Spectra and eigenspaces from regular partitions of Cayley (di)graphs of permutation groups *

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Abstract

In this paper, we present a method to obtain regular (or equitable) partitions of Cayley (di)graphs (that is, graphs, digraphs, or mixed graphs) of permutation groups on n letters. We prove that every partition of the number n gives rise to a regular partition of the Cayley graph. By using representation theory, we also obtain the complete spectra and the eigenspaces of the corresponding quotient (di)graphs. More precisely, we provide a method to find all the eigenvalues and eigenvectors of such (di)graphs, based on their irreducible representations. As examples, we apply this method to the pancake graphs $P(n)$ and to a recent known family of mixed graphs $\Gamma(d,n,r)$ (having edges with and without direction). As a byproduct, the existence of perfect codes in $P(n)$ allows us to give a lower bound for the multiplicity of its eigenvalue -1.

Mathematics Subject Classifications: 05C50, 05C20, 15A18, 20C30.

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1 Preliminaries

In this paper, we study the eigenvalues and eigenvectors of Cayley (di)graphs $\mathrm{Cay}(G,S)$ (in these, we include graphs, digraphs, and mixed graphs), where G is a subgroup of the symmetric group $S_n = \mathrm{Sym}(n)$, and S is the generating set given by some permutations $\pi_1, \pi_2, \ldots, \pi_k$.

Throughout this paper, $\Gamma = (V,E)$ denotes a digraph that, as said before, can be a graph, digraph, or mixed graph, with vertex set V and arc set E. An arc from vertex u to vertex v is denoted by either uv or $u \to v$. The set of vertices adjacent from a vertex $u \in V$ is denoted by $\Gamma^+(u) = \{v \in V : u \to v\}$. We allow *loops* (that is, arcs from a vertex to itself), and *multiple arcs*. A *digon* is a pair of opposite arcs, uv and vu, forming an edge, and it is denoted by $u \sim v$. So from now on, and without loss of generality, we refer to Γ as a digraph, unless stated otherwise. In particular, if Γ contains both edges and arcs, it is usually referred to as a *mixed* (or *partially directed*) graph. For more details, see the comprehensive survey of Miller and Širáň [17].

If Γ has adjacency matrix A, its spectrum

$$\mathrm{sp}\,\Gamma = \mathrm{sp}\,A = \{[\lambda_0]^{m_0}, [\lambda_1]^{m_1}, \ldots, [\lambda_d]^{m_d}\},$$

is constituted by the (possibly complex) distinct eigenvalues with the corresponding algebraic multiplicities m_i, for $i \in [n] = \{1, \ldots, n\}$.

1.1 Regular partitions and their spectra

Let $\Gamma = (V,E)$ be a digraph with adjacency matrix A. A partition $\pi = (V_1, \ldots, V_m)$ of its vertex set V is called *regular* (or *equitable*) whenever, for any $i,j = 1,\ldots,m$, the *intersection numbers* $b_{ij}(u) = |\Gamma^+(u) \cap V_j|$, where $u \in V_i$, do not depend on the vertex u but only on the subsets (usually called *classes* or *cells*) V_i and V_j. In this case, such numbers are simply written as b_{ij}, and the $m \times m$ matrix $B = (b_{ij})$ is referred to as the *quotient matrix* of A with respect to π. This is also represented by the *quotient (weighted) digraph* $\pi(\Gamma)$ (associated with the partition π), with vertices representing the cells, and there is an arc with weight b_{ij} from vertex V_i to vertex V_j if and only if $b_{ij} \neq 0$.

The *characteristic matrix* of a partition π is the $n \times m$ matrix $\boldsymbol{S} = (s_{ui})$ whose i-th column is the characteristic vector of V_i, that is, $s_{ui} = 1$ if $u \in V_i$, and $s_{ui} = 0$ otherwise. In terms of this matrix, we have the following characterization of regular partitions and their spectra (see Godsil [12]).

Lemma 1.1 ([12])**.** *Let* $\Gamma = (V, E)$ *be a digraph with adjacency matrix* A*, and vertex partition* π *with characteristic matrix* S*.*

(*i*) *The partition* π *is regular if and only if there exists an* $m \times m$ *matrix* C *such that* $SC = AS$*. Moreover,* $C = B$*, the quotient matrix of* A *with respect to* π*.*

(*ii*) *If* π *is regular and* x *is an eigenvector of* B*, then* Sx *is an eigenvector of* A*. Consequently, the spectrum of* $\pi(\Gamma)$ *is contained in the spectrum of* Γ*, that is,* $\operatorname{sp} B \subseteq \operatorname{sp} A$.

1.2 Lift digraphs and their spectra

Given a group G with generating set S, a *voltage assignment* of the *base digraph* Γ is a mapping $\alpha : E \to S$. The pair (Γ, α) is often called a *voltage digraph.* The *lifted digraph* (or, simply, *lift*) Γ^α is the digraph with vertex set $V(\Gamma^\alpha) = V \times G$ and arc set $E(\Gamma^\alpha) = E \times G$, where there is an arc from the vertex (u, g) to the vertex (v, h) if and only if $uv \in E$ and $h = \alpha(uv)g$. In this case, we refer to a *regular* lift because of the mapping $\phi : \Gamma^\alpha \to \Gamma$ defined by erasing the second coordinate (that is, $\phi(u, g) = u$ and $\phi(a, g) = a$ for every $u \in N$ and $a \in E$) is a regular ($|G|$-fold) covering, in its usual meaning in algebraic topology (see, for instance, Gross and Tucker [13]).

As a particular case of a lifted graph, notice that the Cayley digraph $\mathrm{Cay}(G, S)$ can be seen as a lift of the base digraph Γ consisting of a vertex with $|S|$ (directed) loops, each of them having assigned, through α, an element of S.

To the pair (Γ, α), we assign the $k \times k$ *base matrix* B, a square matrix whose rows and columns are indexed by the elements of the vertex set of Γ, and whose uv-th element $B_{u,v}$ is determined as follows: If $a_1, \ldots, a_j$ is the set of all the arcs of Γ emanating from u and terminating at v (not excluding the case $u = v$), then

$$B_{u,v} = \alpha(a_1) + \cdots + \alpha(a_j), \tag{1}$$

the sum being an element of the complex group algebra $\mathbb{C}(G)$; otherwise, we let $B_{u,v} = 0$. Given a unitary irreducible representation of G, $\rho \in \mathrm{Irep}(G)$, of dimension d_ρ, let $\rho(B)$ be the $d_\rho k \times d_\rho k$ matrix obtained from B by replacing every entry $B_{u,v} \in \mathbb{C}(G)$ as in (1) by the $d_\rho \times d_\rho$ matrix

$$\rho(B_{u,v}) = \begin{cases} \rho(\alpha(a_1)) + \cdots + \rho(\alpha(a_j)) & \text{if } B_{u,v} \neq 0, \\ O & \text{otherwise,} \end{cases} \tag{2}$$

where O is the all-zero $d_\rho \times d_\rho$ matrix.

The following results from Širáň and the authors [7] (see also [6]) allow us to compute the spectrum of a (regular) lifted digraph from its associated matrix and the irreducible representations of its corresponding group. For more information on representation theory, see James and Liebeck [15] or Burrow [2].

Theorem 1.1 ([7]). *Let $\Gamma = (V, E)$ be a base digraph on k vertices, with a voltage assignment α in a group G, with $|G| = n$. For every irreducible representation $\rho \in \operatorname{Irep} G$, let $\rho(B)$ be the complex matrix whose entries are given by* (2)*. Then,*

$$\operatorname{sp}\Gamma^{\alpha} = \bigcup_{\rho\in \operatorname{Irep}(G)} d_{\rho}\cdot \operatorname{sp}(\rho(B)).$$

The result of Theorem 1.1 can be generalized to deal with the so-called relative voltage assignments and (not necessarily regular) lifts, which are defined as follows. Let $\Gamma = (V, E)$ be the digraph considered above, G a group, and H a subgroup of index n in G. Let G/H denote the set of left cosets of H in G. Furthermore, let $\beta : E \to G$ be a mapping defined on every arc $a \in E$. In this context, one call β a *voltage assignment in G relative to H*, or simply a *relative voltage assignment*. Then, the *relative lift* Γ^{β} has vertex set $V^{\beta} = V \times G/H$ and arc set $E^{\beta} = E \times G/H$. Incidence in the lift is given as expected: If a is an arc from a vertex u to a vertex v in Γ, then for every left coset $J \in G/H$ there is an arc (a, J) from the vertex (u, J) to the vertex $(v, \beta(a)J)$ in Γ^{β}. Notice that a relative voltage assignment β in a group G with subgroup H is equivalent to a regular voltage assignment if and only if H is a normal subgroup of G. In such a case, the relative lift Γ^{β} admits a description in terms of ordinary voltage assignment in the factor group G/H, with voltage $\beta(a)H$ assigned to an arc $a \in E$ with original relative voltage $\beta(a)$. In this context, Pavlíková, Širáň, and the authors [8] proved the following result, which generalizes Theorem 1.1 for relative voltage assignments.

Theorem 1.2 ([8]). *Let Γ be a base digraph of order k, and let β be a voltage assignment on Γ in a group G relative to a subgroup H of index n in G. Given an irreducible representation $\rho \in \operatorname{Irep}(G)$, let us consider the matrix $\rho(H) = \sum_{h\in H} \rho(h)$. Then,*

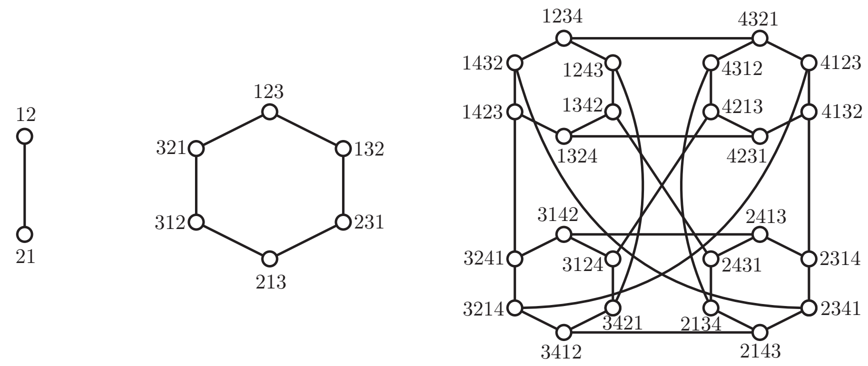
$$\operatorname{sp}\Gamma^{\beta} = \bigcup_{\rho\in \operatorname{Irep}(G)} \operatorname{rank}(\rho(H))\cdot \operatorname{sp}(\rho(B)),$$

where the union must be understood for all $\rho \in \operatorname{Irep}(G)$ such that $\operatorname{rank}(\rho(H)) \neq 0$.

1.3 The pancake graphs

To illustrate our results, we use two families of Cayley graphs: The pancake graphs and a new family of mixed graphs introduced in [5], which can be seen as a generalization of both the pancake graphs and the cycle prefix digraphs. Let us first introduce the pancake graphs, together with some of their basic properties.

The *n-dimensional pancake graph*, proposed by Dweighter [9] (see also Akers and Krishnamuthy [1]), and denoted by $P(n)$, is a graph with the vertex set $V(P(n)) = \{x_1x_2 \dots x_n | x_i \in [n],\ x_i \neq x_j \text{ for } i \neq j\}$. Its adjacencies are as follows:



(a) (b) (c)

Figure 1: Pancakes graphs: (a) $P(2)$, (b) $P(3)$, and (c) $P(4)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
k	0	1	3	4	5	7	8	9	10	11	13	14	15	16	17	18	19

Table 1: The known values of the diameter k of the pancake graph $P(n)$.

$$x_1 x_2 \ldots x_n \sim \begin{cases} x_1 \ldots x_{n-2} x_n x_{n-1}, \\ x_1 \ldots x_{n-3} x_n x_{n-1} x_{n-2}, \\ x_1 \ldots x_{n-4} x_n x_{n-1} x_{n-2} x_{n-3}, \\ \quad \vdots \\ x_n x_{n-1} \ldots x_2 x_1. \end{cases} \tag{3}$$

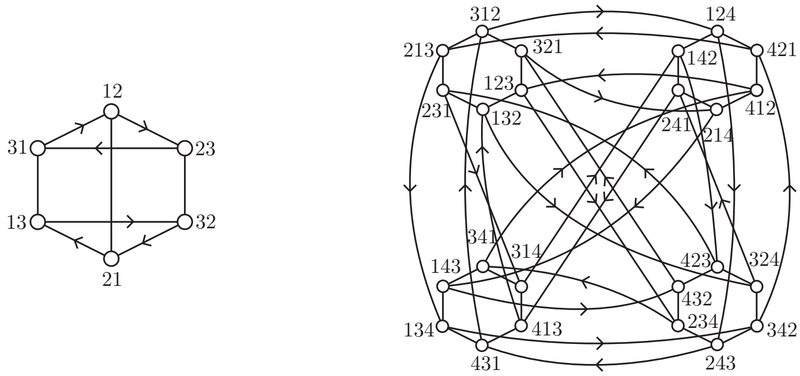
The pancake graph $P(n)$ is a vertex-transitive $(n-1)$-regular graph with $n!$ vertices. It is a Cayley graph $\mathrm{Cay}(G, S)$, where G is the symmetric group $Sym(n)$, and the generating set S corresponds to the permutations of $x_1, x_2, \ldots, x_n$ given by (3). As examples, the pancakes graphs $P(2)$, $P(3)$, and $P(4)$ are shown in Figure 1.

The exact diameters $k = k(n)$ of $P(n)$ are only known for $n \leq 17$, as shown in Table 1 (see Cibulka [3] and Sloane [18]). The best results to our knowledge were given by Gates and Papadimitriou [11], who proved that

$$\frac{17}{16} n \leq k(n) \leq \frac{5n+5}{3},$$

and by Heydari and Sudborough [14], who improved the lower bound to

$$k(n) \geq \frac{15}{14} n.$$



(*a*) (*b*)

Figure 2: New mixed graphs: (*a*) $\Gamma(2,2,1)$ and (*b*) $\Gamma(3,3,2)$.

1.4 The new mixed graphs $\Gamma(d,n,r)$

Recently, the pancake graphs, together with the cyclic prefix digraphs, were used by the first author [5] to propose a new general family of mixed graphs.

The definition of these new mixed graphs is as follows.

Definition 1. *Given the integers $n \geq 2$ and $d, r \geq 1$, with $r < n \leq d+1$, the mixed graph $\Gamma(d,n,r)$ has as vertex set the n-permutations of the $d+1$ symbols $1, 2, \ldots, d, d+1$. Moreover, a vertex $x_1x_2\ldots x_n$ is adjacent, through edges, to the r vertices*

$$x_1x_2\ldots x_n \sim \begin{cases} x_1x_2\ldots x_{n-2}x_nx_{n-1} \\ x_1x_2\ldots x_{n-3}x_nx_{n-1}x_{n-2} \\ \quad\vdots \\ x_1\ldots x_{n-r-1}x_nx_{n-1}\ldots x_{n-r} \end{cases} \tag{4}$$

and adjacent, through arcs, to the $z = d - r$ vertices

$$x_1x_2\ldots x_n \rightarrow \begin{cases} x_2x_3\ldots x_ny, \quad y \neq x_i,\ i = 1,\ldots,n \quad (d-n+1 \text{ vertices}) \\ \left.\begin{array}{l} x_1\ldots x_{n-r-2}x_{n-r}\ldots x_nx_{n-r-1} \\ x_1\ldots x_{n-r-3}x_{n-r-1}\ldots x_nx_{n-r-2} \\ \quad\vdots \\ x_2\ldots x_nx_1 \end{array}\right\} (n-r-1 \text{ vertices}). \end{cases} \tag{5}$$

Thus, the number of vertices of the mixed graph $\Gamma(d,n,r)$ is the number of n-permutations of $d+1$ elements, $N = \frac{(d+1)!}{(d+1-n)!}$. Moreover, $\Gamma(d,n,r)$ is a totally (r,z)-regular mixed graph, and it is also vertex-transitive. In particular, if $n = d+1$ and $r = d$, then $\Gamma(n-1,n,n-1)$ is the pancake graph $P(n)$; and if $r = 1$, then $\Gamma(d,n,1)$ coincides with the so-called cycle prefix digraph $\Gamma_d(n)$ (notice that in this case, we require that $d \geq n$),

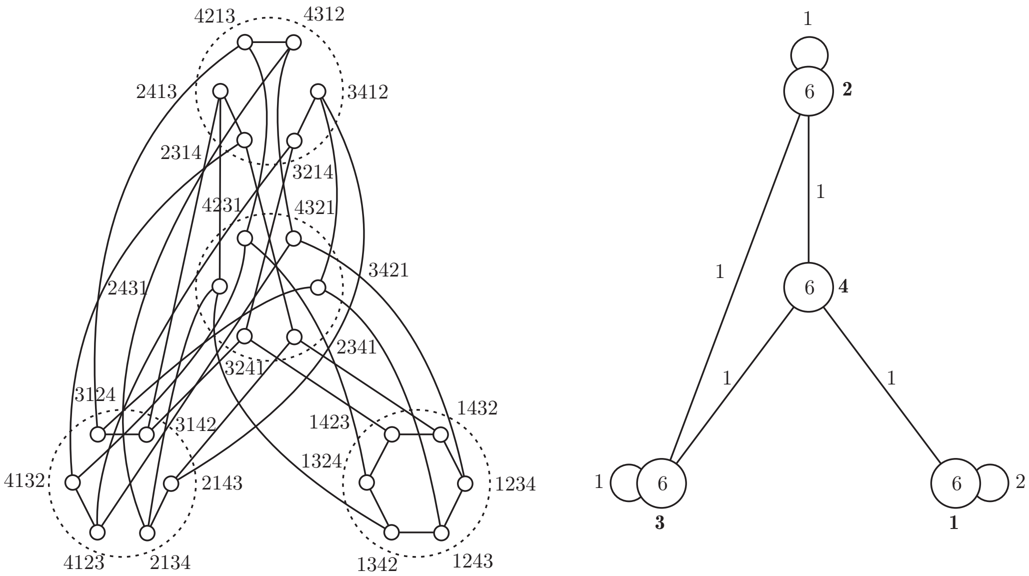


Figure 3: A regular partition of the pancake graph $P(4)$ and its quotient graph. In boldface there is the numbering of the vertices (or classes).

see Faber, Moore, and Chen [10], or Comellas and Fiol [4]. As examples, the new mixed graphs $\Gamma(2,2,1)$ and $\Gamma(3,3,2)$ are depicted in Figure 2.

2 Regular partitions of vertices from number partitions

Given a permutation $\pi : [n] \to [n]$, we denote by $P(\pi) = (p_{ij})$ the $n \times n$ permutation matrix with entries $p_{ij} = 1$ if $\pi(i) = j$, and 0 otherwise (that is, the so-called *column representation*).

Proposition 2.1. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley digraph, where* G *is a subgroup of the symmetric group* $\mathrm{Sym}(n)$ *and its generating set* S *is given by the permutations* $\pi_1, \pi_2, \ldots,$ π_k. *Then,* Γ *has a regular partition* β *with quotient matrix* $B = \sum_{i=1}^{k} P(\pi_i)$.

Proof. Let us show that the cells of the regular partition β are the sets V_i, for $i = 1, \ldots, n$, constituted by the permutations with a given digit, say 1, in the fixed position i, that is $V_i = \{\pi \in S : \pi(i) = 1\}$. Indeed, if $u \in V_i$, the number of vertices $|\Gamma^+(u) \cap V_j|$ (adjacent from u and belonging to V_j) corresponds to the number of the permutations in S that sends 1 from the position i to position j. This is precisely the (i, j)-entry of the matrix B, which is independent of u. □

As a corollary, we have a handy way of obtaining some of the eigenvalues of Γ since, by Lemma 1.1(*ii*), $\mathrm{sp}\, B \subset \mathrm{sp}\, A(\Gamma)$.

Example (**Pancake graph** $P(4)$)**.** *Consider the pancake graph $P(4)$ as the Cayley graph $\mathrm{Cay}(S_4, S)$ with $S = \{(34), (24), (14)(23)\}$. Then, the sum of the corresponding permutation matrices $B = P((34)) + P((24)) + P((14)(23))$ turns out to be*

$$B = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

According to Proposition 2.1, this is the quotient matrix of a regular partition of $P(4)$, as shown in Figure 3, together with its quotient graph. Notice that, as claimed, each class of vertices contains all the permutations with 1 in a fixed position. Moreover, $\mathrm{sp}\, B = \{3, 2, 0, -1\}$, a part of the spectrum of $P(4)$ that, as we show in Section 3, it is

$$\mathrm{sp}\, P(4) = \left\{ [3]^1, [2]^5, \left[\tfrac{-1+\sqrt{17}}{2}\right]^3, [0]^5, \left[\tfrac{-1-\sqrt{17}}{2}\right]^3, [-1]^4, [-2]^3 \right\}. \tag{6}$$

Example (**New mixed graph** $\Gamma(3, 3, 2)$)**.** *Consider the new mixed graph $\Gamma(3, 3, 2)$ as the Cayley graph $\mathrm{Cay}(S_4, S)$ with $S = \{(34), (24), (2341)\}$. Now, the sum of the corresponding permutation matrices $B = P((34)) + P((24)) + P((2341))$ is*

$$B = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix},$$

that corresponds to the quotient matrix of the regular partition shown in Figure 4. Moreover, $\mathrm{sp}\, B = \{[3]^1, [1]^2, [-1]^2\}$. As expected, $\mathrm{sp}\, B \subset \mathrm{sp}\, \Gamma(3, 3, 2)$, since, as shown in Section 3,

$$\mathrm{sp}\, \Gamma(3, 3, 2) = \{[3]^1, [\sqrt{3}]^2, [1]^9, [-1]^9, [-\sqrt{3}]^2, [-3]^1\}. \tag{7}$$

Note that this spectrum is symmetric, in concordance with the fact that $\Gamma(3, 3, 2)$ is a bipartite *(mixed) graph.*

The result of the previous examples can be generalized to obtain some eigenvalues and their associated eigenvectors of the whole family of the pancake graphs $P(n)$ and the new mixed graphs $\Gamma(n, n, n-1)$, as shown in the following results.

Proposition 2.2. *The matrix $B_n = \sum_{i=1}^{n} P(\pi_i)$ of the pancake graph $P(n)$ is the sum $B_n = D_n + T_n$, where $D_n = \mathrm{diag}(n-2, n-1, \ldots, 0, -1)$ and T_n is the 'lower anti-triangular matrix' with entries $(T_n)_{ij} = 1$ if $i + j \geq n + 1$, and $(T_n)_{ij} = 0$ otherwise, with spectrum*

$$\mathrm{sp}\, B_n = \{n-1, n-2, \ldots, 0, -1\} \setminus \{\lfloor (n/2) - 1 \rfloor\}$$

(all the eigenvalues with multiplicity one). Moreover, their associated eigenvectors are, respectively, the all-1 vector $(1, 1, \ldots, 1)^\top$,

$$\left(0, \overset{(r-1)}{\ldots}, 0, n-2r, -1, \overset{(n-2r)}{\ldots}, -1, 0, \overset{(r)}{\ldots}, 0\right)^\top, \text{ for } \begin{cases} r = 1, \ldots, \lfloor n/2 \rfloor & (n \text{ odd}), \\ r = 1, \ldots, \lfloor n/2 \rfloor - 1 & (n \text{ even}), \end{cases} \tag{8}$$

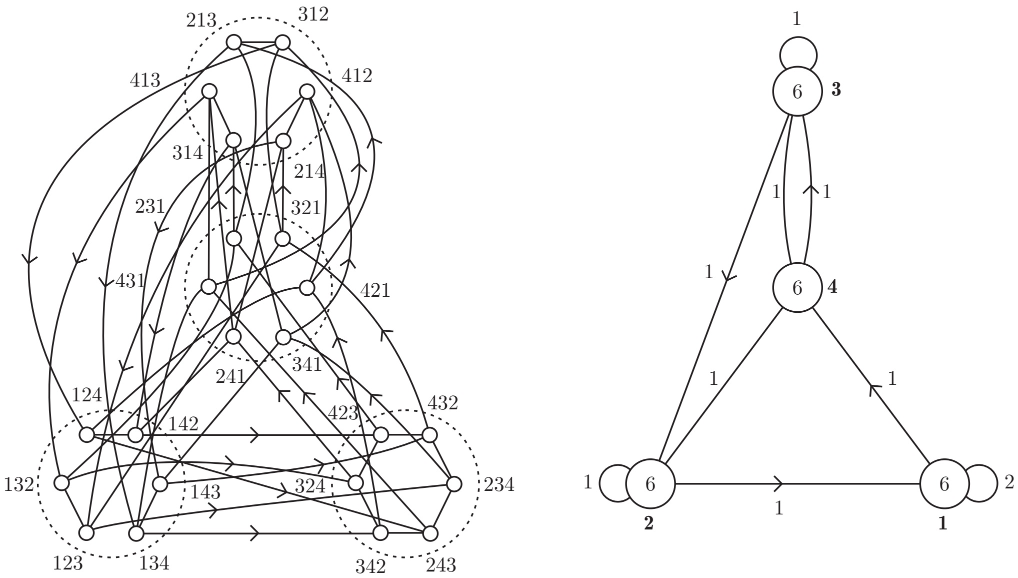


Figure 4: The new mixed graph $\Gamma(3,3,2)$ drawn as a regular partition and its quotient graph. In boldface there is the numbering of the vertices.

and

$$\left(0,\overset{(r-1)}{\ldots},0,-1,\overset{(n-2r+1)}{\ldots},-1,n-2r+1,0,\overset{(r-1)}{\ldots},0\right)^{\top},\quad \textit{for } r=\lfloor n/2\rfloor,\ldots,\ldots,1. \tag{9}$$

Proof. First, it is straightforward to check that the matrix B_n is as claimed. We just have to compute the sum of the involved permutation matrices. For example, for $n=5$, we get

$$B_5=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix}+\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}+\begin{pmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix}$$

$$=\begin{pmatrix}3&0&0&0&1\\0&2&0&1&1\\0&0&2&1&1\\0&1&1&1&1\\1&1&1&1&0\end{pmatrix}=\begin{pmatrix}3&0&0&0&0\\0&2&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&-1\end{pmatrix}+\begin{pmatrix}0&0&0&0&1\\0&0&0&1&1\\0&0&1&1&1\\0&1&1&1&1\\1&1&1&1&1\end{pmatrix}=D_5+T_5.$$

Concerning the eigenpairs, in the case that n is even, let us check that, for every $r=1,\ldots,n/2$,

$$v_r=\left(0,\overset{(r-1)}{\ldots},0,n-2r,-1,\overset{(n-2r)}{\ldots},-1,0,\overset{(r)}{\ldots},0\right)^{\top}$$

is an eigenvector with eigenvalue $\lambda_r=n-r-1$:

$$\begin{aligned}B_n v_k &= D_n v_k + T_n v_k\\ &= \left(0,\overset{(r-1)}{\ldots},0,(n-2r)(n-r-1),-(n-r-2),-(n-r-3),\ldots,-(r-1),0,\overset{(r)}{\ldots},0\right)^\top\\ &+ \left(0,\overset{(r)}{\ldots},0,-1,-2,\ldots,-(n-2r),0,\overset{(r)}{\ldots},0\right)^\top\\ &= \left(0,\overset{(r-1)}{\ldots},0,(n-2r)\lambda_r,-\lambda_r+1,-\lambda_r+2,-\lambda_r+3,\ldots,-\lambda_r+(n-2r),0,\overset{(r)}{\ldots},0\right)^\top\\ &+ \left(0,\overset{(r)}{\ldots},0,-1,-2,\ldots,-(n-2r),0,\overset{(r)}{\ldots},0\right)^\top\\ &= \lambda_r\left(0,\overset{(r-1)}{\ldots},0,n-2r,-1,\overset{(n-2r)}{\ldots},-1,0,\overset{(r)}{\ldots},0\right)^\top = \lambda_r v_r.\end{aligned}$$

The other eigenpairs and the case for odd n can be proved analogously. □

For example, for the case $n = 5$, we obtain $\operatorname{sp} B_5 = \{4, 3, 2, 0, -1\}$, with corresponding matrix of (column) eigenvectors

$$\begin{pmatrix} 1 & 3 & 0 & 0 & -1\\ 1 & -1 & 1 & -1 & -1\\ 1 & -1 & -1 & -1 & -1\\ 1 & -1 & 0 & 2 & -1\\ 1 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Looking at the last column, Lemma 1.1(*ii*) implies that $P(n)$ has the eigenvalue -1 whose eigenvector has entries -1 for the vertices of the classes $1, 2, \ldots, n-1$, and entries $n-1$ for the vertices of the class n (corresponding to the permutations with last symbol 1, as shown in Figure 3 for the case $n = 4$). In this situation, it is known that the graph has a *perfect code* or *efficient dominating set* C (that is, C is an independent vertex set, and each vertex not in C is adjacent to exactly one vertex in C). In other words, to each perfect code C corresponds a (-1)-eigenvector, as described. See, for instance, Godsil [12]. Then, by using the result by Konstantinova [16, Thm. 1], who proved that the pancake graph $P(n)$ contains exactly n perfect codes (the sets of vertices with the same last symbol), we get the following result. First, recall that a circulant matrix $\operatorname{circ}(a_1, a_2, \ldots, a_n)$ has first row $a_1, a_2, \ldots, a_n$ and, for $i = 2, \ldots, n$, its i-th row is obtained from the $(i-1)$-th row by cyclically shifting it to the right one position.

Lemma 2.1. *The pancake graph $P(n)$ has eigenvalue -1 with multiplicity $m(-1) \geq n-1$.*

Proof. Each of the n different perfect codes induces a regular partition with quotient matrix having a (-1)-eigenvector as above. Then, since $\operatorname{rank}\operatorname{circ}(n-1, -1, \overset{(n-1)}{\ldots}, -1) = n-1$, we conclude that $n-1$ of such eigenvectors are independent. □

In the case of the new mixed graph $\Gamma(n, n, n-1)$, we have the following result.

Proposition 2.3. *The matrix* $B'_n = \sum_{i=1}^{n} P(\pi_i)$ *of the new mixed graph* $\Gamma(n,n,n-1)$, $n \geq 3$, *is the sum* $B'_n = D_n + C_n + T'_n$, *where* $D_n = \mathrm{diag}(n-2, n-1, \ldots, 0, -1)$, C_n *is the circulant matrix* $\mathrm{circ}(0,1,0,\ldots,0)$, *and* T'_n *is the 'lower anti-triangular matrix' with entries* $(T'_n)_{ij} = 1$ *if* $i + j > n+1$ *and* $(T'_n)_{ij} = 0$ *otherwise, with eigenvalues* $\{n-1, n-3, -1\} \subset \mathrm{sp}\, B'_n$.

Proof. Again, by computing the sum of the involved permutation matrices, it is easy to check that the matrix B'_n is as claimed. For example, for $n = 5$, we get

$$B'_5 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{pmatrix} + \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0\\ 0&0&1&0&0 \end{pmatrix} + \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&1&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&1\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$$
$$= \begin{pmatrix} 3&0&0&0&1\\ 1&2&0&0&1\\ 0&1&1&1&1\\ 0&0&2&1&1\\ 0&1&1&2&0 \end{pmatrix} = \begin{pmatrix} 3&0&0&0&0\\ 0&2&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&-1 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&1\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&1\\ 0&0&1&1&1\\ 0&1&1&1&1 \end{pmatrix}$$
$$= D_5 + C_5 + T'_5.$$

Concerning the eigenvalues, it is readily checked that the vectors $(1,1,\ldots,1)^\top$ and $(-1,\ldots,-1,n-1)^\top$ are eigenvectors of B'_n with eigenvalues $n-1$ and -1 respectively. Alternatively, to prove that $-1 \in \mathrm{sp}\, B'_n$, we can also check that $\det(-I - B'_n) = (-1)^n \det(I + B'_n) = 0$. Note that this holds since n times the last column of $I + B'_n$ is the sum of the other columns. For instance, for $n = 5$, we get

$$B'_5 + I = \begin{pmatrix} 4&0&0&0&1\\ 1&3&0&0&1\\ 0&1&2&1&1\\ 0&0&2&2&1\\ 0&1&1&2&1 \end{pmatrix}.$$

Finally, $n-3$ is an eigenvalue of B'_n since the first two rows of $(n-3)I - B'_n$ are equal. Namely, $(-1, 0, \ldots, 0, -1)$. □

In fact, the firsts statements of Propositions 2.1 and 2.3 are particular cases of the following result. Let $\mathcal{PR} = PR_n^{n_1,\ldots,n_r}$ denote the set of permutations with repetitions of r symbols $a, b, \ldots$, where a is repeated n_1 times, b is repeated n_2 times, etc. Thus, $|\mathcal{PR}| = \frac{n!}{n_1! \cdots n_r!}$.

Theorem 2.1. *Let* $\Gamma = \mathrm{Cay}(G,S)$ *be a Cayley digraph, where* G *is a subgroup of the symmetric group* $Sym(n)$, *and its generating set* S *is given by the permutations* $\pi_1, \pi_2, \ldots, \pi_k$. *For any partition* $n_1 + n_2 + \cdots + n_r = n$, *there is a regular partition of* Γ *with quotient matrix* B *indexed by the elements of* $\mathcal{PR}$, *and for every* $\sigma, \tau \in \mathcal{PR}$ *the entry* $(B)_{\sigma\tau}$ *is the number (possibly zero) of permutations in* S *that, acting on the symbol positions, map* σ *into* τ.

Proof. Given the partition $n_1 + n_2 + \cdots + n_r = n$, we define the onto mapping $\phi : [n] \to \{a_1, \ldots a_r\}$ such that $|\phi^{-1}(a_i)| = n_i$, for $i = 1, \ldots, r$. Given a permutation $\pi \in Sym(n)$, let $\phi \cdot \pi$ be the permutation with repetition in $\mathcal{PR}$ with i-th symbol $(\phi \cdot \pi)(i) = \phi(\pi(i))$, for $i = 1, \ldots, n$. If we let π to act on the symbol positions (composition of permutations gh being read from left to right), then we can also define the permutation with repetition $\pi \cdot \phi$ such that $(\pi \cdot \phi)(i) = \pi(\phi(i)) = \phi(\pi(i))$ for $i = 1, \ldots, n$ and, hence, $\phi \cdot \pi = \pi \cdot \phi$. Also, it is clear that, for any $g, h \in G$, $(gh) \cdot \phi = g \cdot (h \cdot \phi)$. Let $\phi(G)$ be the set of distinct permutations in $\mathcal{PR}$ of the form $\phi \cdot g$ for a $g \in G$. Now, we claim that Γ has a regular partition ϕ^*, where each class V_σ is represented by the element $\sigma \in \phi(G)$. More precisely, $V_\sigma = \{g \in G : \phi \cdot g = \sigma\}$. Indeed, if $\phi \cdot g = \phi \cdot g'$ and $g \to \pi g$ for some $\pi \in S$, we have

$$\begin{aligned} \phi \cdot (\pi g) &= (\pi g) \cdot \phi = \pi \cdot (g \cdot \phi) = \pi \cdot (\phi \cdot g) \\ &= \pi \cdot (\phi \cdot g') = \pi \cdot (g' \cdot \phi) = (\pi g') \cdot \phi = \phi \cdot (\pi g'). \end{aligned} \tag{10}$$

Thus, ϕ can be interpreted as a homomorphism from Γ to its quotient digraph $\phi^*(\Gamma)$ that preserves the 'colors' (generators) of the arcs. The corresponding quotient matrix B is, then, indexed by the elements of $\phi(G) \subset \mathcal{PR}$, with entries $(B)_{\sigma\tau}$ for every $\sigma, \tau \in \mathcal{PR}$, as claimed. □

Example (**Pancake graph** $P(4)$)**.** *Consider the pancake graph* $P(4)$*. In this case, we have the following partitions:* 4*,* $3+1$*,* $2+2$*,* $2+1+1$*, and* $1+1+1+1$*. According to Theorem 2.1, these partitions yield the regular partitions of* $P(4)$ *in Figure 5, with number of classes* $PR_4^4 = 1$*,* $PR_4^{3,1} = 4$*,* $PR_4^{2,2} = 6$*, and* $PR_4^{2,1,1} = 12$*, respectively. Note that the classes are identified with the corresponding permutations with repetition of the symbols* a, b, c*. Besides, observe that the case of the previous example of* $P(4)$ *corresponds to the partition* $3+1$*. Note that the graph associated with the partition* $1+1+1+1$ *is the whole graph* $P(4)$*, with number of classes (that is, number of vertices)* $PR_4^{1,1,1,1} = 24$ *(see Figure 1(c)).*

Example (**New mixed graph** $\Gamma(3,3,2)$)**.** *Consider the new mixed graph* $\Gamma(3,3,2)$*. We have the partitions* 4*,* $3+1$*,* $2+2$*,* $2+1+1$*, and* $1+1+1+1$ *again. These partitions give the regular partitions of* $\Gamma(3,3,2)$ *in Figure 6, with the same number of classes that in the previous example of* $P(4)$*. The classes are again identified with the corresponding permutations with repetition of the symbols* a, b, c*. Note that the case of the previous example of* $\Gamma(3,3,2)$ *corresponds to the partition* $3+1$*. The graph associated with the partition* $1+1+1+1$ *is the whole graph* $\Gamma(3,3,2)$ *on 24 vertices (see Figure 2(b)).*

3 The spectra of quotient digraphs

In the previous section, we found some eigenvalues, eigenvectors, and regular partitions for whole families of digraphs. In this section, we give a method to find the whole spectrum of a Cayley digraph (on a permutation group) and their quotient digraphs associated with the

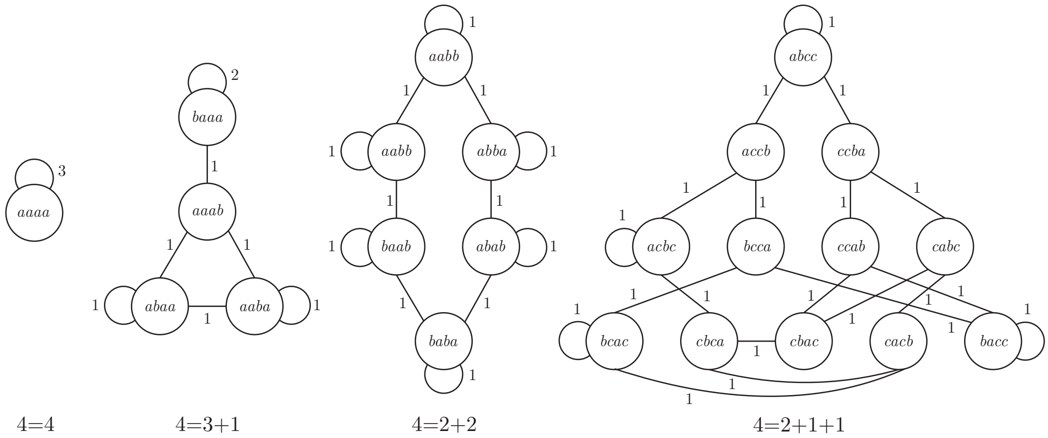


Figure 5: The regular partitions of $P(4)$ corresponding to the number partitions of 4. The partition $4 = 1 + 1 + 1 + 1$ is not represented here since it is the whole graph $P(4)$.

corresponding partitions. We again illustrate the obtained results by using the previous examples: The pancake graph $P(4)$ and the new mixed graph $\Gamma(3, 3, 2)$.

Our results are based on the following lemma, which allows us to apply Theorem 1.2.

Lemma 3.1. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley digraph, where G is a subgroup of the symmetric group $Sym(n)$, with generating set $S = \{\pi_1, \pi_2, \ldots, \pi_k\}$. For a given partition $n_1 + n_2 + \cdots + n_r = n$ induced by the mapping ϕ, the quotient digraph $\phi^*(\Gamma)$ is isomorphic to the relative lift Γ^β with base digraph a singleton with arcs $a_1, \ldots, a_k$, group G, relative voltage assignment β defined by $\beta(a_i) = \pi_i$ for $i = 1, \ldots, k$, and stabilizer subgroup*

$$H = \mathrm{Stab}_G(V_1) \cap \cdots \cap \mathrm{Stab}_G(V_r),$$

where $V_1 \cup \cdots \cup V_r$ is a partition of $[n]$ with $|V_i| = n_i$ for $i = 1, \ldots, r$.

Proof. Let ϕ be the mapping defined in the proof of Theorem 2.1. Let e be the identity element of G, and assume that $\phi \cdot e$ is the permutation with repetition $\varepsilon = a_1 \overset{(n_1)}{\cdots} a_1 \ldots a_r \overset{(n_r)}{\cdots} a_r$. Then, H is constituted by all the elements $h \in G$ such that $\phi \cdot h = \varepsilon$ and, in general, each left coset of H is of the form

$$gH = \{h \in G : \phi \cdot h = \sigma\} \quad \text{if} \quad \phi \cdot g = \sigma.$$

Thus, gH corresponds to the class, or vertex of $\phi^*(\Gamma)$, $V_\sigma = \{g \in G : \phi \cdot g = \sigma\}$ with $\sigma \in \phi(G)$. Moreover, V_σ is adjacent to V_τ through an arc with 'color' $\pi \in S$ if $\tau = \phi \cdot (\pi g) = \pi \cdot \sigma$ (where the second equality comes from (10)). Consequently, in the relative lift Γ^β, the vertex gH is adjacent, through the arc with 'color' π, to $\pi g H$. This proves the claimed isomorphism. □

Example (**Pancake graph** $P(4)$)**.** *Consider the case of the pancake graph $P(4)$ again. We begin computing the spectrum of the whole graph by using Theorem 1.1. We obtained*

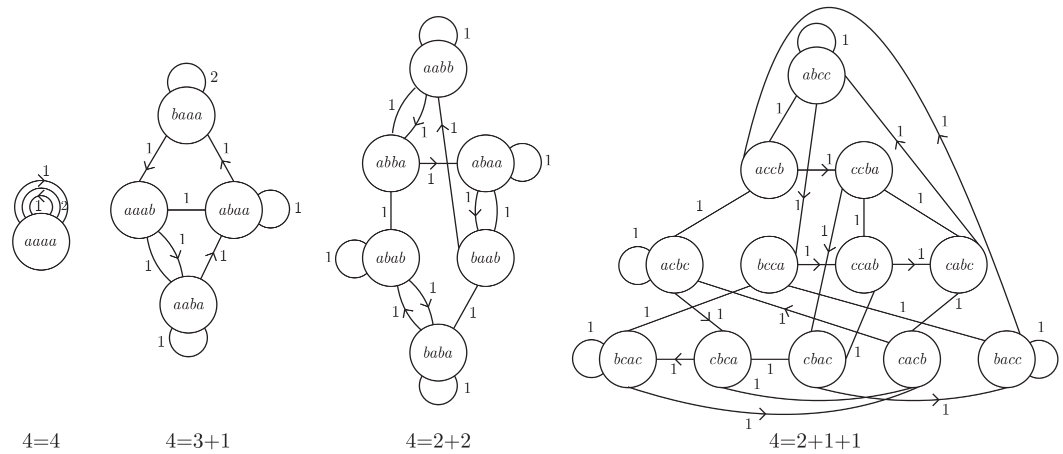


Figure 6: The regular partitions of $\Gamma(3,3,2)$ corresponding to the number partitions of 4.

from SageMath the matrices of the irreducible representations of S_4*, that is,* ρ_1 *(partition* $4=4$*),* ρ_2 *(partition* $4=3+1$*),* ρ_3 *(partition* $4=2+2$*),* ρ_4 *(partition* $4=2+1+1$*), and* ρ_5 *(partition* $4=1+1+1+1$*) shown in Table 2, related to the permutations* $a=1243$*,* $b=1432$*, and* $c=4321$*. Then, from Theorem 1.1, the spectrum of* $P(4)$ *is the union of the following spectra (Note that the dimension of the matrices gives the multiplicities of the corresponding eigenvalues).*

(*i*) $1\cdot \operatorname{sp}\rho_1(B)=\{[3]^1\}$,

(*ii*) $3\cdot \operatorname{sp}\rho_2(B)=\{[2]^3,[0]^3,[-1]^3\}$,

(*iii*) $2\cdot \operatorname{sp}\rho_3(B)=\{[2]^2,[0]^2\}$,

(*iv*) $3\cdot \operatorname{sp}\rho_4(B)=\{[\frac{-1+\sqrt{17}}{2}]^3,[-2]^3,\frac{-1-\sqrt{17}}{2}]^3\}$,

(*v*) $1\cdot \operatorname{sp}\rho_5(B)=\{[-1]^1\}$,

giving $\operatorname{sp}(P_4)$ *as claimed in* (6).

Now to find the spectra of the different quotient graphs, induced from each partition, from Theorem 1.2 and Lemma 3.1, we need to know the ranks of $\rho_i(H_j)$ *for all group stabilizers* H_j*. With this aim, we use the matrices of all irreducible representation shown in Table 3. This gives:*

(*i*) $(4=4)$*:*
$H_1=\operatorname{Stab}_{S_4}(\{1,2,3,4\})=S_4$*, and so* $\rho_i(H_1)=\sum_{g\in S_4}\rho_i(g)$*, for* $i=1,\dots,5$.

(*ii*) $(4=3+1)$*:*
$H_2=\operatorname{Stab}_{S_4}(\{1,2,3\})\cap \operatorname{Stab}_{S_4}(4)=S_3$*, and so* $\rho_i(H_2)=\sum_{g\in S_3}\rho_i(g)$*, for* $i=1,\dots,5$.

Partition: 4=4	a: (1)	b: (1)	c: (1)
$a+b+c$: (3)	Dimension: 1	Eigenvalues: 3	Spectrum: $[3]^1$
Partition: 4=3+1	a: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}$	b: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	c: $\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$a+b+c$: $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	Dimension: 3	Eigenvalues: $2, 0, -1$	Spectrum: $[2]^3, [0]^3, [-1]^3$
Partition: 4=2+2	a: $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	b: $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	c: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$a+b+c$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	Dimension: 2	Eigenvalues: $2, 0$	Spectrum: $[2]^2, [0]^2$
Partition: 4=2+1+1	a: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$	b: $\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	c: $\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$
$a+b+c$: $\begin{pmatrix} 0 & 2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & -3 \end{pmatrix}$	Dimension: 3	Eigenvalues: $\frac{-1+\sqrt{17}}{2}, -2, \frac{-1-\sqrt{17}}{2}$	Spectrum: $[\frac{-1+\sqrt{17}}{2}]^3, [-2]^3, [\frac{-1-\sqrt{17}}{2}]^3$
Partition: 4=1+1+1+1	a: (-1)	b: (-1)	c: (1)
Sum: (-1)	Dimension: 1	Eigenvalues: -1	Spectrum: $[-1]^1$

Table 2: The irreducible matrices of $P(4)$, their sum, and their corresponding eigenvalues.

(*iii*) $(4 = 2+2)$*:*
$H_3 = \mathrm{Stab}_{S_4}(\{1,2\}) \cap \mathrm{Stab}_{S_4}(\{3,4\}) = \{e, (12), (34), (12)(34)\}$*, and so* $\rho_i(H_3) = \sum_{g \in H_3} \rho_i(g)$*, for* $i = 1, \ldots, 5$.

(*iv*) $(4 = 2+1+1)$*:*
$H_4 = \mathrm{Stab}_{S_4}(\{1,2\}) \cap \mathrm{Stab}_{S_4}(3) \cap \mathrm{Stab}_{S_4}(4) = \{e, (12)\}$*, and so* $\rho_i(H_4) = \rho_i(e) + \rho_i((12))$*, for* $i = 1, \ldots, 5$.

(*v*) $(4 = 1+1+1+1)$*:*

$H_5 = \mathrm{Stab}_{S_4}(1) \cap \mathrm{Stab}_{S_4}(2) \cap \mathrm{Stab}_{S_4}(3) \cap \mathrm{Stab}_{S_4}(4) = \{e\}$, *and so* $\rho_i(H_5) = \rho_i(e) = d_{\rho_i}$, *for* $i = 1, \ldots, 5$.

The ranks for the cases (i)–(v)*, together with the corresponding spectra, are shown in Table 4. Notice that, in the last but one row of the same table, we have the spectrum of the whole graph* $P(4)$ *again.*

Example (**New mixed graph** $\Gamma(3,3,2)$)**.** *Consider the mixed graph* $\Gamma(3,3,2)$ *again. Now, by using Theorem 1.1, the spectrum of the whole graph is the union of the following spectra:*

(i) $1 \cdot \mathrm{sp}\,\rho_1(B) = \{[3]^1\}$,

(ii) $3 \cdot \mathrm{sp}\,\rho_2(B) = \{[1]^1, [-1]^1\}$,

(iii) $2 \cdot \mathrm{sp}\,\rho_3(B) = \{[\pm\sqrt{3}]^1\}$,

(iv) $3 \cdot \mathrm{sp}\,\rho_4(B) = \{[1]^2, [-1]^1\}$,

(v) $1 \cdot \mathrm{sp}\,\rho_5(B) = \{[-3]^1\}$,

which gives (7)*. Then, the spectra of the different quotient graphs, induced from each partition, are again computed by using the ranks of* $\rho_i(H_j)$ *for the group stabilizers* H_j*, as in the previous example for* $P(4)$*. The obtained results are shown in Table 5, where we indicate the spectrum of the whole new mixed graph in the last but one row.*

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Permu-tation	Cyclic Notation	4 ρ_1	$3+1$ ρ_2	$2+2$ ρ_3	$2+1+1$ ρ_4	$1+\cdots+1$ ρ_5
1234	e	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1
2134	(12)	1	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	-1
3124	(132)	1	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	1
1324	(23)	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	-1
2314	(123)	1	$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	1
3214	(13)	1	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	-1
3241	(134)	1	$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	1
2341	(1234)	1	$\begin{pmatrix} -1 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$	-1
4321	(14)(23)	1	$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$	1
3421	(1324)	1	$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	-1
2431	(124)	1	$\begin{pmatrix} -1 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	1
4231	(14)	1	$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	-1
4132	(142)	1	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$	1
1432	(24)	1	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	-1
3412	(13)(24)	1	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$	1
4312	(1423)	1	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$	-1
1342	(234)	1	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$	1
3142	(1342)	1	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$	-1
2143	(12)(34)	1	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$	1
1243	(34)	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$	-1
4213	(143)	1	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$	1
2413	(1243)	1	$\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$	-1
1423	(243)	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$	1
4123	(1432)	1	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$	-1

Table 3: The irreducible representations of the symmetric group S_4.

	4 ρ_1	3 + 1 ρ_2	2 + 2 ρ_3	2 + 1 + 1 ρ_4	1 + 1 + 1 + 1 ρ_5	Spectrum
$\text{rank}(\rho(H_1))$	1	0	0	0	0	$\{[3]^1\}$
$\text{rank}(\rho(H_2))$	1	1	0	0	0	$\{[3]^1, [2]^1, [0]^1, [-1]^1\}$
$\text{rank}(\rho(H_3))$	1	1	1	0	0	$\{[3]^1, [2]^2, [0]^2, [-1]^1\}$
$\text{rank}(\rho(H_4))$	1	2	1	1	0	$\{[3]^1, [2]^3, [0]^3, [\frac{-1\pm\sqrt{17}}{2}]^1, [-1]^2, [-2]^1\}$
$\text{rank}(\rho(H_5))$	1	3	2	3	1	$\{[3]^1, [2]^5, [0]^5, [\frac{-1\pm\sqrt{17}}{2}]^3, [0]^5, [-1]^4, [-2]^3\}$
$\text{sp}\,\rho(B)$	$\{[3]^1\}$	$\{[2]^1, [0]^1, [-1]^1\}$	$\{[2]^1, [0]^1\}$	$\{[\frac{-1\pm\sqrt{17}}{2}]^1, [-2]^1\}$	$\{[-1]^1\}$	

Table 4: Spectra of the quotient graphs of $P(4)$.

	4 ρ_1	3 + 1 ρ_2	2 + 2 ρ_3	2 + 1 + 1 ρ_4	1 + 1 + 1 + 1 ρ_5	Spectrum
$\text{rank}(\rho(H_1))$	1	0	0	0	0	$\{[3]^1\}$
$\text{rank}(\rho(H_2))$	1	1	0	0	0	$\{[3]^1, [1]^2, [-1]^1\}$
$\text{rank}(\rho(H_3))$	1	1	1	0	0	$\{[3]^1, [\pm\sqrt{3}]^1, [1]^2, [-1]^1\}$
$\text{rank}(\rho(H_4))$	1	2	1	1	0	$\{[3]^1, [\pm\sqrt{3}]^1, [1]^5, [-1]^4\}$
$\text{rank}(\rho(H_5))$	1	3	2	3	1	$\{[3]^1, [\pm\sqrt{3}]^2, [1]^9, [-1]^9, [-3]^1\}$
$\text{sp}\,\rho(B)$	$\{[3]^1\}$	$\{[1]^1, [-1]^2\}$	$\{[\pm\sqrt{3}]^1\}$	$\{[1]^2, [-1]^1\}$	$\{[-3]^1\}$	

Table 5: Spectra of the quotient graphs of $\Gamma(3, 3, 2)$.