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Abstract: We define a formalism of *equality constraints* and use it to prove the completeness of what we have called *basic* superposition: a restricted form of superposition in which only the subterms *not* originated in previous inferences is superposed upon. We first apply our results to the equational case and to unfailing Knuth-Bendix completion. Second, we extend the techniques to the case of full first-order clauses with equality, proving the refutational completeness of a new simple inference system. Finally, it is briefly outlined how this method can be applied to further restrict inference systems by the use of *ordering constraints*.

1. Introduction

Knuth-Bendix-like completion [KB 70, Rus 87, HR 89, BDP 89, BG 90, NO 91] can be seen as a process that on one hand is refutationally complete and that on the other hand transforms a set of axioms in such a way that, by using the final *complete* set of axioms, efficient *normal form* proofs can be obtained (e.g. rewrite proofs or linear proofs). These procedures are normally based on a form of ordered paramodulation called *superposition*.

In this paper we develop a notion of *equality constraints* that allows to prove the completeness of *basic* superposition. Roughly speaking, the inference rule of basic superposition is the restriction of normal superposition to those occurrences of subterms that have *not* been originated in previous inferences. Consider for example the inference by (equational) superposition

$$\frac{f(g(a)) \simeq a \qquad h(f(x)) \simeq h(x)}{h(a) \simeq h(g(a))}$$

obtained by unifying in $h(f(x))$ the subterm $f(x)$ with $f(g(a))$. Its conclusion is an instance with the unifier $\{x = g(a)\}$ of the equation $h(a) \simeq h(x)$. Therefore, no further basic superposition steps have to be applied to the subterm $g(a)$ of this conclusion. In this paper we will describe this situation by using the equation with equality constraint $h(a) \simeq h(x)\ [\![x = g(a)]\!]$. An alternative notation would be a pair equation-substitution, but we prefer to use constraints, as they allow a uniform treatment with *ordering* constraints (cf. section 5), and we find them more intuitive.

We have called this restriction of superposition "basic" because of its similarity with the one of basic *narrowing* [Hul 80]. Obviously, basic superposition is a considerable improvement over normal superposition, allowing to importantly reduce the search space, and to obtain complete systems in more cases.

In the third section of this paper, after the basic definitions of section 2, we apply our techniques to the particular case of equational logic and to unfailing Knuth-Bendix completion of equations. In section 4 we extend the results to the case of full first-order clauses with equality. We prove the refutational completeness of a basic superposition-based inference system, which moreover uses a simple new factoring rule. In section 5 we very briefly outline how in a similar way our techniques can be applied to further restrict inference systems by the use of *ordering constraints.*

Very recently, from L. Bachmair, we got to know that H. Ganzinger and himself were also working on basic superposition, obtaining results similar to some of ours, but apparently with completely different proof techniques.

2. Basic notions and terminology

We adopt the standard notations and definitions for term rewriting given in [DJ 90, 91].

Furthermore, by an *equation* we mean a multiset $\{s,t\}$, denoted by $s \simeq t$ (or equivalently by $t \simeq s$), where s and t are terms in $\mathcal{T}(\mathcal{F},\mathcal{X})$. In this note, distinct equations are supposed not to share variables.

By *equality constraints* we mean quantifier-free first-order formulae built over the binary predicate symbol $=$ which denotes syntactic equality of terms. An equation with equality constraint is a pair (e,T), denoted $e\,[\![T]\!]$, where e is an equation, and T is an equality constraint. Such a pair can be seen as a shorthand for the set of *ground instances* of $e\,[\![T]\!]$: those ground equations $e\sigma$ such that $T\sigma$ is true.

We consider interpretations that are congruences on ground terms. An interpretation I satisfies $e\,[\![T]\!]$, denoted $I \models e\,[\![T]\!]$, if it satisfies every ground instance of $e\,[\![T]\!]$, i.e. equations with an unsatisfiable constraint are tautologies. It satisfies a set of equations E, denoted by $I \models E$, if I satisfies every equation in E. An equation e *can be deduced* from a set of equations E (denoted by $E \models e$), if e is satisfied by every model of E.

A first-order clause $\Gamma \rightarrow \Delta$ is a pair of (finite) multisets of equations Γ and Δ, called respectively the *antecedent* and the *succedent* of the clause. First-order clauses with equality constraints and their ground instances, satisfiability, etc. are defined as done for equations with equality constraints. For simplicity, we express atoms by equations $P(t_1,\ldots,t_n) \simeq true$, where P is an n-ary predicate symbol, $t_1 \ldots t_n$ are terms, and *true* is a special symbol, i.e. we treat atoms as terms. An interpretation I satisfies a ground clause $\Gamma \rightarrow \Delta$, denoted by $I \models \Gamma \rightarrow \Delta$, if $I \not\supseteq \Gamma$ or else $I \cap \Delta \neq \emptyset$. The empty clause is therefore satisfied by no interpretation.

The symbol $\succ$ denotes a simplification ordering on terms, total on ground terms, where the special symbol *true* is the smallest symbol. We use $\succ_{mul}$ ($\succ_{mul^n}$) to denote its (n-fold) multiset extension.

If E is a set of constrained equations, then a ground term t can be rewritten into a ground term $t[s'\sigma]_u$ by one reductive rewrite step with an equation $s \simeq s'\,[\![T]\!]$ of E, denoted $t \rightarrow_E t[s'\sigma]_u$, if $s \simeq s'\,[\![T]\!]$ has a ground instance $s\sigma \simeq s'\sigma$ such that $t|_u = s\sigma$ and $s\sigma \succ s'\sigma$. We denote by $\rightarrow_E^*$ the reflexive transitive closure of $\rightarrow_E$. A term t' is a *normal form* of t w.r.t. E if $t \rightarrow_E^* t'$ and there is no term t'' s.t. $t' \rightarrow_E t''$. The set E is ground confluent if every ground term has exactly one normal form w.r.t. E.

3. Basic superposition in the equational case

We model basic superposition by using equations with equality constraints. The instantiations caused by previous superpositions are kept in the constraints. Normal superposition can then be used for the equation part:

Definition 1: The inference rule of *basic superposition* is defined as follows:

$$\frac{s \simeq s' \,[\![T']\!] \qquad t \simeq t' \,[\![T]\!]}{t[s']_u \simeq t' \,[\![T \wedge T' \wedge t|_u = s]\!]} \qquad \text{where } t|_u \notin \mathcal{V}ars(t)$$

if, for some ground substitution σ, $[\![T \wedge T' \wedge t|_u = s]\!]\sigma$ is true, $s\sigma \succ s'\sigma$ and $t\sigma \succ t'\sigma$.

In the following section we will also define the rule of (strict) basic superposition for the case of full first order clauses. Note that constraint solving for equality constraints is just unification. In practice, every satisfiable constraint can be kept in a simplest form, which is the corresponding most general unifier.

The difficulty with basic superposition lies in the fact that lifting lemmata like the critical pair lemma do not hold:

Example 2: No inference by basic superposition can be made between the two equations $a \simeq b$ and $f(x) \simeq b\,[\![x = a]\!]$, where $a \succ b$, but there is no rewrite proof for $f(b) \simeq b$.

Another conclusion that we can draw from this example is that basic superposition is *not* complete when starting from an arbitrary set of equations with equality constraints. Therefore, here we will suppose that the equations in the initial set have the trivial constraint $[\![true]\!]$)*. From now on, equations having $[\![true]\!]$ as constraint, will sometimes also be called equations *without* constraint, or equations with trivial constraint.

For simplicity, we will first study basic superposition without simplification. It is proved that the closure under basic superposition of an initial set of equations without constraint is ground confluent. We do this by defining a (canonical) set of ground rewrite rules R_E *generated* from a set E of equations, by selecting ground instances of equations in E that fulfil certain properties. (this is similar to [BG 90], but adapted to equations with equality constraints). Then we show that $R_E \models E$ if E is closed under basic superposition, and we prove that this implies that E is ground confluent.

In order to overcome the problems of the non-existence of a critical pair lemma, we will sometimes consider only instances of equations with *irreducible* substitutions:

Definition 3: A substitution σ is *irreducible* w.r.t. a set of rewrite rules R if $x\sigma$ is irreducible w.r.t. R, for every variable x in the domain of σ. A *normal form* of a substitution σ w.r.t. R is a substitution σ' with the same domain as σ, and such that $x\sigma'$ is a normal form w.r.t. R of $x\sigma$.

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* In fact, this restriction can be slightly weakened

Definition 4: Let $s\sigma \simeq t\sigma$ be a ground instance with $s\sigma \succ t\sigma$ of an equation $s \simeq t\,[\![T]\!]$ in a set of equations E. Then $s\sigma \simeq t\sigma$ *generates* the rule $s\sigma \rightarrow t\sigma$ if $s\sigma$ and σ are irreducible w.r.t. rules generated by ground instances $e\theta$ of equations in E with $s\sigma \simeq t\sigma \succ_{mul} e\theta$.

The set of rules generated by all ground instances of equations in E is denoted by R_E.

Lemma 5: Let E be a set of equations with equality constraints that is closed under basic superposition. Then $R_E \models e\sigma$ for every instance of an equation $e\,[\![T]\!]$ in E such that σ is irreducible w.r.t. R_E.

Proof. Let $t\sigma \simeq t'\sigma$ be a minimal (w.r.t. $\succ_{mul}$) ground instance of a constrained equation $t \simeq t'\,[\![T]\!]$ in E such that σ is irreducible w.r.t. R_E and $R_E \not\models t\sigma \simeq t'\sigma$. We will derive a contradiction from the existence of such an equation.
We can suppose w.l.o.g. that $t\sigma \succ t'\sigma$. Since $R_E \not\models t\sigma \simeq t'\sigma$, the equation has not generated any rule in R_E. Therefore $t\sigma$ must be reducible by R_E, e.g. with a rule $s\sigma' \rightarrow s'\sigma'$ generated by an equation $s\sigma \simeq s'\sigma'$ smaller than $t\sigma \simeq t'\sigma$. Now we have $t\sigma|_u = s\sigma'$, where $t|_u$ cannot be a variable, since σ is irreducible, and therefore the following inference can be made:

$$\frac{s \simeq s'\,[\![T']\!] \qquad t \simeq t'\,[\![T]\!]}{t[s']_u \simeq t'\,[\![T \wedge T' \wedge t|_u = s]\!]}$$

Since E is closed under basic superposition, its conclusion is in E. It has a ground instance d of the form $t\sigma[s'\sigma']_u \simeq t'\sigma$ such that $R_E \not\models d$ (otherwise $R_E \models t\sigma \simeq t'\sigma$). Moreover, d is an instance of this conclusion with a ground substitution θ that is irreducible by R_E, and $t\sigma \simeq t'\sigma \succ_{mul} d$, which contradicts the minimality of $t\sigma \simeq t'\sigma$. ∎

Lemma 6: Let E_0 be a set of equations with trivial constraints, and let E be the closure of E_0 under basic superposition. Then $R_E \models E$.

Proof. First note that $E_0 \models E$, by soundness of basic superposition. Therefore, it suffices to show that $R_E \models E_0$, i.e. $R_E \models e\sigma$ for every instance of an equation $e\,[\![true]\!]$ in E_0. Now let σ' be the normal form of σ w.r.t. R_E. Since $E_0 \subseteq E$, by the previous lemma it holds that $R_E \models e\sigma'$, because σ' is irreducible w.r.t. R_E, and $e\sigma'$ is an existing instance of $e\,[\![true]\!]$. From $R_E \models e\sigma'$ and $R_E \cup \{e\sigma'\} \models e\sigma$ it follows that $R_E \models e\sigma$. ∎

Lemma 7: Let E be a set of constrained equations such that $R_E \models E$. Then E is ground confluent.

Proof. Let s, s' and t be ground terms and let s and s' be normal forms of t w.r.t. E. We prove that s and s' must be syntactically equal. We have $E \models s \simeq s'$, and $R_E \models E$, which implies $R_E \models s \simeq s'$. If s and s' are normal forms w.r.t. E, then they are also normal forms w.r.t. R_E, because R_E is a set of instances of equations of E. Moreover, by construction of R_E, R_E is a canonical set of ground rewrite rules, because there are no overlappings between left hand sides. This implies that s and s' are equal. ∎

Theorem 8: Let E_0 be a set of equations with trivial constraints, and let E be the closure of E_0 under basic superposition. Then E is ground confluent.

3.1. Completion by basic superposition: the equational case

Now we know that if E is the closure under basic superposition of a set of equations without constraints, then E is ground confluent. In this section we show that basic superposition is also the appropriate inference rule for unfailing Knuth-Bendix completion, i.e. for computing such sets E in practice, even when applying the existing powerful simplification and deletion methods that can be used in normal superposition-based completion. However, at first sight there seems to be a problem with simplification:

Example 9: Consider the ordering $f \succ g \succ a \succ b$ and three initial equations:

1) $a \simeq b$
2) $f(g(x)) \simeq g(x)$
3) $f(g(a)) \simeq b$

Furthermore, we obtain:

4) $g(x) \simeq b\,[\![x = a]\!]$ (by basic superposition of 2 and 3)
5) $f(b) \simeq b$ (simplifying 3 by 4)
6) $f(b) \simeq g(x)\,[\![x = a]\!]$ (by basic superposition of 2 and 4)

Now the set $\{1, 2, 4, 5, 6\}$ is closed under basic superposition, but there is no rewrite proof for $g(b) \simeq b$ using instances of this set.

From the previous example we may conclude that, even when starting with equations without constraints, it is incorrect to apply unrestricted simplification. However, as we will see, this problem appears only in (quite special) concrete situations, and can be solved in such a way that basic superposition only in the very worst case may degenerate into normal superposition.

Our notions of completion and redundancy are based on the ones defined by Bachmair and Ganzinger [BG 90,91], where an axiom is redundant if all its ground instances can be deduced from smaller instances of other axioms. Analogously, an *inference* is redundant if, for all its instances, the conclusion can be deduced from instances smaller than the maximal premise. These redundancy notions include, as far as we know, all correct methods that make completion procedures more efficient and terminate in more cases. Here we extend these notions by considering only instances with substitutions that are, in some sense, irreducible. As usual, a completion procedure will be any algorithm computing *fair* derivations:

Definition 10: Let $E_0, E_1, \ldots$ be a sequence of sets of constrained equations.

a) The set E_∞ of *persistent* equations in $E_0, E_1, \ldots$ is defined as $\cup_j(\cap_{k\geq j} E_k)$.

b) We denote by $I(e\,[\![T]\!])$ (resp. $I(E)$) the set of instances of $e\,[\![T]\!]$ (resp. equations $e\,[\![T]\!]$ in E) such that σ is irreducible w.r.t. R_{E_∞}.

c) An equation $e\,[\![T]\!]$ is *redundant* in E_j if for every $e\sigma$ in $I(e\,[\![T]\!])$ there exist instances d_i in $I(E_j)$, for $i = 1 \ldots m$, such that $e\sigma \succ_{mul} d_i$ and $R_{E_\infty} \cup \{d_1, \ldots, d_m\} \models e\sigma$.

Definition 11: A *completion derivation* is a sequence of sets of constrained equations $E_0, E_1, \ldots$ such that T_0 is *true* for every equation $e_0 [\![T_0]\!]$ in E_0 and

$$E_i = E_{i-1} \cup \{e [\![T]\!]\} \quad \text{where } E_{i-1} \models e [\![T]\!], \text{ or}$$
$$E_i = E_{i-1} \setminus \{e [\![T]\!]\} \quad \text{if } e [\![T]\!] \text{ is redundant in } E_{i-1}.$$

Definition 12: Let $E_0, E_1, \ldots$ be a completion derivation, and let π be an inference with premises $e_1 [\![T_1]\!]$ and $e_2 [\![T_2]\!]$, and with conclusion $e [\![T]\!]$.
Then every inference by basic superposition with premises $e_1\sigma$ and $e_2\sigma$, and conclusion $e\sigma$ with $T\sigma =$ true, is a *ground instance* $\pi\sigma$ of π.
The inference π is *redundant* in E_j if for every ground instance $\pi\sigma$ of π with σ irreducible w.r.t. R_{E_∞}, there exist instances d_i in $I(E_j)$, for $i = 1 \ldots m$, such that $max(e_1\sigma, e_2\sigma) \succ_{mul} d_i$ and $R_{E_\infty} \cup \{d_1 \ldots d_m\} \models e\sigma$.

Definition 13: A completion derivation $E_0, E_1, \ldots$ is *fair* if every inference by basic superposition with premises in E_∞ is redundant in some E_j.

In practice, during the computation of a fair completion derivation, one cannot prove the redundancy of equations or inferences in a set E_j, since at that point R_{E_∞} is unknown. Therefore, sufficient conditions for redundancy have to be used. We will define them in detail at the end of this section, and we suppose for the moment that we can indeed compute fair completion derivations.

Definition 14: Let $E_0, E_1, \ldots$ be a completion derivation. Then E_∞ is *complete* if every inference by basic superposition with premises in E_∞ is redundant in E_∞.

Lemma 15: Let $E_0, E_1, \ldots$ be a completion derivation. Then for every set E_j and instance $e\sigma$ in $I(E_j)$, there are instances d_i for $i = 1 \ldots m$ in $I(E_\infty)$, such that $R_{E_\infty} \cup \{d_1, \ldots, d_m\} \models e\sigma$ and $e \succeq_{mul} d_i$.

Proof. We derive a contradiction from the existence of an instance $e\sigma$ that is minimal (w.r.t. $\succ_{mul}$) in all sets $I(E_j)$ such that there are no such instances d_i in $I(E_\infty)$.
The corresponding equation $e [\![T]\!]$ in E_j is not persistent, because otherwise $e\sigma$ is in $I(E_\infty)$. This means that $e [\![T]\!]$ is redundant in some E_k, with $k \geq j$, i.e. there exist instances $\{d'_1, \ldots, d'_n\}$ in $I(E_k)$ such that $R_{E_\infty} \cup \{d'_1, \ldots, d'_n\} \models e\sigma$, with $e\sigma \succ_{mul} d'_j$. However, if the result holds for the instances $d'_1, \ldots, d'_n$ (which must be the case, because $e\sigma$ is minimal), then it also holds for $e\sigma$. ∎

Lemma 16: Let $E_0, E_1, \ldots$ be a completion derivation. If an inference is redundant in some E_j, then it also is in E_∞.

Proof. Let π be an inference with premises $e_1 [\![T_1]\!]$ and $e_2 [\![T_2]\!]$, and with conclusion $e [\![T]\!]$, such that π is redundant in E_j. This means that for every ground instance $\pi\sigma$ of π with σ irreducible w.r.t. R_{S_∞}, there exist instances d_i in $I(E_j)$, for $i = 1 \ldots m$, such that $max(e_1\sigma, e_2\sigma) \succ_{mul} d_i$ and $R_{E_\infty} \cup \{d_1, \ldots, d_m\} \models e\sigma$. By the previous lemma, each of the instances d_i can be deduced from R_{E_∞} and other instances $\{d'_1,\ldots,d'_n\}$ in $I(E_\infty)$ such that $d_i \succeq_{mul} d'_j$. This implies that π is also redundant in E_∞. ∎

Lemma 17: If $E_0, E_1, \ldots$ is a fair completion derivation, then E_∞ is complete.

Proof. By fairness, every inference π with premises in E_∞ is redundant in some E_j. By the previous lemma, then π is also redundant in E_∞, that is, E_∞ is complete. ∎

We now apply the same method as in the previous section to prove that E_∞ is ground confluent. The following lemma states that in fair completion derivations $R_{E_\infty} \models I(E_\infty)$. After this, in lemma 19, we show that $R_{E_\infty} \models E_\infty$, which, as we know by lemma 7, implies that E_∞ is ground confluent.

Lemma 18: Let $E_0, E_1, \ldots$ be a fair completion derivation. Then $R_{E_\infty} \models I(E_\infty)$.

Proof. This proof is an easy extension of that of lemma 5, where the same result is proved for sets E that are *closed* under basic superposition, instead of what we need here: proving it for E_∞ which we only know to be *complete*, i.e. closed *up to redundant inferences.*

Let $t\sigma \simeq t'\sigma$ be a minimal (w.r.t. $\succ_{mul}$) instance in $I(E_\infty)$ such that $R_{E_\infty} \not\models t\sigma \simeq t'\sigma$. We will derive a contradiction from the existence of such an equation.

We can suppose w.l.o.g. that $t\sigma \succ t'\sigma$. Since $R_{E_\infty} \not\models t\sigma \simeq t'\sigma$, the equation has not generated any rule in R_{E_∞}. Therefore $t\sigma$ must be reducible by R_{E_∞}, e.g. with a rule $s\sigma' \rightarrow s'\sigma'$ generated by an equation $s\sigma \simeq s'\sigma'$ smaller than $t\sigma \simeq t'\sigma$. Now we have $t\sigma|_u = s\sigma'$, where $t|_u$ cannot be a variable, since σ is irreducible, and therefore the following inference can be made:

$$\frac{s \simeq s' \; [\![T']\!] \qquad t \simeq t' \; [\![T]\!]}{t[s']_u \simeq t' \; [\![T \wedge T' \wedge t|_u = s]\!]}$$

This inference has a ground instance d of the form $t\sigma[s'\sigma']_u \simeq t'\sigma$ such that $R_{E_\infty} \not\models d$ (otherwise $R_{E_\infty} \models t\sigma \simeq t'\sigma$). Moreover, d is an instance of this conclusion with a ground substitution θ that is irreducible by R_{E_∞}, and $t\sigma \simeq t'\sigma \succ_{mul} d$.

Since E_∞ is complete, the inference must be redundant in E_∞, i.e. there exist instances d_i in $I(E_\infty)$, for $i = 1\ldots m$, such that $t\sigma \simeq t'\sigma \succ_{mul} d_i$ and $R_{E_\infty} \cup \{d_1,\ldots,d_m\} \models d$. But if $R_{E_\infty} \not\models d$ then also $R_{E_\infty} \not\models d_i$ for some d_i, contradicting the minimality of $t\sigma \simeq t'\sigma$. ∎

Lemma 19: Let $E_0, E_1, \ldots$ be a fair completion derivation. Then $R_{E_\infty} \models E_\infty$.

Proof. We have $R_{E_\infty} \models I(E_\infty)$ by the previous lemma. Moreover, $R_{E_\infty} \cup I(E_\infty) \models I(E_0)$ is a direct consequence of lemma 15. Now, since $R_{E_\infty} \cup I(E_0) \models E_0$ holds as in lemma 6 (equations in E_0 have no constraints), and $E_0 \models E_\infty$ holds by soundness of basic superposition, together we have $R_{E_\infty} \models E_\infty$. ∎

Theorem 20: Let $E_0, E_1, \ldots$ be a fair completion derivation. Then E_∞ is ground confluent.

3.2. Redundancy notions for basic superposition

In this section we study in which concrete situations the usual notions of redundancy are incorrect when dealing with basic superposition. It is shown that these situations can be avoided by sometimes slightly *weakening* constraints, in such a way that basic superposition only in the very worst case may degenerate into normal superposition.

The usual notions of redundant axioms and inferences for normal superposition of [BG 91] include most simplification techniques and *critical pair criteria* for proving the redundancy of superpositions. For example, the simplification of an equation e into e' can be modelled in a completion derivation by first adding the consequence e', and then deleting e, which has become redundant, since every instance of e can be deduced from smaller instances of other equations (in this case, of e' and the equation applied in the simplification step).

However, our notion of redundant equation requires every instance *with an irreducible substitution* to be deducible from other smaller instances *with irreducible substitutions*, and also R_{E_∞} may be used.

Example 21: In example 9, the equation $f(g(a)) \simeq b$ is simplified into $f(b) \simeq b$ using $g(x) \simeq b\,[\![x = a]\!]$ with the substitution σ, which is $x = a$.
However, $f(g(a)) \simeq b$ does *not* become redundant by adding $f(b) \simeq b$, because we need $g(x) \simeq b\,[\![x = a]\!]$ instantiated with σ, but σ is *not* irreducible, since R_{E_∞} contains an equation $a \simeq b$, with $a \succ b$.

Definition 22: Let $e\,[\![T]\!]$ be an equation, and let θ be the most general solution of the equality constraint T (i.e. its m.g.u.). Then T *binds* each variable x in $\mathcal{V}ars(e)$ to $x\theta$.

Now let us study when it is correct in our framework to use the normal redundancy notions of [BG 91], i.e. an equation is redundant if all its ground instances can be deduced from smaller instances of other equations, and an *inference* is redundant if, for all its instances, the conclusion can be deduced from instances smaller than the maximal premise.

The lemma below states, roughly speaking, that one can simplify applying an equation $e\,[\![T]\!]$ (or use it in a redundancy proof with the normal notions), if, for every variable x in $\mathcal{V}ars(e)$, x is not bound by T, or else the corresponding position in the equation simplified (or proved) is also a variable.

Example 23: The equation $f(y) \simeq g(y)\,[\![y = h(a)]\!]$ can be simplified by the equation $f(x) \simeq b\,[\![x = h(z)]\!]$ into $g(y) \simeq b\,[\![y = h(a)]\!]$, because, although the variable x is bound to $h(z)$, its corresponding position in $f(y)$ is the variable y.

Lemma 24: Let $E_0, E_1, \ldots$ be a completion derivation. The equation $e\,[\![T]\!]$ is redundant in a set E_j if
(i) it is redundant in the sense of [BG 91], that is, for every ground instance $e\sigma$ of it, there are ground instances $d_i\sigma_i$ for $i = 1 \ldots m$ of equations $d_i\,[\![T_i]\!]$ in E_j such that $\{d_1\sigma_1, \ldots, d_m\sigma_m\} \models e\sigma$ and $e\sigma \succ_{mul} d\sigma_i$, and moreover

(ii) for every i in $1 \ldots m$, and for every x in $\mathcal{V}ars(d_i)$, T_i does not bind x, or else $x\sigma_i = y\sigma$, for some variable y in e.

Proof. We have to prove that the conditions imply that for every $e\sigma$ in $I(e\,[\![T]\!])$ there exist instances d'_k in $I(E_j)$, for $k = 1 \ldots n$, such that $e\sigma \succ_{mul} d'_k$ and $R_{E_\infty} \cup \{d'_1, \ldots, d'_k\} \models e\sigma$.

If every substitution σ_i is irreducible w.r.t. R_{E_∞}, then the result holds. This is certainly the case if for every variable x in every d_i we have $x\sigma_i = y\sigma$, for some variable y in e, since σ is irreducible.

Otherwise, if $x\sigma_i$ is reducible by R_{E_∞}, we can replace $d_i\sigma_i$ by $d_i\theta$, where θ is like σ_i except that $x\theta$ is the normal form w.r.t. R_{E_∞} of $x\sigma$. $d_i\theta$ is an existing instance of d_i, since x is not bound by the corresponding constraint T_i. Moreover, we have $R_{E_\infty} \cup \{d_i\theta\} \models d_i\sigma_i$. By doing so for all these variables x, we obtain the instances d'_k in $I(E_j)$, for $k = 1 \ldots n$, such that $e\sigma \succ_{mul} d'_k$ and $R_{E_\infty} \cup \{d'_1, \ldots, d'_k\} \models e\sigma$. ■

The equivalent lemma for proving the redundancy of *inferences* also holds: it is obtained by using the instance of the maximal premise as upper bound for the instances $d_1, \ldots, d_m$, instead of $e\sigma$, i.e. by replacing in the previous lemma and its proof $e\sigma \succ_{mul} \ldots$ by $max(e_1\sigma, e_2\sigma) \succ_{mul} \ldots$.

Might all the conditions of the previous lemma fail, for some variable x, then we can always *weaken* T for x:

Lemma 25: Let $e\,[\![T]\!]$ be an equation, and let θ be the most general solution of T, with θ of the form $\{x_1 = t_1, \ldots, x_n = t_n\}$. Now let σ be $\{x_1 = t_1\}$. Then the equation $e\sigma\,[\![x_2 = t_2 \wedge \ldots, x_n = t_n]\!]$, obtained by *weakening* $e\,[\![T]\!]$ for x, is logically equivalent to $e\,[\![T]\!]$.

Weakening the constraint of an equation is equivalent to turning basic superposition into normal superposition for the given subterm in the equation (t_1 in the previous lemma), since it becomes again necessary to apply superposition on it, while it was not before weakening.

For simplicity, we have not considered here redundancy of equations by *subsumption*. However, subsumption can easily be included by using a slightly more complicated ordering on instances of equations than the ordering $\succ_{mul}$, comparing pairs $(e\,[\![T]\!], \sigma)$ by a combination of $\succ_{mul}$ and the subsumption ordering.

Practical implementations, such as the one we are working on based on the *Trip*-system [Nie 90, NOR 90], will show whether it pays off to weaken constraints for simplification steps, or whether it is always more efficient to use basic superposition in its full power. For the moment, it seems to us that some mixed strategy has to be used.

4. Completion of first-order clauses by basic superposition

In this section we extend the techniques defined above to the case of full first-order clauses with equality. As done by Bachmair and Ganzinger in [BG 90,91], we obtain an unfailing completion procedure for first order clauses with equality, including powerful notions of redundancy for clauses and inferences. This procedure is refutationally complete and, moreover, very efficient complete strategies can be used for refutational theorem proving with *complete* sets of clauses.

The main new result given here is that our completion procedure, while conserving these properties, uses an inference system that has as main inference rule the one of strict *basic* superposition, instead of normal strict superposition, with the corresponding advantages of a more reduced search space and higher termination probabilities.

Moreover, apart from using basic superposition, the new inference system we define below is also interesting because there is only one inference rule for equality factoring, instead of including, apart from "normal" factoring, inference rules for *merging paramodulation* [BG 90,91] or *equality factoring left* and *equality factoring right* [BG 90]. The fact that we define here this new inference system does not mean that our techniques depend on this specific inference system: all the proofs can be easily adapted for each one of these other systems.

In the following ordering $\succ_C$ on ground clauses, the terms appearing in antecedents of clauses are slightly more complex than the ones in succedents:

Definition 26: The *multiset expression* of an equation $t \simeq t'$ in a clause $\Gamma \rightarrow \Delta$ is

(i) $\{\{t,t\},\{t',t'\}\}$ if $t \simeq t'$ belongs to Γ
(ii) $\{\{t\},\{t'\}\}$ if $t \simeq t'$ belongs to Δ

The ordering $\succ_e$ on ground equations is defined as the ordering $\succ_{mul^2}$ on their multiset expressions.
The ordering $\succ_C$ on ground clauses is defined as the ordering $\succ_{mul^3}$ on the multisets containing the multiset expressions of their equations.

Definition 27: A ground equation e is called *maximal* (resp. *strictly maximal*) in a ground clause C if $e \succeq_e e'$ (resp. $e \succ_e e'$), for every other equation e' in C.

In the following inference system $\mathcal{B}$, inferences take place only in equations of succedents that are strictly maximal and in equations of antecedents that are maximal, for some ground instance. Moreover, only the maximal terms in each equation are used. These conditions imply that, for each ground inference, the conclusion is strictly smaller (w.r.t. $\succ_C$) than the maximal premise.

Definition 28: The inference rules of $\mathcal{B}$ are the following (we always consider maximality of equations in clauses w.r.t. $\succ_e$):

1) *strict basic superposition right:*

$$\frac{\Gamma' \rightarrow \Delta', s \simeq s' \, [\![T']\!] \qquad \Gamma \rightarrow \Delta, t \simeq t' \, [\![T]\!]}{\Gamma', \Gamma \rightarrow \Delta', \Delta, t[s']_u \simeq t' \, [\![T \wedge T' \wedge t|_u = s]\!]} \quad \text{where } t|_u \notin \mathcal{V}ars(t)$$

if $[\![T \wedge T' \wedge t|_u = s]\!]\sigma$ is true for some ground substitution σ such that

a) $t\sigma \succ t'\sigma$, $s\sigma \succ s'\sigma$, and $t\sigma \simeq t'\sigma \succ_e s\sigma \simeq s'\sigma$

b) $s\sigma \simeq s'\sigma$ is strictly maximal in $\Gamma'\sigma \rightarrow \Delta'\sigma, s\sigma \simeq s'\sigma$

c) $t\sigma \simeq t'\sigma$ is strictly maximal in $\Gamma\sigma \rightarrow \Delta\sigma, t\sigma \simeq t'\sigma$.

2) *strict basic superposition left:*

$$\frac{\Gamma' \rightarrow \Delta', s \simeq s' \, [\![T']\!] \qquad \Gamma, t \simeq t' \rightarrow \Delta \, [\![T]\!]}{\Gamma', \Gamma, t[s']_u \simeq t' \rightarrow \Delta', \Delta \, [\![T \wedge T' \wedge t|_u = s]\!]} \quad \text{where } t|_u \notin \mathcal{V}ars(t)$$

if $[\![T \wedge T' \wedge t|_u = s]\!]\sigma$ is true for some ground substitution σ such that

a) $t\sigma \succ t'\sigma$ and $s\sigma \succ s'\sigma$

b) $s\sigma \simeq s'\sigma$ is strictly maximal in $\Gamma'\sigma \rightarrow \Delta'\sigma, s\sigma \simeq s'\sigma$

c) $t\sigma \simeq t'\sigma$ is maximal in $\Gamma\sigma, t\sigma \simeq t'\sigma \rightarrow \Delta\sigma$.

3) *equality resolution:*

$$\frac{\Gamma, t \simeq t' \rightarrow \Delta \, [\![T]\!]}{\Gamma \rightarrow \Delta \, [\![T \wedge t = t']\!]}$$

if $[\![T \wedge t = t']\!]\sigma$ is true for some ground substitution σ such that

a) $t\sigma \simeq t'\sigma$ is maximal in $\Gamma\sigma, t\sigma \simeq t'\sigma \rightarrow \Delta\sigma$.

4) *factoring:*

$$\frac{\Gamma \rightarrow \Delta, t \simeq s, t' \simeq s' \, [\![T]\!]}{\Gamma, s \simeq s' \rightarrow \Delta, t \simeq s \, [\![T \wedge t = t']\!]}$$

if $[\![T \wedge t = t']\!]\sigma$ is true for some ground substitution σ such that

a) $t\sigma \succ s\sigma$ and $t'\sigma \succ s'\sigma$

b) $t\sigma \simeq s\sigma$ is maximal in $\Gamma\sigma \rightarrow \Delta\sigma, t\sigma \simeq s\sigma, t'\sigma \simeq s'\sigma$.

Note that our inference rule for factoring is a generalization to the equality case of "normal" factoring. For instance, if t and t' are atoms, then both s and s' are the symbol *true* and the equation *true* $\simeq$ *true* can be omitted in the antecedent.

In order to prove the correctness of completion procedures based on this inference system $\mathcal{B}$, we will proceed in a similar way as done in the previous section for the equational case. In fact, we will extend almost all the definitions and results to the case of first-order clauses with equality, of which equations are a proper subset. For instance, definitions 29 - 34 are extensions of the equivalent ones in the previous section, and the same thing happens with the lemmata 35 - 37 and 39. Those proofs equal to the corresponding ones in the previous section are omitted.

Now first we associate to a set of constrained clauses S a canonical set of ground rewrite rules R_S. This is done in a little more complicated way that it was done for the equational case (here we use the arrow $\Rightarrow$ for rewrite rules instead of $\rightarrow$, to avoid confusion with the arrow of clauses in sequent notion).

After this, it will be shown that, in a fair completion derivation for first order clauses $S_0, S_1, \ldots$, it holds that $R_{S_\infty} \models S_\infty$ if the empty clause is not in S_∞. This implies (just as $R_{E_\infty} \models E_\infty$ implied the confluence of E_∞), that the completion procedure is refutationally complete. This is true because if the empty clause is not in S_∞, then S_∞ is consistent, since it has a model: the congruence generated by R_{S_∞}.

Definition 29: Let $C\sigma$ be a ground instance $\Gamma \rightarrow \Delta, t \simeq s$ of a clause $C\,[\![T]\!]$ in a set S. Then $C\sigma$ *generates* a rule $t \Rightarrow s$ if the following conditions hold:

(1) $R_C \not\models C\sigma$

(2) $t \simeq s$ is maximal (w.r.t. $\succ_e$) in $C\sigma$ with $t \succ s$

(3) $R_C \not\models s \simeq s'$, for every $t \simeq s'$ in Δ

(4) t is irreducible by R_C

(5) σ is irreducible by R_C

where R_C is the set of rules generated by ground instances smaller than C (w.r.t. $\succ_C$) of clauses in S.
The set of rules generated by all ground instances of clauses in S is denoted by R_S.

Definition 30: Let $S_0, S_1, \ldots$ be a sequence of sets of constrained clauses.

a) The set S_∞ of *persistent* clauses in $S_0, S_1, \ldots$ is defined as $\cup_j(\cap_{k\geq j} S_k)$.

b) We denote by $I(C\,[\![T]\!])$ (resp. $I(S)$) the set of instances of $C\,[\![T]\!]$ (resp. clauses $C\,[\![T]\!]$ in S) such that σ is irreducible w.r.t. R_{S_∞}.

c) A clause $C\,[\![T]\!]$ is *redundant* in S_j if for every $C\sigma$ in $I(C\,[\![T]\!])$ there exist instances D_i in $I(S_j)$, for $i = 1 \ldots m$, such that $C\sigma \succ_{mul} D_i$ and $R_{S_\infty} \cup \{D_1, \ldots, D_m\} \models C\sigma$.

Definition 31: A *completion derivation* is a sequence of sets of constrained clauses $S_0, S_1, \ldots$ such that T_0 is *true* for every clause $C_0\,[\![T_0]\!]$ in S_0 and

$$S_i = S_{i-1} \cup \{C\,[\![T]\!]\} \quad \text{where } S_{i-1} \models C\,[\![T]\!], \text{ or}$$
$$S_i = S_{i-1} \setminus \{C\,[\![T]\!]\} \quad \text{if } C\,[\![T]\!] \text{ is redundant in } S_{i-1}.$$

Definition 32: Let $S_0, S_1, \ldots$ be a theorem proving derivation, and let π be an inference with premises $C_1 [\![T_1]\!], \ldots, C_n [\![T_n]\!]$, and with conclusion $C [\![T]\!]$.
Then every existing inference with premises $C_1\sigma, \ldots, C_n\sigma$, and conclusion $C\sigma$ with $T\sigma =$ true, is a *ground instance* $\pi\sigma$ of π.
The inference π is *redundant* in S_j if for every ground instance $\pi\sigma$ of π with σ irreducible w.r.t. R_{S_∞}, there exist instances D_i in $I(S_j)$, for $i = 1 \ldots m$, such that $max(C_1\sigma, \ldots, C_n\sigma) \succ_{mul} D_i$ and $R_{S_\infty} \cup \{D_1, \ldots, D_m\} \models C\sigma$.

Definition 33: A theorem proving derivation $S_0, S_1, \ldots$ is *fair* if every inference of the inference system $\mathcal{B}$ with premises in S_∞ is redundant in some S_j.

Definition 34: Let $S_0, S_1, \ldots$ be a theorem proving derivation. Then S_∞ is *complete* if every inference of the inference system $\mathcal{B}$ with premises in S_∞ is redundant in S_∞.

The following three lemmata are the extensions to the case of first-order clauses of the ones given in the previous section for the equational case, lemmata 15, 16 and 17. Also their proofs are trivial extensions.

Lemma 35: Let $S_0, S_1, \ldots$ be a theorem proving derivation. Then for every set S_j and instance $C\sigma$ in $I(S_j)$, there are instances D_i for $i = 1 \ldots m$ in $I(S_\infty)$, such that $R_{S_\infty} \cup \{D_1, \ldots, D_m\} \models C\sigma$ and $C \succeq_{mul} D_i$.

Lemma 36: Let $S_0, S_1, \ldots$ be a theorem proving derivation. If an inference is redundant in some S_j, then it also is in S_∞.

Lemma 37: If $S_0, S_1, \ldots$ is a fair theorem proving derivation, then S_∞ is complete.

The only lemma of this section that is significantly different to the equational case is the following one. The reason is that it depends on the inference system used.

Lemma 38: Let $S_0, S_1, \ldots$ be a fair theorem proving derivation, such that S_∞ does not contain the empty clause. Then $R_{S_\infty} \models I(S_\infty)$.

Proof. Let $C\sigma$ be a minimal (w.r.t. $\succ_C$) instance $I(S_\infty)$ of a clause $C [\![T]\!]$ in S_∞, such that $R_{S_\infty} \not\models C\sigma$. We will derive a contradiction from the existence of such a clause. There are several cases to be analyzed, depending on which one is the maximal equation in $C\sigma$:

a) Let $C\sigma$ be a clause $\Gamma\sigma \to \Delta\sigma, t\sigma \simeq t'\sigma$, with a maximal equation $t\sigma \simeq t'\sigma$, and $t\sigma \succ t'\sigma$. Since $R_{S_\infty} \not\models C\sigma$, the clause $C\sigma$ has not generated the rule $t\sigma \Rightarrow t'\sigma$. This must be because one of the conditions 3) or 4) of definition 29 do not hold.

a1) If condition 3) does not hold, then $\Delta\sigma$ must be of the form $\Delta'\sigma, s\sigma \simeq s'\sigma$, where $t\sigma$ is the same term as $s\sigma$ and $R_C \models t'\sigma \simeq s'\sigma$. In this case, consider the following inference

π by factoring

$$\frac{\Gamma \rightarrow \Delta', t \simeq t', s \simeq s' \, [\![T]\!]}{\Gamma, t' \simeq s' \rightarrow \Delta', t \simeq t' \, [\![T \wedge t = s]\!]}$$

Its conclusion has a ground instance D of the form $\Gamma\sigma, t'\sigma \simeq s'\sigma \rightarrow \Delta\sigma, t\sigma \simeq t'\sigma$ that is not deducible from R_{S_∞}. Moreover, D is an instance of this conclusion with a ground substitution that is irreducible by R_{S_∞}.

Since S_∞ is complete, π must be redundant in S_∞. But then there exist instances $D_1, \ldots, D_m$ in $I(S_\infty)$ such that $R_{S_\infty} \cup \{D_1, \ldots, D_m\} \models D$, $C\sigma \succ_C D_i$. The fact that D is not deducible from R_{S_∞} implies that at least one of the D_i is not deducible from R_{S_∞} either, which contradicts the minimality of $C\sigma$.

a2) If condition 4) does not hold, then $t\sigma$ is reducible by R_C, e.g. with a rule $s\sigma' \Rightarrow s'\sigma'$ generated by a clause $C'\sigma'$ smaller than $C\sigma$. Let C' be a clause $\Gamma' \rightarrow \Delta', s \simeq s'$ in S_∞ and $t\sigma|_u = s\sigma'$. Now the following inference π by strict superposition right

$$\frac{\Gamma' \rightarrow \Delta', s \simeq s' \, [\![T']\!] \qquad \Gamma \rightarrow \Delta, t \simeq t' \, [\![T]\!]}{\Gamma', \Gamma \rightarrow \Delta', \Delta, t[s']_u \simeq t' \, [\![T \wedge T' \wedge t|_u = s]\!]}$$

can be made. Its conclusion has a ground instance D of the form $\Gamma'\sigma', \Gamma\sigma \rightarrow \Delta'\sigma', \Delta\sigma, t\sigma[s'\sigma']_u \simeq t'\sigma$, that is not deducible from R_{S_∞}. Moreover, D is an instance of this conclusion with a ground substitution that is irreducible by R_{S_∞}. Since S_∞ is complete, π must again be redundant in S_∞, which, as above, leads to a contradiction with the minimality of $C\sigma$.

b) If $C\sigma$ is a clause $t\sigma \simeq t'\sigma, \Delta\sigma \rightarrow \Gamma\sigma$, where $t\sigma \simeq t'\sigma$ is maximal in $C\sigma$, and $t\sigma$ is $t'\sigma$, then consider the following equality resolution inference:

$$\frac{\Gamma, t \simeq t' \rightarrow \Delta \, [\![T]\!]}{\Gamma \rightarrow \Delta \, [\![T \wedge t = t']\!]}$$

The conclusion of this inference has a ground instance $\Gamma\sigma \rightarrow \Delta\sigma$, that is not deducible from R_{S_∞}. Since the inference is redundant, as above, a contradiction is obtained.

c) The only remaining case is that $C\sigma$ is a clause $\Gamma\sigma, t\sigma \simeq t'\sigma \rightarrow \Delta\sigma$, where $t\sigma \simeq t'\sigma$ is maximal in $C\sigma$ and $t\sigma \succ t'\sigma$. In this case $R_{S_\infty} \models t\sigma \simeq t'\sigma$, because $R_{S_\infty} \not\models C\sigma$. Then $t\sigma$ must be reducible by a rule $s\sigma' \Rightarrow s'\sigma'$ in R_{S_∞} generated by a clause in S_∞ of the form $\Gamma' \rightarrow \Delta', s \simeq s' \, [\![T']\!]$, where $t\sigma|_u = s\sigma'$. The following inference π by strict basic superposition left can then be made:

$$\frac{\Gamma' \rightarrow \Delta', s \simeq s' \, [\![T']\!] \qquad \Gamma, t \simeq t' \rightarrow \Delta \, [\![T]\!]}{\Gamma', \Gamma, t[s']_u \simeq t' \rightarrow \Delta', \Delta \, [\![T \wedge T' \wedge t|_u = s}$$

For the instance $\pi\sigma$ of the inference, $C\sigma$ is the maximal premise, and, as in case a2), its conclusion is not deducible from R_{S_∞}. This implies as before that, since π is redundant, a contradiction is obtained. ∎

Lemma 39: Let $S_0, S_1, \ldots$ be a fair theorem proving derivation. Then $R_{S_\infty} \models S_\infty$.

Theorem 40: Let $S_0, S_1, \ldots$ be a fair theorem proving derivation. Then S_0 is inconsistent if, and only if, the empty clause belongs to some S_j.

Proof. If the empty clause belongs to some S_j, then, by soundness of the inference system, S_0 is inconsistent. For the reverse implication, suppose the empty clause belongs to no S_j. Then it is not in S_∞, and by the previous lemma, $R_{S_\infty} \models S_\infty$. But then S_0 must be consistent, since it has as model the congruence generated by R_{S_∞}. ∎

With respect to the redundancy notions, again the same discussion as in the previous section applies. Completion based on the inference rule of basic superposition strictly improves normal superposition-based completion. The following lemma, equivalent to lemma 24 and with the same proof, tells us when constraint weakening has to be applied in redundancy proofs for first-order clauses:

Lemma 41: Let $S_0, S_1, \ldots$ be a theorem proving derivation. The clause $C\,[\![T]\!]$ is redundant in a set S_j if
(i) it is redundant in the sense of [BG 91], that is, for every ground instance $C\sigma$ of it, there are ground instances $D_i\sigma_i$ for $i = 1 \ldots m$ of clauses $D_i\,[\![T_i]\!]$ in S_j such that $\{D_1\sigma_1, \ldots, D_m\sigma_m\} \models C\sigma$ and $C\sigma \succ_C D\sigma_i$, and moreover
(ii) for every i in $1 \ldots m$, and for every x in $\mathcal{V}ars(D_i)$, T_i does not bind x, or else $x\sigma_i = y\sigma$, for some variable y in C.

The interest of applying basic superposition to completion of first-order clauses with equality lies not only in the gain of efficiency as a consequence of the more reduced search space, but also in the higher probability of obtaining *complete* systems. By using such complete systems S, i.e. sets of clauses in which no more non-redundant inferences can be computed, very efficient complete strategies can be applied for refutational theorem proving, since no new inferences between clauses in S have to be computed. Deducibility from complete systems of some classes of ground clauses can even be shown to be decidable.

5. Further work

Some of the techniques of this paper can be applied to other kinds of constraints. Here we briefly outline some results of our forthcoming paper [NR 91] on the combination of basic superposition modelled by the use of equality constraints, and the notion of *ordering constraints*. The interest of similar ordering constraints has been pointed out earlier, e.g. in [KKR 90], but, as far as we know, no proofs had been found up to now. Below we explain the basic idea.

The inference rule of strict superposition has the advantage that the search space is reduced by selecting only the maximal terms in the maximal literals to paramodulate upon. Therefore, if a clause is obtained in an inference, we are in fact only interested in those ground instances of it for which the literal (and term) selected is really the biggest one. This information can be kept in its constraint. Future choices of maximal literals that are incompatible with this constraint can then be shown to be unnecessary. The inference rule for (equational) superposition with ordering constraints is the following:

$$\frac{s \simeq s' \, [\![T']\!] \qquad t \simeq t' \, [\![T]\!]}{(\ t[s']_u \simeq t' \, [\![T \wedge T' \wedge s \succ s' \wedge t \succ t']\!] \ )\sigma} \quad \text{where } t|_u \notin \mathcal{V}ars(t) \text{ and } \sigma = \text{m.g.u.}(t|_u, s)$$

The inference rules for general clauses with equality are defined analogously. The satisfiability problem for this kind of constraints, i.e. knowing whether a term t can be made bigger than a term s by appropriately instantiating its variables, has recently been shown to be decidable for the lexicographic path ordering by Comon [Com 90] and for the recursive path ordering with status by Jouannoud and Okada [JO 91].

Surprisingly, the known results on completeness of deduction methods with ordering constraints are incorrect. Peterson (in [Pet 90], thm. 4.1), claims that every constrained critical pair between equations in R is joinable iff R is a complete set of reductions. However, the same problem with the critical pair lemma as in the equality constraint case appears here:

Example 42: No inference by superposition can be made between the two equations $a \simeq b$ and $f(x) \simeq b \, [\![x \succeq a]\!]$, where $a \succ b$, but there is no rewrite proof for $f(b) \simeq b$.

In fact, methods similar to the ones explained in this paper can be applied to completion with ordering constraints. Also in this case, the initial sets cannot be arbitrary sets of constrained axioms, as the previous example shows. Even the restrictions on simplification are similar in some sense. Therefore, we think that the appropriate inference rule for dealing with general first-order clauses with equality will be based on a combination of both kinds of constraints. The search space for such theorem proving methods is then much smaller than with normal superposition and it will be possible to obtain complete systems in more cases. The equational version of such a rule would be:

$$\frac{s \simeq s' \, [\![T']\!] \qquad t \simeq t' \, [\![T]\!]}{t[s']_u \simeq t' \, [\![T \wedge T' \wedge s \succ s' \wedge t \succ t' \wedge t|_u = s]\!]} \quad \text{where } t|_u \notin \mathcal{V}ars(t)$$

which, as we can see, also provides a very compact representation for inference rules.

The additional problems that appear in this new combined ordering-equality constrained framework are solved in [NR 91]. For example, important differences appear in the notion of *weakening* this kind of constraints, necessary in order to be able to use the strong redundancy notions. Other problems are related with constraint solving. Since the satisfiability of this kind of constraints depends on the signature, adding new function symbols may cause complete systems to become incomplete. For example, suppose that we want to refute a clause containing new Skolem constants, using a complete system. Now we do not want to compute additional inferences between clauses of the complete system, which may be necessary due to these problems. Of course, similar problems appear when combining complete systems.

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