.1. *There exist b^m-contact traps in any dimension.*

Proof. We only consider the 3-dimensional case, the higher dimensional being similar. Consider a Darboux ball and denote the standard contact form by α_{st}. For convenience, we work in polar coordinates, in which α_{st} writes down as $\alpha_{st} = dz + r^2 d\theta$. The Reeb vector field is given by $R_\alpha = \frac{\partial}{\partial z}$.

We introduce a convex hypersurface and use the existence result, Theorem 4.3.1 of Section 4.3. The vector field $X = 2z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r}$ is a contact vector field as $\mathcal{L}_X \alpha = 2\alpha$ and is transverse to the 2-sphere S^2. Hence S^2 is a convex hypersurface and can be realized as critical set of a b^{2k}-contact manifold. More precisely, introducing in a neighbourhood around S^2 the coordinate t such that $X = \frac{\partial}{\partial t}$, the contact form writes in the Giroux decomposition as follows:

$$\alpha = g(udt + \beta)$$

where $u \in C^\infty(S^2)$, $\beta \in \Omega^1(S^2)$ and g is a smooth function, see Lemma 2.3.7. Note that u and β are independent of the t-coordinate, whereas g is not. We now apply Theorem 4.3.1 to the form α. We obtain a b^{2k}-contact form given by

$$\alpha_\epsilon = g(udf_\epsilon + \beta), \tag{5.1}$$

where f_ϵ is as in Theorem 4.3.1. In order to compute the Reeb dynamics, let us explicitly compute the functions u, g and the 1-form β. We introduce the following change of variables:

$$\begin{cases} \frac{\partial}{\partial t} = 2z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} \\ \frac{\partial}{\partial \xi} = r^2\frac{\partial}{\partial z}. \end{cases}$$

The dual basis is given by

$$\begin{cases} dt = \frac{1}{r}dr \\ d\xi = \frac{1}{r^2}dz - \frac{2z}{r^3}dr. \end{cases}$$

Hence $r = e^t$ and $z = e^{2t}\xi$. Under this change of variable, the Giroux decomposition of the standard contact form is given by

$$\alpha_{st} = e^{2t}(2\xi dt + d\xi + d\theta).$$

The b^{2k}-contact form is given by

$$\alpha_\epsilon = e^{2t}(2\xi df_\epsilon + d\xi + d\theta)$$

and a direct computation yields that the Reeb vector field associated to α_ϵ is given by

$$R_{\alpha_\epsilon} = \frac{f'_\epsilon - 1}{f'_\epsilon} e^{-2t} \frac{\partial}{\partial \theta} + \frac{1}{f'_\epsilon} e^{-2t} \frac{\partial}{\partial \xi}.$$

Close to the singularity, $f'_\epsilon = \frac{1}{t^2}$ so that on the critical set $S^2 = \{t = 0\}$, the Reeb dynamics is given by $R = \frac{\partial}{\partial \theta}$. Furthermore, it follows from the formulas that the orbit entering the Darboux ball at $\theta = 0$ limits to the fixed point on S^2 and hence is trapped. See Figure 5.2, where the dynamics around $Z = S^2$ and the trapped orbit is depicted.

A similar proof holds for the case of b^{2k+1}-contact structures, where we apply Theorem 4.3.5 to produce a b^{2k+1}-contact form where the critical hypersurface is given by two disjoint copies of S^2 to bypass the orientation issues. □

Plugs for the Reeb flow cannot exist by the positive answer to the Weinstein conjecture. In the case of b^m-contact manifolds, we exhibited examples in the last section of compact b^m-contact manifolds without any periodic Reeb orbits away from Z and proved that on Z there are always periodic Reeb orbits in dimension 3. This means that the trap constructed in the present subsection is optimal in the following sense: it has infinitely many periodic orbits on S^2. The dynamics of the trap that we constructed is narrowly related to Example

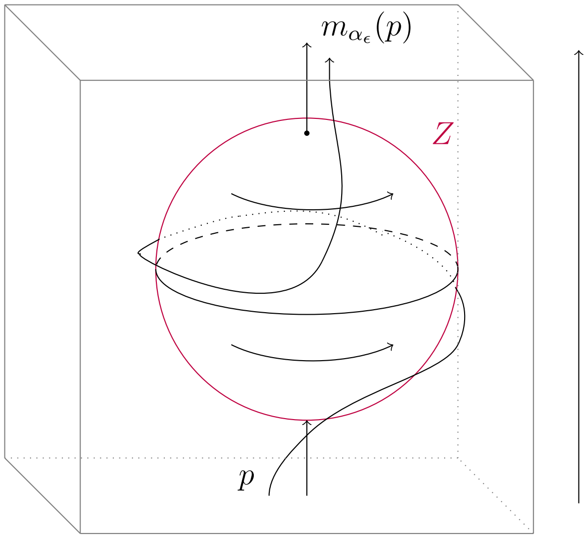


Figure 5.1: The b^{2k}-contact trap. The critical set Z is given by S^{2n-2}. Outside of the ϵ-neighbourhood the dynamics is unchanged. One Reeb orbit is trapped. One orbit is depicted that does not satisfy the entry-exit condition.

5.1.6. The dynamics on Z of this example and on the critical set of the construction in the last proof are the same.

In the last proof, due to the term in $\frac{\partial}{\partial\theta}$, the entry-exit condition is not satisfied, meaning that this construction only yields a trap and not a plug. Due to the local changes of the dynamics in the trap construction, the global dynamics are possibly drastically changed as well. Therefore, in the next sections, we attempt to improve the above construction to get a trap. By this we mean that we attempt to change the Reeb vector field such that it satisfies the entry-exit condition. Let us denote the entry surface of section (where the flow is linear) by $\{-1\} \times D^2$, where D^2 is the 2-dimensional disk. We denote by m_{α_ϵ} the map that sends an entry point of the plug $p \in D^2 \times \{-1\}$ to its exit point following the flow of R_{α_ϵ} given by $m_{\alpha_\epsilon}(p) \in D^2 \times \{1\}$. The map m_{α_ϵ} is called *exit map* associated to the form α_ϵ. As the above construction yields a trap and not a plug, $m_{\alpha_\epsilon} \neq \mathrm{Id}$.

We introduce two attempts to upgrade the trap to a plug of the Reeb flow associated to a b^m-contact form. This attempts will be based on defining a contact form (or b^m-contact form) $\widetilde{\alpha}$ on a flow-box $D^2 \times [0,1]$ whose Reeb vector field $R_{\widetilde{\alpha}}$ agrees with the linear on close to $\partial D \times [0,1]$. This new Reeb vector field induces a new exit map $m^2_{\widetilde{\alpha}} \to D^2$. This map is given by the identity on ∂D^2. The aim is to construct the contact (respectively b^m-contact form $\widetilde{\alpha}$ in such a way that $m_{\alpha_\epsilon} \circ m_{\widetilde{\alpha}} = \mathrm{Id}$. Such a contact form, respectively b^m-contact form, can then be used to concatenate its flow-box with the one associated to α_ϵ. By the condition $m_{\alpha_\epsilon} \circ m_{\widetilde{\alpha}} = \mathrm{Id}$, the entry-exit condition is then satisfied and therefore such a concatenation would yield a plug for the b^m-Reeb flow.

In the next section, we mirror the flow of R_{α_ϵ} to correct the altered dynamics and we analyse whether or not the mirrored flow can be realized as the Reeb vector field associated to a b^m-contact form $\widetilde{\alpha}$.

5.3 An attempt to save the plug I

In this section, we mirror the dynamics of the trap: in fact a mirrored copy undoes the dynamical changes that are occurring when introducing the trap.

In what follows, we focus on the case of b^{2k}-contact forms, the case of odd powers being solved as usual by considering two concentric spheres.

As seen in the last section, the trap changes the standard contact form in a Darboux ball and the linear Reeb vector field $R_{\alpha_\epsilon} = \frac{\partial}{\partial z}$ gets

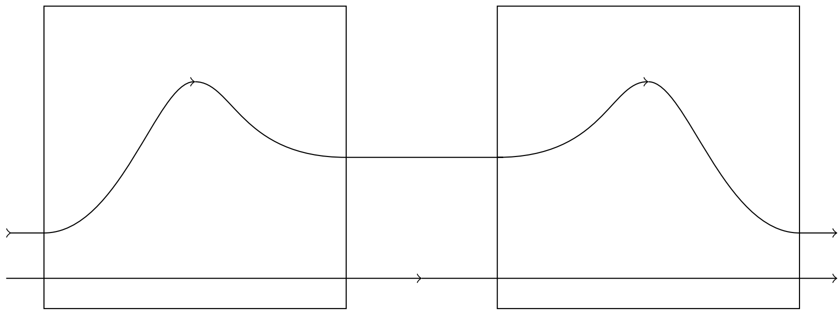


Figure 5.2: Mirroring a trap

changed to

$$R_{\alpha_\epsilon} = \frac{f'_\epsilon - 1}{f'_\epsilon} e^{-2t} \frac{\partial}{\partial \theta} + \frac{1}{f'_\epsilon} e^{-2t} \frac{\partial}{\partial \xi} = R^\theta \frac{\partial}{\partial \theta} + R^\xi \frac{\partial}{\partial \xi}.$$

To mirror the dynamics of the Reeb vector field $R^\theta \frac{\partial}{\partial \theta} + R^\xi \frac{\partial}{\partial \xi}$, we would like to produce a b^{2k}-contact form whose associated Reeb vector field writes

$$\widetilde{R}_{\alpha_\epsilon} = -R^\theta \frac{\partial}{\partial \theta} + R^\xi \frac{\partial}{\partial \xi}.$$

This vector field is given by the linear vector field $\frac{\partial}{\partial z}$ outside of the ϵ-neighbourhood of S^2 and therefore generates an exit map by $\widetilde{m_{\alpha_\epsilon}}$. As before, this map send an entry point $p \in D^2 \times \{-1\}$ to the point $\widetilde{m_{\alpha_\epsilon}}(p) \in D^2 \times \{1\}$ following the flow-lines of $\widetilde{R}_{\alpha_\epsilon}$. We have that near the boundary of D^2, this map is given by the identity. Due to the change of sign in the θ direction, the vector field $\widetilde{R}_{\alpha_\epsilon}$ rotates counter-clockwise, in contrast to the Reeb vector field R_α which rotates clockwise. Therefore, the associated return map satisfies $m_{\alpha_\epsilon} \circ \widetilde{m_{\alpha_\epsilon}} = \mathrm{Id}$.

The aim being to establish a plug for the b^m-Reeb flow, we would investigate in what follows whether or not $\widetilde{R}_{\alpha_\epsilon}$ is the Reeb vector field associated to a b^{2k}-contact form. In view of Lemma 2.2.5, a necessary is that $\widetilde{R}_{\alpha_\epsilon}$ is a contact Hamiltonian vector field for a Hamiltonian function H that is positive. In this case, the Reeb vector field of the contact form $\widetilde{\alpha} = \frac{1}{H}\alpha$ is given by $\widetilde{R}_{\alpha_\epsilon}$. We furthermore ask that $H = 1$ away from the ϵ-neighbourhood of S, denoted by $\mathcal{N}_\epsilon(S^2)$. This condition implies that $\widetilde{\alpha}$ matches the standard contact form.

A direct computation yields that $\alpha_\epsilon(\widetilde{R}_{\alpha_\epsilon}) = \frac{2-f_\epsilon'}{f_\epsilon'} := H_\epsilon$. It follows from the expression of f_ϵ that $H_\epsilon - 1$ is compactly supported in a neighbourhood of S^2. Around S^2, $f_\epsilon(t) = \frac{1}{t^2}$ and therefore, close to the critical hypersurface, $H_\epsilon = 2t^2 - 1$ and $H_\epsilon = 0$ in two isolated points.

Resuming, we proved the following proposition.

Proposition 5.3.1. *Let α_ϵ be the b^{2k}-contact form obtained in the trap construction, as in Theorem 5.2.1. Then $\frac{1}{H_\epsilon}\alpha_\epsilon$, where $H_\epsilon = \frac{2-f_\epsilon'}{f_\epsilon'}$, is a non-smooth b^{2k}-differential form. It is a b^{2k}-contact form away from $H_\epsilon = 0$ and the associated Reeb vector field can be continued to as a smooth vector field over $H_\epsilon = 0$.*

Furthermore, the associated dynamics controls the dynamics of R_{α_ϵ}, meaning that $m_{\alpha_\epsilon} \circ \widetilde{m}$, where $\widetilde{m}$ is the exit map induced by $\widetilde{R}_{\alpha_\epsilon}$.

This proposition does not prove that there exists a plug for the b^m-Reeb flow as the form $\widetilde{\alpha}$ is a non-smooth b^m-form. In the next section, we investigate another approach to the existence of plugs. Instead of controlling the dynamics of α_ϵ, we discuss the possibility to generate a map m using compactly supported deformation of the smooth standard contact form into another smooth contact form to generate the dynamics given by $\widetilde{m}$.

5.4 An attempt to save the plug II

Similar to the previous attempt to correct the dynamics of the exit map of the trap, in this section we concatenate the flow of R_{α_ϵ} by the Reeb flow of a contact form that is obtained through a compactly supported deformation of the standard contact form. As before, this compactly supported change is done in such a way that its Reeb vector field controls the entry-exit condition. In contrast to the last section, this compactly supported change is done through smooth contact forms and not through b^m-contact forms.

Instead of viewing the incoming surface of section as D^2, we regard is as an annulus $\mathbb{A}$: topologically, the plug is given by a direct product of the annulus with an interval. The interior boundary is given by the Reeb orbit that is fixed. On the outside boundary, by the definition of traps, the Reeb vector field is given by the linear vector field $\frac{\partial}{\partial z}$. The Reeb vector field is transverse to the incoming boundary $\mathbb{A} \times \{z = 0\}$. As before, we denote the exit map associated to α_ϵ by m_{α_ϵ}. We consider the standard contact form α on $\mathbb{A} \times [0, K]$ given by $\alpha = dz + \lambda$, where z denotes the coordinate on $[0, K]$ ($K > 0$ is a integer) and $\lambda = r^2 d\theta \in \Omega^1(\mathbb{A})$. We will do a compactly supported deformation of α to obtain a new contact form $\widetilde{\alpha}$. As before, following the flow of the Reeb vector field of $R_{\widetilde{\alpha}}$ until it intersects the out coming boundary $\mathbb{A} \times \{z = K\}$, we obtain the exit map $m_{\widetilde{\alpha}} : \mathbb{A} \times \{0\} \to \mathbb{A} \times \{K\}$ that is area preserving with respect to $d\alpha$. In what follows, we describe a method to this compactly supported change such that $m_{\widetilde{\alpha}} = m_{\alpha_\epsilon}^{-1}$. Hence the concatenation of those two trap construction yields the existence of a a plug. Unfortunately, we will see in the construction that the constant $K > 0$ needs to be big. Therefore, this construction can not be performed in a Darboux ball and therefore does not solve the existence of a plug for the b^m-Reeb flow. Hence, we want to show that starting from an area preserving diffeomorphism $m : \mathbb{A} \to \mathbb{A}$ (that is given by $m_{\alpha_\epsilon}^{-1}$), we prove that there exists a contact form $\widetilde{\alpha}$ on $\mathbb{A} \times [0, K]$ whose associated Reeb flow R_{α_ϵ} induces an exit map $m_{\widetilde{\alpha}}$ satisfies $m_{\widetilde{\alpha}} = m_{\alpha_\epsilon}^{-1}$.

More precisely, we will prove the following theorem.

Theorem 5.4.1. *Consider $\mathbb{A} \times [0, K]$, where $K > 0$ is an integer and consider the contact form $\alpha = dz + \lambda(r, \theta)$ where $\lambda \in \mathbb{A}$ and z denotes the coordinate on the interval. Let m be a compactly supported diffeomorphism of the annulus that preserves the area $d\lambda$. Then there exists an integer $K > 0$ and a deformation of the standard contact form $\alpha \in \Omega^1(\mathbb{A} \times [0, K])$ to a contact form $\widetilde{\alpha}$ such that the exit map associated to the flow of its Reeb vector field $R_{\widetilde{\alpha}}$ is given by m.*

Remark 5.4.2. For the purposes of this section it is actually enough to prove this result for the standard contact form $\lambda = r^2 d\theta$.

We start by proving a small lemma.

Lemma 5.4.3. *Let $(\mathbb{A}, d\lambda)$ be the annulus equipped with the symplectic form and let m_t be a compactly supported symplectic isotopy. Then m_1 is a Hamiltonian isotopy.*

Proof. By Lemma 10.2.7 of [MS17], for an exact symplectic manifold and a compactly supported symplectic isotopy, the flux is given by $[\lambda - m_1^*\lambda]$. To see that this last quantity is zero in the case of $\mathbb{A}$, we evaluate it against the generators of $H_c^1(\mathbb{A}; \mathbb{R})$. As $\mathbb{A}$ is in the interior of the compact manifold $\overline{\mathbb{A}}$ with boundary $\partial\overline{\mathbb{A}}$, we have that $H_c^1(\mathbb{A}; \mathbb{R}) \cong H^1(\overline{\mathbb{A}}, \partial\overline{\mathbb{A}}; \mathbb{R})$. The latter one is generated by the circle.

We take a representative of the circle γ that is close to the boundary in the region where $m_t = \mathrm{Id}$ and hence $m_1^*\lambda(\gamma) - \lambda(\gamma) = 0$. $\square$

We now prove the Theorem 5.4.1.

Proof. Since m is compactly supported in $\mathbb{A}$ we know by the last lemma that it is a Hamiltonian diffeomorphism. We thus obtain a Hamiltonian functions H_t. The vector field X defined by the equation

$$\iota_{X_t} d\lambda = -dH_t$$

has flow Φ_t that such that $m = \Phi$. Let $K > 0$ be a positive integer that we choose later. It is clear adding K to H_t does does not change X_t. We can assume that $\Phi_t = \mathrm{Id}$ for $t < \delta$ and $t > 1 - \delta$, $\delta > 0$ by reparametrization of time. It follows that $X_t = 0$ around $t = 0$ and $t = 1$. We now lift the Hamiltonian diffeomorphism on the base to a Reeb flow on the suspension. We define

$$\begin{aligned} \widetilde{H} : \mathbb{A} \times [0,1] &\to \mathbb{R} \\ (a,t) &\mapsto \widetilde{H}(a,t) = H_t(a) \end{aligned}$$

We define $\overline{\alpha} = (\widetilde{H} + K)dz + \lambda$. We first claim that $\overline{\alpha}$ is contact and we will see later how to interpolate between $\overline{\alpha}$ and α. We compute

$$\overline{\alpha} \wedge d\overline{\alpha} = dz \wedge ((\widetilde{H} + K))d\lambda + \lambda \wedge d\widetilde{H}_z).$$

The first term given by $dz \wedge d\lambda$ is positive by the contact condition. We choose $K > 0$ big enough such that this is positive. We claim that its Reeb vector field is given by a positive multiple of $\frac{\partial}{\partial z} - X_z$ where X_z denotes the vector field given by $X_z(z, r, \theta) = X_z$. We check that the vector field in the kernel of contact form:

$$\iota_{\frac{\partial}{\partial z} - X_z} d\overline{\alpha} = \iota_{\frac{\partial}{\partial z} - X_z}(d\lambda + d\widetilde{H}_z \wedge dz) = -\iota_{X_z} d\lambda - d\widetilde{H}_z - (\iota_{X_z} d\widetilde{H}_z)dz.$$

The first two terms cancel out due to the definition of the Hamiltonian vector field and the last term as well, hence this is zero. Hence the flow of the Reeb vector field is a reparametrization of the flow of $\frac{\partial}{\partial z} - X_z$ and therefore its associated exit map $m_{\overline{\alpha}}$ does the job, that is $m_{\overline{\alpha}} = m$. However it does not match the standard contact form near the boundary.

We now change the contact form $\overline{\alpha}$ to a new contact form $\widetilde{\alpha}$ that matches the initial the contact form α on $\partial(\mathbb{A} \times [0,1])$. First, note that the dynamics of the contact form $\frac{\overline{\alpha}}{K}$ is left unchanged up to reparametrization. It is given by

$$\frac{\overline{\alpha}}{K} = \frac{\widetilde{H} + K}{K} dz + \frac{1}{K} \lambda,$$

so that the first term indeed matches on $\partial(\mathbb{A} \times [0,1])$. We define the 1-form given by

$$\widetilde{\alpha} := \frac{\widetilde{H} + K}{K} dz + \lambda$$

and we claim that this is a contact form such that $\widetilde{\alpha} = \alpha$ on $\partial(\mathbb{A} \times [0,K])$.

$$\widetilde{\alpha} \wedge d\widetilde{\alpha} = (\frac{\widetilde{H} + K}{K} dz + \lambda) \wedge (\frac{1}{K} d\widetilde{H}_z \wedge dz + d\lambda) = \frac{1}{K} dz \wedge ((\widetilde{H} + K) d\lambda + \lambda \wedge d\widetilde{H}_z)$$

so $\widetilde{\alpha}$ is contact. Its Reeb vector field is given by a reparametrization of the vector field $R_{\widetilde{\alpha}} = K\frac{\partial}{\partial z} - X_z$. The induced exit $m_{\widetilde{\alpha}}$ equals the map m and $\widetilde{\alpha} = \alpha$ near $\partial(\mathbb{A} \times [0,K])$. This finishes the proof. □

We highlight once more that this does not prove the existence of the plugs, the reason being the constant $K > 0$. Indeed, if we could assure the constant $K = 1$, the contact form $\overline{\alpha}$ as in the proof would match the standard contact form. The concatenation of the two trap constructions (that is the trap for the b^m-Reeb flow and the one given in the last proof) would then yield the existence of the plug. The reason for this constant to be possibly very big is that the linear vector field $R_{\alpha_{\text{st}}}$ and the Reeb vector field R_{α_ϵ} are not C^1-close.

5.5 Overtwisted disks in b^m-contact manifolds and open contact manifolds

While earlier sections dealt with the non-existence of periodic Reeb orbits away from the critical set, we will now see a sufficient condition to guarantee the existence of periodic orbits away from the critical set. The techniques are based on Hofer's proof of the Weinstein conjecture for overtwisted contact manifolds, recalled in Subsection 2.4.4.

We assume in this section that the manifold is of dimension 3. We consider b^m-contact manifolds that have an overtwisted disk away from the critical set, the first higher dimensional definition of overtwisted disk was given in [Nie06], see also [BEM$^+$15].

Definition 5.5.1. We say that a b^m-contact manifold is overtwisted if there exists an overtwisted disk away from the critical hypersurface Z.

We will prove that in the case of overtwisted b^m-contact manifolds Weinstein conjecture holds, under the supplementary condition that α is $\mathbb{R}^+$-invariant around the critical hypersurface, see Definition 5.5.4. More precisely, we will prove the following theorem.

Theorem 5.5.2. *Let (M, α) be a closed b^m-contact manifold with critical set Z. Assume there exists an overtwisted disk in $M \setminus Z$ and assume that α is $\mathbb{R}^+$-invariant in a tubular neighbourhood around Z.*

Then at least one of the following statement holds:

1. *There exists a periodic Reeb orbit in $M \setminus Z$.*
2. *There exists a family of periodic Reeb orbits approaching the critical set Z.*

Furthermore, the periodic orbits are contractible loops in the symplectization.

Remark 5.5.3. The condition of $\alpha \in {}^{b^m}\Omega^1(M)$ to be $\mathbb{R}^+$-invariant is a non-trivial condition, as is pointed out in Remark 5.5.5.

The proof is based on Hofer's original arguments. The novelty here is that we work in a non-compact set-up, namely on the open manifold $M \setminus Z$. On open manifolds, Hofer's method do not apply directly. However the openness is *gentle* due to the $\mathbb{R}^+$-action. We will see that this theorem is a corollary of a more general statement: in fact, we don't need the geometric structure to be b^m-contact, we only need a contact form on an open manifold that is $\mathbb{R}^+$-invariant in the open ends of the manifold and overtwisted away from the $\mathbb{R}^+$-invariant part, see Theorem 5.5.8.

Under the assumption of $\mathbb{R}^+$-invariance, the manifold decomposes as the union of products of the connected components of the boundary with $\mathbb{R}^+$. We will see that in this decomposition, pseudoholomorphic curves can be translated in the $\mathbb{R}^+$-direction and the compactness of the boundary of the compact set guarantees convergence. With this in mind, we define the following:

Definition 5.5.4. Let $\alpha \in \Omega^1(M)$ be a contact form on an open manifold M. We say that α is $\mathbb{R}^+$-*invariant* in the open ends of M if there exists a compact set $K \subset M$ and a vector field X defined on $M \setminus K$ that satisfies $\mathcal{L}_X\alpha = 0$, meaning that X is a strict contact vector field and such that X is transverse to ∂K and that is complete for positive time. We say that an $\mathbb{R}^+$-invariant contact form α is overtwisted if there is an overtwisted disk contained in K.

Remark 5.5.5. In the light of Definition 5.5.4, we will view b^m-contact manifolds as open contact manifolds by considering the manifold without the critical set. It follows from Lemma 2.3.7 that the $\mathbb{R}^+$-invariance then translates into the fact that the b^m-contact form admits a decomposition given by $\alpha = udz + \beta$, where $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$. Hence the $\mathbb{R}^+$-invariance here means that the b^m-contact form is *convex*, see Definition 4.2.1.

Not every b^m-contact form is of course $\mathbb{R}^+$-invariant: for example the kernel of $g\alpha$, where $g \in C^\infty(M)$ is positive, defines the same contact structure as $\ker\alpha$, but a priori, there is no reason for the function g to be $\mathbb{R}^+$-invariant.

We will describe some examples of $\mathbb{R}^+$-invariant contact forms.

Example 5.5.6. An example of a $\mathbb{R}^+$-invariant b^m-contact for is given by the 3-torus as in Example 3.4.5. The b-contact form is given by $\alpha = \sin\varphi\frac{dx}{\sin x} + \cos\varphi dy$. The vector field given by $\sin x\frac{\partial}{\partial x}$ is a strict contact vector field (that is it satisfied $\mathcal{L}_X\alpha = 0$) and it is transverse to the critical set. However, this example is not overtwisted. Indeed, as we showed in Example 5.1.7 there are no periodic orbits away from the critical set and hence by Theorem 5.5.2, it is not an overtwisted b^m-contact manifold. However, by performing a Lutz twist (see Section 4.3. in [Gei08] for more details on this construction) around a transverse know away from the critical set, we obtain an $\mathbb{R}^+$-invariant overtwisted b^m-contact form on $\mathbb{T}^3$.

This example also shows that the presence of $\mathbb{R}^+$-invariance does not imply that the periodic orbits in Z can be continued away from the critical hypersurface.

Example 5.5.7. The b-contact form exhibited in Example 5.1.6 is not $\mathbb{R}^+$-invariant around the critical set. The b-contact form does not decompose $\alpha = u\frac{dz}{z} + \beta$ where $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$.

The flow of the vector field X generates a $\mathbb{R}^+$-action. We often refer to $M \setminus K$ as the $\mathbb{R}^+$-invariant part of the contact manifold (M, α) and will denote it M_{inv}.

In the $\mathbb{R}^+$-invariant part of the contact manifold M_{inv}, we have coordinates adapted to the action. Indeed, by following the flow of X, the $\mathbb{R}^+$-invariant part is diffeomorphic to $\partial K \times \mathbb{R}^+$, where K is the compact set with boundary as in Definition 5.5.4.

For our purposes, we fix notation for those coordinates in the case of maps from the disk D to M. Consider $u : D \to M$ and assume that for $z \in D$, $u(z) \in M_{inv}$. We will write

$$u(z) = (d(z), w(z)) \tag{5.2}$$

where $d(z) \in \mathbb{R}^+$ and $w(z) \in \partial K$. This notation will appear in the proof of the main theorem of this section, which is given by the following.

Theorem 5.5.8. *Let (M^3, α) be an overtwisted $\mathbb{R}^+$-invariant contact manifold. Then at least one of the two following statement holds:*

1. *There exists a 1-parametric family of periodic Reeb orbits in the*

$\mathbb{R}^+$-invariant part of M.

2. *There exists a periodic Reeb orbit away from the $\mathbb{R}^+$-invariant part.*

Furthermore, the periodic orbits are contractible loops in the symplectization.

It is clear that Theorem 5.5.2 is a straightforward corollary of this theorem.

Remark 5.5.9. As observed before, Theorem 5.5.2 is a proof of the Weinstein conjecture in the non-compact set-up. Another approach to guarantee periodic Reeb orbits on non-compact contact manifolds is by considering additional topological and geometrical conditions on the non-compact level-set of the Hamiltonian in the standard symplectic space. Those Hamiltonian are known as *tentacular Hamiltonians*, see [PVW18].

Similarly, we can generalize Theorem 5.5.2 to the E-contact setting, which is also follows directly from Theorem 5.5.8.

Corollary 5.5.10. *Let (M, α) be*

- *either a compact E-contact manifold*
- *or a compact folded-contact manifold*

that is $\mathbb{R}^+$-invariant around the singular set and assume that there exists an overtwisted disk away from the singular locus, respectively folding hypersurface. Then there exists a 1-parametric family of periodic Reeb orbits in the $\mathbb{R}^+$-invariant part of M or a periodic Reeb orbit away from the $\mathbb{R}^+$-invariant part.

Example 5.5.11. An example of a compact $\mathbb{R}^+$-invariant E-contact manifold is given by the 3-torus with the non-smooth differential form $\alpha = \sin\phi\frac{dx}{\sin x} + \cos\phi\frac{dy}{siny}$. The vector field given by $\sin x\frac{\partial}{\partial x} + \sin y\frac{\partial}{\partial y}$ is a strict contact vector field (it satisfies $\mathcal{L}_X\alpha = 0$) and is transverse to the boundary of a tubular neighbourhood of $\mathbb{T}^3 \cap \{x = 0\} \cap \{y = 0\}$. Therefore it is $\mathbb{R}^+$-invariant. This is not a a b-contact form as $\alpha \wedge d\alpha = \frac{1}{\sin x \sin y} d\phi \wedge dx \wedge dy$, but instead defines a C-contact form, see Example 3.5.5.

We include a remark that will be explained in greater details in the next chapter.

Remark 5.5.12. The author believes that Theorem 5.5.8 can be applied to other examples as well, such as particular cases of E-contact manifolds as for Example 3.5.7. This seems especially alluring in view of the correspondence of rotational Beltrami vector fields and contact structures as is proved in [EG00], consult Section 6.3 for the precise definitions. This correspondence remains valid for non-degenerate singularities in [PA19] and the presence of overtwisted disks is proved. A natural question is therefore to investigate the presence of radial invariance around the non-degenerate singularities, as is explained in Section 6.5.

Before continuing with a sketch of the proof of Theorem 5.5.8, let us highlight the difference between this result and the foliated Weinstein conjecture proved in [dPP18].

Remark 5.5.13. Foliated contact forms are a particular case of E-contact forms, see Example 3.5.8. For foliated contact forms, the authors in [dPP18] consider a compact ambient manifold M that is foliated by 3-dimensional leaves that are contact, but not necessarily compact and prove that if the contact structure is overtwisted in one leaf, there exists a periodic Reeb orbit in the close of the leaf. The overtwisted disk in the leaf is contained, in the language of E-forms, in the singular locus and hence the assumptions of Corollary 5.5.10 do not apply to the particular case of foliated contact forms, see also Remark 3.5.10. Hence we do not recover their result as a corollary of Corollary 5.5.10.

However, as will become clear in the following lines, the proof of the foliated Weinstein conjecture and main theorem of this section follows the same main idea: even though the overtwisted disk lies in a non-compact space (the leaf in the case of [dPP18] or $M \setminus Z$ in the case presented here), Arzela–Ascoli can be applied due to some kind of compactness (more precisely compact ambient space in [dPP18] respectively $\mathbb{R}^+$-invariance here).

We now sketch the proof of the Theorem 5.5.8. As in the standard setting, we study the Bishop family emanating from the overtwisted disk as in Theorem 2.4.11 and we aim to prove the existence of a finite energy plane and conclude by Theorem 2.4.10. As in the standard case, the gradient blows up in the interior of the disk. However, in contrast to the standard proof, we distinguish two different cases.

In the first case, we assume that the sequence where the gradient blows up is contained in a bounded subset of M. In this case, the standard arguments apply, and there a reparametrization of the bubble yields a finite energy plane contained in the symplectization of the bounded subset of M. This yields the existence of a periodic Reeb orbit away from the $\mathbb{R}^+$-invariant part.

In the opposite case, the sequence of points where the gradient blows up is not bounded in M. Loosely speaking this means that the point of blow-up diverges in the non-compact $\mathbb{R}^+$-invariant part. This non-compactness behaviour is settled by translating the J-holomorphic curves in the direction of the $\mathbb{R}^+$-action and therefore, in the decomposition as in Equation 5.2, the first term is being kept constant (so it trivially converges) and the second term is contained in the compact set ∂K, so Arzela–Ascoli theorem applies to this term. We thereby obtain a sequence of J-holomorphic disks, contained in the symplectization of the $\mathbb{R}^+$-invariant part of M, that converge to a non-trivial finite energy plane. This yields a periodic orbit in the $\mathbb{R}^+$-invariant part and by $\mathbb{R}^+$-invariance therefore also a 1-parametric family of periodic Reeb orbits.

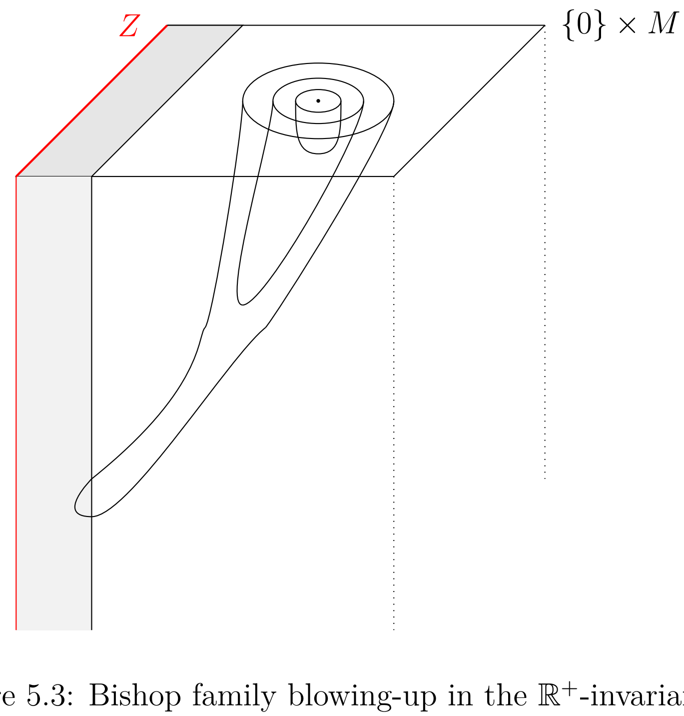


Figure 5.3: Bishop family blowing-up in the $\mathbb{R}^+$-invariant part

First, we begin by collecting the necessary lemmas to prove the main theorem. The proof of Theorem 5.5.8 is then done in the subsequent subsection, Subsection 5.5.2.

5.5.1 Necessary lemmas

We denote the overtwisted disk, that exists away from the $\mathbb{R}^+$-invariant part, by D_{OT} and denote its elliptic singularity by e. As usual D_{OT}^* denotes $D_{OT} \setminus \{e\}$.

Compactness or non-compactness of the Bishop family $\{\widetilde{u}_t\}$ depends on the whether or not the gradient $\nabla\widetilde{u}_t$ is uniformly bounded.

Lemma 5.5.14 (Uniform gradient bound implies compactness of family)**.** *Let (M, α) be a $\mathbb{R}^+$-invariant contact manifold. Assume that*

$$\widetilde{u}_t = (a_t, u_t) : D \to \mathbb{R} \times M, t \in [0, \epsilon)$$

is a Bishop family as in Theorem 2.4.11 satisfying the boundary conditions as in Equation 5.4. Assume furthermore that

$$\sup_{0 \leq t < \epsilon} \|\nabla\widetilde{u}_t\|_{C^0(D)} < \infty. \tag{5.3}$$

Then $\widetilde{u}_t \to \widetilde{u}_\epsilon$ in $C^\infty(D)$ as $t \to \epsilon$ where $\widetilde{u}_\epsilon$ is an embedded pseudo-holomorphic disk satisfying $\widetilde{u}_\epsilon(\partial D) \subset \{0\} \times D_{OT}^*$.

Proof. The proof can be found in Proposition 8.1.2 in [AH19]. □

Let us first do a remark concerning gradient blow-ups.

Remark 5.5.15. As stated in the last lemma, compactness or non-compactness of the Bishop family $\{\widetilde{u}_t\}$ depends on the whether or not the gradient $\nabla\widetilde{u}_t$ is uniformly bounded. Let us remark here that by blow-up of the gradient, we mean that the gradient of J-holomorphic families of disk blows up for a certain parametrization of the disk.

More precisely, it is clear that if $\widetilde{u}_t : D \to \mathbb{R} \times M$ is a family of J-holomorphic disks satisfying the usual boundary conditions

$$\widetilde{u}_t(\partial D) \subset \{0\} \times D^*_{OT} \text{ and } \inf_{0 \leq t < \epsilon} \operatorname{dist}(u_t(\partial D), e) > 0 \tag{5.4}$$

and $\phi : D \to D$ is a conformal automorphism of the unit disk, then $\widetilde{u}_t \circ \phi$ is also a J-holomorphic curve satisfying the same boundary conditions. It is well-known that the conformal automorphism group is of the disk is non-compact and generated by

$$\phi(z) = e^{i\alpha} \frac{a - z}{1 - \bar{a}z},$$

where $\alpha \in [0, 2\pi)$ and $a \in \operatorname{int}(D)$. By choosing a close to ∂D, $\|\nabla \phi\|_{C^0(D)}$ becomes arbitrarily large. Therefore, when we say that the gradient blows up, we mean that it blows up for the infimum of all possible conformal reparametrization of the disk. More explicitly, we mean that the quantity

$$e(\widetilde{u}_t) := \inf_{\phi \in \operatorname{Aut}(D)} \|\nabla(\widetilde{u}_t \circ \phi)\|_{C^0(D)}$$

goes to infinity when $t \to \epsilon$.

As in [AH19], the energy of the family $\widetilde{u}_t$ is bounded above by the $d\alpha$-area of the overtwisted disk.

Lemma 5.5.16 (Universal upper bound on the energy)**.** *Let (M, α) be a $\mathbb{R}^+$invariant contact manifold. Assume that*

$$\widetilde{u}_t = (a_t, u_t) : D \to \mathbb{R} \times M, t \in [0, \epsilon)$$

is a Bishop family as in Theorem 2.4.11 satisfying

$$\inf_{0\leq t<\epsilon} dist(u_t(\partial D), e) > 0.$$

Then there exists a constant $C = C(\alpha, D_{OT}) > 0$ *such that* $E(\widetilde{u}_t) < C$.

Proof. See Lemma 8.1.3 in [AH19]. □

Similar to the arguments in [AH19], the horizontal energy of the family $\{\widetilde{u}_t\}_t$ is bounded below independently of t. Note that a priori, compactness is needed in order to apply Arzela–Ascoli theorem. However, the limit of the Bishop family in the non-compact manifold M is taken care of by the $\mathbb{R}^+$-invariance.

Proposition 5.5.17 (Universal lower bound on the horizontal energy)**.** *Let* (M, α) *be a* $\mathbb{R}^+$*-invariant contact manifold. Assume that*

$$\widetilde{u}_t = (a_t, u_t) : D \to \mathbb{R} \times M, t \in [0, \epsilon)$$

is a Bishop family as in Theorem 2.4.11 satisfying

$$\inf_{0\leq t<\epsilon} dist(u_t(\partial D), e) > 0.$$

Then there exists a constant $c > 0$ *independent of* t *so that*

$$E^h(u_t) := \int_D u_t^* d\alpha \geq c.$$

Proof. The proof follows the strategy of the proof of Proposition 8.1.4 in [AH19]. Let us argue by contradiction. We pick a sequence $\{\widetilde{u}_{t_k}\}_k$ such that $t_k \to \epsilon$ as $k \to \infty$. For convenience, we write $\widetilde{u}_k = \widetilde{u}_{t_k}$. We assume by contradiction that $\int_D u_k^* d\alpha \to 0$ as $k \to \infty$. There are two possibilities: either the gradient is uniformly bounded or it is not.

Case I: the gradient is uniformly bounded.

The case where the gradient is uniformly bounded is as in proof of Proposition 8.1.4: $\widetilde{u}_k$ converges in C^∞ to some $\widetilde{v}$ by Lemma 5.5.14 which satisfies that $E^h(v) = 0$. Following the original proof, this implies that $\widetilde{v}$ is constant, which is a contradiction with $\widetilde{v}$ having non-zero winding number.

Case II: the gradient blows up.

Let us take a sequence $z_k \in D$ such that

$$R_k := |\nabla \widetilde{u}_k(z_k)| \to \infty$$

as $k \to \infty$ and let us assume that $z_k \to z_0 \in D$ (after passing maybe to a subsequence). We will do the bubbling analysis à la Sacks–Uhlenbeck around the point z_0 to derive a contradiction with the assumption that $\int_D u_k^* d\alpha \to 0$. Due to the non-compactness of M, care needs to be taken in the bubbling analysis. Due to the $\mathbb{R}^+$-invariance, the non-compactness is *mild* and we will see that Arzela–Ascoli theorem can still be applied to show convergence to some J-holomorphic plane (in Subcase I) respectively to some J-holomorphic half plane (in Subcase II).

Take a sequence of $\epsilon_k > 0$ such that $\epsilon_k \to 0$ and $R_k \epsilon_k \to \infty$. By Hofer's lemma (Lemma 4.4.4 in [AH19]), we can additionally assume that

$$|\nabla \widetilde{u}_k(z)| \leq 2R_k \quad \text{if } |z - z_k| \leq \epsilon_k. \tag{5.5}$$

We distinguish two subcases.

Subcase I: the gradient blows up in the interior.

More precisely, by this we mean that $R_k \mathrm{dist}(z_k, \partial D) \to \infty$. We will show that in this case a non-trivial finite energy plane bubbles of and has zero horizontal energy, which leads to a contradiction.

We distinguish two further subcases, depending whether or not the gradient blows up in the compact subset $K \subset M$ or the $\mathbb{R}^+$-invariant part $M \setminus K$.

First, let us assume that $u_k(z_k)$ remains in the compact subset K. Then the standard arguments as in Proposition 8.1.4 in [AH19] apply and lead to a non-trivial finite energy plane having zero horizontal energy, which is a contradiction.

Hence, we assume that $u_k(z_k)$ tends to the $\mathbb{R}^+$-invariant part. We use the coordinates introduced in Equation 5.2,

$$\widetilde{u}_k = (a_k, d_k, w_k)$$

and we assume that $d_k(z_k) \to \infty$. If this was not that case, we would be in the case where $u_k(z_k)$ remains in the compact subset K.

We now define the following pseudoholomorphic curves, which are a translation in the invariant direction of a reparametrization of $\widetilde{u}_k$. We define $v_k(z) = u_k(z_k + \frac{z}{R_k})$ so that $v_k(0) = u_k(z_k)$ is the point where the gradient blows up. For $z \in B_{R_k}(-R_k z_k) \cap v_k^{-1}(M_{inv})$, we define

$$\widetilde{v}_k(z) = \Big(a_k\Big(\frac{z}{R_k}+z_k\Big)-a_k(z_k), d_k\Big(z_k+\frac{z}{R_k}\Big)-d_k(z_k)+N, w_k\Big(z_k+\frac{z}{R_k}\Big)\Big)$$

and denote the components by $\widetilde{v}_k := (e_k, f_k, q_k)$ where $(f_k, q_k) \in \partial K \times \mathbb{R}^+$ are the $\mathbb{R}^+$-invariant coordinates. The map (f_k, q_k) is a translation in the $\mathbb{R}^+$-invariant direction of v_k that we do in order to apply Arzela–Ascoli to the function q_k that is contained in the compact space ∂K.

The family J-holomorphic curves $\widetilde{v}_k$ satisfies

1. $|\nabla \widetilde{v}_k(0)| = 1$,

2. $|\nabla \widetilde{v}_k(z)| \leq 2$ if $z \in \Omega_k := B_{\epsilon_k R_k(0)}(0) \cap B_{R_k}(-R_k z_k) \cap v_k^{-1}(M_{inv})$,

3. $e_k(0) = f_k(0) = 0$.

Furthermore

$$\int_{B_{R_k}(-R_k z_k)\cap v^{-1}(M_{inv})} (f_k, q_k)^* d\alpha \leq \int_D u_k^* d\alpha \to 0$$

when $k \to \infty$.

We claim that $\cup_k \Omega_k = \mathbb{C}$. As $z \in B_{\epsilon_k R_k}$, $|z_k - \frac{z}{R_k} - z_k| < \epsilon_k$. As $v_k(z_k) \in M_{inv}$, we thus obtain that for k large, $v_k(\Omega_k) \subset M_{inv}$. Hence for $k > k_0$, where k_0 is large, $\Omega_k = B_{\epsilon_k R_k(0)}(0) \cap B_{R_k}(-R_k z_k) := \widetilde{\Omega}_k$. As $R_k \text{dist}(z_k, \partial D) \to \infty$, we have that $\cup_{k \geq k_0} \widetilde{\Omega}_k = \mathbb{C}$.

By the C^∞_{loc}-bounds (Theorem 4.3.4 in [AH19]), we conclude that (up to choosing a subsequence) $\widetilde{v}_k$ converges in C^∞_{loc} to a J-holomorphic plane

$$\widetilde{v} = (b, v) : \mathbb{C} \to \mathbb{R} \times M_{inv}$$

satisfying $|\nabla \widetilde{v}(0)| = 1$, $\int_{\mathbb{C}} v^* d\alpha = 0$. Furthermore $E(\widetilde{v}) \leq C$. Indeed, let $Q \subset \mathbb{C}$ be a compact subset and assume $k > k_0$ so that $v_k(\Omega_k) \subset M_{inv}$. We compute

$$\sup_{\phi \in \mathcal{C}} \int_Q \widetilde{v}_k^* d(\phi\alpha) \leq \sup_{\phi \in \mathcal{C}} \int_{B_{R_k}(-R_k z_k)} M \widetilde{v}_k^* d(\phi\alpha) \tag{5.6}$$

$$= \sup_{\phi \in \mathcal{C}} \int_D \widetilde{u}_k^* d(\phi\alpha) = E(\widetilde{u}_k) \leq C, \tag{5.7}$$

where $C > 0$ is such as in Lemma 5.5.16. We now take the limit $k \to \infty$ and then the supremum over all compact sets $Q \subset \mathbb{C}$ to obtain $E(\widetilde{v}) < \infty$.

By Proposition 4.4.2 in [AH19], $\widetilde{v}$ is constant, which is in contradiction with $|\nabla \widetilde{v}(0)| = 1$. We conclude that Subcase I cannot happen.

Subcase II: the gradient blows up on the boundary.

More precisely, by this we mean that $R_k \text{dist}(z_k, \partial D) \to l \in [0, \infty)$. We furthermore assume that the gradient only blows up at the boundary, that is that there does not exist another subsequence such that

$R_k\text{dist}(z_k, \partial D) \to \infty$. If this was the case, we would be in Subcase I, and therefore obtain a contradiction. Due to the boundary condition $u_t(\partial D) \subset D_{OT}$, we can assume that the subsequence u_t is therefore contained in the compact subspace $K \subset M$ and therefore the standard arguments of the proof of Proposition 8.1.4 in [AH19] apply and shows bubbling of a non-trivial finite energy half-plane. This is in contradiction with the horizontal energy to be zero.

This finishes the proof of Proposition 5.5.17. □

Finally, the next lemma guarantees that bubbling does not happen at the boundary of the holomorphic disks.

Proposition 5.5.18 (Forbidding bubbling at the boundary)**.** *Let $\{\widetilde{u}_t\}_t$ a Bishop family as before. Assume that the gradient blows up, that is*

$$\sup_{0 \le t \le \epsilon} e(\widetilde{u}_t) = \infty$$

where $e(\widetilde{u}_t)$ is given as in Equation 5.8. If $t \to \epsilon$ and $(z_k)_{k\in\mathbb{N}} \subset D$ are sequences so that $R_k := |\nabla \widetilde{u}_{t_k}(z_k)| \to \infty$. Then the sequence $R_k\, dist(z_k, \partial D)$ is unbounded.

Proof. See Proposition 8.2.1 in [AH19]. □

5.5.2 Proof of Theorem 5.5.8

We continue by proving the main theorem, that is Theorem 5.5.8 using the collection of lemmas and propositions of the last subsection.

Proof of Theorem 5.5.8. Consider the Bishop family $\{\widetilde{u}_t\}_{t\in[0,\epsilon)}$ emanating from the overtwisted disk as in Theorem 2.4.11. Note that the loops $\widetilde{u}_t(\partial D)$ never intersect ∂D_{OT} in view of Lemma 2.4.12. If

$e(\widetilde{u}_t)$ was bounded, then by Lemma 5.5.14 we could continue the family $\{\widetilde{u}_t\}_t$ beyond ϵ which contradicts maximality of the family. Hence $e(\widetilde{u}_t)$ is unbounded. Pick a sequence $\{\widetilde{u}_k\}_k$ such that $\|\nabla \widetilde{u}_k\|_{C^0(D)} \to \infty$ as $k \to \infty$. We know that there are lower and upper bounds for the energy by Lemma 5.5.16 and Proposition 5.5.17:

$$c \leq E(\widetilde{u}_k) \leq C.$$

Pick a sequence of points $(z_k)_{k\in\mathbb{N}} \subset D$ so that $R_k := |\nabla \widetilde{u}_k(z_k)| \to \infty$. By Proposition 5.5.18, bubbling on the boundary cannot happen and we therefore can assume (after passing to a subsequence) that $z_k \to z_0$ and $R_k \text{dist}(z_k, \partial D) \to \infty$ when $k \to \infty$.

Here is where the main difference with Hofer's standard proof occurs: There are two possibilities: either the gradient blows up away from the $\mathbb{R}^+$-invariant part or it blows up inside the $\mathbb{R}^+$-invariant part. This was observed already in Proposition 5.5.17 and we repeat similar arguments here.

For the map $u_k(z_k)$, this is saying that either for k large (up to a subsequence to avoid mixed behaviour), $u_k(z_k)$ can satisfy one of the two cases:

1. $u_k(z_k)$ remains away from the $\mathbb{R}^+$-invariant part. In this case, the standard arguments apply as we can assume that $u_k(D) \subset K$, where K is compact (the set K is as in Definition 5.5.4). Therefore Arzela–Ascoli theorem applies, as well as the rest of Hofer's arguments. This proves that there exists a periodic orbit away from the $\mathbb{R}^+$-invariant part. This proves the first part of the theorem.

2. $u_k(z_k)$ tends to the $\mathbb{R}^+$-invariant part. In this case, as mentioned before, we run into compactness issues and therefore the standard arguments do not apply directly.

In what follows, we hence assume that for $k \to \infty$, $u_k(z_k)$ tends to the $\mathbb{R}^+$-invariant part and we will prove that this implies that there is a 1-parametric family of periodic orbits in the $\mathbb{R}^+$-invariant neighbourhood.

More precisely, we assume that for all $k > k_0$, $u_k(z_k) = (d_k(z_k), w_k(z_k))$ as in Equation 5.2. Furthermore, we assume without loss of generality that $d_k(z_k) \to \infty$. Indeed, if this was false, we could just enlarge the compact set K and we would be in the first case.

We will do now the bubbling analysis around the point z_k as in Hofer and in order to apply Arzela–Ascoli theorem, we do a translation in the $\mathbb{R}^+$-invariant direction.

Take a sequence $\epsilon_k \to 0$ so that $R_k\epsilon_k \to \infty$. We now use the so called Hofer's lemma (Lemma 4.4.4 in [AH19]) to additionally assume that

$$|\nabla \widetilde{u}_k(z)| \leq 2R_k \tag{5.8}$$

for all $z \in D$ with $|z - z_k| \leq \epsilon_k$. We define for $z \in B_{R_k}(-R_k z_k)$ the pseudoholomorphic maps $\overline{v}_k = (b_k, v_k)$ given by

$$\overline{v}_k(z) := \Big(a_k(z_k + \frac{z}{R_k}) - a_k(z_k), u_k(z_k + \frac{z}{R_k})\Big).$$

In the standard case (so also in the first case higher up), these maps are shown to converge to a non-constant finite energy plane using Arzela–Ascoli theorem. However, in this case, Arzela–Ascoli theorem does not apply because u_k is not contained in a compact space. More precisely, $v_k(0) = u_k(z_k)$ and therefore v_k is contained in a M_{inv} around the origin. To overcome this, we define the following pseudoholomorphic maps, which is just a translation of the previous one in the $\mathbb{R}^+$-invariant direction. For $z \in B_{R_k}(-R_k z_k) \cap v_k^{-1}(M_{inv})$, we define:

$$\widetilde{v}_k(z) := \Big(a_k(z_k + \frac{z}{R_k}) - a_k(z_k), d_k(z_k + \frac{z}{R_k}) - d_k(z_k), w_k(z_k + \frac{z}{R_k})\Big).$$

We do this translation in the $\mathbb{R}^+$-invariant direction because $d_k(z_k) \to \infty$. We will now prove that $\widetilde{v}_k$ converges in C^∞_{loc} to a non-trivial finite energy plane.

Let us denote the components of the map $\widetilde{v}_k = (e_k, v_k) = (e_k, f_k, q_k)$

It is clear from the reparametrization that

$$e_k(0) = 0, f_k(0) = 0, \text{ and } |\nabla \widetilde{v}_k(0)| = 1.$$

The advantage of the reparametrization $\widetilde{v}_k$ with respect to the one given by $\overline{v}_k$ is that the convergence of f_k is being taken care of as it is fixed at the origin and Arzela–Ascoli theorem applies to $q_k(z)$ as it belongs to the compact set ∂K. This was not the case for $\overline{v}_k$.

Consider the domains $\Omega_k := B_{R_k}(-R_k z_k) \cap B_{\epsilon_k R_k}(0) \cap v_k^{-1}(M_{inv})$.

For k sufficiently large, we claim that $v_k(\Omega_k) \subset M_{inv}$. Indeed, as $z \in B_{\epsilon_k R_k}$, $|z_k - \frac{z}{R_k} - z_k| < \epsilon_k$. As $v_k(z_k) \in M_{inv}$, we thus obtain that for $k > k_0$, $v_k(\Omega_k) \subset M_{inv}$.

Furthermore it follows from $R_k \text{dist}(z_k, \partial D) \to \infty$ that $\bigcup_{k>k_0} \Omega_k = \mathbb{C}$. The gradient boundedness (Equation 5.8) translates into

$$|\nabla \widetilde{v}_k(z)| \leq 2 \text{ on } \Omega_k.$$

By the C^∞_{loc}-bounds (Theorem 4.3.4 in [AH19]), we conclude that (up to choosing a subsequence) $\widetilde{v}_k$ converges in C^∞_{loc} to a J-holomorphic plane

$$\widetilde{v} = (b, v) : \mathbb{C} \to \mathbb{R} \times M_{inv}$$

which is non-constant because $|\nabla \widetilde{v}(0)| = 1$. We compute that the energy of $\widetilde{v}$ is finite using the standard arguments. Let $Q \subset \mathbb{C}$ be a compact set and take $k > k_0$ large. We obtain

$$\sup_{\phi \in \mathcal{C}} \int_Q \widetilde{v}_k^* d(\phi\alpha) \leq \sup_{\phi \in \mathcal{C}} \int_{B_{R_k}(-R_k z_k)} \widetilde{v}_k^* d(\phi\alpha) \tag{5.9}$$

$$= \sup_{\phi \in \mathcal{C}} \int_D \widetilde{u}_k^* d(\phi\alpha) = E(\widetilde{u}_k) \leq C. \tag{5.10}$$

We now take the limit $k \to \infty$ and take the supremum over all compact $Q \subset \mathbb{C}$ to obtain that $E(\tilde{v}) < \infty$. Moreover the image of v lies in a compact subset of M_{inv}, to be precise in $\{0\} \times \partial K$ since this is true for all the maps v_k.

Hence we found a finite energy plane in the $\mathbb{R}^+$-invariant part of M, and by Theorem 2.4.10 this yields a periodic orbit in M_{inv}. By the $\mathbb{R}^+$-invariance, this yields a 1-parametric family of periodic orbits in every $\{cst\} \times \partial K$.

□

5.6 Do overtwisted contact manifolds admit infinitely many periodic Reeb orbits?

An immediate implication of the conjectural existence of plugs for the b^m-Reeb flow would be examples (in addition to the ones we already exhibited) without any periodic Reeb orbits away from the critical set in any dimension. Indeed, let (M, α) be a contact manifold with only finitely many periodic Reeb orbits. If there exists a plug for the b^m-Reeb flow, it can be introduced in such a way that the periodic orbit entered the plug and gets captured. As plugs satisfy the entry-exit condition, in contrast to traps, the global dynamics are not being altered. The outcome is a closed b^m-contact manifold without any periodic Reeb orbits.

We have already seen that the Weinstein conjecture is not satisfied for b^m-contact manifolds: there are closed examples without periodic orbits away from Z. We will now discuss a more surprising consequence of the conjectural existence of plugs. As already discussed in Section 2.4, the only known examples of closed contact manifolds with finitely many periodic Reeb orbits are given by the irrational ellipsoid and quotients of it to lens spaces, which are both tight contact structures. We therefore raised the question if overtwisted contact manifolds would always admit finitely many periodic Reeb orbits. We claim that this in fact follows from the existence of an $\mathbb{R}^+$-invariant plug for the b^m-Reeb flow. By contradiction, let us assume that there is an overtwisted contact manifold that only admits finitely many periodic Reeb orbits. By applying the plug construction centred on every periodic Reeb orbits to capture it yields a $\mathbb{R}^+$-invariant overtwisted b^m-contact manifold without periodic orbit. This contradicts Theorem 5.5.2 and therefore there must exist an infinite number of periodic Reeb orbits.

To avoid confusion, the result on infinitely many periodic Reeb orbits on overtwisted contact manifolds is based on the existence of plugs for the b^m-Reeb flow, which we have not proved in this thesis. See the discussion in Sections 5.3 and 5.4.

5.7 Singular Weinstein conjecture

We finish this chapter by discussing a singular version of the Weinstein conjecture and also mentioning further interesting lines of research.

As mentioned before, asking for the existence of periodic Reeb orbits away from the critical set does not give rise to a meaningful generalization of Weinstein conjecture.

Indeed, even if overtwistedness for b^m-contact manifolds (Theorem 5.5.2) is a sufficient condition for the manifolds to admit periodic b^m-Reeb orbits away from Z, there are examples of compact b^m-contact manifolds without periodic Reeb orbits away from the critical set. We are going to formulate a conjecture that guarantees the existence of a certain kind of invariant dynamical set, that we call *singular periodic* Reeb orbits.

Definition 5.7.1. Let $\gamma : \mathbb{R} \to M$ be an integral curve of the Reeb vector field R_α associated to a b^m-contact manifold (M, α). We say that γ is a singular periodic Reeb orbits if γ is an orbit that connects two fixed points on Z, where the Reeb vector field vanishes, that is $\lim_{t\to\pm\infty} \gamma(t) = p_\pm \in Z$ where $R_\alpha(p_\pm) = 0$.

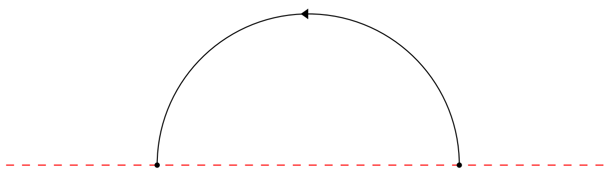


Figure 5.4: A Singular periodic orbit

In the language of dynamical systems, this orbit is a heteroclinic orbit associated to the b^m-Reeb flow. We will now justify the definition of singular periodic orbits by means of the examples without periodic Reeb orbits away from the critical set.

Example 5.7.2. Consider the b-contact manifold $S^3 \subset (\mathbb{R}^4, \omega)$ where $\omega \in {}^b\Omega^2(\mathbb{R}^4)$ is the standard b-symplectic form as in Example 5.1.6. As already observed, the Reeb vector field given by

$$R_\alpha = 2x_1^2 \frac{\partial}{\partial y_1} - x_1 y_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial y_2} - 2y_2 \frac{\partial}{\partial x_2}$$

admits infinitely many periodic orbits on $Z = \{x_1 = 0\}$ but none away from the critical set. However, there is a singular periodic orbits. Indeed, consider the two fixed points on the critical hypersurface given by $(0, \pm 1, 0, 0)$. In the (x_1, y_1)-plane, the Reeb vector field is a singular periodic orbit. It converges to the fixed points. The orbit is topologically given by circle and dynamically speaking, it has two marked fixed points, see also Figure 5.4.

Example 5.7.3. Consider the example of the 3-torus $(\mathbb{T}^3, \sin\varphi \frac{dx}{\sin x} + \cos\varphi dy)$ as in Example 5.1.7. The Reeb vector field is given by

$$R_\alpha = \sin\varphi \sin x \frac{\partial}{\partial x} + \cos\varphi \frac{\partial}{\partial y}.$$

On the critical set $Z = \{x = 0, \pi\}$, the fixed points are given by the 1-dimensional submanifold $\varphi = \pm\frac{\pi}{2}$. Consider the restriction of the Reeb vector field restricted to $\{\varphi = \frac{\pi}{2}\}$. The orbits a given by $R_\alpha = \frac{\sqrt{2}}{2} \sin x \frac{\partial}{\partial x}$, which are all singular periodic Reeb orbits.

Remark 5.7.4. Assume once more that the attempts to save the plug I and II work. The (hypothetical) plug construction morally replaces a periodic Reeb orbit with a singular periodic Reeb orbit. This is saying that the existence of plugs for the b^m-Reeb flow does not contradict the existence of singular periodic Reeb orbits.

With the last examples in mind, we formulate the following conjecture.

Conjecture 5.7.5 (Singular Weinstein conjecture). *Let (M, α) be a closed b^m-contact manifold. Then there always exist singular periodic Reeb orbits.*

In dimension 3, one is tempted to use the desingularization technique (Theorem 4.2.4) in the case of almost convex b^{2k}-contact forms on compact manifolds to prove this conjecture in this case.

Indeed, assume $\alpha \in {}^{b^{2k}}\Omega^1(M)$ an almost convex b^{2k}-contact form, so that Theorem 4.2.4 applies. Hence there exists a family of contact forms α_ϵ that coincide with α outside of a tubular neighbourhood of the critical set, denoted $\mathcal{N}_\epsilon(Z)$. As the manifold M^3 is assumed to be compact, by Taubes complete proof of the Weinstein conjecture [Tau07], there exists a periodic Reeb orbit γ_ϵ on the desingularization (M, α_ϵ) for all $\epsilon > 0$. We distinguish two different cases: the periodic Reeb orbit γ_ϵ is contained in the complement of the desingularization neighbourhood $\mathcal{N}_\epsilon(Z)$, or it intersects it. See Figure 5.5 for a periodic orbit of the desingularized b^{2k}-contact form contained in the complement of the desingularization neighbourhood, denoted by γ_1, and another one intersecting it, given by γ_2.

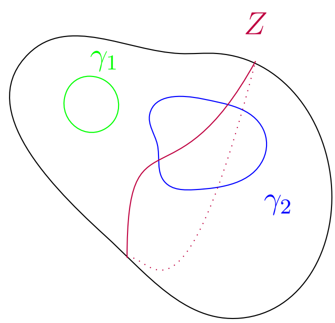


Figure 5.5: Periodic orbits and the desingularization theorem

In the first case, where the periodic orbit does not intersect the desingularization neighbourhood, it corresponds to a periodic Reeb orbit of the initial b^{2k}-contact form as is shown by the following observation.

Lemma 5.7.6. *Let (M, α) be an almost convex b^{2k}-contact manifold.*

Consider the family of contact forms α_ϵ associated to the desingularization. Assume that there exists ϵ such that there is a periodic Reeb orbit of the Reeb vector field R_{α_ϵ} outside of the ϵ-neighbourhood $\mathcal{N}_\epsilon$. Then this orbit corresponds to a periodic orbit of the Reeb vector field R_α.

Proof. The desingularization does not change the dynamics outside of the ϵ-neighbourhood. □

Note that the same would hold for the desingularization of b^{2k+1}-contact structures, where the resulting geometric structure would be of folded-contact type.

The more problematic orbits are those that intersect the desingularization neighbourhood (see γ_2 in Figure 5.5): the desingularization changes the dynamics rather drastically. More precisely, we will see that the Reeb dynamics change whenever the Reeb vector field is not everywhere regular or singular around a connected component of the critical set.

Lemma 5.7.7. *Let (M, α) be a almost-convex b^{2k}-contact manifold. Then in the ϵ-neighbourhood of the critical set, the Reeb flow associated to the desingularization is a reparametrization of the initial Reeb flow if and only if semi-locally, the Reeb vector field is everywhere regular or everywhere singular.*

Proof. As in Theorem 4.2.4, we write $R_\alpha = gz^{2k}\frac{\partial}{\partial z} + X$ and the expression of the desingularized Reeb vector field is given by $R_{\alpha_\epsilon} = g\frac{1}{f_\epsilon'}\frac{\partial}{\partial z} + X$. The flow of the first one is a reparametrization of the second one if and only if $R_\alpha = fR_{\alpha_\epsilon}$ for a smooth function f. This is clearly only the case if the Reeb vector field is everywhere singular or everywhere regular. □

One is tempted to take the limit of $\epsilon \to 0$. However, the continuity of the family of periodic orbits with respect to ϵ cannot be guaranteed. Therefore, limit arguments do not work without any further assumptions on the b^{2k}-contact form. A necessary condition is non-degeneracy for the family of contact forms $\{\alpha_t\}_{t\in]0,\epsilon]}$.

We believe that a first step towards a proof is to verify this conjecture by extending variational calculus to the language of piece-wise smooth loops with marked points (which correspond to the fixed points of the vector field) instead of $C^\infty(S^1, M)$. Using this generalization of variational calculus, it may be possible to prove special cases of this conjecture, as in the smooth case [Rab78b], as for example for convex hypersurfaces in the standard b^m-symplectic Euclidean space $(\mathbb{R}^{2n}, \omega)$.

Furthermore, we saw that there are b^m-symplectic manifolds with proper smooth Hamiltonian whose Hamiltonian flow does not have any periodic orbits away from Z. As before, the construction replaces periodic orbits by singular periodic orbits. We believe that techniques, similar to [Gin97] and [GG03] can be adapted to give examples of level-sets of b^m-symplectic manifolds containing no singular periodic orbit. This would be a counter-example to the singular Hamiltonian Seifert conjecture.

Conjecture 5.7.8. *Let (W, ω) be a b^m-symplectic manifold. Then there exists a $H \in C^\infty(W)$ proper, smooth Hamiltonian whose level-sets do not contain any singular periodic orbits.*

Chapter 6

Applications of b^m-contact geometry: the three body problem and fluid dynamics

It's not always necessary to be strong, but to feel strong.
Jon Krakauer, *Into the Wild*

In this chapter, we discuss applications of b^m-contact geometry to two physical system: the restricted three body problem, as discussed in the introduction, and fluid dynamics.

In the case of the restricted planar circular three body problem, we will see that the dynamics can be described as the Reeb dynamics associated to a b^3-contact form. We apply the results introduced in the former chapters to show existence of periodic orbits on the manifold at infinity, which corresponds to the critical set of the b^3-contact structure.

A part from this example, we will see a narrow connection between fluid dynamics and contact forms, which remains true in the b-setting due to [CMPS19]. The results proved in the last chapter have some direct consequences for the dynamics of the vector fields originating from this correspondence.

6.1 The contact geometry of the restricted three body problem

The three body problem (3BP) describes the motion of three point masses that move under the influence of their own gravity in $\mathbb{R}^3$. The circular restricted three body problem is a simplification of the 3BP as is explained in the following lines.

A first simplification of the problem consists in considering that one body has negligible mass and does not effect the movement of the two other bodies, called the primaries. The primaries move following Keplers law. We assume further that the primaries move on circles around their center of mass and that the small body moves in the plane spanned by the motion of the primaries. This problem is known as the restricted circular three body problem (RC3BP). Typically, we assume that the two primaries are given by the earth and the moon and the body of negligible mass is a satellite, see Figure 1.1. The Hamiltonian of the satellite is given by

$$H(q,p,t) = \frac{|p|^2}{2} - U(q,t), \quad (q,p) \in \mathbb{R}^2 \setminus \{q_E(t), q_M(t)\} \times \mathbb{R}^2,$$

where the potential is given by

$$U(q,t) = \frac{1-\mu}{|q-q_E(t)|} + \frac{\mu}{|q-q_M(t)|}.$$

Here $q_E(t), q_M(t)$ denotes the position of the earth and the moon given by

$$q_E(t) = (\mu \cos t, \mu \sin t)$$

respectively

$$q_M(t) = (-(1-\mu)\cos t, -(1-\mu)\sin t))$$

and $\mu = \frac{m_M}{m_M + m_E} \in [0,1]$ is the relative mass. Instead of the inertial coordinate system, it is convenient to consider the restricted three body problem in rotating coordinates. The Hamiltonian under this time-dependent transformation becomes autonomous but ceases to be the same of kinetic and potential terms: an additional term, whose physical interpretation is given by the Coriolis force, appears. The resulting Hamiltonian is given by

$$H(q,p) = \frac{|p|^2}{2} - \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|} + p_1 q_2 - p_2 q_1, \tag{6.1}$$

where $q_E = (\mu, 0)$ and $q_M = (1-\mu, 0)$. We stress that this Hamiltonian is time independent.

The Hamiltonian in Equation 6.1 is known to have 5 critical points $L_1, \ldots, L_5$ that are called Lagrange points, ordered as follows:

$$H(L_1) < H(L_2) < H(L_3) < H(L_4) = H(L_5).$$

The Hill's region are defined by $\mathcal{K}_c = \pi(\Sigma_c) \subset \mathbb{R}^2 \setminus \{q_E, q_M\}$ where π is the projection to the position coordinates and $\Sigma_c = H^{-1}(c)$ is the level-set of the Hamiltonian.

For $c < H(L_1)$ there are 3 connected components for $\mathcal{K}_c$: two bounded ones (one around the moon, one around the earth that we denote $\mathcal{K}_c^M$, respectively $\mathcal{K}_c^E$), and one unbounded one. We denote the connected component of the level-set that projects to $\mathcal{K}_c^M$ by Σ_c^M. Although $\mathcal{K}_c^M$ is bounded, Σ_c^M is not, due to collision.

Contact manifolds appear as level-set of Hamiltonian whenever there exists a Liouville vector field that is transverse to it. In [AFKP12], the authors prove the following:

Proposition 6.1.1. *The Liouville vector field $X = (q - q_M)\frac{\partial}{\partial q}$ is transverse to Σ_c^M for $c < H(L_1) + \epsilon$, where $\epsilon > 0$ is sufficiently small.*

It follows that Σ_c^M admits an induced contact form and the Hamiltonian dynamics are described by the Reeb vector field. In order to apply Weinstein conjecture in dimension 3, the authors show that Σ_c^M can be compactified using Moser's regularization and that the Liouville vector field extends to this regularization. Therefore by Weinstein conjecture, there exist periodic Reeb orbits in the regularization.

6.2 The b^m-contact geometry of the RPC3BP

Instead of using Moser's regularization, we make use of the McGehee blow-up and will prove that it induces a b^3-contact structure on positive energy level-sets of the Hamiltonian of the CR3BP. The existence of b^3-symplectic structures in the restricted 3BP was already observed by the authors in [DKM17], see also [BDM+19].

We first pass to polar coordinates $(r, \alpha, P_r, P_\alpha)$ through a symplectic change of coordinates. This change of coordinates is given by $q = (r\cos\alpha, r\sin\alpha)$ and $p = (P_r\cos\alpha - \frac{P_\alpha}{r}\sin\alpha, P_r\sin\alpha + \frac{P_\alpha}{r}\cos\alpha)$ and the symplectic form is given by $\omega = \sum_{i=1}^2 dq_i \wedge dp_i = dr\wedge dP_r + d\alpha\wedge dP_\alpha$. We then perform the McGehee blow-up, given by

$$r = \frac{2}{x^2}. \tag{6.2}$$

This is not a symplectic change of coordinates and therefore the symplectic form gives rise to a b^3-symplectic form that writes down

$$-4\frac{dx}{x^3} \wedge dP_r + d\alpha \wedge dP_\alpha. \tag{6.3}$$

We now look at the level-sets of H under those coordinate changes. As the McGehee change of coordinate exchanges infinity with the origin, we consider the level-sets Σ_c such that $\pi(\Sigma_c)$ is unbounded: indeed, we will only consider $c > 0$. This contrasts the work of [AFKP12], where $c < H(L_1) + \epsilon$. Furthermore, we don't consider the Liouville vector field in the position coordinates, that is we don't consider $X = (q - q_M)\frac{\partial}{\partial q}$, but the one given by momenta given by $Y = p\frac{\partial}{\partial p}$. The reason for this is that X is not a b^3-vector field and therefore the contraction $\iota_X \omega$ does not give rise to a b^3-form.

We first check that the Liouville vector field in momenta is everywhere transverse to the positive energy level-sets before doing the McGehee blow-up.

Lemma 6.2.1. *The vector field $Y = p\frac{\partial}{\partial p}$ is a Liouville vector field and is transverse to Σ_c for $c > 0$.*

Proof. The vector field Y is a Liouville vector field as $\mathcal{L}_Y(\sum_{i=1}^2 dp_i \wedge dq_i) = \omega$ and is transverse to Σ_c for $c > 0$. Indeed

$$Y(H) = |p|^2 + p_1 q_2 - p_2 q_1 = \frac{|p|^2}{2} + \frac{1-\mu}{|q - q_E|} + \frac{\mu}{|q - q_M|} + H(q,p).$$

Hence $Y(H)|_{H=c} = \frac{|p|^2}{2} + \frac{1-\mu}{|q-E|} + \frac{\mu}{|q-M|} + c$ which is a sum of positive terms when $c > 0$. □

We now prove that the vector field Y is also transverse to the level-sets of the Hamiltonian at infinity. The strategy of this is to do the

McGehee blow-up and check that the vector field is still transverse to the level-set of the Hamiltonian.

Theorem 6.2.2. *After the McGehee blow-up, the Liouville vector field $Y = p\frac{\partial}{\partial p}$ is a b^3-vector field that is everywhere transverse to Σ_c for $c > 0$ and the level-sets $(\Sigma_c, \iota_Y \omega)$ for $c > 0$ are b^3-contact manifolds. Topologically, the critical set is a cylinder and the Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.*

Proof. Let us compute the Hamiltonian given by Equation 6.1 first in polar coordinates and then perform the McGehee blow-up. The polar coordinates are defined by the position $q = (r\cos\alpha, r\sin\alpha)$, $(r,\theta) \in \mathbb{R}^+ \times S^1$, and the momenta $p = (P_r\cos\alpha - \frac{P_\alpha}{r}\sin\alpha, P_r\sin\alpha + \frac{P_\alpha}{r}\cos\alpha)$, $(P_r, P_\alpha) \in \mathbb{R}^2$. Under this coordinate change, the resulting Hamiltonian is given by the following expression:

$$\begin{aligned} &H(r,\alpha,P_r,P_\alpha)\\ =&\frac{1}{2}(P_r^2 - (\frac{P_\alpha}{r})^2) - \frac{1-\mu}{r^2 - 2\mu r\cos\alpha + \mu^2}\\ &- \frac{\mu}{r^2 - 2(1-\mu)r\cos\alpha + (1-\mu)^2} - P_\alpha. \end{aligned}$$

The coordinate change is symplectic and therefore the symplectic form is given by $dr \wedge d\alpha + dP_r \wedge dP_\alpha$ and the Liouville vector field writes down $Y = P_r\frac{\partial}{\partial P_r} + P_\alpha\frac{\partial}{\partial P_\alpha}$.

After the McGehee blow-up $r = \frac{2}{x^2}$, the Hamiltonian is given by

$$\begin{aligned} &H(x,\alpha,P_r,P_\alpha)\\ =&\frac{1}{2}(P_r^2 - \frac{1}{4}x^4P_\alpha^2) - x^4\frac{1-\mu}{4 - 4\mu x^2\cos\alpha + \mu^2x^4}\\ &- x^4\frac{\mu}{4 - 4x^2(1-\mu)\cos\alpha + (1-\mu)^2x^4} - P_\alpha. \end{aligned}$$

The Liouville vector field does not change under the McGehee blow-up, but instead of a symplectic form, the underlying geometric structure is a b^3-symplectic structure with critical set given by $\{x = 0\}$ given by $\omega = -4\frac{dx}{x^3} \wedge dPr + d\alpha \wedge dP_\alpha$. We already checked that the Liouville vector field is everywhere transverse to the level-set of H and we now check that it is also transverse at the critical set.

On the critical set, the Hamiltonian is given by $H = \frac{1}{2}P_r^2 - P_\alpha$, so that $Y(H) = P_r^2 - P_\alpha$. On the level-set $H = c > 0$, we obtain $Y(H) = \frac{1}{2}P_r^2 + c > 0$. Hence it is transverse to the critical set as well, and therefore the induced b^3-contact form on the critical set is given by $\alpha = (P_r\frac{dx}{x^3} + P_\alpha d\alpha)|_{H=c}$.

The critical set of the b^3-contact manifold is given by

$$Z = \{(x, \alpha, P_r, P_\alpha)|x = 0, \frac{1}{2}P_r^2 - P_\alpha = c\}.$$

Topologically, the critical set of the b^3-contact manifold is given by $Z = \{(x, \alpha, P_r, P_\alpha)|x = 0, \frac{1}{2}P_r^2 - P_\alpha\}$. Topologically, the critical set is a cylinder, as solutions for $\frac{1}{2}P_r^2 - P_\alpha = c$ are given by $P_\alpha = \frac{1}{2}P_r^2 - c := P_\alpha(P_r)$. The cylinder is described by $Z = \{0, \alpha, P_r, P_\alpha(P_r)\}$ and hence non-compact.

According to the decomposition lemma, the b^3-contact form decomposes as $\alpha = u\frac{dx}{x^3} + \beta$ and by Theorem 3.1.4, the Reeb vector field on the critical set is Hamiltonian for the Hamiltonian function $-u$. The Hamiltonian function here is given by P_r. As the Hamiltonian vector field is contained in the level-set of the Hamiltonian, we obtain that both cylinder are foliated by non-trivial periodic orbits away from $P_r = 0$. □

A reformulation of Theorem 6.2.2 from a view-point of dynamical system is the following:

Corollary 6.2.3. *After the McGehee blow-up in the RPC3BP, there are infinitely many non-trivial periodic orbits at the manifold at infinity for energy values of $H = c > 0$ (that is hyperbolic motion).*

Periodic orbits at infinity have been studied in the past to successfully show oscillatory motions in the RPC3BP, see [GMS16], as well as to show global instability, see [DKdlRS19]. The result presented here in fact generalizes the result on the existence of periodic orbits in [DKdlRS19], where the authors consider parabolic motions. As we consider positive energy level-sets, the motion considered here is classically known as hyperbolic motion. The author believes that it could be interesting to understand not only the dynamics at the manifold at infinity, as is presented in the last result, but also away from the critical set by applying perturbation methods (continuation methods, KAM theory,...) to the set-up.

6.3 Fluid dynamics and contact geometry

Euler equations model the dynamics of an inviscid and incompressible fluid flow. Their viscid counterpart yield the Navier-Stokes equations.

Euler equations can be generalized from the Euclidean to the general Riemannian case as follows: On a Riemannian 3-manifold (M^3, g) they can be described by

$$\begin{cases} \frac{\partial X}{\partial t} + \nabla_X X = -\nabla P \\ \operatorname{div} X = 0 \end{cases} \tag{6.4}$$

where X is the velocity, ∇ the Riemannian gradient and P the pressure. The Bernoulli function is given by $B = P + \frac{1}{2}g(X, X)$. We can take advantage of the metric g to identify several classical concepts using Riemannian duality as follows: The vorticity vector ω is defined as

$$\iota_\omega \mu = d\alpha$$

where $\alpha = \iota_X g$ and μ the Riemannian volume.

Even though the classical work in the subject use the language of vector calculus, we use here the reformulation of those equations in the language of differential forms, see for instance [PS16] for the derivation of those equations.

When the flow does not depend on time we obtain the so-called *stationary solutions.*

In terms of $\alpha = \iota_X g$, stationary Euler equations can be written as

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0. \end{cases} \tag{6.5}$$

An important class of stationary solutions are given by Beltrami fields.

Definition 6.3.1. A Beltrami vector field is a solution to Equations (6.5) that satisfy

$$\operatorname{curl} X = fX, \text{ with } f \in C^\infty(M).$$

When $f \neq 0$, the vector field is called *rotational* Beltrami vector field.

Contact forms and Beltrami vector fields are narrowly related, giving rise to a fruitful interplay between contact geometry and hydrodynamics, with important benefits for both fields. Indeed, if X is a non-vanishing rotational Beltrami then $\alpha = \iota_X g$ is a contact form. In order to prove this note that the Beltrami equation in the language of forms described above can be written as $d\alpha = f\iota_X\mu$. Since f is strictly positive and X is not vanishing we obtain: $\alpha \wedge d\alpha = f\alpha \wedge \iota_X\mu > 0$, thus proving that α is a contact form.

Further, the vectorfield X satisfies $\iota_X(d\alpha) = \iota_X\iota_X\mu = 0$ so $X \in \ker d\alpha$. This implies that it is a reparametrization of the Reeb vector field by the function $\alpha(X) = g(X, X)$.

This proves one of the implications of the theorem below proved in [EG00]:

Theorem 6.3.2 ([EG00]). *Any non-singular rotational Beltrami field is a reparametrization of a Reeb vector field for some contact form and conversely any reparametrization of a Reeb vector field of a contact structure is a non-singular rotational Beltrami field for some metric and volume form.*

6.4 b^m-contact geometry and fluid dynamics

In view of the correspondence in Theorem 6.3.2, it is natural to generalize this to the context of b^m-contact manifolds.

In [CMPS19] contact manifolds with boundary having a singular contact structure on the boundary of *b-type* are identified with contact manifolds with boundary where the boundary is pushed to infinity (or manifolds with cylindrical ends). Using this identification, it is proved in [CMPS19] that the correspondence contact–Beltrami can be extended to the singular set up thus extending the previous geometrical picture on Beltrami fields to 3-dimensional manifolds with boundary.

Theorem 6.4.1 ([CMPS19]). *Let M^3 be a b-manifold. Any rotational Beltrami field and non-vanishing as a section of ${}^bTM$ on M is a Reeb vector field (up to rescaling) for some b-contact form on M. Conversely given a b-contact form α with Reeb vector field X then any non-zero rescaling of X is a rotational Beltrami field for some b-metric and b-volume form on M.*

We call the vector field X obtained in this correspondence *b-Beltrami vector field.*

We will prove the following proposition[1] that shows that the dynamics of the vector fields obtained in the correspondence of Theorem 6.4.1 considerably contrast rotational Beltrami vector fields. The proof follows the lines as the proof of Proposition 27 in [PS16].

Proposition 6.4.2. *Consider a closed surface $\Sigma \subset M$. Assume that Σ is invariant by a smooth vector field X . Then if X is a rotational Beltrami, its restriction $X|_\Sigma$ cannot be Hamiltonian.*

Proof. Let us denote by $j : \Sigma \to M$ the inclusion and let us assume the opposite, that is that j^*X is Hamiltonian, i.e., $j^*X = X_H$ for $H \in C^\infty(\Sigma)$. Then by compactness, H attains its extrema on Σ and furthermore the zeros of j^*X are non-degenerate and therefore isolated.

To see this, let us first denote as before $\alpha = \iota_X g$. Note that $j^*\alpha$ is closed because $\iota_X d\alpha = 0$ and therefore locally exact, hence there exists a function $F \in C^\infty(\Sigma)$ such that $j^*\alpha = dF$. As $\alpha = g(X, \cdot)$, this is saying that the vector field j^*X is the gradient of F. As X is divergence free, F is in fact harmonic and therefore the critical points

[1] The author would like to thank Daniel Peralta–Salas for explaining this result during the Inauguration of the VLAB virtual Lab meeting.

of F are isolated.

There exists hence contractible periodic orbits of X around the extrema of H on Σ. Let us denote by γ one of these orbits and the disk supporting γ by D. Let σ be an area form on the disk. By Stokes theorem and using the definition of Beltrami vector fields (that is $\operatorname{curl} X = fX$),

$$0 < \int_\gamma X ds = \int_D \operatorname{curl} X \cdot N d\sigma = \int_D fX \cdot N d\sigma = 0$$

because X is tangent to Σ. This is a contradiction and hence j^*X cannot be Hamiltonian. □

This result comes as a surprise in view of the following: As an outcome of Theorem 3.1.4 in the 3-dimensional b-contact case the Reeb vector field is tangent to the critical set Z and Hamiltonian along Z. Now consider the b-Beltrami case Theorem 6.4.1. The critical set of the associated b-contact structure is an invariant manifold which is Hamiltonian along Z thus proving that new interesting dynamics emerge from the existence of the critical hypersurface.

The dynamical results proved in Chapter 5 can be applied to b-Beltrami vector fields. For instance, the next corollary follows immediately from Proposition 5.1.1:

Corollary 6.4.3. *Let X be a b-Beltrami vector field on a compact* 3-*dimensional manifold. Then there exist infinitely many periodic orbits on the critical hypersurface.*

In particular, if the singular Weinstein conjecture holds true in this new singular set-up, then we obtain the following corollary.

Corollary 6.4.4 (Corollary of singular Weinstein conjecture, [CMPS19])**.** *Any Beltrami field in a manifold with a cylindrical end has at least one* *of the two:*

1. *a periodic orbit.*

2. *an orbit that goes to infinity for $t \to +\infty$ and $t \to -\infty$.*

Both situations are illustrated below in Figure 6.1.

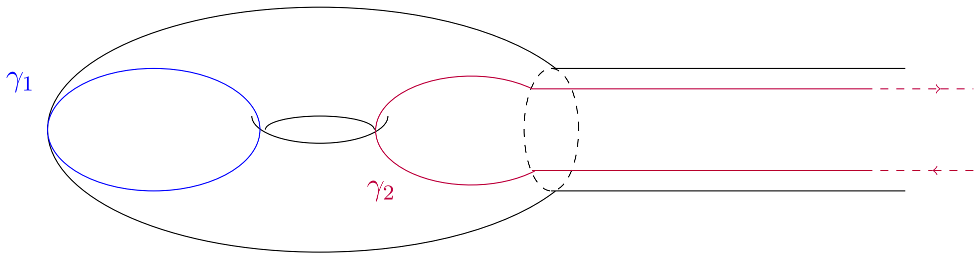


Figure 6.1: Periodic orbits on the regular part and periodic orbits going to infinity.

6.5 And more

Contrary to considering b-manifolds, the authors in [PA19] prove that rotational Beltrami vector fields that have non-degenerate singularities induce contact forms with non-degenerate zeros and thereby generalizing one implication of the correspondence theorem (Theorem 6.3.2). Furthermore, the authors prove the presence of overtwisted disk around those non-degenerate singularities. Let us denote the set of non-degenerate singularities by Γ.

This result is especially alluring in view of the Theorem 5.5.8 as the manifold $M \setminus \Gamma$ is non-compact, but overtwisted. A natural question is therefore to investigate the presence of radial invariance around the non-degenerate singularities to satisfy the assumptions of Theorem 5.5.8. Potentially, this can be achieved by changing the contact form around the singularities, while controlling the Reeb dynamics of the new contact form, as is being proposed in Section 5.3 and Section 5.4.

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