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Critical velocity in kink-defect interaction models: Rigorous results

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Abstract

In this work we study a model of interaction of kinks of the sine-Gordon equation with a weak defect. We obtain rigorous results concerning the so-called critical velocity derived in [7] by a geometric approach. More specifically, we prove that a heteroclinic orbit in the energy level 0 of a 2-dof Hamiltonian H_ε is destroyed giving rise to heteroclinic connections between certain elements (at infinity) for exponentially small (in ε) energy levels. In this setting Melnikov theory does not apply because there are exponentially small phenomena.

Keywords: Hamiltonian systems; Exponentially small phenomena; Heteroclinic connections; Sine-Gordon equation; Kink; Critical velocity

Contents

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1. Introduction

Given an evolutionary partial differential equation, a solitary wave is a solution which travels with constant speed and localized in space. There are several types of solitary waves which are important in modeling physical phenomena. In particular, kinks are solitary waves which travel from one asymptotic state to another. In the last years, kinks have attracted the focus of researchers due to their significant role in many scientific fields as optical fibers, fluid dynamics, plasma physics and others (see [11,15,17,18,21] and references therein).

In this work, we study a model of interaction between kinks of the sine-Gordon equation and a weak defect. The defect is modeled as a small perturbation given by a Dirac delta function. Such interaction has also been studied for the nonlinear Schrödinger equation in [13,14].

We consider the finite-dimensional reduction of the equation given by a 2-degrees of freedom Hamiltonian H proposed by Fei, Kivshar and Vazquez [5] (see also [7]). Following a geometric approach, we give conditions on the energy of the system to admit "kink-like" solutions in this reduced model.

1.1. The model

The sine-Gordon equation is a nonlinear hyperbolic partial differential equation given by

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = 0, \tag{1}$$

which presents a family of kinks $u_{\mathtt{k}}(x,t)$ given by

$$u_{\mathtt{k}}(x,t) = 4\arctan\left(\exp\left(\frac{x - vt - x_0}{\sqrt{1-v^2}}\right)\right), \tag{2}$$

where the parameter v represents the velocity of the kink.

In this work, we perturb this equation by a localized nonlinear defect at the origin

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = \varepsilon\delta(x)\sin(u), \tag{3}$$

where $\delta(x)$ is the Dirac delta function. This equation was studied in [5,7] where the authors consider finite-dimensional reductions of it to understand the kink-like dynamics. As a first step, they consider solutions u of small amplitude of (3), which can be approximated by solutions of the linear partial differential equation

$$\partial_t^2 u - \partial_x^2 u + u = \varepsilon\delta(x)u, \tag{4}$$

which has a family of wave solutions $u_{\mathtt{im}}(x,t)$ given by

$$u_{\mathtt{im}}(x,t) = a(t)e^{-\varepsilon|x|/2}, \tag{5}$$

where $a(t) = a_0\cos(\Omega t + \theta_0)$, $\Omega = \sqrt{1-\varepsilon^2/4}$ and $\mathtt{im}$ stands for impurity. The solution $u_{\mathtt{im}}$ is not a traveling wave, but it is spatially localized at $x = 0$.

In order to study the interaction of kinks of the sine-Gordon equation with the defect considered in (3), [5,7] use variational approximation techniques to obtain the equations which describe the evolution of the kink position X and the defect mode amplitude a. To derive such equations, they consider the ansatz

$$u(x,t) = 4\arctan(\exp(x - X(t))) + a(t)e^{-\varepsilon|x|/2}. \tag{6}$$

Notice that (6) combines the traveling property of the family of kinks (2) with the localized shape of (5).

Using the ansatz (6) in (3) and considering terms up to order 2 in ε, Fei, Kivshar and Vazquez [5] (see also [7]) obtain the system of Euler-Lagrange equations

$$\begin{aligned} 8\ddot{X} + \varepsilon U'(X) + \varepsilon a F'(X) &= 0, \\ \ddot{a} + \Omega^2 a + \frac{1}{2}\varepsilon^2 F(X) &= 0, \end{aligned} \tag{7}$$

where

$$U(X) = -2\,\mathrm{sech}^2(X),\;\; F(X) = -2\tanh(X)\,\mathrm{sech}(X) \text{ and } \Omega = \sqrt{1-\frac{\varepsilon^2}{4}}, \tag{8}$$

which describes approximately the evolution of the kink position X and the defect mode amplitude a. More details of this approach and its applications can be found in [5,7,16]. It is worth to mention that the finite dimensional reduction of PDE problems to ODE systems via an adequate ansatz and variational methods has been considered in an extensive range of works (see [4,6,8–10,23,24]).

It remains as an open problem to prove that the solutions of the reduced system rigorously approximate the PDE solutions. Nevertheless there are numerical evidences ensuring this reasoning (see [19,20]). In particular, in [22], the authors analyze numerically the simulations done in [7] for the perturbed sine-Gordon equation (3).

From (6), if $X(t)$ and $a(t)$ satisfy $X(t) \to \pm\infty$, $\dot{X}(t) \to C^{\pm}$ and $a(t) \to 0$ as $t \to \pm\infty$, then the associated $u(x,t)$ (see (6)) can be seen as an "approximation" for a kink of (3), since it transitions from an asymptotic state to another when $x - X(t) \to \pm\infty$. Moreover, abusing the language, we say that $v_i = C^-$ and $v_f = C^+$ are the **initial velocity** and **final velocity** of the kink.

If $X(t)$ satisfies $X(t) \to \pm\infty$, $\dot{X}(t) \to C^{\pm}$ and $a(t)$ is asymptotic to a periodic function with small amplitude when $t \to +\infty$ or $t \to -\infty$, then the associated $u(x,t)$ can be seen as an approximation for a kink of (3) with asymptotically periodic oscillations. In this case, one can define their initial and final velocities in the same way. In addition, we also look for solutions $(X(t), a(t))$ such that $a(t) \to 0$ as $t \to -\infty$ and $a(t)$ is asymptotically periodic as $t \to +\infty$, which can be seen as an approximation of a kink with exponential decay as $t \to -\infty$ and asymptotic periodic oscillations as $t \to +\infty$.

In this paper we perform a rigorous study of such solutions of the finite-dimensional reduction (7) of the partial differential equation (3).

1.2. The reduced model

Consider the change of variables $(X, \dot{X}, a, \dot{a}) \to (X, Z, b, B)$, where

$$X = X,\;\; Z = \frac{8\dot{X}}{\sqrt{\varepsilon}},\;\; b = \sqrt{\frac{2\Omega}{\varepsilon}}\varepsilon^{-1/4}a,\;\; B = \sqrt{\frac{\varepsilon}{2\Omega}}\varepsilon^{-1/4}\frac{2}{\varepsilon}\dot{a}, \tag{9}$$

and the time rescaling $\tau = \sqrt{\varepsilon}t$. Then, denoting $' = d/d\tau$, the evolution equations of (7) are equivalent to

$$\begin{cases} X' = \dfrac{Z}{8}, \\ Z' = -U'(X) - \dfrac{\varepsilon^{3/4}}{\sqrt{2\Omega}}F'(X)b, \\ b' = \dfrac{\Omega}{\sqrt{\varepsilon}}B, \\ B' = -\dfrac{\Omega}{\sqrt{\varepsilon}}b - \dfrac{\varepsilon^{3/4}}{\sqrt{2\Omega}}F(X). \end{cases} \tag{10}$$

Notice that (10) is a Hamiltonian system with respect to

$$H(X,Z,b,B;\varepsilon)=\frac{Z^2}{16}+U(X)+\frac{\Omega}{2\sqrt{\varepsilon}}(B^2+b^2)+\frac{\varepsilon^{3/4}}{\sqrt{2\Omega}}F(X)b, \tag{11}$$

which can be split as $H=H_{\mathrm{p}}+H_{\mathrm{osc}}+R$, where

$$\begin{cases} H_{\mathrm{p}}(X,Z)=\dfrac{Z^2}{16}+U(X),\\ H_{\mathrm{osc}}(b,B)=H_{\mathrm{osc}}(b,B;\varepsilon)=\dfrac{\Omega}{2\sqrt{\varepsilon}}(B^2+b^2),\\ R(X,b)=R(X,b;\varepsilon)=\dfrac{\varepsilon^{3/4}}{\sqrt{2\Omega}}F(X)b. \end{cases} \tag{12}$$

Thus the Hamiltonian H is the sum of a pendulum-like Hamiltonian H_{p} with an oscillator H_{osc} coupled by the term R.

Remark 1.1. Applying the change of variables $Y=4\arctan(e^X)$, the Hamiltonian system (10) is brought into

$$\begin{cases} \dot{Y}=2\sin(Y/2)Z/8,\\ \dot{Z}=2\sin(Y/2)\left(\sin(Y)-\dfrac{\varepsilon^{3/4}}{\sqrt{2\Omega}}\cos(Y)b\right),\\ \dot{b}=\dfrac{\Omega}{\sqrt{\varepsilon}}B,\\ \dot{B}=-\dfrac{\Omega}{\sqrt{\varepsilon}}b-\dfrac{\varepsilon^{3/4}}{\sqrt{2\Omega}}\sin(Y). \end{cases}$$

When $Y=0$ and $Y=2\pi$, this system has parabolic critical points and periodic orbits which have invariant manifolds. These hyperplanes $Y=0$ and $Y=2\pi$ correspond to $X=-\infty$ and $X=+\infty$ of (10) respectively. For this reason, even if they are not solutions of the system, they can be seen as asymptotic solutions at infinity. Thus, abusing notation, we denote $f(\pm\infty)$ as $\lim_{X\to\pm\infty} f(X)$ when it is well defined.

System (10) inherits many properties of the sine-Gordon equation. In fact, the functions U and F have exponential decay when $|X|\to+\infty$, therefore, for large values of X the system becomes decoupled. Nevertheless, when $X=\mathcal{O}(1)$, the equations are coupled and the Hamiltonians H_{p} and H_{osc} may exchange energy. This will result in interesting global phenomena.

If $F=0$ (i.e. $R=0$), then each energy level $H=h\geq 0$ of system (7) contains a unique heteroclinic orbit between critical points with $\dot{X}\geq 0$ and all the other solutions are heteroclinic orbits to periodic orbits (with the same oscillation in both tails).

In this paper, we prove that the unique heteroclinic orbit between critical points in $H=h$ breaks down for low energies (see Theorem A) and we obtain a **critical energy** h_c (with associated critical initial velocity $v_c=4\sqrt{h_c}$) such that the energy level $H=h$ (h small) contains a heteroclinic connection between a critical point and a periodic orbit (continuation of the unperturbed "point to point" heteroclinic) if and only if $h \geq h_c$. In addition we give an asymptotic formula for h_c (see Theorem C) which happens to be exponentially small in the parameter ε. We also find an energy $0 < h_s < h_c$ such that the energy level $H = h$ (h small) has a heteroclinic connection between periodic orbits if and only if $h \geq h_s$ (see Theorem B).

In [7], the authors present numerical and formal arguments for the existence of the critical velocity v_c and they conjecture that the final velocity v_f in an energy level $h \geq h_c$ (h small) is given by $v_f \approx (v_i - v_c)^{1/2}$, where $v_i \geq v_c$ is its initial velocity. Our results prove the validity of the asymptotic formula for v_c and the conjecture for v_f (see Theorem D).

We emphasize that the rigorous approach presented in this work is necessary to validate the conclusions obtained in [7]. In fact, their results rely on the computation of a Melnikov integral as a leading-order approximation for the total loss of energy ΔE over the separatrix of (10) with $\delta = 0$ (or more precisely of the transfer of energy from the pendulum-like Hamiltonian H_p to the oscillator H_osc, see (12)). Nevertheless, Melnikov theory cannot be applied in this case due the exponential smallness in the parameter ε of the Melnikov function. In this paper we prove that the Melnikov function is indeed a leading-order approximation of ΔE. Note that this is not always the case: often, in problems presenting exponentially small phenomena, the Melnikov integral is not the dominant part of the total loss of energy over a separatrix of a Hamiltonian system (see [2]). In this paper, we relate the loss of energy ΔE of [7] with the exponentially small transversal intersection of the invariant manifolds $W^{u,s}$ of certain objects (critical points and periodic orbits) at infinity.

2. Mathematical formulation and main goal

2.1. The unperturbed problem

Consider system (10) for $F = 0$. Then $H = H_\mathrm{p} + H_\mathrm{osc}$ consists simply of two uncoupled integrable systems.

In the XZ-plane, the solutions are contained in the level curves $H_\mathrm{p}(X, Z) = \kappa$. This system can be transformed into a degenerate (parabolic) pendulum by a change of coordinates (see Remark 1.1). As one can see in Fig. 1, for $\kappa < 0$, $H_\mathrm{p} = \kappa$ is diffeomorphic to a circle. For $\kappa \geq 0$, $H_\mathrm{p} = \kappa$ contains the points $q_\kappa^\pm = (\pm\infty, 4\sqrt{\kappa})$ which behave as "fixed points" and are connected by a heteroclinic orbit Υ_κ given by the graph of

$$Z_\kappa(X) = 4\sqrt{\kappa - U(X)} = 4\sqrt{\kappa + \frac{2}{\cosh^2(X)}}, \quad X \in \mathbb{R}. \tag{13}$$

Notice that Υ_0 is a separatrix. Analogously, $(\pm\infty, -4\sqrt{\kappa}) \in \{H_\mathrm{p} = \kappa\}$ are fixed points at infinity connected by the heteroclinic orbit given by the graph of $-Z_\kappa(X)$. From now on, we focus on the heteroclinic orbits contained in $Z > 0$, since all the results of this paper can be obtained for the orbits in $Z < 0$ in an analogous way.

In the bB-plane, the solutions of (10) for $F = 0$ are, for $\kappa \geq 0$,

$$P_\kappa = \{H_\mathrm{osc} = \kappa\} = \left\{(b, B);\ b^2 + B^2 = 2\kappa\sqrt{\varepsilon}/\Omega\right\} \text{ (see Fig. 2).} \tag{14}$$

Combining (13) and (14) in the energy level $H = h$, $h \geq 0$, we define

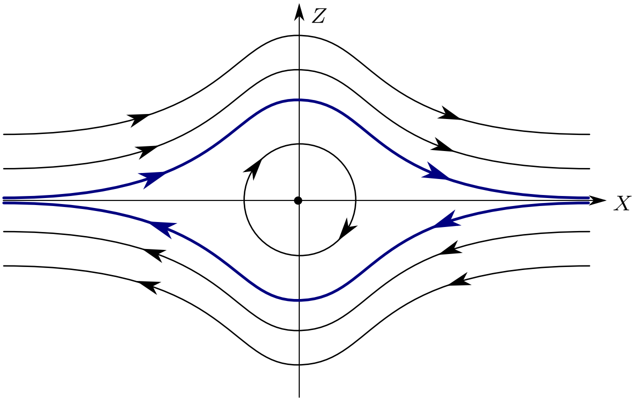

Fig. 1. Projection of the phase space of the unperturbed system in the XZ-plane (level curves of H_{p}). Positive energy corresponds to heteroclinic connections "between $X = +\infty$ and $X = -\infty$".

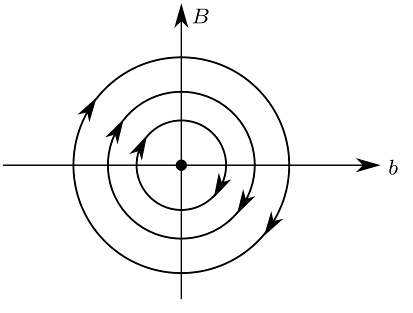

Fig. 2. Projection of the phase space of the unperturbed system in the bB-plane (level curves of H_{osc}).

$$\Lambda^{\pm}_{\kappa_1,\kappa_2} = q^{\pm}_{\kappa_1} \times P_{\kappa_2} = \left\{ (\pm\infty, 4\sqrt{\kappa_1}, b, B);\ b^2 + B^2 = 2\kappa_2\sqrt{\varepsilon}/\Omega \right\}, \tag{15}$$

for every $\kappa_1, \kappa_2 \geq 0$ such that $\kappa_1 + \kappa_2 = h$. Notice that

- If $\kappa_2 = 0$, then $\Lambda^{\pm}_{h,0}$ is a degenerate saddle (parabolic) point of (10);
- If $\kappa_2 > 0$, then $\Lambda^{\pm}_{\kappa_1,\kappa_2}$ are degenerate saddle (parabolic) periodic orbits of (10).

For simplicity, we denote the limit cases $\kappa_1 = 0$ and $\kappa_2 = 0$ by

$$\begin{aligned} \Lambda^{\pm}_{h} &= \Lambda^{\pm}_{0,h} = \left\{ (\pm\infty, 0, b, B), b^2 + B^2 = 2h\sqrt{\varepsilon}/\Omega \right\}, \\ p^{\pm}_{h} &= \Lambda^{\pm}_{h,0} = (\pm\infty, 4\sqrt{h}, 0, 0), \end{aligned} \tag{16}$$

respectively. We stress that $p^{\pm}_{h}$ are points and $\Lambda^{\pm}_{h}$ are periodic orbits, both contained in the planes $X = \pm\infty$ and in the energy level $H = h$.

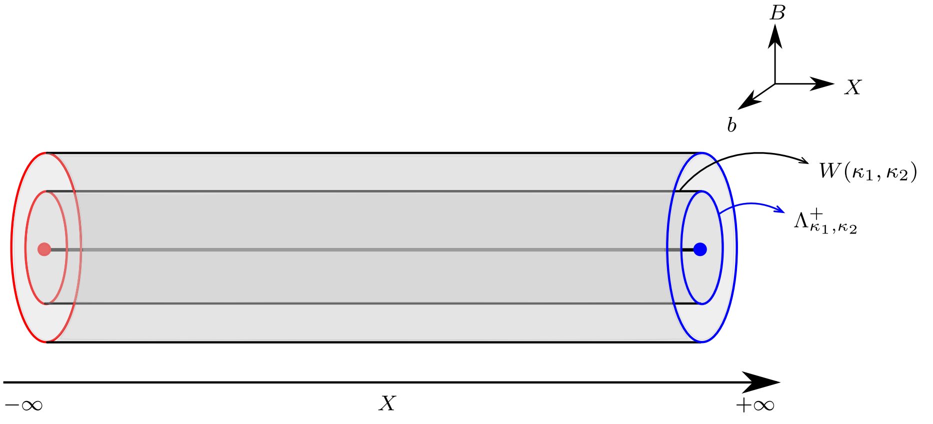

Fig. 3. Projection of the heteroclinic manifolds $W(\kappa_1,\kappa_2)$ in the bXB-space. In the figure, the most external cylinder is the projection of $W(0,h)$ and the straight line represents the projection of $W(h,0)$.

These invariant objects have invariant manifolds. Denote

$$W(\kappa_1,\kappa_2) = \Upsilon_{\kappa_1} \times P_{\kappa_2} = \left\{(X,Z,b,B);\ Z = 4\sqrt{\kappa_1 - U(X)} \text{ and } b^2 + B^2 = 2\kappa_2\sqrt{\varepsilon}/\Omega\right\}, \tag{17}$$

for each $\kappa_1,\kappa_2 \geq 0$ such that $\kappa_1+\kappa_2 = h$.

- **(1D-0)** $W(0,0) = W_0^u(p_0^-) = W_0^s(p_0^+)$ is a 1-dimensional heteroclinic connection (separatrix) between the points p_0^- and p_0^+;
- **(1D-h)** If $h>0$, $W(h,0) = W_0^u(p_h^-) = W_0^s(p_h^+)$ is a 1-dimensional heteroclinic connection between the points p_h^- and p_h^+;
- **(2D-0)** If $h>0$, then $W(0,h) = W_0^u(\Lambda_h^-) = W_0^s(\Lambda_h^+)$ is a 2-dimensional heteroclinic manifold (separatrix) between Λ_h^- and Λ_h^+;
- **(2D-κ_1)** If $\kappa_1,\kappa_2>0$, then $W(\kappa_1,\kappa_2)$ is a 2-dimensional heteroclinic manifold between $\Lambda^-_{\kappa_1,\kappa_2}$ and $\Lambda^+_{\kappa_1,\kappa_2}$.

For $h>0$ fixed, the level energy $H=h$ is a 3-dimensional manifold. Using the conserved Hamiltonian one can eliminate the variable Z, and the manifolds $W(\kappa_1,\kappa_2)$ project into the bXB-space as horizontal cylinders centered along the X-axis as shown in Fig. 3.

In this unperturbed case, there is no exchange of energy between the pendulum and the oscillator through the heteroclinic connections of $W(\kappa_1,\kappa_2)$, i.e. $H_{\rm p}$ and $H_{\rm osc}$ are first integrals. In the perturbed case (10) ($F\neq 0$) the coupling term R (see (11)) goes to 0 as $X\to\pm\infty$ and, thus, the system is uncoupled at $X=\pm\infty$. As a consequence, $\Lambda^{\pm}_{\kappa_1,\kappa_2}$ are orbits of system (10) in the sense of Remark 1.1. Nevertheless, the system may exchange energy between the pendulum and the oscillator when X varies, through the appearance of heteroclinic connections between different $\Lambda^-_{\kappa_1,\kappa_2}$ and $\Lambda^+_{\kappa_1',\kappa_2'}$ such that $\kappa_1+\kappa_2=\kappa_1'+\kappa_2'=h$.

Recall that, in particular, we are interested in solutions $(X(t),Z(t),b(t),B(t))$ which are heteroclinic connections between a critical point at $X=-\infty$ and a periodic orbit at $X=+\infty$ (see Section 1.1). Therefore, they satisfy

$$\lim_{t\to-\infty} X(t) = -\infty, \quad \lim_{t\to-\infty} Z(t) = v_i, \quad \lim_{t\to-\infty} b(t) = \lim_{t\to-\infty} B(t) = 0, \tag{18}$$

$$\lim_{t\to+\infty} X(t) = +\infty, \quad \lim_{t\to+\infty} Z(t) = v_f, \tag{19}$$

where $v_i \geq 0$ is the initial velocity and $v_f \geq 0$ is the final velocity and $(b(t), B(t))$ are asymptotic to periodic functions as $X \to +\infty$. For such solutions

$$h_i = H(X(t), Z(t), b(t), B(t)) = \frac{v_i^2}{16}, \quad \text{for every } t \in \mathbb{R}.$$

Thus, considering $h_i = v_i^2/16$ and defining $\kappa_f = v_f^2/16$, we have that, the solution $(X(t), Z(t), b(t), B(t))$ satisfying (18) and (19) is a heteroclinic connection between the 1-dimensional unstable manifold of $p_{h_i}^-$ and the 2-dimensional stable manifold of $\Lambda^+_{\kappa_f, h_i-\kappa_f}$.

2.2. Main results

Our goal is to look for solutions traveling from $X = -\infty$ to $X = +\infty$. More concretely, we prove the existence of $v_c > 0$ such that the solutions X of (7) coming with velocity v_i from $X = -\infty$ escape the defect location and continue traveling towards $X = +\infty$ with (asymptotic) final velocity v_f, provided $v_i \geq v_c$.

Therefore, the critical energy h_c is characterized as the lowest energy level $h_c = v_c^2/16$ such that for any $h \geq h_c$, there exist $\kappa_1, \kappa_2 > 0$ with $\kappa_1 + \kappa_2 = h$ such that $W^u_\varepsilon(p_h^-) \subset W^s_\varepsilon(\Lambda^+_{\kappa_1,\kappa_2})$.

Notice that $W^u_\varepsilon(p_h^-) \subset W^s_\varepsilon(\Lambda^+_{\kappa_1,\kappa_2})$ implies that the final velocity of the corresponding orbit $X(t)$ (which has initial velocity $4\sqrt{h}$) is given by $v_f = 4\sqrt{\kappa_1}$.

To analyze the existence of heteroclinic orbits between the invariant objects at $X = \pm\infty$, we consider the section $X = 0$, which is transversal to the flow. Restricting to the energy level $H = h$, eliminating the variable Z and using (8), this section becomes the disk

$$\Sigma_h = \left\{(0, b, B) : b^2 + B^2 \leq \frac{(4+2h)\sqrt{\varepsilon}}{\Omega}\right\}. \tag{20}$$

We compute intersections between unstable and stable manifolds in Σ_h.

In the unperturbed case $F = 0$, the one-dimensional heteroclinic connection between the "infinity points" p_h^+ and p_h^-, $W(h, 0) = W^u_0(p_h^-) = W^s_0(p_h^+)$ intersect Σ_h at the point $(0, 0)$. In the following theorem, we show that this heteroclinic connection breaks down when $F \neq 0$ if $h \geq 0$ is small enough. In Fig. 4 (a) we show, on the right, the invariant manifolds $W^{u,s}_\varepsilon(p_0^\mp)$ and, on the left, their intersection with Σ_0: $P_0^{u,s}$. Analogously, in Fig. 4 (b) the red and blue curves on the right are the invariant manifolds $W^{u,s}_\varepsilon(p_h^\mp)$ and their intersection with Σ_h are the red/blue points on the left (the disks and "tubes" in the same figures are explained below since correspond to invariant manifolds of periodic orbits).

Theorem A *(Breakdown of the* 1*-dimensional connection* $W(h, 0)$*). Consider system* (10)*. There exists* $\varepsilon_0 > 0$ *and* $h_0 > 0$ *sufficiently small such that, for every* $0 < \varepsilon \leq \varepsilon_0$ *and* $0 \leq h \leq h_0$*, the invariant manifolds* $W^{u,s}_\varepsilon(p_h^\mp)$ *intersect* Σ_h *(given in* (20)*). Denoting by* $P_h^{u,s}$ *the first intersection points,*

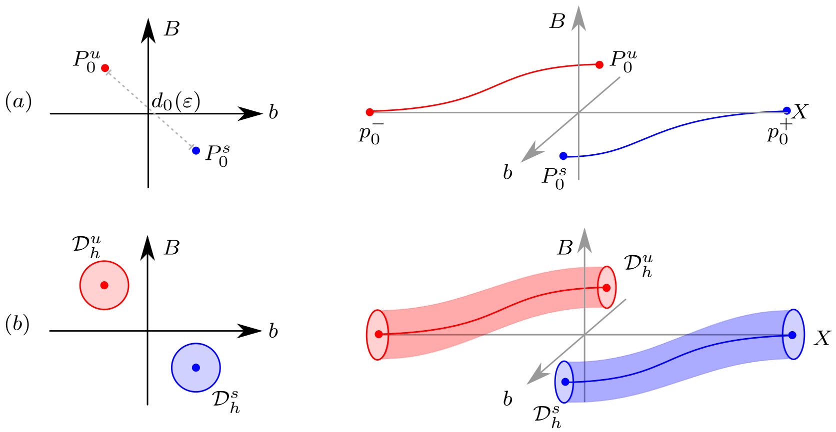

Fig. 4. On the right: First intersection of the invariant manifolds W^u and W^s, contained in the energy level h, with the section Σ_h. On the left: Projection of the invariant manifolds W^u and W^s, contained in the energy level h, in the bXB-space. The case $h = 0$ and the case $h > 0$ (small) are illustrated in (a) and (b), respectively. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\left|P_0^u - P_0^s\right| = d_0(\varepsilon) = \frac{2\pi\varepsilon^{3/4}}{\sqrt{\Omega}} e^{-\Omega\sqrt{2/\varepsilon}} + \mathcal{O}\left(\varepsilon^{7/4} e^{-\Omega\sqrt{2/\varepsilon}}\right), \tag{21a}$$

$$\left|P_h^u - P_h^s\right| = d_0(\varepsilon) + \mathcal{O}(\varepsilon^{7/4}\sqrt{h}), \tag{21b}$$

where Ω *is the constant introduced in* (8).

This theorem is proven in two steps. First in Sections 4.1.1 (for $h = 0$) and 4.1.3 (for $h > 0$) we obtain parameterizations of the invariant manifolds. Then, in Section 4.2 we complete the proof by analyzing their difference.

Remark 2.1. In the asymptotic formulas (21a) and (21b), we could write $\Omega = 1$. Nevertheless, we keep $\Omega = \sqrt{1 - \varepsilon^2/4}$ in order to compare our results with [7]. The same remark holds for Theorems B, C and D below.

When $F = 0$, the energy level $h > 0$ has a family of heteroclinic manifolds $W(\kappa_1, \kappa_2)$, with $\kappa_1 + \kappa_2 = h$, $\kappa_1, \kappa_2 > 0$, connecting the periodic orbits $\Lambda^{\pm}_{\kappa_1,\kappa_2}$, as can be seen in Fig. 3.

Each one intersects Σ_h at a circle centered at $(0, 0)$ with radius $\sqrt{2\kappa_2\sqrt{\varepsilon}/\Omega}$, which generates a disk of radius $\sqrt{2h\sqrt{\varepsilon}/\Omega}$ when we vary $0 < \kappa_2 \leq h$ (see (13) and (14)).

We show that, for the perturbed case, $W_\varepsilon^u(\Lambda^-_{\kappa_1,\kappa_2})$ and $W_\varepsilon^s(\Lambda^+_{\kappa_1,\kappa_2})$ also intersect Σ_h in closed curves which are close to circles of radius $\sqrt{2\kappa_2\sqrt{\varepsilon}/\Omega}$ centered at P_h^u and P_h^s. Thus, varying $0 \leq \kappa_2 \leq h$, we can see that $W_\varepsilon^{u,s}(\Lambda^{\pm}_{\kappa_1,\kappa_2})$ intersect Σ_h in topological disks $\mathcal{D}_h^u$ and $\mathcal{D}_h^s$ which are close to the disks of radius $\sqrt{2h\sqrt{\varepsilon}/\Omega}$ centered at P_h^u and P_h^s, respectively. Fig. 4 (b) shows the invariant manifolds $W_\varepsilon^{u,s}(\Lambda^{\mp}_{\kappa_1,\kappa_2})$ and their intersections with Σ_h. Those intersections fill up the topological disks $\mathcal{D}_h^{u,s}$ when one varies κ_2 from 0 to h.

The existence of heteroclinic connections continuation of the unperturbed ones corresponds to intersections between the disks $\mathcal{D}_h^u$ and $\mathcal{D}_h^s$. Even if in the energy level $h = 0$, there is no (first round) heteroclinic connections between the points at $X = \pm\infty$ (p_0^- and p_0^+), the heteroclinic

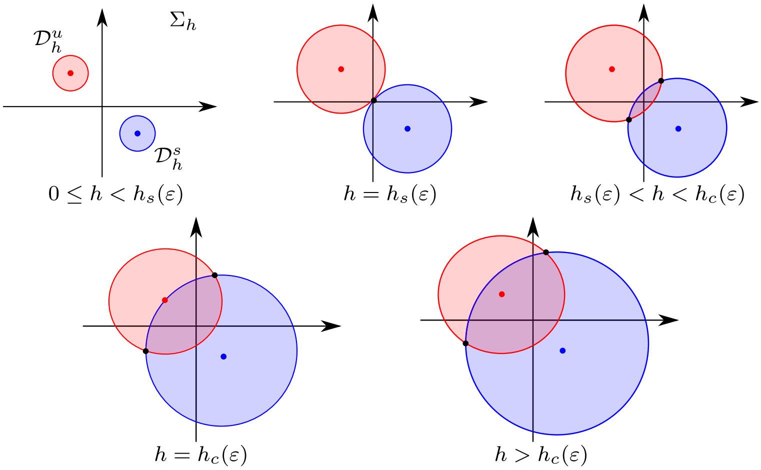

Fig. 5. Relative position of the disks $\mathcal{D}_h^u$ and $\mathcal{D}_h^s$ in the section Σ_h in function of the energy level h. Top-left: The perturbed system has no heteroclinic orbits (passing a unique time through Σ_h). Top-center: h_s is the first energy level for which the system has a heteroclinic orbit between the biggest periodic orbits (at the infinity). Top-left: The system has uncountable heteroclinic orbits between periodic orbits. Bottom-left: h_c is the first energy level for which the system has a heteroclinic orbit between a point and a periodic orbit. Bottom-right: for $h > h_c$ (small) the system still presents a heteroclinic connection between a point and a periodic orbit. In the bottom figures, the system also possesses heteroclinic orbits between periodic orbits.

connections between the periodic orbits $\Lambda^{\pm}_{\kappa_1,\kappa_2}$ may certainly exist when $h > 0$, since the two disks may intersect for some values of h. The lowest energy level $h_s > 0$ for which these heteroclinic connections exist is reached when the boundaries of these disks are tangent (see Fig. 5 top-center). Equivalently, when $W^u_\varepsilon(\Lambda_h^-)$ intersects $W^s_\varepsilon(\Lambda_h^+)$ in the energy level $h_s = h_s(\varepsilon)$.

Theorem B *(Lowest energy level with 2-dimensional heteroclinic connection). Fix $h_0 > 0$. There exists $\varepsilon_0 > 0$ sufficiently small such that, for every $0 < \varepsilon < \varepsilon_0$ and $0 \leq h \leq h_0$, the invariant manifolds $W^u_\varepsilon(\Lambda_h^-)$, $W^s_\varepsilon(\Lambda_h^+)$ intersect Σ_h (given in (20)). The first intersection is given by closed curves, which we denote by $\partial\mathcal{D}_h^{u,s}$. Then, there exists*

$$h_s(\varepsilon) = \frac{\varepsilon\pi^2 e^{-2\Omega\sqrt{2/\varepsilon}}}{2}(1 + \mathcal{O}(\varepsilon)),$$

where Ω is the constant introduced in (8), such that the following statements hold for system (10).

(1) *If $0 \leq h < h_s(\varepsilon)$, the closed curves $\partial\mathcal{D}_h^{u,s}$ do not intersect each other.*
(2) *If $h_s(\varepsilon) \leq h \leq h_0$, the closed curves $\partial\mathcal{D}_h^{u,s}$ intersect at least once.*

Furthermore, given $\mu > 1$, there exists $\varepsilon_\mu > 0$ and

$$h_\mu(\varepsilon) = \frac{\varepsilon\pi^2 e^{-2\Omega\sqrt{2/\varepsilon}}}{2}(\mu + \mathcal{O}(\varepsilon))^2 \geq h_s(\varepsilon),$$

such that, for $0 < \varepsilon < \varepsilon_\mu$ *and* $h_\mu(\varepsilon) \le h \le h_0$*, the closed curves* $\partial\mathcal{D}_h^{u,s}$ *have at least two intersections.*

This theorem is proven in several steps. First in Section 4.1 we obtain parameterizations of the invariant manifolds involved in the theorem and in Section 4.2.2 we obtain estimates for the difference between those invariant manifolds and $W_\varepsilon^u(p_0^-)$, $W_\varepsilon^s(p_0^+)$. Finally, in Section 4.3 we complete the proof by obtaining the asymptotic formula for $h_s(\varepsilon)$ and the other statements of the theorem.

Theorem B ensures that, for $h > h_s$, there is a family of heteroclinic connections between elements of $X = \pm\infty$ which are contained in the energy level h. (See Fig. 5 top-right.) Actually, we prove that, in the energy level $H = h_s$, $\partial\mathcal{D}_{h_s}^u$ and $\partial\mathcal{D}_{h_s}^s$ intersect (tangentially) at least once, and for this reason, $\partial\mathcal{D}_{h_s}^u \cap \partial\mathcal{D}_{h_s}^s$ may have more than one point. Also, our methods show that, for $h > h_s$, $\partial\mathcal{D}_h^u \cap \partial\mathcal{D}_h^s$ has at least two points and $\mathcal{D}_h^u \cap \mathcal{D}_h^s$ has at least one connected component with positive Lebesgue measure (see Fig. 5 top-right).

2.2.1. *The critical energy level h_c*

From our approach and the definitions of Section 1.2, the critical energy level occurs for the smallest h such that $W_\varepsilon^u(p_h^-) \subset W_\varepsilon^s(\Lambda_{\kappa_1,\kappa_2}^+)$, for some κ_1, κ_2 satisfying $\kappa_1 + \kappa_2 = h$. Thus, h_c occurs when $W_\varepsilon^u(p_{h_c}^-) \subset W_\varepsilon^s(\Lambda_{h_c}^+)$.

Geometrically speaking, h_c is characterized as the energy level such that $P_{h_c}^u$ belongs to the boundary of the (topological) disk $\mathcal{D}_{h_c}^s$ "centered" at $P_{h_c}^s$ (see Fig. 5 bottom-left). In the next theorem, whose proof is given in Section 4.4 we compute $h_c = h_c(\varepsilon)$.

Theorem C *(Existence of heteroclinic connections between p_h^- and Λ_h^+). Consider system* (10). *There exist* $\varepsilon_0 > 0$*,* $h_0 > 0$ *and a function*

$$h_c(\varepsilon) = 2\pi^2 \varepsilon e^{-2\Omega\sqrt{2/\varepsilon}}(1 + \mathcal{O}(\varepsilon)), \quad \text{with } 0 < \varepsilon < \varepsilon_0,$$

such that, for every $0 < \varepsilon < \varepsilon_0$ *and* $0 < h < h_0$*, the invariant manifolds* $W_\varepsilon^u(p_h^-)$*,* $W_\varepsilon^s(\Lambda_h^+)$ *intersect* Σ_h *(given in* (20)*). The first intersection of* $W_\varepsilon^u(p_h^-)$*,* $W_\varepsilon^s(\Lambda_h^+)$ *with* Σ_h *is given by a point and a closed curve, denoted by* P_h^u *and* $\partial\mathcal{D}_h^s$*, respectively. Then,* $P_h^u \in \partial\mathcal{D}_h^s$ *if, and only if* $h = h_c(\varepsilon)$*.*

Theorem C also holds if we change p_h^- and Λ_h^+ by p_h^+ and Λ_h^-, respectively.

Now, given $h \ge h_c$, we compute the radius $\kappa_2 = \kappa_2(h)$ of the periodic orbit $\Lambda_{\kappa_1,\kappa_2}^+$ such that p_h^- connects to $\Lambda_{\kappa_1,\kappa_2}^+$ through a heteroclinic orbit.

Theorem D. *There exist* $\varepsilon_0 > 0$*,* $h_0 > 0$ *sufficiently small such that, for each* $0 < \varepsilon < \varepsilon_0$ *and* $h_c(\varepsilon) \le h < h_c(\varepsilon) + 2\pi^2\varepsilon e^{-2\Omega\sqrt{2/\varepsilon}}h_0$*, where* $h_c(\varepsilon)$ *is given by Theorem C, there exists a function*

$$\kappa : \left(h_c(\varepsilon), h_c(\varepsilon) + 2\pi^2\varepsilon e^{-2\Omega\sqrt{2/\varepsilon}}h_0\right) \to \mathbb{R},$$

such that

(1) $0 < \kappa(h) < h$ *and* $\lim_{h \to h_c(\varepsilon)^+} \kappa(h) = 0$*;*

(2) *For system* (10)*,* $W_\varepsilon^u(p_h^-) \subset W_\varepsilon^s(\Lambda_{\kappa(h),h-\kappa(h)}^+)$*;*

(3) *There exists an orbit of* (10) *with input velocity* $v_i = 4\sqrt{h}$ *and output velocity* $v_f = 4\sqrt{\kappa(h)}$. *Furthermore, define* $v_c = 4\sqrt{h_c}$, *then*

$$v_f = \sqrt{2v_c c_\varepsilon}\sqrt{v_i - v_c} + \mathcal{O}((v_i - v_c)^{3/2}), \tag{22}$$

where $c_\varepsilon = 1 + \mathcal{O}(\varepsilon)$.

The last item of Theorem D proves the conjecture $v_f \approx \mathcal{O}\left((v_i - v_c)^{1/2}\right)$ raised in [7]. This theorem is proved in Section 4.5.

3. Heuristics of the proof

We devote this section to give the main ideas of the proofs of Theorems A, B, C and D. As we will see, the most delicate part of the proof is to obtain (21a) in Theorem A, where we give an asymptotic formula for the difference between the one-dimensional unstable manifold of p_0^- and the stable one of p_0^+ at the section Σ_0. Once this item is proved, the remaining results will follow studying the dependence on h of the stable and unstable manifolds of the different objects $\Lambda^{\pm}_{\kappa_1,\kappa_2}$ considered.

To obtain the results in Theorem A, it will be more convenient to work in coordinates $\Gamma = B + ib$ and $\Theta = B - ib$. In these coordinates, System (10) becomes

$$\begin{cases} X' = \dfrac{Z}{8}, \\ Z' = -U'(X) - \dfrac{\delta}{\sqrt{2\Omega}}F'(X)\dfrac{(\Gamma - \Theta)}{2i}, \\ \Gamma' = \omega i \Gamma - \dfrac{\delta}{\sqrt{2\Omega}}F(X), \\ \Theta' = -\omega i \Theta - \dfrac{\delta}{\sqrt{2\Omega}}F(X), \end{cases} \quad \text{with} \quad \begin{cases} \delta = \varepsilon^{3/4}, \\ \omega = \dfrac{\Omega}{\sqrt{\varepsilon}}, \\ \Omega = \sqrt{1 - \dfrac{\varepsilon^2}{4}}. \end{cases} \tag{23}$$

This system is Hamiltonian with respect to

$$\mathcal{H}(X, Z, \Gamma, \Theta) = \frac{Z^2}{16} + U(X) + \frac{\delta}{\sqrt{2\Omega}}F(X)\frac{\Gamma - \Theta}{2i} + \frac{\omega}{2}\Gamma\Theta, \tag{24}$$

and the symplectic form $dX \wedge dZ + \dfrac{1}{2i}d\Gamma \wedge d\Theta$.

In the next section, we summarize the information we need about the unperturbed case $F = 0$.

3.1. Decoupled system ($F = 0$)

We parameterize the invariant manifolds $W(\kappa_1, \kappa_2)$ (see (17)) of the decoupled system (23) (with $\delta = 0$) in the coordinates (X, Z, Γ, Θ).

Lemma 3.1. *The one-dimensional invariant manifold* $W(h, 0) = W_0^u(p_h^-) = W_0^s(p_h^+)$ *is parameterized in the coordinate system* (X, Z, Γ, Θ) *by*

$$N_{h,0}(v) = (X_h(v), Z_h(v), 0, 0), \ v \in \mathbb{R}, \tag{25}$$

such that

(1) *If* $h = 0$, *then*

$$\begin{aligned} X_0(v) &= \operatorname{arcsinh}\left(\frac{\sqrt{2}}{2}v\right), \\ Z_0(v) &= 8(X_0)'(v) = \frac{8}{\sqrt{v^2+2}}. \end{aligned} \tag{26}$$

(2) *If* $h > 0$, *then*

$$\begin{aligned} X_h(v) &= \operatorname{arcsinh}\left(\sqrt{\frac{2+h}{h}}\sinh\left(v\sqrt{h}/2\right)\right), \\ Z_h(v) &= 8(X_h)'(v) = \frac{4\cosh(v\sqrt{h}/2)}{\sqrt{\frac{1}{2+h} + \frac{\sinh^2(v\sqrt{h}/2)}{h}}}. \end{aligned} \tag{27}$$

A simple application of the L'Hospital rule shows us that $X_h(v) \to X_0(v)$, pointwise, as $h \to 0$. Nevertheless, the decay of X_h as $v \to \pm\infty$ is significantly different from X_0 (for $h = 0$ the decay is polynomial whereas for $h > 0$ is exponential). Notice that $N_{0,0}(v)$ has poles at the points $\pm\sqrt{2}i$, whereas the poles of $N_{h,0}(v)$ are all contained in the imaginary axis and the closest to the real line are $\pm\sqrt{2}i + \mathcal{O}(h)$.

Lemma 3.2. *The two-dimensional invariant manifold* $W(\kappa_1, \kappa_2) = W_0^u(\Lambda^-_{\kappa_1,\kappa_2}) = W_0^s(\Lambda^+_{\kappa_1,\kappa_2})$, *with* $\kappa_1 \geq 0$, $\kappa_2 > 0$ *and* $\kappa_1 + \kappa_2 = h$ *is parameterized in the coordinate system* (X, Z, Γ, Θ) *by*

$$N_{\kappa_1,\kappa_2}(v, \tau) = (X_{\kappa_1}(v), Z_{\kappa_1}(v), \Gamma_{\kappa_2}(\tau), \Theta_{\kappa_2}(\tau)), \tag{28}$$

with $v \in \mathbb{R}$ *and* $\tau \in \mathbb{T}$, *such that*

$$\Gamma_{\kappa_2}(\tau) = \sqrt{\frac{2\kappa_2}{\omega}}e^{i\tau} \ \ \text{and} \ \ \Theta_{\kappa_2}(\tau) = \sqrt{\frac{2\kappa_2}{\omega}}e^{-i\tau}, \tag{29}$$

and X_{κ_1}, Z_{κ_1} *are given in* (26) *(*$\kappa_1 = 0$*) and* (27) *(*$\kappa_1 > 0$*).*

Notice that $N_{\kappa_1,\kappa_2}(v, \tau) \to N_{\kappa_1,0}(v)$ as $\kappa_2 \to 0$ uniformly, and thus the dependence of N_{κ_1,κ_2} is regular at $\kappa_2 = 0$.

Remark 3.3. Notice that, if $\kappa_2 = 0$, then N_{κ_1,κ_2} depends on one variable and if $\kappa_2 > 0$, it depends on two variables.

Remark 3.4. The functions $N_{\kappa_1,\kappa_2}(v, \tau)$, with v or τ fixed, do not parameterize the solutions of (23) for $\delta = 0$. Nevertheless, if $\phi_t^0(\cdot)$ denotes the flow of (23) for $\delta = 0$, we have

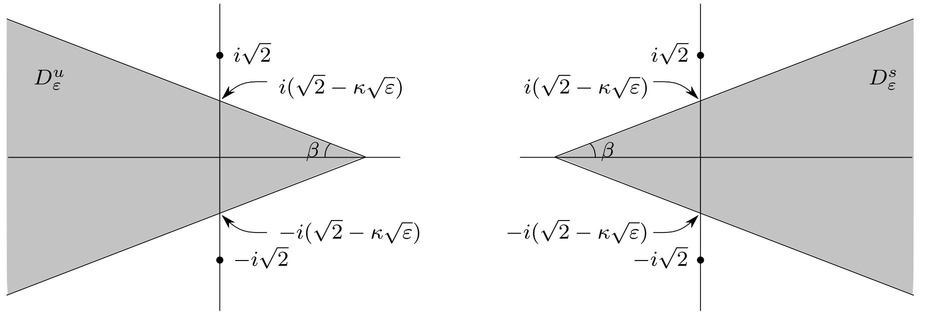

Fig. 6. Complex domains D^u_ε and D^s_ε.

$$\phi^0_t(N_{\kappa_1,\kappa_2}(v,\tau)) = N_{\kappa_1,\kappa_2}(v+t,\tau+\omega t),$$

and therefore the manifolds parameterized by $N_{\kappa_1,\kappa_2}(v,\tau)$ are invariant by the flow.

3.2. The invariant manifolds $W^u_\varepsilon(p^-_0)$ and $W^s_\varepsilon(p^+_0)$ and its difference

We devote this section to give the main ideas of the proof of the first item of Theorem A. First, in Section 4.1 we find parameterizations of the perturbed invariant manifolds of the different objects $\Lambda^{\pm}_{\kappa_1,\kappa_2}$ in terms of the parameters v and τ (see Lemma 3.2) in suitable complex domains.

More specifically, in Section 4.1.1 (see Theorem 4.1) we study parameterizations of the one-dimensional invariant manifolds $W^u_\varepsilon(p^-_0)$ and $W^s_\varepsilon(p^+_0)$ of the form

$$N^\star_{0,0}(v) = (X_0(v), Z^\star_0(v), \Gamma^\star_0(v), \Theta^\star_0(v)),\ \star = u, s. \tag{30}$$

That is, we parameterize the invariant manifolds as graphs. Those parameterizations are close to $N_{0,0}$ (see (25), (26)), in the complex domains

$$\begin{aligned} D^u_\varepsilon &= \{v \in \mathbb{C};\ |\operatorname{Im}(v)| < -\tan\beta \operatorname{Re}(v) + \sqrt{2} - \sqrt{\varepsilon}\},\\ D^s_\varepsilon &= \{v \in \mathbb{C};\ -v \in D^u_\varepsilon\}, \end{aligned} \tag{31}$$

where $0 < \beta < \pi/4$ is a fixed angle independent of ε (see Fig. 6). Observe that the parameterization $N_{0,0}(v)$ in (26) has singularities only at $\pm\sqrt{2}i$, thus $N_{0,0}$ is analytic in $D^{u,s}_\varepsilon$. The reason why we look for the parameterizations in complex domains instead of just for real values of the parameters will be clear later. Indeed, the analytic continuation of the parameterizations of the invariant manifolds into these complex domains is fundamental in obtaining the exponentially small estimates for the distance between the invariant manifolds.

Both parameterizations $N^{u,s}_{0,0}(v)$ are defined in the complex domain $\mathcal{D}_\varepsilon = D^u_\varepsilon \cap D^s_\varepsilon$, which contains 0 (see Fig. 7). To compute the difference between the invariant manifolds in the section Σ_0 (see (20)), we analyze its difference $\Delta\xi(v)$ given by

$$\Delta\xi(v) = \begin{pmatrix} \Delta_\Gamma(v) \\ \Delta_\Theta(v) \end{pmatrix} = \begin{pmatrix} \Gamma^u_0(v) - \Gamma^s_0(v) \\ \Theta^u_0(v) - \Theta^s_0(v) \end{pmatrix},$$

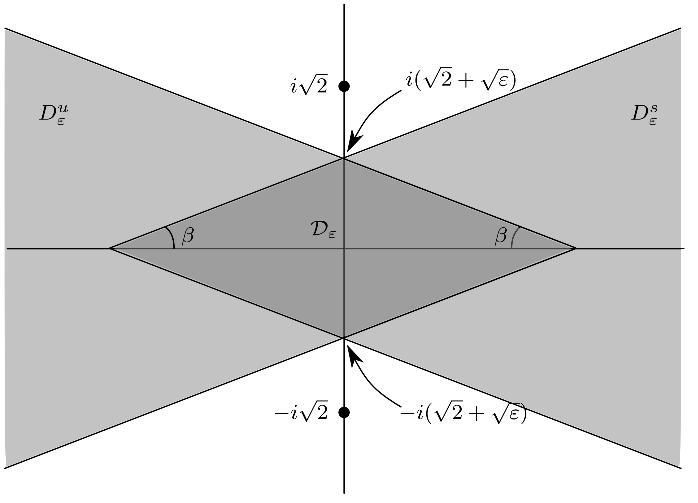

Fig. 7. Domain $\mathcal{D}_\varepsilon$ where both parameterizations $N^*_{0.0}$ are defined.

for $v \in \mathcal{I}_\varepsilon = \mathcal{D}_\varepsilon \cap \mathbb{R}$. Note that the invariant manifolds are parameterized as graphs and therefore these quantities just measure the distance between the invariant manifolds on the section $v =$ constant (equivalently $X =$ constant) projected into the Γ and Θ direction. The distance in the Z direction can be deduced from the energy conservation.

As both parameterizations are solutions of the same equation, using also the energy conservation, one can see that its difference $\Delta\xi$ satisfies a linear equation

$$\Delta\xi' = \begin{pmatrix} \omega i & 0 \\ 0 & -\omega i \end{pmatrix} \Delta\xi + B(v)\Delta\xi, \tag{32}$$

where the entries of the matrix B are functions of order $\mathcal{O}(\varepsilon^{8/3})$ in $\mathcal{D}_\varepsilon$.

The main observation is that, if $B \equiv 0$, then $\Delta\xi$ is the analytic function

$$\Delta\xi(v) = \begin{pmatrix} e^{\omega i(v-v_0)}\Delta_\Gamma(v_0) \\ e^{-\omega i(v-v_1)}\Delta_\Theta(v_1) \end{pmatrix},$$

for any $v_0, v_1 \in \mathcal{D}_\varepsilon$. Thus, choosing $v_0 = -i(\sqrt{2} - \sqrt{\varepsilon}) \in \mathcal{D}_\varepsilon$ and $v_1 = i(\sqrt{2} - \sqrt{\varepsilon}) \in \mathcal{D}_\varepsilon$, we have that, for $v \in \mathcal{I}_\varepsilon$,

$$|\Delta_\Gamma(v)| \simeq e^{-\sqrt{2}\omega}|\Delta_\Gamma(v_0)| \simeq e^{-\sqrt{\frac{2}{\varepsilon}}}|\Delta_\Gamma(v_0)|,$$

$$|\Delta_\Theta(v)| \simeq e^{-\sqrt{2}\omega}|\Delta_\Theta(v_1)| \simeq e^{-\sqrt{\frac{2}{\varepsilon}}}|\Delta_\Theta(v_1)|.$$

Now, using the results of Theorem 4.1 below, we have that $|\Delta_\Gamma(v_0)|$ and $|\Delta_\Theta(v_1)|$ are of order $\varepsilon^{4/3}$ obtaining that

$$\Delta\xi(v) \simeq e^{-\sqrt{\frac{2}{\varepsilon}}}\varepsilon^{4/3}, \ v \in \mathcal{I}_\varepsilon = \mathcal{D}_\varepsilon \cap \mathbb{R},$$

and therefore it is exponentially small with respect to ε.

In Section 6.2, we implement these ideas in the proof of the formula (21a) in Theorem A to obtain the asymptotic formula for $\Delta\xi$ using that B is small.

3.3. The invariant manifolds $W_\varepsilon^{u,s}(\Lambda^\pm_{\kappa_1,\kappa_2})$

Next step is to study the stable and unstable manifolds of the rest of invariant objects $\Lambda^\pm_{\kappa_1,\kappa_2}$, for $\kappa_1+\kappa_2=h>0$, including the special case $\kappa_1=h$, $\kappa_2=0$ which correspond to the points $p_h^\pm$. For these manifolds we perform a perturbative analysis in κ_1 and κ_2 and therefore we need not to face exponentially small phenomena. For this reason, it is enough to analyze them in the consider complex domains

$$\begin{aligned} D^u &= \left\{ v\in\mathbb{C};\ |\mathrm{Im}(v)|\leq -\tan(\beta)\,\mathrm{Re}(v)+\sqrt{2}/2 \right\},\\ D^s &= \{v\in\mathbb{C};\ -v\in D^u\}, \end{aligned} \tag{33}$$

for some $0<\beta<\pi/4$ fixed. Note that these domains are independent of ε, (in particular, the distance of the singularities of $N_{0,0}$ to those domains is independent of ε). We also consider

$$\mathbb{T}_\sigma=\{\tau\in\mathbb{C};\ |\mathrm{Im}(\tau)|<\sigma \text{ and } \mathrm{Re}(\tau)\in\mathbb{T}\}. \tag{34}$$

The parameterizations of the invariant manifolds $W_\varepsilon^{u,s}(\Lambda^\pm_{\kappa_1,\kappa_2})$ are obtained in Theorems 4.3, 4.5 and 4.6. In Theorem 4.8 we compare these parameterizations with $N^*_{0,0}$, $*=u,s$. Then, to obtain information about the possible intersections between the unstable manifold $W_\varepsilon^u(\Lambda^-_{\kappa_1,\kappa_2})$ and the stable manifold $W_\varepsilon^s(\Lambda^+_{\kappa_1',\kappa_2'})$, where $h=\kappa_1+\kappa_2=\kappa_1'+\kappa_2'>0$ we just use the estimates (21a) in Theorem A and the estimates of Theorem 4.8.

More concretely, to prove the estimate (21b) in Theorem A, we use Theorem 4.8 to approximate the points $P_h^{u,s}$ by the points $P_0^{u,s}$, and the result will follow from estimate (21a). The results of Theorem B are obtained looking for the minimum value of $h=h_s$ that ensures that there are intersections between $W_\varepsilon^u(\Lambda_h^-)$ and $W_\varepsilon^s(\Lambda_h^+)$. Analogously, the results of Theorem C are obtained looking for the minimum value of $h=h_c$ where there are intersections between $W_\varepsilon^u(p_h^-)$ and $W_\varepsilon^s(\Lambda_h^+)$, which, in fact, means that $W_\varepsilon^u(p_h^-)\subset W_\varepsilon^s(\Lambda_h^+)$. Finally, in Theorem D, we prove that if the energy $h=h_c$ of the system is slightly increased, then there also exists a heteroclinic connection between the point p_h^- and a periodic orbit $\Lambda^+_{\kappa_1,\kappa_2}$ at $X=+\infty$. More specifically, for $h>h_c$, $W_\varepsilon^u(p_h^-)\subset W_\varepsilon^s(\Lambda^+_{\kappa(h),h-\kappa(h)})$, where $\kappa(h)$ is a small number between 0 and h, in such a way that, if $h\to h_c$, then $\Lambda^+_{\kappa(h),h-\kappa(h)}\to\Lambda^+_{h_c}$. Furthermore, we obtained an estimate for the final velocity $v_f=4\sqrt{\kappa(h)}$ of the heteroclinic in terms of the initial one.

4. Proofs of Theorems A, B, C and D

4.1. Parameterizations of the invariant manifolds $W^u(\Lambda^-_{\kappa_1,\kappa_2})$, $W^s(\Lambda^+_{\kappa_1,\kappa_2})$

We devote this section to build and analyze suitable parameterizations of the invariant manifolds of the periodic orbits $\Lambda^\pm_{\kappa_1,\kappa_2}$. The construction of the manifolds for the different cases is analogous: we find them through a Perron-like method suitable on complex domains. Nevertheless, as explained in Section 3, the complex domains are significantly different in the case

$\kappa_1 = \kappa_2 = 0$, that is, when we construct $W^u_\varepsilon(p_0^-)$ and $W^s_\varepsilon(p_0^+)$. In this case, to be able to obtain the results about their difference in Theorem A, we need to construct these manifolds in the domains (31), which reach a neighborhood of size $\sqrt{\varepsilon}$ of the singularities $\pm\sqrt{2}i$ of $N(0,0)$. For the rest of the invariant manifolds we will use the domains (33) whose points are always at a distance of these singularities independent of ε.

4.1.1. Parameterizations of the invariant manifolds $W^u_\varepsilon(p_0^-)$, $W^s_\varepsilon(p_0^+)$

In this section we consider parameterizations $N^\star_{0,0}(v)$, $\star = u, s$, given by (30), of the invariant manifolds $W^u_\varepsilon(p_0^-)$ and $W^s_\varepsilon(p_0^+)$ near $N_{0,0}$, in the complex domains D^u_ε and D^s_ε given by (31), respectively. The parameterization $N_{0,0}(v)$ in (26) has singularities only at $\pm\sqrt{2}i$, thus $N_{0,0}$ is analytic in $D^{u,s}_\varepsilon$.

We state all the results for the unstable case, since it is analogous for the stable one. Based on a fixed point argument, we prove the following theorem in Section 5.

Theorem 4.1. *Given $\nu > 0$. There exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \le \varepsilon_0$, the one-dimensional manifold $W^u_\varepsilon(p_0^-)$ is parameterized by*

$$N^u_{0,0}(v) = (X_0(v), Z^u_0(v), \Gamma^u_0(v), \Theta^u_0(v)), \tag{35}$$

with $v \in D^u_\varepsilon$, where X_0 is given in (26), $Z^u_0(v)$ is obtained from $\mathcal{H}(N^u_{0,0}(v)) = 0$ ($\mathcal{H}$ given in (24)) and

$$\begin{cases} \Gamma^u_0(v) = Q^0(v) + \gamma^u_0(v), \\ \Theta^u_0(v) = -Q^0(v) + \theta^u_0(v), \end{cases} \tag{36}$$

with

$$Q^0(v) = -i\frac{\delta}{\omega\sqrt{2\Omega}}F(X_0(v)) = \frac{\delta}{\omega}\frac{2iv}{\sqrt{2\Omega}(2+v^2)}. \tag{37}$$

Furthermore, $\gamma^u_0(v), \theta^u_0(v)$ are analytic functions such that $\theta^u_0(v) = \overline{\gamma^u_0(v)}$, for every $v \in \mathbb{R} \cap D^u_\varepsilon$, and there exists a constant $M > 0$ independent of ε such that

(1) $\left|\gamma^u_0(v)\right|, \left|\theta^u_0(v)\right| \le M\dfrac{\delta}{\omega^2}\dfrac{1}{|v|^2}$, *for each $v \in D^u_\varepsilon$, $|\mathrm{Re}(v)| \le \nu$;*

(2) $\left|\gamma^u_0(v)\right|, \left|\theta^u_0(v)\right| \le M\dfrac{\delta}{\omega^2}\dfrac{1}{|v^2+2|^2}$, *for each $v \in D^u_\varepsilon$, $|\mathrm{Re}(v)| \ge \nu$.*

Remark 4.2. Notice the points $p_0^\pm$ behave as degenerate-saddles at infinity, and thus the existence of local invariant manifolds for the perturbed system is not standard. Nevertheless, these singularities at infinity behave as parabolic points (see Remark 1.1) and Theorem 4.1 gives the existence of their invariant manifolds.

Next sections are devoted to find parameterizations of the invariant manifolds $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ and $W^s_\varepsilon(\Lambda^+_{\kappa_1,\kappa_2})$, for $\kappa_1, \kappa_2 \ge 0$ and $\kappa_1 + \kappa_2 = h > 0$. Even if one theorem could contain all the results for $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$, we state three separate theorems, Theorem 4.3 ($\kappa_1 = 0$), Theorem 4.5 ($\kappa_2 = 0$) and Theorem 4.6 ($\kappa_1, \kappa_2 > 0$), to clarify the exposition (and the structure of the corresponding proofs).

4.1.2. Zero energy for the pendulum (separatrix case $\kappa_1 = 0$ and $\kappa_2 = h > 0$)

We look for parameterizations of the 2-dimensional invariant manifolds $W^u_\varepsilon(\Lambda_h^-)$ and $W^s_\varepsilon(\Lambda_h^+)$,

$$N^\star_{0,h}(v,\tau) = \left(X_0(v), Z_0(v) + Z^\star_{0,h}(v,\tau), \Gamma_h(\tau) + \Gamma^\star_{0,h}(v,\tau), \Theta_h(\tau) + \Theta^\star_{0,h}(v,\tau)\right), \star = u, s,$$

as perturbations of $W(0,h)$ (see Lemma 3.2) in the domains $D^\star \times \mathbb{T}_\sigma$, $\star = u, s$ (see (33) and (34)).

We prove the following theorem in Section 7.

Theorem 4.3. *Fix $\sigma > 0$ and $h_0 > 0$. There exists $\varepsilon_0 > 0$ sufficiently small such that, for $0 < \varepsilon \leq \varepsilon_0$ and $0 < h \leq h_0$, $W^u_\varepsilon(\Lambda_h^-)$ is parameterized by*

$$N^u_{0,h}(v,\tau) = (X_0(v), Z_0(v) + Z^u_{0,h}(v,\tau), \Gamma_h(\tau) + \Gamma^u_{0,h}(v,\tau), \Theta_h(\tau) + \Theta^u_{0,h}(v,\tau)), \tag{38}$$

with $v \in D^u$ (see (33)) and $\tau \in \mathbb{T}_\sigma$, where $X_0, Z_0, \Gamma_h, \Theta_h$ are given by (26) and (29),

$$\begin{cases} Z^u_{0,h}(v,\tau) = Z_{0,h}(v,\tau) + z^u_{0,h}(v,\tau), \\ \Gamma^u_{0,h}(v,\tau) = Q^0(v) + \gamma^u_{0,h}(v,\tau), \\ \Theta^u_{0,h}(v,\tau) = -Q^0(v) + \theta^u_{0,h}(v,\tau), \end{cases} \tag{39}$$

where Q^0 is given by (37), and

$$Z_{0,h}(v,\tau) = \frac{\delta}{\omega\sqrt{2\Omega}} F'(X_0(v)) \frac{\Gamma_h(\tau) + \Theta_h(\tau)}{2} = \mathcal{O}\left(\frac{\delta\sqrt{h}}{\omega^{3/2}} \frac{1}{\sqrt{v^2+2}}\right). \tag{40}$$

Furthermore, $z^u_{0,h}$ is a real-analytic function and $\gamma^u_{0,h}, \theta^u_{0,h}$ are analytic functions satisfying

$$\theta^u_{0,h}(v,\tau) = \overline{\gamma^u_{0,h}(v,\tau)}, \ (v,\tau) \in \mathbb{R}^2 \cap D^u \times \mathbb{T}_\sigma,$$

such that there exists a constant $M > 0$ independent of ε and h such that, for $(v,\tau) \in D^u \times \mathbb{T}_\sigma$,

$$|z^u_{0,h}(v,\tau)|, |\gamma^u_{0,h}(v,\tau)|, |\theta^u_{0,h}(v,\tau)| \leq M \frac{\delta}{\omega} \frac{1}{|\sqrt{v^2+2}|}. \tag{41}$$

Remark 4.4. We stress that the estimates in (40) and (41) and are only valid for $v^2 + 2 > 1/2$ and therefore do not give any information about the behavior of $N^u_{0,h}(v,\tau)$ near the singularities $v = \pm i\sqrt{2}$. The role of these bounds is to show how the functions decay as $v \to \pm\infty$.

4.1.3. Positive energy for the pendulum

This section is devoted to study the invariant manifolds of the periodic orbits $\Lambda^{\mp}_{\kappa_1,\kappa_2}$ for $\kappa_1>0$. First, we consider the case $\kappa_1=h$ and $\kappa_2=0$. In this case $\Lambda^{\mp}_{h,0}=p_h^{\mp}$ is a point. We apply the same ideas of Section 4.1.1 to parameterize $W^u_\varepsilon(p_h^-)$ as

$$N^u_{h,0}(v)=(X_h(v),Z^u_{h,0}(v),\Gamma^u_{h,0}(v),\Theta^u_{h,0}(v)),$$

where $X_h(v)$ has been introduced in (27). The main difference is that we need to take into account the singular dependence on the parameter h at $h=0$. Indeed, one has to be careful of the analysis of the parameterizations of the invariant manifolds as $v\to\pm\infty$ since X_h decays exponentially for $h>0$ and polynomially for $h=0$.

As in Theorem 4.3, for our purposes it is sufficient to parameterize the manifolds in the domains $D^{u,s}$ (see (33)). We prove the following theorem in Section 8.

Theorem 4.5. *There exist $\varepsilon_0>0$ and $h_0>0$ sufficiently small such that, for $0<\varepsilon\le\varepsilon_0$ and $0<h\le h_0$, $W^u_\varepsilon(p_h^-)$ is parameterized by*

$$N^u_{h,0}(v)=(X_h(v),Z^u_{h,0}(v),\Gamma^u_{h,0}(v),\Theta^u_{h,0}(v)),\ \ v\in D^u,$$

where X_h is given by (27), $Z^u_{h,0}(v)$ is obtained from $\mathcal{H}(N^u_{h,0}(v))=h$ ($\mathcal{H}$ given in (24)) and

$$\begin{cases}\Gamma^u_{h,0}(v)=Q^h(v)+\gamma^u_{h,0}(v),\\ \Theta^u_{h,0}(v)=-Q^h(v)+\theta^u_{h,0}(v),\end{cases} \tag{42}$$

with

$$Q^h(v)=-i\frac{\delta}{\omega\sqrt{2\Omega}}F(X_h(v))=\mathcal{O}\left(\frac{\delta}{\omega}\frac{1}{\sqrt{v^2+2}}\right). \tag{43}$$

Furthermore, $\gamma^u_{h,0}(v),\theta^u_{h,0}(v)$ are analytic functions satisfying $\theta^u_{h,0}(v)=\overline{\gamma^u_{h,0}(v)}$ for $v\in\mathbb{R}\cap D^u$ such that there exists a constant $M>0$ independent of ε such that for $v\in D^u$

$$\left|\gamma^u_{h,0}(v)\right|,\left|\theta^u_{h,0}(v)\right|\le M\frac{\delta}{\omega^2}\frac{1}{|v^2+2|}. \tag{44}$$

Finally we deal with the case $\kappa_1,\kappa_2>0$. Next theorem, proven in Section 9, gives the parameterizations of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$.

Theorem 4.6. *Fix $\sigma>0$. There exist $\varepsilon_0>0$ and $h_0>0$ sufficiently small such that, for $0<\varepsilon\le\varepsilon_0$, $0<h\le h_0$, and $\kappa_1>0$, $\kappa_2\ge0$ with $\kappa_1+\kappa_2=h$, the invariant manifold $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ is parameterized by*

$$N^u_{\kappa_1,\kappa_2}(v,\tau)=(X_{\kappa_1}(v),Z_{\kappa_1}(v)+Z^u_{\kappa_1,\kappa_2}(v,\tau),\Gamma_{\kappa_2}(\tau)+\Gamma^u_{\kappa_1,\kappa_2}(v,\tau),\Theta_{\kappa_2}(\tau)+\Theta^u_{\kappa_1,\kappa_2}(v,\tau)),$$

for $(v,\tau)\in D^u\times\mathbb{T}_\sigma$, where X_{κ_1}, $Z_{\kappa_1},\Gamma_{\kappa_2},\Theta_{\kappa_2}$ are given by (27) and (29),

$$\begin{cases} Z^u_{\kappa_1,\kappa_2}(v,\tau) = Z_{\kappa_1,\kappa_2}(v,\tau) + z^u_{\kappa_1,\kappa_2}(v,\tau), \\ \Gamma^u_{\kappa_1,\kappa_2}(v,\tau) = Q^{\kappa_1}(v) + \gamma^u_{\kappa_1,\kappa_2}(v,\tau), \\ \Theta^u_{\kappa_1,\kappa_2}(v,\tau) = -Q^{\kappa_1}(v) + \theta^u_{\kappa_1,\kappa_2}(v,\tau), \end{cases} \tag{45}$$

where Q^{κ_1} is given in (43) and

$$Z_{\kappa_1,\kappa_2}(v,\tau) = \frac{\delta}{\omega\sqrt{2\Omega}} F'(X_{\kappa_1}(v)) \frac{\Gamma_{\kappa_2}(\tau) + \Theta_{\kappa_2}(\tau)}{2} = \mathcal{O}\left(\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}} \frac{1}{\sqrt{v^2+2}}\right).$$

Furthermore, $z^u_{\kappa_1,\kappa_2}$ is a real-analytic function and $\gamma^u_{\kappa_1,\kappa_2}, \theta^u_{\kappa_1,\kappa_2}$ are analytic functions satisfying $\theta^u_{\kappa_1,\kappa_2}(v,\tau) = \overline{\gamma^u_{\kappa_1,\kappa_2}(v,\tau)}$ for $(v,\tau) \in \mathbb{R}^2 \cap D^u \times \mathbb{T}_\sigma$ such that there exists a constant $M > 0$ independent of ε, κ_1 and κ_2 such that, for $(v,\tau) \in D^u \times \mathbb{T}_\sigma$ (see (33)),

$$|z^u_{\kappa_1,\kappa_2}(v,\tau)|, |\gamma^u_{\kappa_1,\kappa_2}(v,\tau)|, |\theta^u_{\kappa_1,\kappa_2}(v,\tau)| \le M \frac{\delta}{\omega} \frac{1}{|\sqrt{v^2+2}|}. \tag{46}$$

4.2. Proof of Theorem A

We devote this section to complete the proofs of Theorem A. That is, to obtain (21a) and (21b). Note that these two asymptotic formulas are of significantly different nature. The first one is an asymptotic formula in ε. Since the leading-order approximation is exponentially small, one has to follow the techniques explained in Section 3. This is done in Section 4.2.1. On the contrary, (21b) is an asymptotic formula in h (which is not exponentially small) and therefore the analysis is done just for the real parameterizations. This is done in Section 4.2.2.

4.2.1. Splitting of $W^u_\varepsilon(p_0^-)$ and $W^s_\varepsilon(p_0^+)$

By Theorem 4.1, both parameterizations $N^{u,s}_{0,0}(v)$ are defined in the complex domain $\mathcal{D}_\varepsilon = D^u_\varepsilon \cap D^s_\varepsilon$, which contains 0 (see Fig. 7). As explained in Section 3.2, to compute the difference between the invariant manifolds in the section Σ_0 (see (20)), we analyze $\Delta\xi(v)$ given by

$$\Delta\xi(v) = \begin{pmatrix} \Gamma^u_0(v) - \Gamma^s_0(v) \\ \Theta^u_0(v) - \Theta^s_0(v) \end{pmatrix},$$

for $v \in \mathcal{I}_\varepsilon = \mathcal{D}_\varepsilon \cap \mathbb{R}$.

In Section 6.2, we prove that $\Delta\xi$ satisfies (32) and we also prove the following theorem, which provides an asymptotic formula for the difference $\Delta\xi$. This theorem implies the estimate (21a) in Theorem A.

Theorem 4.7. *Consider system (23). Given any compact interval $\mathcal{I} \subset \mathbb{R}$ containing 0, there exists $\varepsilon_0 > 0$ sufficiently small such that, for every $0 < \varepsilon < \varepsilon_0$, the parameterizations $N^\star_{0,0}(v)$, $\star = u, s$, given in (30), are defined for $v \in \mathcal{I}$ and satisfy*

$$\begin{cases} \Gamma^u_0(0) - \Gamma^s_0(0) = -i\dfrac{2\pi\delta}{\sqrt{\Omega}} e^{-\sqrt{2}\omega} + \mathcal{O}(\omega\delta^3 e^{-\sqrt{2}\omega}), \\ \Theta^u_0(0) - \Theta^s_0(0) = i\dfrac{2\pi\delta}{\sqrt{\Omega}} e^{-\sqrt{2}\omega} + \mathcal{O}(\omega\delta^3 e^{-\sqrt{2}\omega}). \end{cases} \tag{47}$$

4.2.2. Approximation of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ by $W^u_\varepsilon(p^-_0)$ in the section Σ_h

Recall that for the unperturbed case, we have that

$$W(\kappa_1,\kappa_2)\cap\Sigma_h=\{(Z,b,B);\ Z=4\sqrt{2+\kappa_1}\text{ and }b^2+B^2=2\kappa_2/\omega\},$$

(see (17)). Thus, the sets $W(\kappa_1,\kappa_2)\cap\Sigma_h$ and $W(0,0)\cap\Sigma_0$ are $(\kappa_1+\sqrt{\kappa_2})$-close. Since the perturbed invariant manifolds are close to the unperturbed ones (see Theorems 4.3, 4.5, 4.6), in the next theorem we approximate $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ by $W^u_\varepsilon(p^-_0)$ for κ_1,κ_2 small. Using energy conservation and the fact that Γ and Θ are complex conjugate for real values of the variables, it is enough to compare the invariant manifolds only in the variable Γ. We define the projection $\pi_\Gamma(X,Z,\Gamma,\Theta)=\Gamma$.

Theorem 4.8. *Consider $\kappa_1,\kappa_2\geq 0$, $\kappa_1+\kappa_2=h$, and the parameterization of $N^u_{\kappa_1,\kappa_2}$ of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ obtained in Theorems 4.1, 4.3, 4.5, and 4.6. Then, there exist $\varepsilon_0>0$ and $h_0>0$ sufficiently small such that for $0<h\leq h_0$ and $0<\varepsilon\leq\varepsilon_0$*

$$\pi_\Gamma N^u_{\kappa_1,\kappa_2}(0,\tau)-\pi_\Gamma N^u_{0,0}(0)=\Gamma_{\kappa_2}(\tau)+\mathcal{O}\left(\frac{\delta\sqrt{\kappa_1}}{\omega^2}+\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}\right),\ \tau\in\mathbb{T},$$

where $\Gamma_{\kappa_2}(\tau)$ has been introduced in (29).

The proof of this theorem is done in Section 10. The result of this theorem for $\kappa_1=h$ and $\kappa_2=0$ implies equation (21b) in Theorem A (note that we are abusing notation since, in this case, the function $N^u_{\kappa_1,\kappa_2}$ does not depend on τ).

4.3. Proof of Theorem B

Theorems 4.3 and 4.8 provide, for $0\leq h\leq h_0$, $\varepsilon\leq\varepsilon_0$, the existence of the invariant manifolds $W^u_\varepsilon(\Lambda^-_h)$ and $W^s_\varepsilon(\Lambda^+_h)$ and their approximation by $W^u_\varepsilon(p^-_h)$ and $W^s_\varepsilon(p^+_h)$. The invariant manifolds $W^u_\varepsilon(\Lambda^-_h)$ and $W^s_\varepsilon(\Lambda^+_h)$ are parameterized by

$$N^{u,s}_{0,h}(v,\tau)=\begin{pmatrix}X_0(v)\\ Z_0(v)+Z_{0,h}(v,\tau)+z^{u,s}_{0,h}(v,\tau)\\ \Gamma_h(\tau)+\Gamma^{u,s}_0(v)+F^{u,s}(v,\tau,h,\varepsilon)\\ \Theta_h(\tau)+\Theta^{u,s}_0(v)+\overline{F^{u,s}(v,\tau,h,\varepsilon)}\end{pmatrix},\ (v,\tau)\in(D^{u,s}\cap\mathbb{R})\times\mathbb{T},$$

where X_0, Z_0 are given in (26), $Z_{0,h}$ and $z^{u,s}_{0,h}$ are given by (39), Γ_h and Θ_h are given in (29), $\Gamma^{u,s}_0$, $\Theta^{u,s}_0$ are given in (36) and $F^{u,s}$ are analytic functions such that

$$F^{u,s}(v,\tau,h,\varepsilon)=\mathcal{O}\left(\frac{\delta\sqrt{h}}{\omega^{3/2}}\right).$$

Consider the section Σ_h (which corresponds to $v=0\in D^u\cap D^s$). Then, $W^u_\varepsilon(\Lambda^-_h)$ and $W^s_\varepsilon(\Lambda^+_h)$ intersect along a heteroclinic orbit if and only if there exist τ^u, τ^s in $[-\pi,\pi)$ such that

$N^u_{0,h}(0,\tau^u) = N^s_{0,h}(0,\tau^s)$. Moreover, using energy conservation, $N^u_{0,h}(0,\tau^u) = N^s_{0,h}(0,\tau^s)$ if, and only if,

$$\Gamma_h(\tau^u) + \Gamma^u_0(0) + F^u(0,\tau^u,h,\varepsilon) = \Gamma_h(\tau^s) + \Gamma^s_0(0) + F^s(0,\tau^s,h,\varepsilon),$$
$$\Theta_h(\tau^u) + \Theta^u_0(0) + \overline{F^u(0,\tau^u,h,\varepsilon)} = \Theta_h(\tau^s) + \Theta^s_0(0) + \overline{F^u(0,\tau^s,h,\varepsilon)}.$$

Since $\tau^u, \tau^s \in \mathbb{R}$, using Theorem 4.7, the expression of Γ_h in (29), the equations above are equivalent to

$$\begin{aligned} \sqrt{\frac{2h}{\omega}}(\cos(\tau^u) - \cos(\tau^s)) + M_1(\varepsilon) + F_1(\tau^u,\tau^s,h,\varepsilon) = 0, \\ \sqrt{\frac{2h}{\omega}}(\sin(\tau^u) - \sin(\tau^s)) - \frac{2\pi\delta}{\sqrt{\Omega}}e^{-\sqrt{2}\omega} + M_2(\varepsilon) + F_2(\tau^u,\tau^s,h,\varepsilon) = 0, \end{aligned} \tag{48}$$

where $0 < \varepsilon \leq \varepsilon_0$, $0 < h \leq h_0$ and M_1, M_2, F_1, F_2 are real-analytic functions such that

$$M_1, M_2 = \mathcal{O}(\omega\delta^3 e^{-\sqrt{2}\omega}) \text{ and } F_1, F_2 = \mathcal{O}\left(\frac{\delta\sqrt{h}}{\omega^{3/2}}\right).$$

We change the parameter $h \geq 0$

$$h = \frac{\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{2\Omega}\mu^2, \text{ for } \mu \geq 0. \tag{49}$$

Then, since $0 < h \leq h_0$, it is sufficient to consider

$$0 < \mu \leq \mu_0 = \frac{1}{\delta_0\pi}\sqrt{\frac{2\Omega_0 h_0}{\omega_0}}e^{\sqrt{2}\omega_0},$$

where $\Omega_0 = \sqrt{1 - \varepsilon_0^2/4}$, $\omega_0 = \Omega_0/\sqrt{\varepsilon_0}$ and $\delta_0 = \varepsilon_0^{3/4}$. Considering $\varepsilon_0 > 0$ sufficiently small, we can assume that $\mu_0 > 1$. Replacing h in (48) and multiplying the equation by $\frac{\sqrt{\Omega}}{\pi\delta}e^{\sqrt{2}\omega} > 0$, we may rewrite (48) as

$$\begin{aligned} \mu(\cos(\tau^u) - \cos(\tau^s)) + \widetilde{M}_1(\varepsilon) + \widetilde{F}_1(\tau^u,\tau^s,\mu,\varepsilon) = 0, \\ \mu(\sin(\tau^u) - \sin(\tau^s)) - 2 + \widetilde{M}_2(\varepsilon) + \widetilde{F}_2(\tau^u,\tau^s,\mu,\varepsilon) = 0, \end{aligned} \tag{50}$$

where $\widetilde{M}_1, \widetilde{M}_2, \widetilde{F}_1, \widetilde{F}_2$, are real-analytic functions such that

$$\widetilde{M}_1, \widetilde{M}_2 = \mathcal{O}(\omega\delta^2) \text{ and } \widetilde{F}_1, \widetilde{F}_2 = \mathcal{O}\left(\frac{\delta}{\omega}\mu\right).$$

Define the function $G = (G_1, G_2) : [-\pi,\pi]^2 \times (0,\mu_0] \times [0,\varepsilon_0] \to \mathbb{R}^2$ corresponding to the left-hand side of system (50). Recalling that $\delta = \varepsilon^{3/4}$ and $\omega = \Omega/\sqrt{\varepsilon}$, it is clear that

$$G(\tau^u,\tau^s,\mu,\varepsilon)=\begin{pmatrix}\mu(\cos(\tau^u)-\cos(\tau^s))+\mathcal{O}(\varepsilon)\\ \mu(\sin(\tau^u)-\sin(\tau^s))-2+\mathcal{O}(\varepsilon)\end{pmatrix}. \tag{51}$$

The equation $G(\tau^u,\tau^s,\mu,0)=(0,0)$ has a unique family of solutions

$$\mathcal{S}_0=\{(\alpha,-\alpha,1/\sin(\alpha),0);\ \arcsin(1/\mu_0)\leq\alpha\leq\pi-\arcsin(1/\mu_0)\}.$$

We find zeroes of G using the Implicit Function Theorem around every solution of the family $\mathcal{S}_0$. Denote $\alpha_0=\arcsin(1/\mu_0)$ and fix $0<\alpha_0\leq\alpha\leq\pi-\alpha_0$. Then,

(1) $G(\alpha,-\alpha,1/\sin(\alpha),0)=(0,0)$,
(2) $\det\left(\dfrac{\partial(G_1,G_2)}{\partial(\mu,\tau^s)}\right)(\alpha,-\alpha,1/\sin(\alpha),0)=2\sin(\alpha)\neq 0$.

Thus, it follows from the Implicit Function Theorem that there exist $\varepsilon_\alpha>0$ and unique functions $\tau^s_\alpha:(\alpha-\varepsilon_\alpha,\alpha+\varepsilon_\alpha)\times[0,\varepsilon_\alpha)\to[-\pi,\pi]$, $\mu_\alpha:(\alpha-\varepsilon_\alpha,\alpha+\varepsilon_\alpha)\times[0,\varepsilon_\alpha)\to(0,\mu_0]$ such that

$$G(\tau^u,\tau^s_\alpha(\tau^u,\varepsilon),\mu_\alpha(\tau^u,\varepsilon),\varepsilon)=(0,0).$$

Furthermore

$$\begin{cases}\tau^s_\alpha(\tau^u,\varepsilon)=-\alpha+\mathcal{O}(\tau^u-\alpha,\varepsilon),\\ \mu_\alpha(\tau^u,\varepsilon)=1/\sin(\alpha)+\mathcal{O}(\tau^u-\alpha,\varepsilon),\end{cases}\qquad \tau^u\in(\alpha-\varepsilon_\alpha,\alpha+\varepsilon_\alpha). \tag{52}$$

Consider the compact set $K=[\alpha_0,\pi-\alpha_0]$. We can find $n\in\mathbb{N}$, $\alpha_1,\cdots,\alpha_n$ with respective $\varepsilon_{\alpha_1},\cdots,\varepsilon_{\alpha_n}$, previously found, such that the intervals $(\alpha_i-\varepsilon_{\alpha_i},\alpha_i+\varepsilon_{\alpha_i})$, $i=1,\cdots,n$ form a finite cover of K. Using the uniqueness of solutions obtained from the Implicit Function Theorem, it is possible to conclude that there exist $\varepsilon_1>0$ sufficiently small and functions

$$\begin{aligned}\tau^s_*\left(\tau^u,\varepsilon\right)&=-\tau^u+\mathcal{O}(\varepsilon),\\ \mu_*\left(\tau^u,\varepsilon\right)&=1/\sin(\tau^u)+\mathcal{O}(\varepsilon),\end{aligned}$$

defined for every $\varepsilon<\varepsilon_1$ and $\tau^u\in K$, such that

$$G\left(\tau^u,\tau^s_*\left(\tau^u,\varepsilon\right),\mu_*\left(\tau^u,\varepsilon\right),\varepsilon\right)=(0,0).$$

This implies that there exists at least one heteroclinic connection in the energy level

$$h=\frac{\pi^2\omega\delta^2e^{-2\sqrt{2}\omega}}{2\Omega}(\mu_*(\tau^u,\varepsilon))^2,\ \tau^u\in K.$$

Moreover, $(\mu_*(\tau^u,0))^2\geq(\mu_*(\pi/2,0))^2=1$, for every $\tau^u\in K$. Thus $(\mu_*(\tau^u,\varepsilon))^2\geq 1+\mathcal{O}(\varepsilon)$ for $\tau^u\in K$ and $\varepsilon<\varepsilon_1$. Therefore, since $\mu_*(\pi/2,\varepsilon)=1+\mathcal{O}(\varepsilon)$, there must exist a curve $\tau^u_{\min}(\varepsilon)$, such that

$$(\mu_*(\tau^u,\varepsilon))^2\geq(\mu_*(\tau^u_{\min}(\varepsilon),\varepsilon))^2,$$

for $\tau^u \in K$, $\varepsilon < \varepsilon_1$, and $\mu_*(\tau^u_{\min}(\varepsilon), \varepsilon) = 1 + \mathcal{O}(\varepsilon)$.

Thus, defining

$$h_s(\varepsilon) = \frac{\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{2\Omega}(\mu_*(\tau^u_{\min}(\varepsilon), \varepsilon))^2 = \frac{\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{2\Omega}(1 + \mathcal{O}(\varepsilon)),$$

system (23) has one heteroclinic orbit between the periodic orbits Λ_h^- and Λ_h^+ in the energy level $0 < h \le h_0$ if, and only if $h \ge h_s(\varepsilon)$.

It only remains to prove the last statement of Theorem B. Given $\mu_1 > 1$, let $\tau_1^u = \arcsin(\mu_1^{-1}) \in [\alpha_0, \pi/2) \subset K$, and consider the function $g(\tau^u, \varepsilon) = \mu_*(\tau^u, \varepsilon) - \mu_*(\tau_1^u, \varepsilon)$. Applying the Implicit Function Theorem to $g = 0$ at the point $(\pi - \tau_1^u, 0)$, there exist $\varepsilon_{\mu_1} > 0$ and a unique curve $\tau_2^u = \tau_2^u(\tau_1^u, \varepsilon)$, defined for $0 \le \varepsilon < \varepsilon_{\mu_1}$, such that $\mu_*(\tau_2^u, \varepsilon) = \mu_*(\tau_1^u, \varepsilon)$ and $\tau_2^u(\tau_1^u, \varepsilon) = \pi - \tau_1^u + \mathcal{O}(\varepsilon)$. Moreover, taking ε_{μ_1} small enough $\tau_1^u \ne \tau_2^u$ for $\varepsilon < \varepsilon_{\tau_1^u}$. Thus, in the energy level

$$h_{\mu_1} = \frac{\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{2\Omega}(\mu_*(\tau_1^u, \varepsilon))^2,$$

where $\mu_*(\tau_1^u, \varepsilon) = \mu_1 + \mathcal{O}(\varepsilon)$, there exist two heteroclinic connections corresponding to τ_1^u and τ_2^u.

This completes the proof of Theorem B.

Remark 4.9. Notice that $g(\pi/2, 0) = \partial_{\tau^u} g(\pi/2, 0) = 0$ and $\partial^2_{\tau^u} g(\pi/2, 0) \ne 0$. Unfortunately, the characterization of the bifurcation of zeros for $\varepsilon > 0$ becomes impossible, since there is no information on $\partial_\varepsilon g(\pi/2, 0)$, and its computation requires complicated second order expansions which are beyond the objectives of this work. Nevertheless, under some non-degenericity condition, for example $\partial_\varepsilon g(\pi/2, 0) \ne 0$, it is possible to detect a saddle-node bifurcation.

4.4. Proof of Theorem C

Following the same lines of Section 4.3, we use Theorems 4.3 (for the invariant manifold $W^s_\varepsilon(\Lambda_h^+)$), 4.5 (for the invariant manifold $W^u_\varepsilon(p_h^-)$) and 4.8 (to compare them to $W^s_\varepsilon(p_0^+)$ and $W^u_\varepsilon(p_0^-)$). Then, we can see that $W^u_\delta(p_h^-) \subset W^s_\delta(\Lambda_h^+)$, if and only if

$$\begin{cases} -\sqrt{\dfrac{2h}{\omega}}\cos(\tau^s) + M_1(\varepsilon) + F_1(\tau^s, h, \varepsilon) = 0, \\ -\sqrt{\dfrac{2h}{\omega}}\sin(\tau^s) - \dfrac{2\pi\delta}{\sqrt{\Omega}} e^{-\sqrt{2}\omega} + M_2(\varepsilon) + F_2(\tau^s, h, \varepsilon) = 0, \end{cases} \tag{53}$$

has solutions $\tau^u, \tau^s \in [-\pi, \pi]$, $0 < \varepsilon \le \varepsilon_0$, $0 < h \le h_0$ where h_0 is given in Theorem 4.5. The functions M_j, F_j are real-analytic and satisfy

$$M_j = \mathcal{O}(\omega\delta^3 e^{-\sqrt{2}\omega}) \text{ and } F_j = \mathcal{O}\left(\frac{\delta\sqrt{h}}{\omega^{3/2}} + \frac{\delta\sqrt{h}}{\omega^2}\right), \; j = 1, 2.$$

In order to look for solutions of (53), we consider the change

$$h = \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}\mu^2, \quad 0 < \mu \leq \mu_0 = \frac{\sqrt{\Omega_0 h_0}}{\delta_0 \pi\sqrt{2\omega_0}e^{-\sqrt{2}\omega_0}}. \tag{54}$$

Considering $\varepsilon_0 > 0$ sufficiently small, we can assume that $\mu_0 > 1$. Replacing h in (53) and multiplying it by $\frac{\sqrt{\Omega}}{2\pi\delta e^{-\sqrt{2}\omega}} > 0$, we may rewrite this system as

$$\begin{aligned} -\mu\cos(\tau^s) + \widetilde{M}_1(\varepsilon) + \widetilde{F}_1(\tau^s, \mu, \varepsilon) = 0, \\ -\mu\sin(\tau^s) - 1 + \widetilde{M}_2(\varepsilon) + \widetilde{F}_2(\tau^s, \mu, \varepsilon) = 0, \end{aligned} \tag{55}$$

where $\widetilde{M}_j$, $\widetilde{F}_j$, are real-analytic functions such that

$$\widetilde{M}_j = \mathcal{O}(\omega\delta^2) \text{ and } \widetilde{F}_j = \mathcal{O}\left(\frac{\delta}{\omega}\mu\right), \; j = 1, 2.$$

Define the function $G : [-\pi, \pi] \times (0, \mu_0] \times [0, \varepsilon_0] \to \mathbb{R}^2$ as the left-hand side of system (55). Recalling that $\delta = \varepsilon^{3/4}$ and $\omega = \Omega/\sqrt{\varepsilon}$, we can see that

$$G(\tau^s, \mu, \varepsilon) = \begin{pmatrix} -\mu\cos(\tau^s) + \mathcal{O}(\varepsilon) \\ -\mu\sin(\tau^s) - 1 + \mathcal{O}(\varepsilon) \end{pmatrix}. \tag{56}$$

Since,

(1) $G(-\pi/2, 1, 0) = (0, 0)$,
(2) $\det\left(\frac{\partial(G_1, G_2)}{\partial(\tau^s, \mu)}\right)(-\pi/2, 1, 0) = 1$,

we can apply the Implicit Function Theorem to obtain $\varepsilon_* > 0$ and functions $\tau^s_* : [0, \varepsilon_*) \to [-\pi, \pi]$, $\mu_* : [0, \varepsilon_*) \to (0, \mu_0]$ such that $G(\tau^s_*(\varepsilon), \mu_*(\varepsilon), \varepsilon) = 0$ for $0 \leq \varepsilon \leq \varepsilon_*$. Furthermore, $\tau^s_*(\varepsilon) = -\pi/2 + \mathcal{O}(\varepsilon)$ and $\mu_*(\varepsilon) = 1 + \mathcal{O}(\varepsilon)$.

Defining

$$h_c(\varepsilon) = \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon))^2 = \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}(1 + \mathcal{O}(\varepsilon)),$$

and reducing ε_0 to ε_*, Theorem C follows directly from these facts.

4.5. *Proof of Theorem D*

Following the same lines of Section 4.3, we use Theorems 4.5 (for the invariant manifold $W^u_\varepsilon(p^-_h)$), 4.6 (for the invariant manifold $W^s_\varepsilon(\Lambda^+_{\kappa_1,\kappa_2})$), and 4.8 (to compare them to $W^s_\varepsilon(p^+_0)$ and $W^u_\varepsilon(p^-_0)$). We can see that $W^u_\delta(p^-_h) \subset W^s_\delta(\Lambda^+_{\kappa_1,\kappa_2})$, if and only if

$$
\begin{aligned}
-\sqrt{\frac{2\kappa_2}{\omega}}\cos(\tau^s)+M_1(\varepsilon)+F_1(\tau^s,h,\varepsilon)=0,\\
-\sqrt{\frac{2\kappa_2}{\omega}}\sin(\tau^s)-\frac{2\pi\delta}{\sqrt{\Omega}}e^{-\sqrt{2}\omega}+M_2(\varepsilon)+F_2(\tau^s,h,\varepsilon)=0,
\end{aligned}
\tag{57}
$$

has a solution $\tau^s\in[-\pi,\pi]$ for $0<\varepsilon\le\varepsilon_0$, $0<h\le h_0$. The functions M_j, F_j are real-analytic and

$$
M_j=\mathcal{O}(\omega\delta^3e^{-\sqrt{2}\omega})\text{ and }F_j=\mathcal{O}\left(\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}+\frac{\delta\sqrt{\kappa_1}}{\omega^2}+\frac{\delta\sqrt{h}}{\omega^2}\right),\ j=1,2,\ \kappa_1+\kappa_2=h.
$$

We consider the change of parameters and variables

$$
h=\frac{2\pi^2\omega\delta^2e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon)+\mu)^2, \tag{58}
$$

$$
\kappa_2=\frac{2\pi^2\omega\delta^2e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon)+\mu-\xi)^2, \tag{59}
$$

$$
\tau^s=\tau^s_*(\varepsilon)+\tau, \tag{60}
$$

where $(\mu_*(\varepsilon),\tau^s_*(\varepsilon))$ is the solution of (55). Since $\kappa_2\le h$, $\mu_*(\varepsilon)=1+\mathcal{O}(\varepsilon)$ and we are looking for solutions with $\mu,\xi,\tau\approx 0$, we have that $\xi\ge 0$ and $(\mu_*(\varepsilon)+\mu-\xi)^2\le(\mu_*(\varepsilon)+\mu)^2$.

Replacing h, κ_2 and κ_1 and multiplying it by $\dfrac{\sqrt{\Omega}}{2\pi\delta e^{-\sqrt{2}\omega}}>0$, system (57) as

$$
\begin{aligned}
-(\mu_*(\varepsilon)+\mu-\xi)\cos(\tau^s_*(\varepsilon)+\tau)+\widetilde{M}_1(\varepsilon)+\widetilde{F}_1(\tau,\mu,\xi,\varepsilon)=0,\\
-(\mu_*(\varepsilon)+\mu-\xi)\sin(\tau^s_*(\varepsilon)+\tau)-1+\widetilde{M}_2(\varepsilon)+\widetilde{F}_2(\tau,\mu,\xi,\varepsilon)=0,
\end{aligned}
\tag{61}
$$

where $\widetilde{M}_j$, $\widetilde{F}_j$, are real-analytic functions such that $\widetilde{M}_j=\mathcal{O}(\omega\delta^2)$ and

$$
\widetilde{F}_j=\mathcal{O}\left(\frac{\delta}{\omega}\left((\mu_*(\varepsilon)+\mu-\xi)+\frac{\sqrt{(\mu_*(\varepsilon)+\mu)^2+(\mu_*(\varepsilon)+\mu-\xi)^2}}{\omega^{1/2}}+\frac{(\mu_*(\varepsilon)+\mu)}{\omega^{1/2}}\right)\right),
$$

$j=1,2.$

Define the function $G:[-\chi_0,\chi_0]\times[0,\chi_0]\times[-\chi_0,\chi_0]\times[0,\chi_0]\to\mathbb{R}^2$ as the left hand side of system (55) and fix $\chi_0>0$ small enough. Recalling that $\delta=\varepsilon^{3/4}$ and $\omega=\Omega/\sqrt{\varepsilon}$, we can see that

$$
G(\tau,\mu,\xi,\varepsilon)=\begin{pmatrix}-(\mu_*(\varepsilon)+\mu-\xi)\cos(\tau^s_*(\varepsilon)+\tau)+\mathcal{O}(\varepsilon)\\-(\mu_*(\varepsilon)+\mu-\xi)\sin(\tau^s_*(\varepsilon)+\tau)-1+\mathcal{O}(\varepsilon)\end{pmatrix}.
$$

From Section 4.4, $\mu_*(0)=1$ and $\tau^s_*(0)=-\pi/2$. Thus $G(\tau,\mu,\xi,0)=(0,0)$ has a solution $\tau=0$ and $\mu=\xi$. Since, we are looking for solutions with $\mu,\xi\approx 0$, we consider the solution $\mu=\xi=0$. Then, since

(1) $G(0,0,0,0) = (0,0)$,
(2) $\det\left(\dfrac{\partial(G_1, G_2)}{\partial(\tau,\xi)}\right)(0,0,0,0) = 1$,

we can apply the Implicit Function Theorem to obtain $\varepsilon_0 > 0$ and unique functions $\overline{\tau} : [0,\varepsilon_0) \times [0,\varepsilon_0) \to [-\chi_0, \chi_0]$, $\overline{\xi} : [0,\varepsilon_0) \times [0,\varepsilon_0) \to [-\chi_0, \chi_0]$ such that $G(\overline{\tau}(\mu,\varepsilon), \mu, \overline{\xi}(\mu,\varepsilon), \varepsilon) = 0$. Furthermore $\overline{\tau}(\mu,\varepsilon) = \mathcal{O}(\mu,\varepsilon)$ and $\overline{\xi}(\mu,\varepsilon) = \mathcal{O}(\mu,\varepsilon)$. For $\varepsilon = 0$, we have that $\xi = \mu$ and $\tau = 0$ is a solution of $G(\tau,\mu,\xi,\varepsilon) = (0,0)$. Thus $\overline{\xi}(\mu,0) = \mu$ and, for ε small enough, $\overline{\xi}(\mu,\varepsilon) = \mu + \mathcal{O}(\varepsilon)$.

Finally, if $\xi = 0$, then $\kappa_2 = h$, $\kappa_1 = 0$ and therefore (57) becomes (53). Thus, considering the different scalings done in the systems and the uniqueness of solutions of (53) obtained in Section 4.4, we conclude that $\overline{\xi}(0,\varepsilon) = \overline{\tau}(0,\varepsilon) \equiv 0$.

These facts, allows us to see that

$$\overline{\xi}(\mu,\varepsilon) = c_\varepsilon \mu + \mathcal{O}(\mu^2), \quad \text{with } c_\varepsilon = 1 + \mathcal{O}(\varepsilon).$$

Hence, for $\mu \geq 0$ sufficiently small, in the energy level

$$h_\mu = \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon) + \mu)^2,$$

there exists a unique heteroclinic connection between $p^-_{h_\mu}$ and $\Lambda^-_{h_\mu}(\kappa_1^\mu, \kappa_2^\mu)$, where

$$\kappa_2^\mu = \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon) + \mu - \overline{\xi}(\mu,\varepsilon))^2,$$

and $\kappa_1^\mu = h_\mu - \kappa_2^\mu$. Moreover, if $-\mu_*(\varepsilon) < \mu < 0$ there is no heteroclinic connections in the energy level h_μ.

We set $v_i = \sqrt{h_\mu}$, $v_f = \sqrt{\kappa_1^\mu}$ and $v_c = \sqrt{h_c}$, where

$$h_c(\varepsilon) = \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon))^2.$$

In what follows we give an asymptotic formula to the output velocity v_f of orbits with incoming velocity $v_i \approx v_c$. We omit the dependence of v_i, v_f on μ in order to simplify the notation. For $\mu \geq 0$ sufficiently small, we have

$$\begin{aligned}
v_f^2 &= \kappa_1^\mu = h_\mu - \kappa_2^\mu \\
&= \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}\left((\mu_*(\varepsilon) + \mu)^2 - (\mu_*(\varepsilon) + \mu - \overline{\xi}(\mu,\varepsilon))^2\right) \\
&= \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}\left(\overline{\xi}(\mu,\varepsilon)(2(\mu_*(\varepsilon) + \mu) - \overline{\xi}(\mu,\varepsilon))\right) \\
&= \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}(c_\varepsilon\mu + \mathcal{O}(\mu^2))(2\mu_*(\varepsilon) + (2 - c_\varepsilon)\mu + \mathcal{O}(\mu^2))
\end{aligned}$$

$$= \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}(2\mu_*(\varepsilon)c_\varepsilon\mu + \mathcal{O}(\mu^2)).$$

Notice that

$$v_i - v_c = \sqrt{\frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}\mu}.$$

Thus

$$v_f^2 = 2v_c c_\varepsilon(v_i - v_c) + \mathcal{O}((v_i - v_c)^2).$$

Finally, we obtain that

$$v_f = \sqrt{2v_c c_\varepsilon}\sqrt{v_i - v_c} + \mathcal{O}((v_i - v_c)^{3/2}).$$

Theorem D follow directly from these facts.

5. Proof of Theorem 4.1

The strategy to prove the existence of $W^u_\varepsilon(p_0^-)$ and $W^s_\varepsilon(p_0^+)$ when $\delta \neq 0$ (see (23)), is to look for a parameterization $N^u_{0,0}(v)$ of $W^u_\varepsilon(p_0^\pm)$ as a perturbation of $N_{0,0}(v)$.

As in the unperturbed case $W(0,0)$ is parameterized as a graph over X (see (26)), we look for $N^u_{0,0}$ as

$$N^u_{0,0}(v) = (X_0(v), Z^u_0(v), \Gamma^u_0(v), \Theta^u_0(v)). \tag{62}$$

Next lemma, which is straightforward, gives the equation $N^u_{0,0}(v)$ has to satisfy to be invariant by the flow of (23).

Lemma 5.1. *The invariant manifold $W^u_\delta(p_0^-)$, with $\delta \neq 0$, is parameterized by $N^u_{0,0}(v)$ if and only if $(\Gamma^u_0(v), \Theta^u_0(v))$ satisfy*

$$\begin{aligned}
\frac{d\Gamma}{dv}(v) - \omega i\Gamma(v) &= -\frac{\delta}{\sqrt{2\Omega}}F(X_0(v)) + \left(\frac{Z_0(v)}{\widetilde{\eta}_0(v,\Gamma,\Theta)} - 1\right)\left(\omega i\Gamma(v) - \frac{\delta}{\sqrt{2\Omega}}F(X_0(v))\right),\\
\frac{d\Theta}{dv}(v) + \omega i\Theta(v) &= -\frac{\delta}{\sqrt{2\Omega}}F(X_0(v)) + \left(\frac{Z_0(v)}{\widetilde{\eta}_0(v,\Gamma,\Theta)} - 1\right)\left(-\omega i\Theta(v) - \frac{\delta}{\sqrt{2\Omega}}F(X_0(v))\right),\\
\lim_{v\to-\infty}\Gamma(v) &= \lim_{v\to-\infty}\Theta(v) = 0,
\end{aligned} \tag{63}$$

where

$$\widetilde{\eta}_0(v,\Gamma,\Theta) = 4\sqrt{-U(X_0(v)) - \frac{\delta}{\sqrt{2\Omega}}F(X_0(v))\frac{\Gamma(v)-\Theta(v)}{2i} - \frac{\omega}{2}\Gamma(v)\Theta(v)}, \tag{64}$$

with X_0 given in (26), U, F given in (8), and $Z^u_0(v) = \widetilde{\eta}_0(v, \Gamma^u_0(v), \Theta^u_0(v))$.

The term $\frac{\delta}{\sqrt{2\Omega}}F(X_0(v))$ decays as $1/v$ as $v \to \infty$. To have integrability, we consider the change of variables (36) to system (63). Then, (γ_0^u, θ_0^u) satisfy

$$\begin{aligned} &\frac{d}{dv}\gamma - \omega i\gamma = \omega i\gamma(\eta_0(v,\gamma,\theta)-1) - (Q^0)'(v), \\ &\frac{d}{dv}\theta + \omega i\theta = -\omega i\theta(\eta_0(v,\gamma,\theta)-1) + (Q^0)'(v), \\ &\lim_{v\to-\infty}\gamma(v) = \lim_{v\to-\infty}\theta(v) = 0, \end{aligned} \tag{65}$$

where Q^0 is given by (37) and

$$\eta_0(v,\gamma,\theta) = \left(1 + \frac{4\delta^2}{\Omega\omega}\left(\frac{F(X_0(v))}{Z_0(v)}\right)^2 - 8\omega\frac{\gamma\theta}{(Z_0(v))^2}\right)^{-1/2}. \tag{66}$$

To prove Theorem 4.1, it is sufficient to find a solution of (65).

Proposition 5.2. *Fix $\nu > 0$. There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, the equation (65) has a solution $(\gamma_0^u(v), \theta_0^u(v))$ defined in the domain $D_\varepsilon^u \subset \mathbb{C}$ (see (31)) such that $\theta_0^u(v) = \overline{\gamma_0^u(v)}$, for every $v \in D_\varepsilon^u \cap \mathbb{R}$. Furthermore, both γ_0^u, θ_0^u satisfy bounds (1) and (2) of Theorem 4.1.*

We look for a fixed point (γ_0^u, θ_0^u) of the operator

$$\mathcal{G}_{\omega,0} = \mathcal{G}_\omega \circ \mathcal{F}_0, \tag{67}$$

where

$$\mathcal{G}_\omega(\gamma,\theta)(v) = \begin{pmatrix} \displaystyle\int_{-\infty}^{v} e^{\omega i(v-r)}\gamma(r)dr \\ \displaystyle\int_{-\infty}^{v} e^{-\omega i(v-r)}\theta(r)dr \end{pmatrix}, \tag{68}$$

$$\mathcal{F}_0(\gamma,\theta)(v) = \begin{pmatrix} \omega i\gamma(v)(\eta_0(v,\gamma(v),\theta(v))-1) - (Q^0)'(v) \\ -\omega i\theta(v)(\eta_0(v,\gamma(v),\theta(v))-1) + (Q^0)'(v) \end{pmatrix}, \tag{69}$$

and Q^0, η_0 are given in (37) and (66), respectively.

5.1. Banach spaces and technical lemmas

In this section, we introduce a Banach space which will be used to find a fixed point of $\mathcal{G}_{\omega,0}$.

Consider the complex domain D_ε^u given in (31). For each analytic function $f : D_\varepsilon^u \to \mathbb{C}$, $\nu > 0$, $\alpha \geq 0$, we consider:

$$\|f\|_{\alpha,\nu} = \sup_{v\in D_\varepsilon^u\cap\{\mathrm{Re}(v)\leq-\nu\}} |v^2 f(v)| + \sup_{v\in D_\varepsilon^u\cap\{\mathrm{Re}(v)>-\nu\}} |(v^2+2)^\alpha f(v)|. \tag{70}$$

For any $\nu > 0$, and $\alpha > 0$ fixed, the function space

$$\mathcal{X}_{\alpha,\nu} = \{f : D^u_\varepsilon \to \mathbb{C};\ f \text{ is an analytic function such that, } \|f\|_{\alpha,\nu} < \infty\} \tag{71}$$

is a Banach space with respect to the norm $\|\cdot\|_{\alpha,\nu}$.

We also consider the product space

$$\mathcal{X}^2_{\alpha,\nu} = \left\{(f,g) \in \mathcal{X}_{\alpha,\nu} \times \mathcal{X}_{\alpha,\nu};\ g(v) = \overline{f(v)} \text{ for every } v \in D^u_\varepsilon \cap \mathbb{R}\right\},$$

endowed with the norm

$$\|(f,g)\|_{\alpha,\nu} = \|f\|_{\alpha,\nu} + \|g\|_{\alpha,\nu}.$$

Proposition 5.3. *Given $\nu > 0$, $\alpha > 0$ fixed, and $(f,g) \in \mathcal{X}^2_{\alpha,\nu}$, we have that $\mathcal{G}_\omega(f,g) \in \mathcal{X}^2_{\alpha,\nu}$. Furthermore, there exists a constant $M > 0$ independent of ε such that*

$$\|\mathcal{G}_\omega(f,g)\|_{\alpha,\nu} \leq \frac{M}{\omega}\,\|(f,g)\|_{\alpha,\nu},$$

for every $(f,g) \in \mathcal{X}^2_{\alpha,\nu}$.

The proof of Proposition 5.3 follows from [12].

Proposition 5.4. *Let η_0 be the function given in (66), and $\mathcal{F}_0$ given in (69). Given $\nu > 0$ and $K > 0$, there exist $\varepsilon_0 > 0$ and $M > 0$ such that:*

For $0 < \varepsilon \leq \varepsilon_0$ and $(\gamma_j, \theta_j) \in \mathcal{B}_0(R) \subset \mathcal{X}^2_{2,\nu}$ where $R = K\dfrac{\delta}{\omega^2}$ and $j = 1, 2$, the following statements hold for $v \in D^u_\varepsilon$.

(1) $\left|\eta_0(v, \gamma_j(v), \theta_j(v)) - 1\right| \leq M\delta^2$;
(2) $|\eta_0(v, \gamma_1(v), \theta_1(v)) - \eta_0(v, \gamma_2(v), \theta_2(v))| \leq M\delta\omega^2\|(\gamma_1,\theta_1) - (\gamma_2,\theta_2)\|_{2,\nu}$;
(3) $\mathcal{F}_0(\gamma_j, \theta_j) \in \mathcal{X}^2_{2,\nu}$;
(4) $\|\mathcal{F}_0(\gamma_1,\theta_1) - \mathcal{F}_0(\gamma_2,\theta_2)\|_{2,\nu} \leq M\delta^2\omega\|(\gamma_1,\theta_1) - (\gamma_2,\theta_2)\|_{2,\nu}$.

Proof. Replacing the expressions of F, X_0 and Z_0 given in (8) and (26) in (66), we obtain

$$\eta_0(v,\gamma,\theta) = \left(1 + \frac{\delta^2}{4\Omega\omega}\frac{v^2}{v^2+2} - (v^2+2)\omega\frac{\gamma\theta}{8}\right)^{-1/2}. \tag{72}$$

Taking $\gamma, \theta \in \mathcal{B}_0(R)$, the first statement of the proposition comes from the following inequalities

$$\left|\frac{\delta^2}{4\Omega\omega}\frac{v^2}{v^2+2} - (v^2+2)\omega\frac{\gamma\theta}{8}\right| \leq M\frac{\delta^2}{\omega}, \qquad \text{if } \mathrm{Re}(v) \leq -\nu,$$

$$\left|\frac{\delta^2}{4\Omega\omega}\frac{v^2}{v^2+2} - (v^2+2)\omega\frac{\gamma\theta}{8}\right| \leq M\delta^2, \qquad \text{if } \mathrm{Re}(v) \geq -\nu.$$

We observe that

$$|\eta_0(v,\gamma_1,\theta_1)-\eta_0(v,\gamma_2,\theta_2)|\leq M\omega|(v^2+2)\gamma_1(v)||\theta_1(v)-\theta_2(v)| \\ +M\omega|(v^2+2)\theta_2(v)||\gamma_1(v)-\gamma_2(v)| \tag{73}$$

Thus, if $\mathrm{Re}(v)\leq -\nu$, then

$$\left|(v^2+2)\gamma_1(v)(\theta_1(v)-\theta_2(v))\right|\leq R\left|\frac{v^2+2}{v^2}\right|\frac{\|\theta_1-\theta_2\|_{2,\nu}}{|v|^2}\leq M\frac{\delta}{\omega^2}\|\theta_1-\theta_2\|_{2,\nu}, \tag{74}$$

whereas, if $\mathrm{Re}(v)\geq -\nu$,

$$\left|(v^2+2)\gamma_1(v)(\theta_1(v)-\theta_2(v))\right|\leq M\frac{\delta}{\sqrt{\varepsilon}}\|\theta_1-\theta_2\|_{2,\nu}. \tag{75}$$

Recalling that $\omega=\Omega/\sqrt{\varepsilon}$ and joining (74) and (75), we obtain that estimate (75) holds in D^u_ε. The other term in (73) is bounded in an analogous way. Thus, statement (2) holds.

If $(\gamma_j,\theta_j)\in\mathcal{X}^2_{2,\nu}$, then $\eta_0(v,\gamma_j,\theta_j)\in\mathbb{R}$, for each $v\in D^u_\varepsilon\cap\mathbb{R}$, thus, it is clear that $\mathcal{F}_0(\gamma_j,\theta_j)\in\mathcal{X}^2_{2,\nu}$. Finally, for $v\in D^u_\varepsilon$,

$$\begin{aligned}|\pi_1\circ\mathcal{F}_0(\gamma_1,\theta_1)(v)-\pi_1\circ\mathcal{F}_0(\gamma_2,\theta_2)(v)|&=\omega\,|\gamma_1(v)(\eta_0(v,\gamma_1,\theta_1)-1)-\gamma_2(v)(\eta_0(v,\gamma_2,\theta_2)-1)|\\ &\leq M\delta^2\left(\frac{1}{\omega}+1\right)\omega|\gamma_1(v)-\gamma_2(v)|\\ &\quad+M\delta\omega^3\|(\gamma_1,\theta_1)-(\gamma_2,\theta_2)\|_{2,\nu}|\gamma_2(v)|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\pi_1\circ\mathcal{F}_0(\gamma_1,\theta_1)-\pi_1\circ\mathcal{F}_0(\gamma_2,\theta_2)\|_{2,\nu}&\leq M\delta^2\left(\frac{1}{\omega}+1\right)\omega\|\gamma_1-\gamma_2\|_{2,\nu}\\ &\quad+MR\delta\omega^3\|(\gamma_1,\theta_1)-(\gamma_2,\theta_2)\|_{2,\nu}\\ &\leq M\delta^2\omega\|(\gamma_1,\theta_1)-(\gamma_2,\theta_2)\|_{2,\nu}.\end{aligned}$$

We can prove the same bound for the second coordinate of $\mathcal{F}_0$ analogously. □

Proposition 5.5. *Consider the operator $\mathcal{G}_{\omega,0}=\mathcal{G}_\omega\circ\mathcal{F}_0$, where $\mathcal{G}_\omega$ and $\mathcal{F}_0$ are given in (68) and (69). Given $\nu>0$, there exists a constant $M>0$ independent of ε, such that*

$$\left\|\mathcal{G}_{\omega,0}(0,0)\right\|_{2,\nu}\leq M\frac{\delta}{\omega^2}.$$

Proof. Recall that $\mathcal{F}_0(0,0)=(-(Q^0)'(v),(Q^0)'(v))$, where Q^0 is given by (37). Thus $\pi_1\circ\mathcal{F}_0(0,0)(v)=\pi_2\circ\mathcal{F}_0(0,0)(v)$, for each $v\in D^u_\varepsilon\cap\mathbb{R}$ and

$$\|\mathcal{F}_0(0,0)\|_{2,\nu}=2\frac{\delta}{\omega\sqrt{2\Omega}}\|F(X_0)'\|_{2,\nu}.$$

A straightforward computation shows that

$$F(X_0(v))' = \frac{2\sqrt{2}(v^2-2)}{(v^2+2)^2}.$$

Then,

$$\begin{aligned} |v^2 F(X_0(v))'| &\le M, && \text{for } \mathrm{Re}(v) \le -\nu, \\ |(v^2+2)^2 F(X_0(v))'| &\le M|v^2+2| \le M, && \text{for } \mathrm{Re}(v) \ge -\nu. \end{aligned}$$

The result follows directly from these bounds and Proposition 5.3. □

5.2. *The fixed point argument*

Finally, we prove the existence of a fixed point of $\mathcal{G}_{\omega,0}$.

Proposition 5.6. *Given $\nu > 0$ fixed. There exists $\varepsilon_0 > 0$ such that for $\varepsilon \le \varepsilon_0$, the operator $\mathcal{G}_{\omega,0}$ has a fixed point (γ_0^u, θ_0^u) in $\mathcal{X}_{2,\nu}^2$. Furthermore, there exists a constant $M > 0$ independent of ε such that*

$$\|(\gamma_0^u, \theta_0^u)\|_{2,\nu} \le M\frac{\delta}{\omega^2}.$$

Proof. From Proposition 5.5, there exists a constant $b_1 > 0$ independent of h and ε such that

$$\left\|\mathcal{G}_{\omega,0}(0,0)\right\|_{2,\nu} \le \frac{b_1}{2}\frac{\delta}{\omega^2}.$$

Given $(\gamma_1, \theta_1), (\gamma_2, \theta_2) \in \mathcal{B}_0(b_1\delta/\omega^2) \subset \mathcal{X}_{2,\nu}^2$, we can use Propositions 5.3, 5.4 (with $K = b_1$) and the linearity of the operator $\mathcal{G}_\omega$ to see that

$$\begin{aligned} \left\|\mathcal{G}_{\omega,0}(\gamma_1,\theta_1) - \mathcal{G}_{\omega,0}(\gamma_2,\theta_2)\right\|_{2,\nu} &\le \frac{M}{\omega}\|\mathcal{F}_0(\gamma_1,\theta_1) - \mathcal{F}_0(\gamma_2,\theta_2)\|_{2,\nu} \\ &\le M\delta^2\|(\gamma_1,\theta_1) - (\gamma_2,\theta_2)\|_{2,\nu}. \end{aligned}$$

Thus, choosing ε_0 sufficiently small, we have that $\mathrm{Lip}(\mathcal{G}_{\omega,0}) \le 1/2$. Also, it follows that $\overline{\pi_1 \circ \mathcal{G}_{\omega,0}(\gamma,\theta)(v)} = \pi_2 \circ \mathcal{G}_{\omega,0}(\gamma,\theta)(v)$, for each $v \in D_\varepsilon^u \cap \mathbb{R}$ and $(\gamma,\theta) \in \mathcal{B}_0(b_1\delta/\omega^2)$.

Therefore $\mathcal{G}_{\omega,0}$ sends the ball $\mathcal{B}_0(b_1\delta/\omega^2)$ into itself and it is a contraction. Thus, it has a unique fixed point $(\gamma_0^u, \theta_0^u) \in \mathcal{B}_0(b_1\delta/\omega^2)$. □

Proposition 5.2 is a consequence of Proposition 5.6.

6. Proof of Theorem 4.7

6.1. *The difference map*

In Proposition 5.6, we have found complex functions $\Gamma_0^\star = Q^0 + \gamma_0^\star$ and $\Theta_0^\star = -Q^0 + \theta_0^\star$ defined in the complex domains $D_\varepsilon^\star$, respectively, such that,

$$N_{0,0}^\star(v) = (X_0(v), Z_0^\star(v), \Gamma_0^\star(v), \Theta_0^\star(v)),$$

are parameterizations of $W^{\star}_{\delta}(p_0^{\mp})$ of (23). Both (Γ_0^u, Θ_0^u) and (Γ_0^s, Θ_0^s) are defined in the complex domain

$$\mathcal{D}_\varepsilon = D^u_\varepsilon \cap D^s_\varepsilon. \tag{76}$$

Note that $0 \in \mathcal{I}_\varepsilon := \mathcal{D}_\varepsilon \cap \mathbb{R}$. To prove that the heteroclinic connection between p_0^- and p_0^+ of (23) is broken for $\varepsilon > 0$ sufficiently small, it is sufficient to show that

$$\left|N^u_{0,0}(v) - N^s_{0,0}(v)\right| \geq \left|(\Gamma_0^u, \Theta_0^u)(v) - (\Gamma_0^s, \Theta_0^s)(v)\right| > 0, \tag{77}$$

for some $v \in \mathcal{I}_\varepsilon$. To this end, we study the difference map

$$\Delta\xi(v) = \begin{pmatrix} \Gamma_0^u(v) - \Gamma_0^s(v) \\ \Theta_0^u(v) - \Theta_0^s(v) \end{pmatrix} = \begin{pmatrix} \gamma_0^u(v) - \gamma_0^s(v) \\ \theta_0^u(v) - \theta_0^s(v) \end{pmatrix}, \tag{78}$$

where $(\gamma_0^\star, \theta_0^\star)$, $\star = u, s$, are given by Proposition 5.6.

Proposition 6.1. *The difference map $\Delta\xi$ satisfies the differential equation:*

$$\Delta\xi' = A\Delta\xi + B(v)\Delta\xi, \tag{79}$$

where

$$A = \begin{pmatrix} \omega i & 0 \\ 0 & -\omega i \end{pmatrix} \text{ and } B(v) = \begin{pmatrix} b_{1,1}(v) & b_{1,2}(v) \\ b_{2,1}(v) & b_{2,2}(v) \end{pmatrix}, \tag{80}$$

and there exists a constant M independent of ε, such that for $v \in \mathcal{D}_\varepsilon$,

$$|b_{j,k}(v)| \leq M\omega\delta^2, \quad j, k = 1, 2. \tag{81}$$

Proof. Recall that both $(\gamma_0^{u,s}, \theta_0^{u,s})$ satisfy (65) and therefore

$$\begin{pmatrix} \gamma' - \omega i \gamma \\ \theta' + \omega i \theta \end{pmatrix} = \mathcal{F}_0(\gamma, \theta),$$

where $\mathcal{F}_0$ is given in (69). Therefore $\Delta\xi$ satisfies

$$\Delta\xi' = A\Delta\xi + G(v), \tag{82}$$

where $G(v) = g(v, \gamma_0^u(v), \theta_0^u(v)) - g(v, \gamma_0^s(v), \theta_0^s(v))$, with

$$g(v, z_1, z_2) = \begin{pmatrix} i\omega z_1(\eta_0(v, z_1, z_2) - 1) \\ -i\omega z_2(\eta_0(v, z_1, z_2) - 1) \end{pmatrix}.$$

Notice that $G(v)$ is a known function, since $(\gamma_0^{u,s}, \theta_0^{u,s})$ are given by Proposition 5.6. We apply the Integral Mean Value Theorem to obtain

$$g(v,\gamma_0^u,\theta_0^u)-g(v,\gamma_0^s,\theta_0^s)=\begin{pmatrix} b_{1,1}(v) & b_{1,2}(v)\\ b_{2,1}(v) & b_{2,2}(v)\end{pmatrix}\cdot\begin{pmatrix}\gamma_0^u-\gamma_0^s\\ \theta_0^u-\theta_0^s\end{pmatrix}, \tag{83}$$

where $b_{j,k}$ are analytic functions, $j,k=1,2$. Estimate (81) follows from Propositions 5.4 and 5.6. □

6.2. Exponentially small splitting of $W_\varepsilon^u(p_0^-)$ and $W_\varepsilon^s(p_0^+)$

We study the solutions of (79). Notice that, if $B=0$, then any analytic solution of (79) which is bounded in $\mathcal{D}_\varepsilon$ is exponentially small with respect to ε for real values $v\in\mathcal{I}_\varepsilon$. In this section, we follow ideas from [3] to prove that the same holds for solutions of the full equation (79) using that B (given in (80)) is small for ε small enough.

We are interested in obtaining an asymptotic expression for $\Delta\xi$ given in (78). From Proposition 5.6, we have that $(\gamma_0^{u,s},\theta_0^{u,s})$ is obtained as a fixed point of $\mathcal{G}_{\omega,0}^{u,s}$. Thus, the difference map can be expressed as

$$\Delta\xi=\mathcal{G}_{\omega,0}^u(\gamma_0^u,\theta_0^u)-\mathcal{G}_{\omega,0}^s(\gamma_0^s,\theta_0^s).$$

Therefore, as $\gamma_0^{u,s},\theta_0^{u,s}$ are small, it suggests that the dominant part of $\Delta\xi$ should be given by $\mathcal{M}=\mathcal{G}_{\omega,0}^u(0,0)-\mathcal{G}_{\omega,0}^s(0,0)$. For this reason, we decompose

$$\Delta\xi=\mathcal{M}+\Delta\xi_1, \tag{84}$$

where $\mathcal{M}=(\mathcal{M}_\Gamma,\mathcal{M}_\Theta)$ is given by the Melnikov integrals

$$\begin{aligned}\mathcal{M}_\Gamma(v)&=ie^{i\omega v}\int_{-\infty}^{\infty}e^{-i\omega r}\frac{2\delta(r^2-2)}{\omega\sqrt{\Omega}(r^2+2)^2}dr=c_1^0e^{i\omega v},\\ \mathcal{M}_\Theta(v)&=-ie^{-i\omega v}\int_{-\infty}^{\infty}e^{i\omega r}\frac{2\delta(r^2-2)}{\omega\sqrt{\Omega}(r^2+2)^2}dr=c_2^0e^{-i\omega v},\end{aligned} \tag{85}$$

and $\Delta\xi_1=(\Delta_\Gamma^1,\Delta_\Theta^1)$.

A straightforward computation proves the following lemma.

Lemma 6.2. *The constants c_1^0 and c_2^0 are given by*

$$c_1^0=-i\frac{2\pi\delta}{\sqrt{\Omega}}e^{-\sqrt{2}\omega},\ \text{and } c_2^0=\overline{c_1^0}. \tag{86}$$

Theorem 4.7 is equivalent to the following theorem. The remainder of Section 6.2 is devoted to prove it.

Theorem 6.3. *There exists $\varepsilon_0>0$ sufficiently small such that for $v\in\mathcal{I}_\varepsilon\subset\mathbb{R}$, $0<\varepsilon\le\varepsilon_0$,*

$$\Delta\xi(v)=\mathcal{M}(v)+\mathcal{O}\left(\omega\delta^3e^{-\sqrt{2}\omega}\right), \tag{87}$$

where $\mathcal{M} = (\mathcal{M}_\Gamma, \mathcal{M}_\Theta)$ is the Melnikov vector defined in (85).

6.2.1. A fixed point argument for the error $\Delta\xi_1$

We write $\Delta\xi_1$ in (84) as solution of a fixed point equation in the functional space

$$\mathcal{E} = \left\{ f : \mathcal{D}_\varepsilon \to \mathbb{C}^2;\ f \text{ is analytic and } \|f\|_{\mathcal{E}} < \infty \right\}, \tag{88}$$

where

$$\|f\|_{\mathcal{E}} = \sum_{j=1}^{2} \sup_{v \in \mathcal{D}_\varepsilon} |(v^2+2)^2 \pi_j \circ f(v)|. \tag{89}$$

We also consider the linear operator $\mathcal{H}_0$ given by

$$\mathcal{H}_0(g)(v) = \begin{pmatrix} e^{\omega i v} \displaystyle\int_{v^*}^{v} e^{-i\omega r} \pi_1(B(r)\cdot g(r))dr \\ e^{-\omega i v} \displaystyle\int_{\overline{v*}}^{v} e^{i\omega r} \pi_2(B(r)\cdot g(r))dr \end{pmatrix}, \tag{90}$$

where $v^* = -(\sqrt{2} - \sqrt{\varepsilon})i$ and B is the matrix given (80).

Using (81), the operator $\mathcal{H}_0$ is well-defined from $\mathcal{E}_\varepsilon$ to itself. To simplify the notation, we introduce the function

$$I(k_1, k_2)(v) = e^{Av} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} e^{i\omega v} k_1 \\ e^{-i\omega v} k_2 \end{pmatrix}, \tag{91}$$

where $k_j \in \mathbb{C},\ j = 1, 2,\ v \in \mathcal{D}_\varepsilon$ and A is the matrix given by (80). Notice that $\mathcal{M}(v) = I(c_1^0, c_2^0)(v)$.

Lemma 6.4. *The difference map $\Delta\xi$ belongs to $\mathcal{E}_\varepsilon$ and $\|\Delta\xi\|_{\mathcal{E}} \leq M\varepsilon$. Furthermore, there exist $(c_1, c_2) \in \mathbb{C}^2$ such that:*

$$\Delta\xi_1(v) = I(c_1 - c_1^0, c_2 - c_2^0)(v) + \mathcal{H}_0(\Delta\xi_1)(v) + \mathcal{H}_0(\mathcal{M})(v), \tag{92}$$

and $|c_j - c_j^0| \leq M\delta^3 e^{-\sqrt{2}\omega}$, $j = 1, 2$, where M is a constant independent of ε.

Proof. Since $(\gamma_0^{u,s}, \theta_0^{u,s}) \in \mathcal{X}_{2,\nu}^2$, it is clear to see that $\Delta\xi \in \mathcal{E}_\varepsilon$. In addition, from Proposition 5.6,

$$\|\Delta\xi\|_{\mathcal{E}} \leq 2(\|(\gamma_0^u, \theta_0^u)\|_{2,\nu} + \|(\gamma_0^s, \theta_0^s)\|_{2,\nu}) \leq M\frac{\delta}{\omega^2},$$

where M is a constant independent of ε.

Since $\Delta\xi$ is a solution of (79), the method of variation of parameters implies that, given $v_1, v_2 \in \mathcal{D}_\varepsilon$, there exist $c_1, c_2 \in \mathbb{C}$ such that

$$\Delta\xi(v)=\begin{pmatrix} e^{\omega i v}c_1+e^{\omega i v}\displaystyle\int_{v_1}^{v} e^{-i\omega r}\pi_1(B(r)\cdot\Delta\xi(r))dr \\ e^{-\omega i v}c_2+e^{-\omega i v}\displaystyle\int_{v_2}^{v} e^{i\omega r}\pi_2(B(r)\cdot\Delta\xi(r))dr \end{pmatrix}. \tag{93}$$

We take $v_1=v^*$, $v_2=\overline{v^*}$, with $v^*=-(\sqrt{2}-\sqrt{\varepsilon})i$. Thus,

$$\Delta\xi(v)=I(c_1,c_2)(v)+\mathcal{H}_0(\Delta\xi)(v).$$

Using that $\Delta\xi=\mathcal{M}+\Delta\xi_1$, $\mathcal{M}(v)=I(c_1^0,c_2^0)(v)$ and $\mathcal{H}_0$ is linear,

$$\Delta\xi_1(v)=I(c_1-c_1^0,c_2-c_2^0)(v)+\mathcal{H}_0(\Delta\xi_1)(v)+\mathcal{H}_0(\mathcal{M})(v).$$

Now, we bound $|c_j-c_j^0|$, $j=1,2$. By (84) and Proposition 5.6,

$$\begin{aligned}\|\Delta\xi_1\|_{2,\nu}&=\|\Delta\xi-\mathcal{M}\|_{2,\nu}\\ &=\|(\gamma_0^u,\theta_0^u)-(\gamma_0^s,\theta_0^s)-(\mathcal{G}^u_{\omega,0}(0,0)-\mathcal{G}^s_{\omega,0}(0,0))\|_{2,\nu}\\ &=\|\mathcal{G}^u_{\omega,0}(\gamma_0^u,\theta_0^u)-\mathcal{G}^u_{\omega,0}(0,0)-(\mathcal{G}^s_{\omega,0}(\gamma_0^s,\theta_0^s)-\mathcal{G}^s_{\omega,0}(0,0))\|_{2,\nu}\\ &\le M\delta^2(\|(\gamma_0^u,\theta_0^u)\|_{2,\nu}+\|(\gamma_0^s,\theta_0^s)\|_{2,\nu})\\ &\le M\frac{\delta^3}{\omega^2}.\end{aligned}$$

Thus,

$$|\pi_j(\Delta\xi_1(v))|\le M\frac{\delta^3}{\omega^2|v^2+2|^2}\le M\delta^3,\ \text{for each } v\in\mathcal{D}_\varepsilon,\ \ j=1,2.$$

In particular, replacing $v=v^*$ in the first component of (92), we obtain that

$$|e^{\omega i v^*}(c_1-c_1^0)|\le M\delta^3 \Leftrightarrow |c_1-c_1^0|\le M\delta^3e^{\omega\sqrt{\varepsilon}}e^{-\sqrt{2}\omega}\le 2M\delta^3e^{-\sqrt{2}\omega}.$$

Analogously, taking $v=\overline{v^*}$ in the second component of (92), we obtain that $|c_2-c_2^0|\le 2M\delta^3e^{-\sqrt{2}\omega}$. □

6.2.2. Exponential smallness of $\Delta\xi_1$

Consider the function space

$$\mathcal{Z}=\{f:\mathcal{D}_\varepsilon\to\mathbb{C}^2;\ f \text{ is analytic and } \|f\|_{\mathcal{Z}}<+\infty\}, \tag{94}$$

where

$$\|f\|_{\mathcal{Z}}=\sum_{j=1}^{2}\sup_{v\in\mathcal{D}_\varepsilon}\left|e^{\omega(\sqrt{2}-|\mathrm{Im}(v)|)}\pi_j\circ f(v)\right|. \tag{95}$$

In order to prove Theorem 6.3, it is enough to check that $\Delta\xi_1$ belongs to $\mathcal{Z}$ and that $\|\Delta\xi_1\|_{\mathcal{Z}} \leq M\omega\delta^3$. Our strategy to achieve these results is to prove that both $I(c_1 - c_1^0, c_2 - c_2^0)$ and $\mathcal{H}_0(\mathcal{M})$ belong to $\mathcal{Z}$ and that the operator $\mathrm{Id} - \mathcal{H}_0$ is invertible in $\mathcal{Z}$.

Lemma 6.5. *There exists $\varepsilon_0 > 0$, such that the linear operator $\mathrm{Id} - \mathcal{H}_0$ is invertible in $\mathcal{Z}$ for $\varepsilon \leq \varepsilon_0$. Furthermore, there exists $M > 0$ independent of ε such that $\|\mathcal{H}_0\|_{\mathcal{Z}} \leq M\omega\delta^2$ and hence*

$$\|(\mathrm{Id} - \mathcal{H}_0)^{-1}\|_{\mathcal{Z}} \leq (1 - \|\mathcal{H}_0\|_{\mathcal{Z}})^{-1} \leq 1 + M\omega\delta^2. \tag{96}$$

Proof. Since $\mathcal{H}_0$ is a linear operator, to prove this lemma, it is sufficient to show that $\|\mathcal{H}_0\|_{\mathcal{Z}} \leq M\omega\delta^2 < 1$.

Let $h \in \mathcal{Z}$ and denote by M any constant independent of ε. Using (81) and (95), we have that for $v \in \mathcal{D}_\varepsilon$ and $j = 1, 2$,

$$|\pi_j(B(v)\cdot h(v))| \leq \sum_{k=1}^{2} |b_{j,k}(v)\pi_k(h(v))| \leq M\omega\delta^2 e^{-\omega(\sqrt{2}-|\mathrm{Im}(v)|)}\|h\|_{\mathcal{Z}}.$$

Thus

$$\begin{aligned}
|e^{\omega(\sqrt{2}-|\mathrm{Im}(v)|)}\pi_1(\mathcal{H}_0(h)(v))| &= \left| e^{\sqrt{2}\omega} \int_{v^*}^{v} e^{-i\omega(r-v-i|\mathrm{Im}(v)|)}\pi_1(B(r)\cdot h(r))dr \right| \\
&\leq M\omega\delta^2 e^{-\sqrt{2}\omega} e^{\sqrt{2}\omega}\|h\|_{\mathcal{Z}} \int_{v^*}^{v} \left| e^{-i\omega(r-v-i|\mathrm{Im}(v)|)} \right| e^{\omega|\mathrm{Im}(r)|}dr \\
&\leq M\omega\delta^2\|h\|_{\mathcal{Z}} \int_{v^*}^{v} e^{\omega(\mathrm{Im}(r)+|\mathrm{Im}(r)|-\mathrm{Im}(v)-|\mathrm{Im}(v)|)}dr.
\end{aligned}$$

Since $\mathrm{Im}(v^*) \leq \mathrm{Im}(r) \leq \mathrm{Im}(v)$, we have that $\mathrm{Im}(r) + |\mathrm{Im}(r)| - \mathrm{Im}(v) - |\mathrm{Im}(v)| \leq 0$, then

$$\left| \int_{v^*}^{v} e^{\omega(\mathrm{Im}(r)+|\mathrm{Im}(r)|-\mathrm{Im}(v)-|\mathrm{Im}(v)|)}dr \right| \leq M.$$

Analogously, we have that

$$|e^{\omega(\sqrt{2}-|\mathrm{Im}(v)|)}\pi_2(\mathcal{H}_0(h)(v))| \leq M\omega\delta^2\|h\|_{\mathcal{Z}},$$

and thus $\|\mathcal{H}_0(h)\|_{\mathcal{Z}} \leq M\omega\delta^2\|h\|_{\mathcal{Z}}$. Since, $\|\mathcal{H}_0\|_{\mathcal{Z}} < 1$, for ε sufficiently small, the linear operator $\mathrm{Id} - \mathcal{H}_0$ is invertible and satisfies (96). □

Now, recall that $\mathcal{M} = I(c_1^0, c_2^0)$, where I is given by (91) and c_1^0, c_2^0 are given by (86). Moreover, from Lemma 6.4, we have that

$$(\mathrm{Id}-\mathcal{H}_0)\Delta\xi_1 = I(c_1-c_1^0, c_2-c_2^0)+\mathcal{H}_0(I(c_1^0,c_2^0)). \tag{97}$$

Since $\mathrm{Id}-\mathcal{H}_0$ is invertible in $\mathcal{Z}$, it only remains to show that $I(c_1-c_1^0, c_2-c_2^0)$ and $I(c_1^0,c_2^0)$ belong to $\mathcal{Z}$.

Lemma 6.6. *Given $k_1, k_2 \in \mathbb{C}$, then the function I given in (91) satisfies*

$$\|I(k_1,k_2)\|_{\mathcal{Z}} \le Me^{\sqrt{2}\omega}(|k_1|+|k_2|), \tag{98}$$

where M is a constant independent of ε.

To prove Lemma 6.6 it is enough to recall the definitions of $\|\cdot\|_{\mathcal{Z}}$ in (95) and I in (91).

Lemma 6.7. *The error vector $\Delta\xi_1$ given in (84) belongs to $\mathcal{Z}$ and it is determined by*

$$\Delta\xi_1 = (\mathrm{Id}-\mathcal{H}_0)^{-1}\left(I(c_1-c_1^0, c_2-c_2^0)\right)+(\mathrm{Id}-\mathcal{H}_0)^{-1}\left(\mathcal{H}_0(\mathcal{M})\right). \tag{99}$$

Furthermore, there exists a constant $M>0$ independent of ε such that

$$\|\Delta\xi_1\|_{\mathcal{Z}} \le M\omega\delta^3. \tag{100}$$

Proof. From Lemmas 6.4 and 6.2, we have that $|c_j-c_j^0|\le M\delta^3e^{-\sqrt{2}\omega}$, and $|c_j^0|\le M\delta e^{-\sqrt{2}\omega}$, $j=1,2$. Therefore, it follows from Lemma 6.6 that $I(c_1-c_1^0, c_2-c_2^0)\in\mathcal{Z}$, and $\mathcal{M} = I(c_1^0,c_2^0)\in\mathcal{Z}$. Furthermore

$$\|I(c_1-c_1^0, c_2-c_2^0)\|_{\mathcal{Z}} \le M\delta^3 \text{ and } \|\mathcal{M}\|_{\mathcal{Z}}\le M\delta.$$

As $\mathrm{Id}-\mathcal{H}_0$ is invertible in $\mathcal{Z}$ by Lemma 6.5, formula (99) is equivalent to (92). Therefore, $\Delta\xi_1\in\mathcal{Z}$ and, using again Lemma 6.5,

$$\begin{aligned}\|\Delta\xi_1\|_{\mathcal{Z}} &\le \|(\mathrm{Id}-\mathcal{H}_0)^{-1}\|_{\mathcal{Z}}\left(\|I(c_1-c_1^0, c_2-c_2^0)\|_{\mathcal{Z}}+\|\mathcal{H}_0(\mathcal{M})\|_{\mathcal{Z}}\right)\\ &\le M\delta^3+M\|\mathcal{H}_0\|_{\mathcal{Z}}\|\mathcal{M}\|_{\mathcal{Z}}\\ &\le M\omega\delta^3. \quad\square\end{aligned}$$

Proof of Theorem 6.3. Finally, we prove that $\Delta\xi_1$ is exponentially small and we obtain an asymptotic formula for $\Delta\xi$. From (100) and the definition of the norm (95), we have

$$|e^{\omega(\sqrt{2}-|\mathrm{Im}(v)|)}\pi_j\circ\Delta\xi_1(v)|\le M\omega\delta^3, \text{ for } v\in\mathcal{D}_\varepsilon, \text{ and } j=1,2. \tag{101}$$

In particular, if $v\in\mathcal{I}_\varepsilon=\mathcal{D}_\varepsilon\cap\mathbb{R}$, $|\Delta\xi_1(v)|\le M\omega\delta^3e^{-\sqrt{2}\omega}$, for $j=1,2$. The result follows directly from this bound and (84). $\square$

7. Proof of Theorem 4.3

In this section we look for parameterizations of the invariant manifolds $W^u_\varepsilon(\Lambda_h^-)$ of the periodic orbits Λ_h^- of the form

$$N^u_{0,h}(v,\tau) = (X_0(v), Z_0(v) + Z^u_{0,h}(v,\tau), \Gamma_h(\tau) + \Gamma^u_{0,h}(v,\tau), \Theta_h(\tau) + \Theta^u_{0,h}(v,\tau)), \tag{102}$$

where Z_0, Γ_h, Θ_h are given in (26) and (29), as a perturbation of $N_{0,h}(v,\tau)$ (see (28)).

Lemma 7.1. *The invariant manifold $W^u_\delta(\Lambda_h^-)$, with $\delta \neq 0$, can be parameterized by $N^u_{0,h}(v,\tau)$ in (102) if $(Z^u_{0,h}(v,\tau), \Gamma^u_{0,h}(v,\tau), \Theta^u_{0,h}(v,\tau))$ satisfy the following system of partial differential equations*

$$\begin{cases} \partial_v Z + \omega\partial_\tau Z + \dfrac{Z_0'(v)}{Z_0(v)} Z = -\dfrac{Z}{Z_0(v)}\partial_v Z - \dfrac{\delta}{\sqrt{2\Omega}} F'(X_0(v))\dfrac{\Gamma - \Theta}{2i} \\ \qquad - \dfrac{\delta}{\sqrt{2\Omega}} F'(X_0(v))\dfrac{\Gamma_h(\tau) - \Theta_h(\tau)}{2i}, \\ \partial_v \Gamma + \omega\partial_\tau \Gamma = -\dfrac{Z}{Z_0(v)}\partial_v\Gamma + \omega i\Gamma - \dfrac{\delta}{\sqrt{2\Omega}} F(X_0(v)), \\ \partial_v \Theta + \omega\partial_\tau \Theta = -\dfrac{Z}{Z_0(v)}\partial_v\Theta - \omega i\Theta - \dfrac{\delta}{\sqrt{2\Omega}} F(X_0(v)), \\ \lim\limits_{v\to-\infty} Z(v,\tau) = \lim\limits_{v\to-\infty}\Gamma(v,\tau) = \lim\limits_{v\to-\infty}\Theta(v,\tau) = 0, \text{ for each } \tau \in [0,2\pi], \end{cases} \tag{103}$$

and $Z^u_{0,h}, \Gamma^u_{0,h}, \Theta^u_{0,h}$ are 2π-periodic in the variable τ.

In contrast to the 1-dimensional case, for technical reasons, we do not use that $\mathcal{H}(W^u_\varepsilon(\Lambda_h^-)) = h$ to obtain $Z = Z(X, \Gamma, \Theta)$. Thus, we deal with the problem in dimension 3.

As in the 1-dimensional case (63), if we set $Z = \Gamma = \Theta = 0$, the right-hand side of (103) decays as $1/|v|$ as $v \to -\infty$. To have quadratic decay as $|v| \to \infty$ to have integrability, we perform with the change (39) to system (103). Then, $(z^u_{0,h}, \gamma^u_{0,h}, \theta^u_{0,h})$ satisfy

$$\begin{aligned} \partial_v z + \omega\partial_\tau z + \frac{Z_0'(v)}{Z_0(v)} z &= f_1^h(v,\tau) - \frac{z + Z_{0,h}(v,\tau)}{Z_0(v)}\partial_v z - \frac{\partial_v Z_{0,h}(v,\tau)}{Z_0(v)} z \\ &\quad - \frac{\delta}{\sqrt{2\Omega}} F'(X_0(v))\frac{\gamma - \theta}{2i}, \\ \partial_v\gamma + \omega\partial_\tau\gamma - \omega i\gamma &= f_2^h(v,\tau) - \frac{(Q^0)'(v)}{Z_0(v)} z - \frac{z + Z_{0,h}(v,\tau)}{Z_0(v)}\partial_v\gamma, \\ \partial_v\theta + \omega\partial_\tau\theta + \omega i\theta &= -f_2^h(v,\tau) + \frac{(Q^0)'(v)}{Z_0(v)} z - \frac{z + Z_{0,h}(v,\tau)}{Z_0(v)}\partial_v\theta, \\ \lim_{v\to-\infty} z(v,\tau) &= \lim_{v\to-\infty}\gamma(v,\tau) = \lim_{v\to-\infty}\theta(v,\tau) = 0, \end{aligned} \tag{104}$$

where

$$\begin{aligned} f_1^h(v,\tau) =& -\partial_v Z_{0,h}(v,\tau) - \frac{Z_0'(v)}{Z_0(v)} Z_{0,h}(v,\tau) \\ & - \frac{\delta}{\sqrt{2\Omega}} F'(X_0(v)) \frac{Q^0(v)}{i} - \frac{Z_{0,h}(v,\tau)\partial_v Z_{0,h}(v,\tau)}{Z_0(v)}, \\ f_2^h(v,\tau) =& -(Q^0)'(v) - \frac{Z_{0,h}(v,\tau)(Q^0)'(v)}{Z_0(v)}, \end{aligned} \tag{105}$$

and Q^0, $Z_{0,h}$ are given by (37), (40), respectively.

We consider equation (104) with $(v,\tau) \in D^u \times \mathbb{T}_\sigma$ (see (33) and (34)), and asymptotic conditions $\lim_{\mathrm{Re}(v)\to-\infty} z(v,\tau) = \lim_{\mathrm{Re}(v)\to-\infty} \gamma(v,\tau) = \lim_{\mathrm{Re}(v)\to-\infty} \theta(v,\tau) = 0$, for every $\tau \in \mathbb{T}_\sigma$.

Proposition 7.2. *Fix $\sigma > 0$ and $h_0 > 0$. There exists $\varepsilon_0 > 0$ sufficiently small such that for $0 < \varepsilon \le \varepsilon_0$ and $0 \le h \le h_0$, equation (104) has a solution $(z_{0,h}^u, \gamma_{0,h}^u, \theta_{0,h}^u)$ defined in $D^u \times \mathbb{T}_\sigma$ such that $z_{0,h}^u$ is real-analytic, $\gamma_{0,h}^u, \theta_{0,h}^u$ are analytic, $\theta_{0,h}^u(v,\tau) = \overline{\gamma_{0,h}^u(v,\tau)}$ for each $(v,\tau) \in \mathbb{R}^2$, and*

$$\lim_{\mathrm{Re}(v)\to-\infty} z_{0,h}^u(v,\tau) = \lim_{\mathrm{Re}(v)\to-\infty} \gamma_{0,h}^u(v,\tau) = \lim_{\mathrm{Re}(v)\to-\infty} \theta_{0,h}^u(v,\tau) = 0,$$

for every $\tau \in \mathbb{T}_\sigma$. Furthermore, $(z_{0,h}^u, \gamma_{0,h}^u, \theta_{0,h}^u)$ satisfy the bounds in (41).

We devote the rest of this section to prove Proposition 7.2. Equation (104) can be written as the functional equation

$$\mathcal{L}_\omega(z,\gamma,\theta) = \mathcal{P}_h(z,\gamma,\theta), \tag{106}$$

where $\mathcal{L}_\omega$ and $\mathcal{P}_h$ are the operators

$$\mathcal{L}_\omega(z,\gamma,\theta) = \begin{pmatrix} \partial_v z + \omega\partial_\tau z + \dfrac{Z_0'(v)}{Z_0(v)} z \\ \partial_v \gamma + \omega\partial_\tau \gamma - \omega i \gamma \\ \partial_v \theta + \omega\partial_\tau \theta + \omega i \theta \end{pmatrix}, \tag{107}$$

and

$$\mathcal{P}_h(z,\gamma,\theta) = \begin{pmatrix} f_1^h(v,\tau) - \dfrac{z + Z_{0,h}(v,\tau)}{Z_0(v)} \partial_v z - \dfrac{\partial_v Z_{0,h}}{Z_0(v)} z - \dfrac{\delta}{\sqrt{2\Omega}} F'(X_0(v)) \dfrac{\gamma - \theta}{2i} \\ f_2^h(v,\tau) - \dfrac{(Q^0)'(v)}{Z_0(v)} z - \dfrac{z + Z_{0,h}(v,\tau)}{Z_0(v)} \partial_v \gamma \\ -f_2^h(v,\tau) + \dfrac{(Q^0)'(v)}{Z_0(v)} z - \dfrac{z + Z_{0,h}(v,\tau)}{Z_0(v)} \partial_v \theta \end{pmatrix}. \tag{108}$$

7.1. Banach spaces and technical results

For analytic functions $f : D^u \to \mathbb{C}$ and $g : D^u \times \mathbb{T}_\sigma \to \mathbb{C}$ and $\alpha > 0$, we define

$$\|f\|_\alpha = \sup_{v\in D^u} |(v^2+2)^{\alpha/2} f(v)|,$$
$$\|g\|_{\alpha,\sigma} = \sum_{k\in\mathbb{Z}} \|g^{[k]}\|_\alpha e^{|k|\sigma}, \tag{109}$$

where $g(v,\tau) = \sum_{k\in\mathbb{Z}} g^{[k]}(v)e^{ik\tau}$.

Remark 7.3. Notice that there exists a constant $d > 0$ independent of ε such that the distance between each $v \in D^u$ (given in (33)) and the poles $\pm i\sqrt{2}$ of $N_{0,h}(v,\tau)$ (given in (28)) is greater than d. The weight $|v^2+2|^{\alpha/2}$ in the norm $\|\cdot\|_\alpha$ is chosen to control the behavior as $\mathrm{Re}\, v \to -\infty$ and to have it well-defined for $v = 0 \in D^u$. In fact, at infinity this norm is equivalent to the norm with weight $|v|^\alpha$.

We also define

$$[\![g]\!]_{\alpha,\sigma} = \max\{\|g\|_{\alpha,\sigma}, \|\partial_\tau g\|_{\alpha,\sigma}, \|\partial_v g\|_{\alpha+1,\sigma}\}, \tag{110}$$

and the Banach spaces

$$\mathcal{X}_{\alpha,\sigma} = \{g : D^u \times \mathbb{T}_\sigma \to \mathbb{C} \text{ is an analytic function, such that } \|g\|_{\alpha,\sigma} < \infty\},$$
$$\mathcal{Y}_{\alpha,\sigma} = \{g : D^u \times \mathbb{T}_\sigma \to \mathbb{C} \text{ is an analytic function, such that } [\![g]\!]_{\alpha,\sigma} < \infty\}.$$

Consider the product spaces

$$\mathcal{X}^3_{\alpha,\sigma} = \Big\{(f,g,h) \in \mathcal{X}_{\alpha,\sigma} \times \mathcal{X}_{\alpha,\sigma} \times \mathcal{X}_{\alpha,\sigma};\ f \text{ is real-analytic, } g(v,\tau) = \overline{h(v,\tau)}, \text{ for every } v \in D^u \cap \mathbb{R}, \tau \in \mathbb{T}\Big\},$$
$$\mathcal{Y}^3_{\alpha,\sigma} = \Big\{(f,g,h) \in \mathcal{Y}_{\alpha,\sigma} \times \mathcal{Y}_{\alpha,\sigma} \times \mathcal{Y}_{\alpha,\sigma};\ f \text{ is real-analytic, } g(v,\tau) = \overline{h(v,\tau)}, \text{ for every } v \in D^u \cap \mathbb{R}, \tau \in \mathbb{T}\Big\},$$

endowed with the norms

$$\|(f,g,h)\|_{\alpha,\sigma} = \|f\|_{\alpha,\sigma} + \|g\|_{\alpha,\sigma} + \|h\|_{\alpha,\sigma},$$
$$[\![(f,g,h)]\!]_{\alpha,\sigma} = [\![f]\!]_{\alpha,\sigma} + [\![g]\!]_{\alpha,\sigma} + [\![h]\!]_{\alpha,\sigma},$$

respectively. We present some properties of the norm $\|\cdot\|_{\alpha,\sigma}$, which are proven in [2].

Lemma 7.4. *Given real-analytic functions $f : \mathbb{C} \to \mathbb{C}$, $g, h : D^u \times \mathbb{T}_\sigma \to \mathbb{C}$, the following statements hold*

(1) *If* $\alpha_1 \geq \alpha_2 \geq 0$*, then*

$$\|h\|_{\alpha_2,\sigma} \leq \|h\|_{\alpha_1,\sigma}.$$

(2) *If* $\alpha_1, \alpha_2 \geq 0$*, and* $\|g\|_{\alpha_1,\sigma}, \|h\|_{\alpha_2,\sigma} < \infty$*, then*

$$\|gh\|_{\alpha_1+\alpha_2,\sigma} \leq \|g\|_{\alpha_1,\sigma}\|h\|_{\alpha_2,\sigma}.$$

(3) *If* $\|g\|_{\alpha,\sigma}, \|h\|_{\alpha,\sigma} \leq R_0/4$*, where* R_0 *is the convergence ratio of* f' *at* 0*, then*

$$\|f(g) - f(h)\|_{\alpha,\sigma} \leq M\|g-h\|_{\alpha,\sigma}.$$

7.2. *The operators* $\mathcal{L}_\omega$ *and* $\mathcal{G}_\omega$

Let f, g, and h be analytic functions defined in $D^u \times \mathbb{T}_\sigma$. We define

$$\begin{aligned} F^{[k]}(f)(v) &= \int_{-\infty}^{v} \frac{e^{\omega i k(r-v)} Z_0(r)}{Z_0(v)} f^{[k]}(r)dr, \\ G^{[k]}(g)(v) &= \int_{-\infty}^{v} e^{\omega i(k-1)(r-v)} g^{[k]}(r)dr, \\ H^{[k]}(h)(v) &= \int_{-\infty}^{v} e^{\omega i(k+1)(r-v)} h^{[k]}(r)dr, \end{aligned} \tag{111}$$

and consider the linear operator $\mathcal{G}_\omega$ given by

$$\mathcal{G}_\omega(f,g,h) = \begin{pmatrix} \sum_k F^{[k]}(f)(v)e^{ik\tau} \\ \sum_k G^{[k]}(g)(v)e^{ik\tau} \\ \sum_k H^{[k]}(h)(v)e^{ik\tau} \end{pmatrix}. \tag{112}$$

Lemma 7.5. *Fix* $\alpha \geq 1$ *and* $\sigma > 0$*, the operator*

$$\mathcal{G}_\omega : \mathcal{X}^3_{\alpha+1,\sigma} \to \mathcal{Y}^3_{\alpha,\sigma},$$

given in (112) *is well-defined and the following statements hold:*

(1) $\mathcal{G}_\omega$ *is an inverse of the operator* $\mathcal{L}_\omega : \mathcal{Y}^3_{\alpha,\sigma} \to \mathcal{X}^3_{\alpha+1,\sigma}$ *given in* (107)*, i.e.* $\mathcal{G}_\omega \circ \mathcal{L}_\omega = \mathcal{L}_\omega \circ \mathcal{G}_\omega = \mathrm{Id}$*;*
(2) $[\![\mathcal{G}_\omega(f,g,h)]\!]_{\alpha,\sigma} \leq M\|(f,g,h)\|_{\alpha+1,\sigma}$*;*
(3) *If* $f^{[0]} = g^{[1]} = h^{[-1]} = 0$*, then* $[\![\mathcal{G}_\omega(f,g,h)]\!]_{\alpha,\sigma} \leq \dfrac{M}{\omega}[\![(f,g,h)]\!]_{\alpha,\sigma}$.

The proof of Lemma 7.5 can be found in [2].

To find a solution of (104), it is sufficient to find a fixed point of the operator

$$\overline{\mathcal{G}}_{\omega,h} = \mathcal{G}_{\omega} \circ \mathcal{P}_h, \tag{113}$$

where $\mathcal{G}_{\omega}$ is given by (112) and $\mathcal{P}_h$ is given by (108).

7.3. The operator $\mathcal{P}_h$

We show some properties of the operator $\mathcal{P}_h$ defined in (108).

Lemma 7.6. *Fix $\sigma > 0,\ h_0 > 0$. For $0 \le h \le h_0$, the operator $\mathcal{P}_h$ defined in (108) satisfies*

$$\|\mathcal{P}_h(0,0,0)\|_{2,\sigma} \le M\frac{\delta}{\omega}.$$

Proof. Notice that $\mathcal{P}_h(0,0,0) = (f_1^h, f_2^h, -f_2^h)$, where f_1^h and f_2^h are given by (105), and involve the functions $F'(X_0),\ Z_0,\ Z_0',\ Q^0,\ Q_0',\ Z_{0,h}, \partial_v Z_{0,h}$. By (8), (26), (37) and (40), we can see that

$$\|Q^0\|_{1,\sigma}, \|(Q^0)'\|_{2,\sigma} \le M\frac{\delta}{\omega},$$

$$\left\|Z_{0,h}\right\|_{1,\sigma}, \|\partial_v Z_{0,h}\|_{2,\sigma} \le M\frac{\delta\sqrt{h}}{\omega^{3/2}},$$

$$\|Z_0\|_{1,\sigma}, \|Z_0'\|_{2,\sigma}, \|F'(X_0)\|_{1,\sigma} \le M.$$

It follows from these bounds and Lemma 7.4 that

$$\|f_1^h\|_{2,\sigma} \le M\max\left\{\frac{\delta\sqrt{h}}{\omega^{3/2}}, \frac{\delta^2}{\omega}, \frac{\delta^2}{\omega^3}h\right\} = M\max\left\{\frac{\delta\sqrt{h}}{\omega^{3/2}}, \frac{\delta^2}{\omega}\right\},$$

$$\|f_2^h\|_{2,\sigma} \le M\max\left\{\frac{\delta}{\omega}, \frac{\delta^2}{\omega^{5/2}}\sqrt{h}\right\} = M\frac{\delta}{\omega}. \quad \square$$

Lemma 7.7. *Fix $\sigma > 0,\ h_0 > 0$ and $K > 0$. If $0 \le h \le h_0$, the operator*

$$\mathcal{P}_h : \mathcal{Y}_{1,\sigma}^3 \to \mathcal{X}_{2,\sigma}^3,$$

is well defined. Moreover, given $(z_j, \gamma_j, \theta_j) \in \mathcal{B}_0(K\delta/\omega) \subset \mathcal{Y}_{1,\sigma}^3,\ j = 1, 2,$

$$\|\mathcal{P}_h(z_1,\gamma_1,\theta_1) - \mathcal{P}_h(z_2,\gamma_2,\theta_2)\|_{2,\sigma} \le M\left(\delta + \frac{\delta}{\omega^{3/2}}\sqrt{h}\right)[\![(z_1,\gamma_1,\theta_1) - (z_2,\gamma_2,\theta_2)]\!]_{1,\sigma}, \tag{114}$$

where M is a constant independent of ε and h.

Proof. It is straightforward to see that $\mathcal{P}_h$ is well defined. Denote $\mathcal{P}_h^j = \pi_j \circ \mathcal{P}_h$. We show the bound of the difference for $\mathcal{P}_h^1$ and $\mathcal{P}_h^2$, since the bound of $\mathcal{P}_h^3$ can be obtained in exactly the same way as $\mathcal{P}_h^2$.

Notice that

$$\begin{aligned}\mathcal{P}_h^1(z_1,\gamma_1,\theta_1)-\mathcal{P}_h^1(z_2,\gamma_2,\theta_2) = &-\frac{\delta}{\sqrt{2\Omega}}F'(X_0(v))\frac{(\gamma_1-\gamma_2)-(\theta_1-\theta_2)}{2i}\\ &-\frac{\partial_v Z_{0,h}(v,\tau)}{Z_0(v)}(z_1-z_2)-\partial_v z_2\frac{z_1-z_2}{Z_0(v)}\\ &-\frac{z_1+Z_{0,h}(v,\tau)}{Z_0(v)}(\partial_v z_1-\partial_v z_2).\end{aligned}$$

Using the bounds contained in the proof of Lemma 7.6 and that Z_0 is lower bounded in D^u by a positive constant independent of ε, one can see that

$$\left\|\mathcal{P}_h^1(z_1,\gamma_1,\theta_1)-\mathcal{P}_h^1(z_2,\gamma_2,\theta_2)\right\|_{2,\sigma} \le M\max\left\{\delta,\frac{\delta}{\omega^{3/2}}\sqrt{h}\right\}[\![(z_1,\gamma_1,\theta_1)-(z_2,\gamma_2,\theta_2)]\!]_{1,\sigma}.$$

Now,

$$\begin{aligned}\mathcal{P}_h^2(z_1,\gamma_1,\theta_1)-\mathcal{P}_h^2(z_2,\gamma_2,\theta_2) = &-\frac{(Q^0)'(v)}{Z_0(v)}(z_1-z_2)-\partial_v\gamma_2\frac{z_1-z_2}{Z_0(v)}\\ &-\frac{z_1+Z_{0,h}(v,\tau)}{Z_0(v)}(\partial_v\gamma_1-\partial_v\gamma_2),\end{aligned}$$

which, proceeding analogously,

$$\left\|\mathcal{P}_h^2(z_1,\gamma_1,\theta_1)-\mathcal{P}_h^2(z_2,\gamma_2,\theta_2)\right\|_{2,\sigma} \le M\max\left\{\frac{\delta}{\omega},\frac{\delta}{\omega^{3/2}}\sqrt{h}\right\}[\![(z_1,\gamma_1,\theta_1)-(z_2,\gamma_2,\theta_2)]\!]_{1,\sigma}.$$

□

7.4. The fixed point theorem

Now, we write Proposition 7.2 in terms of Banach spaces and we prove it through a fixed point argument applied to the operator $\overline{\mathcal{G}}_{\omega,h}$ given by (113).

Proposition 7.8. *Fix $\sigma > 0$ and $h_0 > 0$. There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$, the operator $\overline{\mathcal{G}}_{\omega,h}$ in (113) has a fixed point $(z_{0,h}^u, \gamma_{0,h}^u, \theta_{0,h}^u) \in \mathcal{Y}_{1,\sigma}^3$. Furthermore, there exists a constant $M > 0$ independent of ε and h such that*

$$[\![(z_{0,h}^u,\gamma_{0,h}^u,\theta_{0,h}^u)]\!]_{1,\sigma} \le M\frac{\delta}{\omega}.$$

Proof. From Lemmas 7.5 and 7.6, there exists a constant $b_2 > 0$ independent of ε and h such that

$$[\![\overline{\mathcal{G}}_{\omega,h}(0,0,0)]\!]_{1,\sigma} \leq M\|\mathcal{P}_h(0,0,0)\|_{2,\sigma} \leq \frac{b_2}{2}\frac{\delta}{\omega}.$$

Consider the operator $\overline{\mathcal{G}}_{\omega,h} = \mathcal{G}_\omega \circ \mathcal{P}_h : \mathcal{B}_0(b_2\delta/\omega) \subset \mathcal{Y}_{1,\sigma} \to \mathcal{Y}_{1,\sigma}$. Notice that Lemmas 7.5 and 7.7 imply that it is well defined in these spaces.

To show that $\overline{\mathcal{G}}_{\omega,h}$ sends $\mathcal{B}_0(b_2\delta/\omega)$ into itself, consider $K = b_2$ in Lemma 7.7 and $(z_j, \gamma_j, \theta_j) \in \mathcal{B}_0(b_2\delta/\omega)$, $j = 1, 2$. It follows from Lemmas 7.5, 7.7 and the fact that $\mathcal{G}_\omega$ is a linear operator that

$$\begin{aligned}[\![\overline{\mathcal{G}}_{\omega,h}(z_1,\gamma_1,\theta_1) - \overline{\mathcal{G}}_{\omega,h}(z_2,\gamma_2,\theta_2)]\!]_{1,\sigma} &\leq M\,\|\mathcal{P}_h(z_1,\gamma_1,\theta_1) - \mathcal{P}_h(z_2,\gamma_2,\theta_2)\|_{2,\sigma} \\ &\leq M\delta\,[\![(z_1,\gamma_1,\theta_1) - (z_2,\gamma_2,\theta_2)]\!]_{1,\sigma}.\end{aligned}$$

Choosing ε_0 sufficiently small such that $\mathrm{Lip}(\overline{\mathcal{G}}_{\omega,h}) < 1/2$, $\overline{\mathcal{G}}_{\omega,h}$ sends $\mathcal{B}_0(b_2\delta/\omega)$ into itself and it is a contraction. Thus, it has a unique fixed point $(z^u_{0,h}, \gamma^u_{0,h}, \theta^u_{0,h}) \in \mathcal{B}_0(b_2\delta/\omega)$. □

8. Proof of Theorem 4.5

The strategy used to prove Theorem 4.5 is analogous to the one of Theorem 4.1 taking into account that all the expressions appearing become singular as $h \to 0$. We write

$$N^u_{h,0}(v) = (X_h(v), Z^u_{h,0}(v), \Gamma^u_{h,0}(v), \Theta^u_{h,0}(v)). \tag{115}$$

Lemma 8.1. *Given $h > 0$, the invariant manifold $W^u_\delta(p^-_h)$, with $\delta \neq 0$, is parameterized by $N^u_{h,0}(v)$ if and only if $(\Gamma^u_{h,0}(v), \Theta^u_{h,0}(v))$ satisfy*

$$\begin{aligned}\frac{d\Gamma}{dv}(v) &= \frac{Z_h(v)}{\widetilde{\eta}_h(v,\Gamma,\Theta)}\left(\omega i\Gamma(v) - \frac{\delta}{\sqrt{2\Omega}}F(X_h(v))\right),\\ \frac{d\Theta}{dv}(v) &= \frac{Z_h(v)}{\widetilde{\eta}_h(v,\Gamma,\Theta)}\left(-\omega i\Theta(v) - \frac{\delta}{\sqrt{2\Omega}}F(X_h(v)))\right),\\ \lim_{v\to-\infty}\Gamma(v) &= \lim_{v\to-\infty}\Theta(v) = 0,\end{aligned} \tag{116}$$

and

$$\widetilde{\eta}_h(v,\Gamma,\Theta) = 4\sqrt{h - U(X_h(v)) - \frac{\delta}{\sqrt{2\Omega}}F(X_h(v))\frac{\Gamma(v)-\Theta(v)}{2i} - \frac{\omega}{2}\Gamma(v)\Theta(v)},$$

with X_h given in (26), U, F given in (8), and $Z^u_{h,0}(v) = \widetilde{\eta}_h(v, \Gamma^u_{h,0}(v), \Theta^u_{h,0}(v))$.

As in Section 5, we compute an explicit term of $(\Gamma^u_{h,0}, \Theta^u_{h,0})$. Thus, the solution of (116) can be written as (42) and $(\gamma^u_{h,0}, \theta^u_{h,0})$ satisfy

$$\begin{aligned}
&\frac{d}{dv}\gamma - \omega i\gamma = \omega i\gamma(\eta_h(v,\gamma,\theta)-1) - (Q^h)'(v),\\
&\frac{d}{dv}\theta + \omega i\theta = -\omega i\theta(\eta_h(v,\gamma,\theta)-1) + (Q^h)'(v),\\
&\lim_{v\to-\infty}\gamma(v) = \lim_{v\to-\infty}\theta(v) = 0,
\end{aligned} \tag{117}$$

where Q^h is given in (43) and

$$\eta_h(v,\gamma,\theta) = \left(1 + \frac{4\delta^2}{\Omega\omega}\left(\frac{F(X_h(v))}{Z_h(v)}\right)^2 - 8\omega\frac{\gamma\theta}{(Z_h(v))^2}\right)^{-1/2}. \tag{118}$$

We prove Theorem 4.5 by finding a solution of (117) in the next proposition.

Proposition 8.2. *There exists $\varepsilon_0 > 0$ and $h_0 > 0$ such that for $0 < h \le h_0$ and $0 < \varepsilon \le \varepsilon_0$, equation (117) has a solution $(\gamma^u_{h,0}(v), \theta^u_{h,0}(v))$ defined in D^u (see (33)) such that $\theta^u_{h,0}(v) = \overline{\gamma^u_{h,0}(v)}$ for every $v \in \mathbb{R}$. Furthermore, $(\gamma^u_{h,0}, \theta^u_{h,0})$ satisfy the bound (44).*

To prove Proposition 8.2, it is sufficient to find a fixed point $(\gamma^u_{h,0}, \theta^u_{h,0})$ of the operator

$$\mathcal{G}_{\omega,h} = \mathcal{G}_\omega \circ \mathcal{F}_h, \tag{119}$$

where $\mathcal{G}_\omega$ is given in (68) and

$$\mathcal{F}_h(\gamma,\theta)(v) = \begin{pmatrix} \omega i\gamma(v)(\eta_h(v,\gamma(v),\theta(v))-1) - (Q^h)'(v) \\ -\omega i\theta(v)(\eta_h(v,\gamma(v),\theta(v))-1) + (Q^h)'(v)\end{pmatrix}, \tag{120}$$

and Q^h, η_h are given in (43) and (118), respectively.

The rest of this section is devoted to find a fixed point of (119).

8.1. Banach spaces and technical lemmas

By (8), (27) and (43)

$$Q^h(v) = \frac{2\delta i}{\omega\sqrt{2\Omega}}\left(\frac{\sqrt{\frac{2+h}{h}}\sinh(v\sqrt{h}/2)}{1+\frac{2+h}{h}\sinh^2(v\sqrt{h}/2)}\right), \tag{121}$$

which has poles at

$$s^{\pm,j}_{h,k} = i\frac{2}{\sqrt{h}}\left(\delta_{j,1}\pi \pm \arcsin\left(\sqrt{\frac{h}{2+h}}\right) + 2k\pi\right), \tag{122}$$

where $\delta_{j,1}$ is the delta of Kronecker, $j = 0, 1$ and $k \in \mathbb{Z}$. All these singularities are contained in the imaginary axis and satisfy

$$s_{h,k}^{\pm,j} = i\left(\pm\sqrt{2} + \mathcal{O}(h) + \frac{2}{\sqrt{h}}\left(\delta_{j,1}\pi + 2k\pi\right)\right).$$

Thus, for h sufficiently small $\left|s_{h,k}^{\pm,j}\right| \geq 3\sqrt{2}/4$, $j = 0, 1$ and $k \in \mathbb{Z}$.

Therefore, we can consider the same domain D^u in (33). It satisfies the following property, whose proof is straightforward.

Lemma 8.3. *If $v \in D^u$ is such that $|\operatorname{Re}(v)| \geq \chi_0$, for some $\chi_0 > 0$, then*

$$|\operatorname{Im}(v)| \leq \frac{\chi_0 + 1}{\chi_0} |\operatorname{Re}(v)|.$$

For $\alpha \geq 0$, we consider the Banach space

$$\mathcal{X}_\alpha = \left\{ f : D^u \to C;\ f \text{ is analytic and } \|f\|_\alpha < \infty \right\}, \tag{123}$$

endowed with the norm

$$\|f\|_\alpha = \sup_{v \in D^u} |(v^2 + 2)^{\alpha/2} f(v)|, \tag{124}$$

and the product space

$$\mathcal{X}_\alpha^2 = \left\{ (f, g) \in \mathcal{X}_\alpha \times \mathcal{X}_\alpha;\ g(v) = \overline{f(v)} \text{ for every } v \in \mathbb{R} \right\},$$

endowed with the norm $\|(f, g)\|_\alpha = \|f\|_\alpha + \|g\|_\alpha$. Remark 7.3 and Lemma 7.4 also apply to $\|\cdot\|_\alpha$.

Lemma 8.4. *Given $0 < h_0 \leq 1$, there exists a constant $M^* > 0$ such that, for each $v \in D^u$ and $0 < h \leq h_0$,*

$$\left|\sinh(v\sqrt{h}/2)\right| \geq M^* \sqrt{h} |v|, \qquad \left|\cosh(v\sqrt{h}/2)\right| \geq M^*.$$

The following Lemma is proved in [1].

Lemma 8.5. *Let $1/2 < \beta < \pi/4$ be fixed. The following statements hold*

(1) *There exists $\beta_0 > 0$ sufficiently small such that $D^u \subset D^u(\beta_0)$, where*

$$D^u(\beta_0) = \left\{ v \in \mathbb{C};\ |\operatorname{Im}(v)| \leq -\tan(\beta + \beta_0) \operatorname{Re}(v) + 2\sqrt{2}/3 \right\}.$$

(2) *Given $\alpha > 0$, if $f : D^u(\beta_0) \to \mathbb{C}$ is a real-analytic function such that*

$$m_\alpha(f) = \sup_{v \in D^u(\beta_0)} |(v^2 + 2)^{\alpha/2} f(v)| < \infty,$$

then, for any $n \in \mathbb{N}$

$$\|f^{(n)}\|_{\alpha+n} \leq M m_\alpha(f).$$

In the remaining of this paper, all the Landau symbols $\mathcal{O}(f(v,h,\varepsilon))$ denote a function dependent on v, h and ε such that there exists a constant $M > 0$ independent of h and ε such that $|\mathcal{O}(f(v,h,\varepsilon))| \leq M|f(v,h,\varepsilon)|$, for every (v,h,ε) in the domain considered.

Lemma 8.6. *There exist $h_0 \in (0,1)$ and a constant $M > 0$ such that, for $v \in D^u$ and $0 < h \leq h_0$,*

(1) $|F(X_h(v))| \leq \dfrac{M}{|\sqrt{v^2+2}|}$;

(2) $|F(X_h(v))'| \leq \dfrac{M}{|v^2+2|}$,

where X_h given in (27) and $F(X)$ in (8).

Proof. By (43) and (121), we have that

$$F(X_h(v)) = -2\sqrt{\frac{h}{2+h}}\frac{1}{\sinh(v\sqrt{h}/2)}\left(\frac{1}{1+\frac{h}{2+h}\frac{1}{\sinh^2(v\sqrt{h}/2)}}\right).$$

Then, Lemma 8.4 implies

$$|F(X_h(v))| \leq M\sqrt{h}\frac{1}{\sqrt{h}|v|}\left(\frac{1}{\left|1+\frac{h}{2+h}\frac{1}{\sinh^2(v\sqrt{h}/2)}\right|}\right).$$

Notice that

$$\left|1+\frac{h}{2+h}\frac{1}{\sinh^2(v\sqrt{h}/2)}\right| \geq 1-\frac{h}{2+h}\left|\frac{1}{\sinh^2(v\sqrt{h}/2)}\right|,$$

and, by Lemma 8.4,

$$\frac{h}{2+h}\left|\frac{1}{\sinh^2(v\sqrt{h}/2)}\right| \leq \frac{h}{2+h}\frac{1}{(M^*)^2h|v|^2} \leq \frac{1}{2(M^*)^2|v|^2}.$$

Thus, for $|v| \geq (M^*)^{-1}$,

$$\left|1+\frac{h}{2+h}\frac{1}{\sinh^2(v\sqrt{h}/2)}\right| \geq 1/2. \tag{125}$$

We also know that, if $|v| \geq (M^*)^{-1}$, $|\sqrt{v^2+2}| \leq \sqrt{1+2M^*}|v|$. Hence

$$|(\sqrt{v^2+2})F(X_h(v))| \leq M\frac{|\sqrt{v^2+2}|}{|v|} \leq M.$$

Now, assume that $|v| \leq (M^*)^{-1}$. Hence $|v\sqrt{h}/2| \leq M$ and expanding $\sinh(z)$ at 0 we obtain

$$F(X_h(v)) = -2\frac{\sqrt{\frac{2+h}{h}}\left(v\sqrt{h}/2 + \mathcal{O}(h^{3/2}v^3)\right)}{1 + \frac{2+h}{h}\left(hv^2/4 + \mathcal{O}(h^2v^4)\right)}$$
$$= -2\frac{\sqrt{2+h}(v/2 + \mathcal{O}(h))}{1 + v^2/2 + \mathcal{O}(h)}.$$

Since $v \in D^u$, we have that there exists $M > 0$ such that

$$|1 + v^2/2 + \mathcal{O}(h)| \geq |1 + v^2/2| - \mathcal{O}(h) \geq M - \mathcal{O}(h).$$

Therefore, for $h > 0$ sufficiently small, we have that $|F(X_h(v))| \leq M$, for $|v| \leq (M^*)^{-1}$, and since $|\sqrt{v^2+2}|$ is inferiorly and superiorly bounded by nonzero constants in this domain, we have that

$$|(\sqrt{v^2+2})F(X_h(v))| \leq M \quad \text{for } |v| \leq (M^*)^{-1}.$$

This concludes the proof of the first item. One can obtain item 2 using Lemma 8.5. □

Lemma 8.7. *Given $0 < h_0 \leq 1$, there exists a constant $M > 0$ such that, for $v \in D^u$ and $0 < h \leq h_0$,*

$$\left|\frac{1}{Z_h^2(v)}\frac{1}{v^2+2}\right| \leq M,$$

where Z_h in (27).

The proof is analogous to the one of Lemma 8.6.

8.2. The fixed point theorem

Now, we study the operator $\mathcal{G}_{\omega,h}$ in order to find a fixed point in $\mathcal{X}_2^2$. Recall the definition of $\mathcal{G}_{\omega,h} = \mathcal{G}_\omega \circ \mathcal{F}_h$ in (119), and notice that $\mathcal{G}_\omega$ is the same operator of the case $h = 0$. Thus, Proposition 5.3 still holds for functions in the Banach space $\mathcal{X}_2^2$.

Proposition 8.8. *Given $(f, g) \in \mathcal{X}_2^2$, we have that $\mathcal{G}_\omega(f, g) \in \mathcal{X}_2^2$. Furthermore, there exists a constant $M > 0$ independent of ε such that*

$$\|\mathcal{G}_\omega(f, g)\|_2 \leq \frac{M}{\omega}\|(f, g)\|_2 .$$

We proceed by studying the operator $\mathcal{F}_h$ in (120).

Proposition 8.9. *There exists $h_0 > 0$, $\varepsilon_0 > 0$ and a constant $M > 0$ such that for, $0 < \varepsilon \le \varepsilon_0$ and $0 < h \le h_0$,*

$$\left\| \mathcal{G}_{\omega,h}(0,0) \right\|_2 \le M \frac{\delta}{\omega^2}.$$

Proof. Notice that $\mathcal{F}_h(0,0) = (-(Q^h)'(v), (Q^h)'(v))$ (see (43)), which implies

$$\|\mathcal{F}_h(0,0)\|_2 = 2\frac{\delta}{\omega\sqrt{2\Omega}} \|F(X_h)'\|_2.$$

Thus, it is enough to apply Lemma 8.6 and Proposition 8.8. □

Proposition 8.10. *There exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that for $0 < \varepsilon \le \varepsilon_0$, $0 < h \le h_0$:*

Let η_h be given in (118) and take $(\gamma_j, \theta_j) \in \mathcal{B}_0(R) \subset \mathcal{X}_2^2$ with $j = 1, 2$ and $R = K\dfrac{\delta}{\omega^2}$, where K is a constant independent of h and ε, the following statements hold.

(1) $\left|\eta_h(v, \gamma_j(v), \theta_j(v)) - 1\right| \le M\dfrac{\delta^2}{\omega}$;

(2) $|\eta_h(v, \gamma_1(v), \theta_1(v)) - \eta_h(v, \gamma_2(v), \theta_2(v))| \le M\dfrac{\delta}{\omega}\|(\gamma_1, \theta_1) - (\gamma_2, \theta_2)\|_0$;

(3) $\|\mathcal{F}_h(\gamma_1, \theta_1) - \mathcal{F}_h(\gamma_2, \theta_2)\|_2 \le M\delta^2\|(\gamma_1, \theta_1) - (\gamma_2, \theta_2)\|_2$.

Proof. Lemmas 8.6 and 8.7 and the fact that $(\gamma, \theta) \in \mathcal{B}_0(R)$ imply

$$\left| \frac{4\delta^2}{\Omega\omega} \left(\frac{F(X_h(v))}{Z_h(v)} \right)^2 - 8\omega \frac{\gamma\theta}{(Z_h(v))^2} \right| \le M\frac{\delta^2}{\omega}.$$

Thus, using (118), it follows that

$$|\eta_h(v, \gamma, \theta) - 1| \le M \left| \frac{4\delta^2}{\Omega\omega} \left(\frac{F(X_h(v))}{Z_h(v)} \right)^2 - 8\omega \frac{\gamma\theta}{(Z_h(v))^2} \right| \le M\frac{\delta^2}{\omega},$$

and using also Lemma 8.7, we have

$$\begin{aligned} |\eta_h(v, \gamma_1, \theta_1) - \eta_h(v, \gamma_2, \theta_2)| &\le M\omega \left| \frac{\gamma_1\theta_1 - \gamma_2\theta_2}{(Z_h(v))^2} \right| \\ &\le MR\omega \left(\frac{|\theta_1 - \theta_2|}{|(Z_h(v))^2(v^2+2)|} + \frac{|\gamma_1 - \gamma_2|}{|(Z_h(v))^2(v^2+2)|} \right) \\ &\le M\frac{\delta}{\omega}\|(\gamma_1, \theta_1) - (\gamma_2, \theta_2)\|_0. \end{aligned}$$

Finally, it follows from items (1) and (2) of this proposition and (120) that

$$
\begin{aligned}
\|\pi_1 \circ \mathcal{F}_h(\gamma_1,\theta_1) - \pi_1 \circ \mathcal{F}_h(\gamma_2,\theta_2)\|_2 \leq & \omega \,\|\eta_h(v,\gamma_1,\theta_1) - 1\|_0 \,\|\gamma_1 - \gamma_2\|_2 \\
& + \omega\|\gamma_2\|_2 \,\|\eta_h(v,\gamma_1,\theta_1) - \eta_h(v,\gamma_2,\theta_2)\|_0 \\
\leq & M\delta^2 \|\gamma_1 - \gamma_2\|_2 + M\omega R \frac{\delta}{\omega} \|(\gamma_1,\theta_1) - (\gamma_2,\theta_2)\|_0 \\
\leq & M\delta^2 \|(\gamma_1,\theta_1) - (\gamma_2,\theta_2)\|_2 .
\end{aligned}
$$

Analogously, we obtain the same inequality for the second component of $\mathcal{F}_h$. □

Finally, we are able to prove Proposition 8.2 (and thus Theorem (4.5)) by a fixed point argument.

Proposition 8.11. *There exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that for $0 < h \leq h_0$ and $\varepsilon \leq \varepsilon_0$, the operator $\mathcal{G}_{\omega,h}$ (given in (119)) has a fixed point $(\gamma^u_{h,0}, \theta^u_{h,0})$ in $\mathcal{X}_2^2$ which satisfies*

$$
\|(\gamma^u_{h,0}, \theta^u_{h,0})\|_2 \leq M \frac{\delta}{\omega^2}.
$$

Proof. From Proposition 8.9, there exists a constant $b_3 > 0$ independent of h and ε such that

$$
\left\|\mathcal{G}_{\omega,h}(0,0)\right\|_2 \leq \frac{b_3}{2} \frac{\delta}{\omega^2}.
$$

Now, given (γ_1,θ_1) and (γ_2,θ_2) in $\mathcal{B}_0(b_3\delta/\omega^2)$, we can use Propositions 8.10 (with $K = b_3$) and 8.8 and the linearity of the operator $\mathcal{G}_\omega$ to see that

$$
\begin{aligned}
\left\|\mathcal{G}_{\omega,h}(\gamma_1,\theta_1) - \mathcal{G}_{\omega,h}(\gamma_2,\theta_2)\right\|_2 &\leq \frac{M}{\omega} \,\|\mathcal{F}_h(\gamma_1,\theta_1) - \mathcal{F}_h(\gamma_2,\theta_2)\|_2 \\
&\leq M \frac{\delta^2}{\omega} \|(\gamma_1,\theta_1) - (\gamma_2,\theta_2)\|_2 .
\end{aligned}
$$

Choosing ε_0 sufficiently small, we have that $\mathrm{Lip}(\mathcal{G}_{\omega,h}) \leq 1/2$. Therefore $\mathcal{G}_{\omega,h}$ sends the ball $\mathcal{B}_0(b_3\delta/\omega^2)$ into itself and it is a contraction. Thus, it has a unique fixed point $(\gamma^u_{h,0}, \theta^u_{h,0}) \in \mathcal{B}_0(b_3\delta/\omega^2)$. □

9. Proof of Theorem 4.6

In this section we prove the existence of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$, with $\delta \neq 0$. As in the previous sections, we look for parameterizations $N^u_{\kappa_1,\kappa_2}$ of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ as graphs

$$
N^{u,s}_{\kappa_1,\kappa_2}(v,\tau) = (X_{\kappa_1}(v), Z_{\kappa_1}(v) + Z^{u,s}_{\kappa_1,\kappa_2}(v,\tau), \Gamma_{\kappa_2}(\tau) + \Gamma^{u,s}_{\kappa_1,\kappa_2}(v,\tau), \Theta_{\kappa_2}(\tau) + \Theta^{u,s}_{\kappa_1,\kappa_2}(v,\tau)), \tag{126}
$$

where $X_{\kappa_1}, Z_{\kappa_1}$ are given in (26) and $\Gamma_{\kappa_2}, \Theta_{\kappa_2}$ are given in (29).

Following the same lines of Section 8 we have a characterization of $N^u_{\kappa_1,\kappa_2}$.

Lemma 9.1. *Write* $Z^u_{\kappa_1,\kappa_2}(v,\tau)=Z_{\kappa_1,\kappa_2}(v,\tau)+z^u_{\kappa_1,\kappa_2}(v,\tau)$*,* $\Gamma^u_{\kappa_1,\kappa_2}(v,\tau)=Q^{\kappa_2}(v)+\gamma^u_{\kappa_1,\kappa_2}(v,\tau)$*,* $\Theta^u_{\kappa_1,\kappa_2}(v,\tau)=-Q^{\kappa_1}(v)+\theta^u_{\kappa_1,\kappa_2}(v,\tau)$*, where* Q^{κ_1} *is given by* (43) *and*

$$Z_{\kappa_1,\kappa_2}(v,\tau)=\frac{\delta}{\omega\sqrt{2\Omega}}F'(X_{\kappa_1}(v))\frac{\Gamma_{\kappa_2}(\tau)+\Theta_{\kappa_2}(\tau)}{2}, \tag{127}$$

with $\Gamma_{\kappa_1},\Theta_{\kappa_1}$ *given by* (29)*. Then,* $N^u_{\kappa_1,\kappa_2}(v,\tau)$*, given in* (126)*, with* $\kappa_1,\kappa_2\geq 0$ *and* $\kappa_1+\kappa_2=h$*, parameterizes* $W^u(\Lambda^-_{\kappa_1,\kappa_2})$ *provided* $(z^u_{\kappa_1,\kappa_2},\gamma^u_{\kappa_1,\kappa_2},\theta^u_{\kappa_1,\kappa_2})$ *satisfy*

$$\begin{aligned}
\partial_v z+\omega\partial_\tau z+\frac{Z'_{\kappa_1}(v)}{Z_{\kappa_1}(v)}z=&f_1^{\kappa_1,\kappa_2}(v,\tau)-\frac{z+Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)}\partial_v z-\frac{\partial_v Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)}z\\
&-\frac{\delta}{\sqrt{2\Omega}}F'(X_{\kappa_1}(v))\frac{\gamma-\theta}{2i},\\
\partial_v\gamma+\omega\partial_\tau\gamma-\omega i\gamma=&f_2^{\kappa_1,\kappa_2}(v,\tau)-\frac{(Q^{\kappa_1})'(v)}{Z_{\kappa_1}(v)}z-\frac{z+Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)}\partial_v\gamma,\\
\partial_v\theta+\omega\partial_\tau\theta+\omega i\theta=&-f_2^{\kappa_1,\kappa_2}(v,\tau)+\frac{(Q^{\kappa_1})'(v)}{Z_{\kappa_1}(v)}z-\frac{z+Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)}\partial_v\theta,\\
\lim_{v\to-\infty}z(v,\tau)=&\lim_{v\to-\infty}\gamma(v,\tau)=\lim_{v\to-\infty}\theta(v,\tau)=0,
\end{aligned} \tag{128}$$

where

$$\begin{aligned}
f_1^{\kappa_1,\kappa_2}(v,\tau)=&-\partial_v Z_{\kappa_1,\kappa_2}(v,\tau)-\frac{Z'_{\kappa_1}(v)}{Z_{\kappa_1}(v)}Z_{\kappa_1,\kappa_2}(v,\tau)-\frac{\delta}{\sqrt{2\Omega}}F'(X_{\kappa_1}(v))\frac{Q^{\kappa_1}(v)}{i}\\
&-\frac{Z_{\kappa_1,\kappa_2}(v,\tau)\partial_v Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)},\\
f_2^{\kappa_1,\kappa_2}(v,\tau)=&-(Q^{\kappa_1})'(v)-\frac{Z_{\kappa_1,\kappa_2}(v,\tau)(Q^{\kappa_1})'(v)}{Z_{\kappa_1}(v)}.
\end{aligned} \tag{129}$$

We consider the equation (128) with $(v,\tau)\in D^u\times\mathbb{T}_\sigma$ with the asymptotic conditions $\lim_{\mathrm{Re}(v)\to-\infty}z(v)=\lim_{\mathrm{Re}(v)\to-\infty}\gamma(v)=\lim_{\mathrm{Re}(v)\to-\infty}\theta(v)=0$, for every $\tau\in\mathbb{T}_\sigma$.

Theorem (4.6) is a consequence of the following proposition.

Proposition 9.2. *Fix* $\sigma>0$*. There exist* $h_0>0$ *and* $\varepsilon_0>0$ *sufficiently small such that for* $0<\varepsilon\leq\varepsilon_0$*,* $0<h\leq h_0$ *and* $\kappa_1,\kappa_2\geq 0$ *with* $\kappa_1+\kappa_2=h$*, system* (128) *has an analytic solution* $(z^u_{\kappa_1,\kappa_2},\gamma^u_{\kappa_1,\kappa_2},\theta^u_{\kappa_1,\kappa_2})$ *defined in* $D^u\times\mathbb{T}_\sigma$ *(see* (33) *and* (34)*) such that* $z^u_{\kappa_1,\kappa_2}$ *is real-analytic,* $\theta^u_{\kappa_1,\kappa_2}(v,\tau)=\overline{\gamma^u_{\kappa_1,\kappa_2}(v,\tau)}$ *for each* $(v,\tau)\in D^u\times\mathbb{T}_\sigma\cap\mathbb{R}^2$ *and*

$$\lim_{\mathrm{Re}(v)\to-\infty}z^u_{\kappa_1,\kappa_2}(v,\tau)=\lim_{\mathrm{Re}(v)\to-\infty}\gamma^u_{\kappa_1,\kappa_2}(v,\tau)=\lim_{\mathrm{Re}(v)\to-\infty}\theta^u_{\kappa_1,\kappa_2}(v,\tau)=0,$$

for every $\tau\in\mathbb{T}_\sigma$*. Furthermore,* $(z^u_{\kappa_1,\kappa_2},\gamma^u_{\kappa_1,\kappa_2},\theta^u_{\kappa_1,\kappa_2})$ *satisfies the bounds in* (46)*.*

Equation (128) can be written as the functional equation

$$\mathcal{L}_{\omega,\kappa_1}(z,\gamma,\theta)=\mathcal{P}_{\kappa_1,\kappa_2}(z,\gamma,\theta), \tag{130}$$

where $\mathcal{L}_{\omega,\kappa_1}$ and $\mathcal{P}_{\kappa_1,\kappa_2}$ are the functional operators given by

$$\mathcal{L}_{\omega,\kappa_1}(z,\gamma,\theta)=\begin{pmatrix} \partial_v z+\omega\partial_\tau z+\dfrac{Z'_{\kappa_1}(v)}{Z_{\kappa_1}(v)}z \\ \partial_v\gamma+\omega\partial_\tau\gamma-\omega i\gamma \\ \partial_v\theta+\omega\partial_\tau\theta+\omega i\theta \end{pmatrix}, \tag{131}$$

and

$$\mathcal{P}_{\kappa_1,\kappa_2}(z,\gamma,\theta)=\begin{pmatrix} f_1^{\kappa_1,\kappa_2}(v,\tau)-\dfrac{z+Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)}\partial_v z-\dfrac{\partial_v Z_{\kappa_1,\kappa_2}}{Z_{\kappa_1}(v)}z-\dfrac{\delta}{\sqrt{2\Omega}}F'(X_{\kappa_1}(v))\dfrac{\gamma-\theta}{2i} \\ f_2^{\kappa_1,\kappa_2}(v,\tau)-\dfrac{(Q^{\kappa_1})'(v)}{Z_{\kappa_1}(v)}z-\dfrac{z+Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)}\partial_v\gamma \\ -f_2^{\kappa_1,\kappa_2}(v,\tau)+\dfrac{(Q^{\kappa_1})'(v)}{Z_{\kappa_1}(v)}z-\dfrac{z+Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)}\partial_v\theta \end{pmatrix}. \tag{132}$$

We show the existence of an inverse $\mathcal{G}_\omega^{\kappa_1}$ of $\mathcal{L}_{\omega,\kappa_1}$ in the Banach spaces $\mathcal{X}^3_{\alpha,\sigma}$ and $\mathcal{Y}^3_{\alpha,\sigma}$ introduced in Section 7.1.

Given analytic functions f, g, and h defined in $D^u\times\mathbb{T}_\sigma$, consider

$$F_{\kappa_1}^{[k]}(f)(v)=\int_{-\infty}^{v}\frac{e^{\omega ik(r-v)}Z_{\kappa_1}(r)}{Z_{\kappa_1}(v)}f^{[k]}(r)dr, \tag{133}$$

and $G^{[k]}(g)$, $H^{[k]}(h)$ given in (111). Then, we define the linear operator $\mathcal{G}_\omega^{\kappa_1}$

$$\mathcal{G}_\omega^{\kappa_1}(f,g,h)=\begin{pmatrix} \sum_k F_{\kappa_1}^{[k]}(f)(v)e^{ik\tau} \\ \sum_k G^{[k]}(g)(v)e^{ik\tau} \\ \sum_k H^{[k]}(h)(v)e^{ik\tau} \end{pmatrix}. \tag{134}$$

Lemma 9.3. *Fix $\alpha\geq 1$ and $\sigma>0$. There exists $\kappa_1^0>0$ sufficiently small, such that, for $0<\varepsilon\leq\varepsilon_0$ and $0<\kappa_1\leq\kappa_1^0$, the operator*

$$\mathcal{G}_\omega^{\kappa_1}:\mathcal{X}^3_{\alpha+1,\sigma}\to\mathcal{Y}^3_{\alpha,\sigma}$$

is well-defined and satisfies:

(1) *$\mathcal{G}_\omega^{\kappa_1}$ is an inverse of the operator $\mathcal{L}_{\omega,\kappa_1}:\mathcal{Y}^3_{\alpha,\sigma}\to\mathcal{X}^3_{\alpha+1,\sigma}$, i.e. $\mathcal{G}_\omega^{\kappa_1}\circ\mathcal{L}_{\omega,\kappa_1}=\mathcal{L}_{\omega,\kappa_1}\circ\mathcal{G}_\omega^{\kappa_1}=\mathrm{Id}$;*

(2) $[\![\mathcal{G}_\omega^{\kappa_1}(f,g,h)]\!]_{\alpha,\sigma} \le M\|(f,g,h)\|_{\alpha+1,\sigma}$;

(3) *If* $f^{[0]} = g^{[1]} = h^{[-1]} = 0$, *then* $[\![\mathcal{G}_\omega^{\kappa_1}(f,g,h)]\!]_{\alpha,\sigma} \le \dfrac{M}{\omega}[\![(f,g,h)]\!]_{\alpha,\sigma}$,

where M *is a constant independent of* κ_1 *and* ε.

The proof of the following lemma is analogous to that in Lemma 10.3 below.

Lemma 9.4. *Let* F, X_{κ_1}, Z_{κ_1} *be given by* (8) *and* (27). *There exist* $\kappa_1^0 > 0$ *and a constant* $M > 0$ *such that, for* $v \in D^u$ *and* $0 < \kappa_1 \le \kappa_1^0$,

(1) $|F(X_{\kappa_1}(v))''| \le \dfrac{M}{|v^2+2|^{3/2}}$;

(2) $\left|\dfrac{Z'_{\kappa_1}(v)}{Z_{\kappa_1}(v)}\right| \le \dfrac{M}{|\sqrt{v^2+2}|}$.

Lemma 9.5. *Fix* $\sigma > 0$ *and* $K > 0$. *There exist* $\varepsilon_0 > 0$ *and* $h_0 > 0$ *sufficiently small such that, for* $0 < \varepsilon < \varepsilon_0$, $0 \le h \le h_0$ *and* $\kappa_1, \kappa_2 \ge 0$ *with* $\kappa_1 + \kappa_2 = h$, *the operator* $\mathcal{P}_{\kappa_1,\kappa_2} : \mathcal{Y}^3_{1,\sigma} \to \mathcal{X}^3_{2,\sigma}$, *is well defined and there exists a constant* $M > 0$ *such that*

$$\|\mathcal{P}_{\kappa_1,\kappa_2}(0,0,0)\|_{2,\sigma} \le M\frac{\delta}{\omega}.$$

Moreover, given $(z_j, \gamma_j, \theta_j) \in \mathcal{B}_0(K\delta/\omega) \subset \mathcal{Y}^3_{1,\sigma}$, $j = 1, 2$,

$$\left\|\mathcal{P}_{\kappa_1,\kappa_2}(z_1,\gamma_1,\theta_1) - \mathcal{P}_{\kappa_1,\kappa_2}(z_2,\gamma_2,\theta_2)\right\|_{2,\sigma} \le M\left(\delta + \frac{\delta}{\omega^{3/2}}\sqrt{h}\right)[\![(z_1,\gamma_1,\theta_1) - (z_2,\gamma_2,\theta_2)]\!]_{1,\sigma}. \tag{135}$$

Proof. Recall that $\mathcal{P}_{\kappa_1,\kappa_2}(0,0,0) = (f_1^{\kappa_1,\kappa_2}, f_2^{\kappa_1,\kappa_2}, -f_2^{\kappa_1,\kappa_2})$, where $f_1^{\kappa_1,\kappa_2}$, $f_2^{\kappa_1,\kappa_2}$ are given in (129), respectively, and involve the functions $F'(X_{\kappa_1})$, $Z'_{\kappa_1}/Z_{\kappa_1}$, Q^{κ_1}, $(Q^{\kappa_1})'$, Z_{κ_1,κ_2}, $\partial_v Z_{\kappa_1,\kappa_2}$ which can be computed using the expressions in (8), (27), (37), and (40). By Lemmas 8.6, 8.7 and 9.4, we have

$$\|Q^{\kappa_1}\|_{1,\sigma}, \|(Q^{\kappa_1})'\|_{2,\sigma} \le M\frac{\delta}{\omega},$$
$$\left\|Z_{\kappa_1,\kappa_2}\right\|_{1,\sigma}, \|\partial_v Z_{\kappa_1,\kappa_2}\|_{2,\sigma} \le M\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}},$$
$$\|Z'_{\kappa_1}/Z_{\kappa_1}\|_{1,\sigma}, \|F'(X_{\kappa_1})\|_{1,\sigma} \le M.$$

Therefore, using also Lemma 7.4, one has

$$\|f_1^{\kappa_1,\kappa_2}\|_{2,\sigma} \le M\max\left\{\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}, \frac{\delta^2}{\omega}, \frac{\delta^2}{\omega^3}\kappa_2\right\} = M\max\left\{\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}, \frac{\delta^2}{\omega}\right\},$$
$$\|f_2^{\kappa_1,\kappa_2}\|_{2,\sigma} \le M\max\left\{\frac{\delta}{\omega}, \frac{\delta^2}{\omega^{5/2}}\sqrt{\kappa_2}\right\} = M\frac{\delta}{\omega}.$$

Thus, $\|\mathcal{P}_{\kappa_1,\kappa_2}(0,0,0)\|_{2,\sigma} \leq M\delta/\omega$.

Following the lines of the proof of Lemma 7.7 one can complete the proof of Lemma 9.5. □

Now, we write Proposition 9.2 in terms of Banach spaces. Then, it can be proved in the same way as Proposition 7.8 by considering the operator $\overline{\mathcal{G}}_{\omega,\kappa_1,\kappa_2} = \mathcal{G}_\omega^{\kappa_1} \circ \mathcal{P}_{\kappa_1,\kappa_2}$.

Proposition 9.6. *Fix $\sigma > 0$. There exist $h_0 > 0$ and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, $0 < h \leq h_0$ and $\kappa_1, \kappa_2 \geq 0$ with $\kappa_1 + \kappa_2 = h$, the operator $\overline{\mathcal{G}}_{\omega,\kappa_1,\kappa_2} = \mathcal{G}_\omega^{\kappa_1} \circ \mathcal{P}_{\kappa_1,\kappa_2}$, with $\mathcal{G}_\omega^{\kappa_1}$ and $\mathcal{P}_{\kappa_1,\kappa_2}$ given in (134) and (132), respectively, has a fixed point $(z^u_{\kappa_1,\kappa_2}, \gamma^u_{\kappa_1,\kappa_2}, \theta^u_{\kappa_1,\kappa_2}) \in \mathcal{Y}^3_{1,\sigma}$. Furthermore, there exists a constant $M > 0$ independent of ε, κ_1 and κ_2 such that*

$$[\![(z^u_{\kappa_1,\kappa_2}, \gamma^u_{\kappa_1,\kappa_2}, \theta^u_{\kappa_1,\kappa_2})]\!]_{1,\sigma} \leq M\frac{\delta}{\omega}.$$

This completes the proof of Theorem 4.6.

10. Proof of Theorem 4.8

We compare the parameterizations of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ obtained in Sections 7, 8 and 9, respectively, with the parameterization (62) of $W^u_\varepsilon(p^-_0)$ obtained in Section 5.

10.1. Approximation of $W^u_\varepsilon(\Lambda^-_h)$ by $W^u_\varepsilon(p^-_0)$

We compare the parameterizations $N^u_{0,h}$ and $N^u_{0,0}$ of $W^u_\varepsilon(\Lambda^-_h)$ and $W^u_\varepsilon(p^-_0)$, obtained in Theorems 4.3 and 4.1, respectively.

Proposition 10.1. *Let $\Gamma^u_0(v)$, $\Theta^u_0(v)$ and $\Gamma^u_{0,h}(v,\tau)$, $\Theta^u_{0,h}(v,\tau)$ be given in (36) and (39), respectively. Given $h_0 > 0$, there exists $\varepsilon_0 > 0$ and a constant $M > 0$, such that for $v \in D^u \cap \mathbb{R}$, $\tau \in \mathbb{T}$, $0 \leq \varepsilon \leq \varepsilon_0$ and $0 \leq h \leq h_0$,*

$$\begin{aligned} \left|\partial_\tau(\Gamma^u_{0,h}(v,\tau) - \Gamma^u_0(v))\right|, \left|\Gamma^u_{0,h}(v,\tau) - \Gamma^u_0(v)\right| &\leq M\frac{\delta\sqrt{h}}{\omega^{3/2}}, \\ \left|\partial_\tau(\Theta^u_{0,h}(v,\tau) - \Theta^u_0(v))\right|, \left|\Theta^u_{0,h}(v,\tau) - \Theta^u_0(v)\right| &\leq M\frac{\delta\sqrt{h}}{\omega^{3/2}}. \end{aligned} \tag{136}$$

Proof. Considering $h = 0$ in Theorem 4.3, it follows that $N^u_{0,0}(v,\tau)$ is also a parameterization of $W^u_\varepsilon(p^-_0)$. Since $W^u_\varepsilon(p^-_0)$ is parameterized by both $N^u_{0,0}(v)$ (from Theorem 4.1) and $N^u_{0,0}(v,\tau)$ (from Theorem 4.3) and both have the same first coordinate, these parameterizations coincide. Therefore $\gamma^u_{0,0}$ and $\theta^u_{0,0}$ given in Theorem 4.3 with $h = 0$ depend only on the variable v and we can write

$$\begin{aligned} \Gamma^u_0(v) &= Q^0(v) + \gamma^u_{0,0}(v), \\ \Theta^u_0(v) &= -Q^0(v) + \theta^u_{0,0}(v). \end{aligned}$$

Based on these arguments, we can use Theorem 4.3 and Proposition 7.8 to see that

$$\begin{pmatrix} \Gamma^u_{0,h}(v,\tau) - \Gamma^u_0(v) \\ \Theta^u_{0,h}(v,\tau) - \Theta^u_0(v) \end{pmatrix} = \begin{pmatrix} \gamma^u_{0,h}(v,\tau) - \gamma^u_{0,0}(v) \\ \theta^u_{0,h}(v,\tau) - \theta^u_{0,0}(v) \end{pmatrix},$$

where $(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0})$ and $(z^u_{0,h}, \gamma^u_{0,h}, \theta^u_{0,h})$ are fixed points of the operators $\overline{\mathcal{G}}_{\omega,0}$ and $\overline{\mathcal{G}}_{\omega,h}$ given in (113), respectively.

Denoting

$$\mathcal{E} = (z^u_{0,h} - z^u_{0,0}, \gamma^u_{0,h} - \gamma^u_{0,0}, \theta^u_{0,h} - \theta^u_{0,0}),$$

we compute $\|\mathcal{E}\|_{1,\sigma}$.

Notice that

$$\begin{aligned}\mathcal{E} &= (z^u_{0,h} - z^u_{0,0}, \gamma^u_{0,h} - \gamma^u_{0,0}, \theta^u_{0,h} - \theta^u_{0,0}) \\ &= \overline{\mathcal{G}}_{\omega,h}(z^u_{0,h}, \gamma^u_{0,h}, \theta^u_{0,h}) - \overline{\mathcal{G}}_{\omega,h}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) \\ &\quad + \overline{\mathcal{G}}_{\omega,h}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) - \overline{\mathcal{G}}_{\omega,0}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}).\end{aligned}$$

For $0 \le h \le h_0$, $(z^u_{0,h}, \gamma^u_{0,h}, \theta^u_{0,h}) \in \mathcal{B}_0(M\delta/\omega)$ and $\overline{\mathcal{G}}_{\omega,h}$ is Lipschitz in this ball with $\mathrm{Lip}(\overline{\mathcal{G}}_{\omega,h}) \le M\delta$. Then,

$$[\![\overline{\mathcal{G}}_{\omega,h}(z^u_{0,h}, \gamma^u_{0,h}, \theta^u_{0,h}) - \overline{\mathcal{G}}_{\omega,h}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0})]\!]_{1,\sigma} \le M\delta[\![\mathcal{E}]\!]_{1,\sigma}.$$

Choosing ε_0 sufficiently small such that $\mathrm{Lip}(\overline{\mathcal{G}}_{\omega,h}) < 1/2$, we obtain

$$[\![\mathcal{E}]\!]_{1,\sigma} \le M[\![\overline{\mathcal{G}}_{\omega,h}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) - \overline{\mathcal{G}}_{\omega,0}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0})]\!]_{1,\sigma}.$$

Now, denoting $\mathcal{P}_h(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) - \mathcal{P}_0(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) = \Delta^0_h$, where $\mathcal{P}_h$ is given in (108), and using that $\left[\!\!\left[(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0})\right]\!\!\right]_{1,\sigma} \le M\delta/\omega$, we have that $\|\Delta^0_h\|_{2,\sigma} \le M\frac{\delta\sqrt{h}}{\omega^{3/2}}$.

It follows from the linearity of $\mathcal{G}_\omega$ and Lemma 7.5 that

$$\left[\!\!\left[\overline{\mathcal{G}}_{\omega,h}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) - \overline{\mathcal{G}}_{\omega,0}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0})\right]\!\!\right]_{1,\sigma} \le M\frac{\delta\sqrt{h}}{\omega^{3/2}}.$$

Thus, we conclude that $[\![\mathcal{E}]\!]_{1,\sigma} \le M\dfrac{\delta\sqrt{h}}{\omega^{3/2}}$. □

10.2. Approximation of $W^u_\varepsilon(p^-_h)$ by $W^u_\varepsilon(p^-_0)$

We compare the parameterizations $N^u_{0,0}$ and $N^u_{h,0}$ of $W^u_\varepsilon(p^-_0)$ and $W^u_\varepsilon(p^-_h)$, obtained in Theorems 4.1 and 4.5, respectively.

Proposition 10.2. *Let $\Gamma^u_0(v)$, $\Theta^u_0(v)$ and $\Gamma^u_{h,0}(v)$, $\Theta^u_{h,0}(v)$ be given in (36) and (42), respectively. There exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that, for $0 < \varepsilon \le \varepsilon_0$ and $0 \le h \le h_0$,*

(1) $\left|\Gamma^u_{h,0}(0) - \Gamma^u_0(0)\right| \le M\dfrac{\delta\sqrt{h}}{\omega^2}$;

(2) $\left|\Theta^u_{h,0}(0) - \Theta^u_0(0)\right| \leq M\dfrac{\delta\sqrt{h}}{\omega^2}$.

10.2.1. Technical lemmas

To prove Proposition 10.2, we first state some lemmas.

Lemma 10.3. *Let X_0, Z_0, X_h, Z_h, Q^0, and Q^h be given in (26), (27), (37) and (43) and fix $M_0 > 0$. There exist $h_0 > 0$ and a constant $M > 0$ such that, for $0 \leq h \leq h_0$ and $v \in D^u$ with $|h^{1/4}v| \leq M_0$,*

(1) $|F(X_h(v)) - F(X_0(v))| \leq \dfrac{M\sqrt{h}}{|\sqrt{v^2+2}|}$;

(2) $|Z_h(v) - Z_0(v)| \leq \dfrac{M\sqrt{h}}{|\sqrt{v^2+2}|}$;

(3) $\left|\dfrac{1}{Z_h(v)} - \dfrac{1}{Z_0(v)}\right| \dfrac{1}{|\sqrt{v^2+2}|} \leq M\sqrt{h}$;

(4) $\left|(Q^h)'(v) - (Q^0)'(v)\right| \leq \dfrac{M\delta\sqrt{h}}{\omega|v^2+2|}$.

Proof. Using the formulas (8), (26) and (27), we obtain

$$F(X_h(v)) - F(X_0(v)) = -2\left(\frac{\sqrt{\frac{2+h}{h}}\sinh(v\sqrt{h}/2)}{1+\frac{2+h}{h}\sinh^2(v\sqrt{h}/2)} - \sqrt{2}\frac{v}{v^2+2}\right).$$

Since $|vh^{1/4}| \leq M_0$, it follows that $|v\sqrt{h}/2| \leq Mh^{1/4} \ll 1$.

Expanding $\sinh(z)$ at 0, we have

$$\begin{aligned}\frac{\sqrt{\frac{2+h}{h}}\sinh(v\sqrt{h}/2)}{1+\frac{2+h}{h}\sinh^2(v\sqrt{h}/2)} &= \frac{\sqrt{\frac{2+h}{h}}\left(\frac{v\sqrt{h}}{2} + \mathcal{O}(h^{3/2}|v|^3)\right)}{1+\frac{2+h}{h}\left(\frac{v^2h}{4} + \mathcal{O}(h^2|v|^4)\right)} \\ &= \frac{\sqrt{2}v + \mathcal{O}(\sqrt{h}|v|)}{v^2+2+\mathcal{O}(\sqrt{h}|v|^2)} \\ &= \frac{\sqrt{2}v}{v^2+2}\left(1+\mathcal{O}(\sqrt{h})\right).\end{aligned}$$

Item (1) follows directly from this expression, considering h sufficiently small. Items (2) and (3) can be computed in an analogous way.

Formulas (37) and (43) imply

$$\left|(Q^h)'(v) - (Q^0)'(v)\right| \leq M\frac{\delta}{\omega}|F'(X_h(v))Z_h(v) - F'(X_0(v))Z_0(v)|.$$

Thus, it is enough to apply the bounds in items (1) and (2) to obtain item (4). □

Lemma 10.4. *Let η_0 and η_h be given in (66) and (118), respectively, and consider the functions (γ_0^u, θ_0^u) obtained in Proposition 5.6. Fix $M_0 > 0$. There exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, $0 \leq h \leq h_0$ and $v \in D^u$ with $|h^{1/4}v| \leq M_0$,*

$$\left|\eta_h(v, \gamma_0^u, \theta_0^u) - \eta_0(v, \gamma_0^u, \theta_0^u)\right| \leq \frac{M\delta\sqrt{h}}{\omega}.$$

Proof. Using the expression of η_h in (118) and that $\left\|(\gamma_0^u, \theta_0^u)\right\|_2 \leq M\delta/\omega^2 \ll 1$, it follows from Lemmas 8.6, 8.7 and 10.3 that

$$\begin{aligned}\left|\eta_h(v, \gamma_0^u, \theta_0^u) - \eta_0(v, \gamma_0^u, \theta_0^u)\right| &\leq M\frac{\delta}{\omega}\left|\left(\frac{F(X_h)}{Z_h}\right)^2 - \left(\frac{F(X_0)}{Z_0}\right)^2\right| + M\omega\left|\gamma_0^u\theta_0^u\right|\left|\frac{1}{Z_h^2} - \frac{1}{Z_0^2}\right| \\ &\leq \frac{M\delta\sqrt{h}}{\omega}. \quad \square\end{aligned}$$

10.2.2. Proof of Proposition 10.2

The domain D^u defined in (33) is contained in the domain D^u_ε defined in (31). Therefore, the restriction of the fixed point obtained in Section 5 can be seen as an element of the space $\mathcal{X}_2^2$ with the same bound.

Proposition 10.5. *Consider (γ_0^u, θ_0^u) and $(\gamma_{h,0}^u, \theta_{h,0}^u)$ obtained in Theorems 5.6 and 8.11, respectively, and the operator $\mathcal{G}_{\omega,h}$ given by (119). Then, there exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that for $0 \leq h \leq h_0$ and $0 < \varepsilon \leq \varepsilon_0$,*

$$\left\|\mathcal{G}_{\omega,h}(\gamma_{h,0}^u, \theta_{h,0}^u) - \mathcal{G}_{\omega,h}(\gamma_0^u, \theta_0^u)\right\|_0 \leq M\frac{\delta^2}{\omega}\left\|(\gamma_{h,0}^u, \theta_{h,0}^u) - (\gamma_0^u, \theta_0^u)\right\|_0.$$

Proof. By Proposition 8.10, we have

$$\left|\eta_h(v, \gamma_{h,0}^u, \theta_{h,0}^u) - \eta_h(v, \gamma_0^u, \theta_0^u)\right| \leq M\frac{\delta}{\omega}\left\|(\gamma_{h,0}^u, \theta_{h,0}^u) - (\gamma_0^u, \theta_0^u)\right\|_0.$$

Thus, using the expression of $\mathcal{F}_h$ in (120) and Proposition 8.10,

$$\begin{aligned}\left\|\pi_1(\mathcal{F}_h(\gamma_{h,0}^u, \theta_{h,0}^u) - \mathcal{F}_h(\gamma_0^u, \theta_0^u))\right\|_0 &\leq \omega\left\|\eta_h(v, \gamma_{h,0}^u, \theta_{h,0}^u) - 1\right\|_0\left\|\gamma_{h,0}^u - \gamma_0^u\right\|_0 \\ &\quad + \omega\left\|\gamma_0^u\right\|_0\left\|\eta_h(v, \gamma_{h,0}^u, \theta_{h,0}^u) - \eta_h(v, \gamma_0^u, \theta_0^u)\right\|_0 \\ &\leq M\delta^2\left\|\gamma_{h,0}^u - \gamma_0^u\right\|_0 \\ &\quad + M\delta\left\|\gamma_0^u\right\|_2\left\|(\gamma_{h,0}^u, \theta_{h,0}^u) - (\gamma_0^u, \theta_0^u)\right\|_0 \\ &\leq M\left(\delta^2 + \frac{\delta^2}{\omega^2}\right)\left\|(\gamma_{h,0}^u, \theta_{h,0}^u) - (\gamma_0^u, \theta_0^u)\right\|_0.\end{aligned}$$

The same bound can be obtained for the second coordinate of $\mathcal{F}_h$. Thus

$$\left\|\mathcal{F}_h(\gamma_{h,0}^u, \theta_{h,0}^u) - \mathcal{F}_h(\gamma_0^u, \theta_0^u)\right\|_0 \leq M\delta^2\left\|(\gamma_{h,0}^u, \theta_{h,0}^u) - (\gamma_0^u, \theta_0^u)\right\|_0.$$

Now, denote $\Delta_h^j = \pi_j \left(\mathcal{F}_h(\gamma_{h,0}^u, \theta_{h,0}^u) - \mathcal{F}_h(\gamma_0^u, \theta_0^u) \right)$, $j = 1, 2$, and $\Delta_h = (\Delta_h^1, \Delta_h^2)$. Then,

$$\left| \pi_1 \left(\mathcal{G}_{\omega,h}(\gamma_{h,0}^u, \theta_{h,0}^u) - \mathcal{G}_{\omega,h}(\gamma_0^u, \theta_0^u) \right)(v) \right| = \left| \int_{-\infty}^{0} e^{\omega i s} \Delta_h^1(s+v) ds \right|.$$

Since $\Delta_h \in \mathcal{X}_2^2$, we can change the path of integration to obtain

$$\begin{aligned}
\left| \int_{-\infty}^{0} e^{\omega i s} \Delta_h^1(s+v) ds \right| &= \left| \int_{-\infty}^{0} e^{\omega i e^{-i\beta} \xi} \Delta_h^1(e^{-i\beta}\xi + v) e^{i\beta} d\xi \right| \\
&\leq \int_{-\infty}^{0} e^{\omega \sin(\beta)\xi} |\Delta_h^1(e^{-i\beta}\xi + v)| d\xi \\
&\leq \|\Delta_h\|_0 \int_{-\infty}^{0} e^{\omega \sin(\beta)\xi} d\xi \\
&\leq \frac{M}{\omega} \|\Delta_h\|_0.
\end{aligned}$$

The same argument holds for the second coordinate of $\mathcal{G}_{\omega,h}(\gamma_{h,0}^u, \theta_{h,0}^u) - \mathcal{G}_{\omega,h}(\gamma_0^u, \theta_0^u)$. □

Lemma 10.6. *Let $\mathcal{F}_0$ and $\mathcal{F}_h$ be given in (69) and (120), respectively, and consider the functions (γ_0^u, θ_0^u) obtained in Theorem 5.6. Given $M_0 > 0$ fixed, there exist ε_0, $h_0 > 0$ and a constant $M > 0$ such that for $0 \leq h \leq h_0$, $0 < \varepsilon \leq \varepsilon_0$ and $v \in D^u$ with $|h^{1/4} v| \leq M_0$,*

$$\left| \pi_j \circ \mathcal{F}_h(\gamma_0^u, \theta_0^u)(v) - \pi_j \circ \mathcal{F}_0(\gamma_0^u, \theta_0^u)(v) \right| \leq \frac{M\delta\sqrt{h}}{\omega|v^2+2|}, \qquad j = 1, 2.$$

Proof. Lemmas 10.3 and 10.4 imply

$$\begin{aligned}
\left| \pi_1(\mathcal{F}_h(\gamma_0^u, \theta_0^u)(v) - \mathcal{F}_0(\gamma_0^u, \theta_0^u))(v) \right| &\leq |(Q^h)'(v) - (Q^0)'(v)| \\
&\quad + \omega \left|\gamma_0^u\right| \left| \eta_h(v, \gamma_0^u, \theta_0^u) - \eta_0(v, \gamma_0^u, \theta_0^u) \right| \\
&\leq M \frac{\delta\sqrt{h}}{\omega|v^2+2|}.
\end{aligned}$$

The same holds for the second coordinate. □

Proposition 10.7. *Consider the functions (γ_0^u, θ_0^u) obtained in Proposition 5.6 and the operators $\mathcal{G}_{\omega,0}$ and $\mathcal{G}_{\omega,h}$ given in (67) and (119), respectively. There exist $\varepsilon_0 >$, $h_0 > 0$ and a constant $M > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$ and $0 < h \leq h_0$*

$$\left\| \mathcal{G}_{\omega,h}(\gamma_0^u, \theta_0^u) - \mathcal{G}_{\omega,0}(\gamma_0^u, \theta_0^u) \right\|_0 \leq \frac{M\delta\sqrt{h}}{\omega^2}.$$

Proof. It follows from the proof of Proposition 8.11 that the Lipschitz constant of $\mathcal{G}_{\omega,h}$ in a ball $\mathcal{B}_0(K\delta/\omega^2)$, for some $K>0$ fixed, satisfies $\mathrm{Lip}(\mathcal{G}_{\omega,h})\leq M\delta^2/\omega$. Moreover, $\|\mathcal{G}_{\omega,h}(0,0)\|_2\leq M\delta/\omega^2$ and $\|(\gamma_0^u,\theta_0^u)\|_2\leq M\delta/\omega^2$. Thus

$$\|\mathcal{G}_{\omega,h}(\gamma_0^u,\theta_0^u)\|_2\leq\|\mathcal{G}_{\omega,h}(\gamma_0^u,\theta_0^u)-\mathcal{G}_{\omega,h}(0,0)\|_2+\|\mathcal{G}_{\omega,h}(0,0)\|_2\leq M\frac{\delta}{\omega^2}.$$

Moreover, $\|\mathcal{G}_{\omega,0}(\gamma_0^u,\theta_0^u)\|_2=\|(\gamma_0^u,\theta_0^u)\|_2\leq M\delta/\omega^2$.

Let $v\in D^u$ and first assume that $|h^{1/4}v|\geq 1$, hence

$$\begin{aligned}\left|\pi_j(\mathcal{G}_{\omega,h}(\gamma_0^u,\theta_0^u)(v)-\mathcal{G}_{\omega,0}(\gamma_0^u,\theta_0^u)(v))\right|&\leq\frac{\left\|\mathcal{G}_{\omega,h}(\gamma_0^u,\theta_0^u)\right\|_2}{|v^2+2|}+\frac{\left\|\mathcal{G}_{\omega,0}(\gamma_0^u,\theta_0^u)\right\|_2}{|v^2+2|}\\&\leq M\frac{\delta}{\omega^2||v|^2-2|}\\&\leq M\frac{\delta}{\omega^2(1/\sqrt{h}-2)}\\&\leq M\frac{\delta}{\omega^2}\sqrt{h},\end{aligned}$$

for $h>0$ sufficiently small, $j=1,2$.

Now, assume that $|h^{1/4}v|<1$, and denote $\Delta_h^j=\pi_j(\mathcal{F}_h(\gamma_0^u,\theta_0^u)-\mathcal{F}_0(\gamma_0^u,\theta_0^u))$, $j=1,2$.

Consider the path $s=e^{-i\beta}\xi$ (since $\Delta_h\in\mathcal{X}_2^2$) and let $\xi_0(v)\in\mathbb{R}$ be such that $v_0(v)=v+e^{-i\beta}\xi_0(v)$ is the unique point of intersection between the curve $\gamma(\xi)=v+e^{-i\beta}\xi$ and the circle S_h of radius $h^{-1/4}$ centered at the origin.

$$\begin{aligned}\left|\pi_1(\mathcal{G}_{\omega,h}(\gamma_0^u,\theta_0^u)-\mathcal{G}_{\omega,0}(\gamma_0^u,\theta_0^u))(v)\right|&=\left|\int_{-\infty}^{0}e^{\omega is}\Delta_h^1(s+v)ds\right|\\&=\left|\int_{-\infty}^{0}e^{-\omega ie^{-i\beta}\xi}\Delta_h^1(v+e^{-i\beta}\xi)e^{-i\beta}d\xi\right|\\&\leq\left|\int_{-\infty}^{\xi_0(v)}e^{-\omega ie^{-i\beta}\xi}\Delta_h^1(v+e^{-i\beta}\xi)e^{-i\beta}d\xi\right|\\&\quad+\left|\int_{\xi_0(v)}^{0}e^{-\omega ie^{-i\beta}\xi}\Delta_h^1(v+e^{-i\beta}\xi)e^{-i\beta}d\xi\right|.\end{aligned}$$

Notice that the points in the path $\gamma(\xi)=v+e^{-i\beta}\xi$ satisfy that $|\gamma(\xi)h^{1/4}|\geq 1$ for every $\xi\leq\xi_0(v)$ and $|\gamma(\xi)h^{1/4}|<1$ for every $\xi_0(v)<\xi<0$. Also, let $v_0^*(v)=e^{i\beta}v_0(v)$, and notice that $\mathrm{Im}(v_0^*(v))=\mathrm{Im}(v)$ and $|h^{1/4}v_0^*(v)|=1$. See Fig. 8.

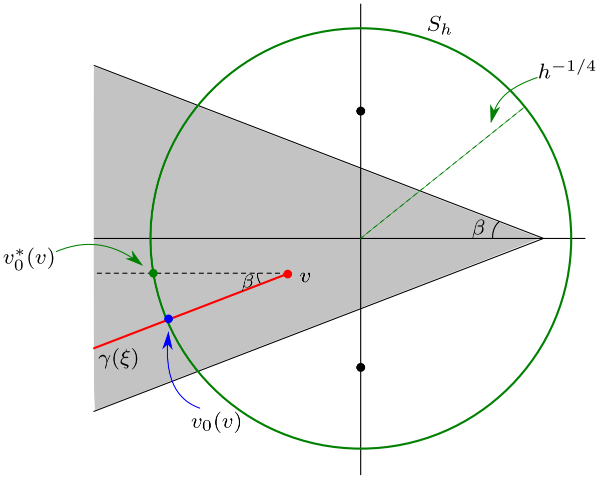

Fig. 8. Definition of the points $v_0(v)$ and $v_0^*(v)$.

Thus the first integral satisfies that

$$\begin{aligned}\left|\int_{-\infty}^{\xi_0(v)} e^{-\omega i e^{-i\beta}\xi}\Delta_h^1(v+e^{-i\beta}\xi)e^{-i\beta}d\xi\right| &= \left|\int_{-\infty}^{v_0^*(v)} e^{\omega i(v-r)}\Delta_h^1(r)dr\right| \\ &= \left|e^{\omega i(v-v_0^*(v))}\int_{-\infty}^{v_0^*(v)} e^{\omega i(v_0^*(v)-r)}\Delta_h^1(r)dr\right| \\ &= \left|\pi_1(\mathcal{G}_{\omega,h}(\gamma_0^u,\theta_0^u)(v_0^*(v)) - \mathcal{G}_{\omega,0}(\gamma_0^u,\theta_0^u)(v_0^*(v)))\right| \\ &\le M\frac{\delta\sqrt{h}}{\omega^2}.\end{aligned}$$

Now, since $|\gamma(\xi)h^{1/4}| < 1$ for every $\xi_0(v) < \xi < 0$, we can use Lemma 10.6 to see that the second integral satisfies

$$\begin{aligned}\left|\int_{\xi_0(v)}^{0} e^{-\omega i e^{-i\beta}\xi}\Delta_h^1(v+e^{-i\beta}\xi)e^{-i\beta}d\xi\right| &\le \int_{\xi_0(v)}^{0} e^{\omega\sin(\beta)\xi}|\Delta_h^1(v+e^{-i\beta}\xi)|d\xi \\ &\le \frac{M\delta\sqrt{h}}{\omega}\int_{-\infty}^{0} e^{\omega\sin(\beta)\xi}\frac{1}{|(v+e^{-i\beta}\xi)^2+2|}d\xi \\ &\le \frac{M\delta\sqrt{h}}{\omega|v^2+2|}\int_{-\infty}^{0} e^{\omega\sin(\beta)\xi}d\xi\end{aligned}$$

$$\leq \frac{M\delta\sqrt{h}}{\omega^2|v^2+2|}.$$

The result follows from these bounds. $\square$

Now, define $\mathcal{E}(v) = (\gamma^u_{h,0}(v) - \gamma^u_0(v), \theta^u_{h,0}(v) - \theta^u_0(v))$ and notice that

$$\begin{pmatrix} \Gamma^u_{h,0}(0) - \Gamma^u_0(0) \\ \Theta^u_{h,0}(0) - \Theta^u_0(0) \end{pmatrix} = \begin{pmatrix} Q^h(0) - Q^0(0) \\ -Q^h(0) + Q^0(0) \end{pmatrix} + \mathcal{E}(0)^T.$$

Using (37) and (43), we have $Q^h(0) = Q^0(0) = 0$. Hence, to prove Proposition 10.2, it is enough to bound $\|\mathcal{E}\|_0$. Since $(\gamma^u_{h,0}, \theta^u_{h,0})$ and (γ^u_0, θ^u_0) are fixed points of $\mathcal{G}_{\omega,h}$ and $\mathcal{G}_{\omega,0}$, respectively,

$$\begin{aligned} \mathcal{E} &= (\gamma^u_{h,0}, \theta^u_{h,0}) - (\gamma^u_0, \theta^u_0) \\ &= \mathcal{G}_{\omega,h}(\gamma^u_{h,0}, \theta^u_{h,0}) - \mathcal{G}_{\omega,h}(\gamma^u_0, \theta^u_0) + \mathcal{G}_{\omega,h}(\gamma^u_0, \theta^u_0) - \mathcal{G}_{\omega,0}(\gamma^u_0, \theta^u_0). \end{aligned}$$

It follows from Propositions 10.5 and 10.7 that

$$\begin{aligned} \|\mathcal{E}\|_0 &\leq \|\mathcal{G}_{\omega,h}(\gamma^u_{h,0}, \theta^u_{h,0}) - \mathcal{G}_{\omega,h}(\gamma^u_0, \theta^u_0)\|_0 + \|\mathcal{G}_{\omega,h}(\gamma^u_0, \theta^u_0) - \mathcal{G}_{\omega,0}(\gamma^u_0, \theta^u_0)\|_0 \\ &\leq M\delta^2\|\mathcal{E}\|_0 + \frac{M\delta\sqrt{h}}{\omega^2}. \end{aligned}$$

Thus, for ε_0 sufficiently small, we have that $\|\mathcal{E}\|_0 \leq 2\dfrac{M\delta\sqrt{h}}{\omega^2}$. This completes the proof.

10.3. Approximation of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ by $W^u_\varepsilon(p^-_0)$

In this section, we obtain an approximation of $N^u_{\kappa_1,\kappa_2}$ by $N^u_{0,0}$, by approximating $N^u_{\kappa_1,\kappa_2}$ by $N^u_{\kappa_1,0}$ and $N^u_{\kappa_1,0}$ by $N^u_{0,0}$.

Proceeding as for Proposition 10.1 and Lemma 9.5, one can obtain the next result.

Proposition 10.8. *Let $\Gamma^u_{\kappa_1,0}(v)$, $\Theta^u_{\kappa_1,0}(v)$ and $\Gamma^u_{\kappa_1,\kappa_2}(v,\tau)$, $\Theta^u_{\kappa_1,\kappa_2}(v,\tau)$ be given in (42) and (45), respectively. There exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that, for $v \in D^u \cap \mathbb{R}$, $\tau \in \mathbb{T}$, $0 \leq \varepsilon \leq \varepsilon_0$, $0 \leq h \leq h_0$ $\kappa_1, \kappa_2 \geq 0$ with $\kappa_1 + \kappa_2 = h$,*

$$\begin{aligned} \left|\partial_\tau(\Gamma^u_{\kappa_1,\kappa_2}(v,\tau) - \Gamma^u_{\kappa_1,0}(v))\right|, \left|\Gamma^u_{\kappa_1,\kappa_2}(v,\tau) - \Gamma^u_{\kappa_1,0}(v)\right| &\leq M\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}, \\ \left|\partial_\tau(\Theta^u_{\kappa_1,\kappa_2}(v,\tau) - \Theta^u_{\kappa_1,0}(v))\right|, \left|\Theta^u_{\kappa_1,\kappa_2}(v,\tau) - \Theta^u_{\kappa_1,0}(v)\right| &\leq M\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}. \end{aligned} \tag{137}$$

Notice that Proposition 10.2 allows us to approximate $N^u_{\kappa_1,0}$ by $N^u_{0,0}$, for κ_1 sufficiently small. Thus, we can combine this fact with Proposition 10.8 to obtain the following proposition.

Proposition 10.9. *Let $\Gamma^u_0(v)$, $\Theta^u_0(v)$ and $\Gamma^u_{\kappa_1,\kappa_2}(v,\tau)$, $\Theta^u_{\kappa_1,\kappa_2}(v,\tau)$ be given in (36) and (45), respectively. There exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that, for $v \in D^u \cap \mathbb{R}$, $\tau \in \mathbb{T}$, $0 \leq \varepsilon \leq \varepsilon_0$, $0 \leq h \leq h_0$ and $\kappa_1, \kappa_2 \geq 0$ with $\kappa_1 + \kappa_2 = h$,*

$$\begin{aligned} &\left|\Gamma^u_{\kappa_1,\kappa_2}(v,\tau)-\Gamma^u_0(v)\right|, \left|\Theta^u_{\kappa_1,\kappa_2}(v,\tau)-\Theta^u_0(v)\right| \le M\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}+M\frac{\delta\sqrt{\kappa_1}}{\omega^2}, \\ &\left|\partial_\tau(\Gamma^u_{\kappa_1,\kappa_2}(v,\tau)-\Gamma^u_0(v))\right|, \left|\partial_\tau(\Theta^u_{\kappa_1,\kappa_2}(v,\tau)-\Theta^u_0(v))\right| \le M\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}. \end{aligned} \tag{138}$$

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