Controller design for point to point trajectory generation through feedback linearization of nonlinear control systems

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In a control system, in order to go from any initial point to any final point a necessary and sufficient condition is needed: controllability. However, in a general controllable nonlinear system, it still is an open problem to find a constructive algorithm to generate the desired trajectory. However, this problem is solved for systems that are feedback linearizable: that is, systems which we can transform to equivalent linear systems for which a control law is easy to design. Via the inverse coordinate change, one can take back the designed trajectory in the linear system to obtain a trajectory for the original problem. In this paper, the algorithm for dynamic feedback linearization is studied, and it is applied to the specific example of the rolling disk. This example is then simulated using Matlab, where the good performance of the design is illustrated.

I. INTRODUCTION

Feedback linearization of nonlinear control systems has been actively studied over the last two decades [1] [2]. Its importance lies in the fact that it transfers the properties of linear systems to nonlinear ones. By means of a diffeomorphism and a feedback law it allows to obtain the Brunovsky form of the system, where control laws are easily implemented.

For the cases in which the system is not static feedback linearizable, one can try to extend it with new variables to obtain an overall linearizable system. In general, the prolongation problem is not solved and can be quite complicated [3]. However for two-input driftless systems, the method requires few computations and is explained in [4].

In this paper, dynamic feedback linearization is applied to the rolling disk motion problem. After obtaining the Brunovsky form, an algorithm is implemented to automatically obtain the feedback laws that guarantee a stable trajectory of the disk between two arbitrary points. Finally, the behaviour of the system is simulated with various initial and final conditions.

The paper is divided into three main parts: in Section III dynamic feedback linearization is applied to the rolling disk problem after the corresponding mathematical background has been established in Section II. In Section IV the method is tested through various simulations and the obtained results are discussed.

II. THEORETICAL FRAMEWORK

A. Definitions

Given a scalar function \( \lambda : \mathbb{R}^n \rightarrow \mathbb{R} \) and a vector field \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \), the Lie derivative of \( \lambda \) along \( f \) is a scalar function defined as:

\[
L_f \lambda (x) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(x)
\]

When the Lie derivative is iterated \( k \) times along \( f \), the following notation is used:

\[
L_f^k \lambda (x) = L_f \left( L_f^{k-1} \lambda (x) \right);
\]

\[
L_f^0 \lambda (x) = \lambda (x).
\]

Given two vector fields \( f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n \), the Lie bracket between them is a vector field defined as:

\[
[f, g] (x) = \frac{\partial g(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g(x)
\]

When the Lie bracket is iterated \( k \) times, the following notation is used:

\[
ad_f^k g = [f, ad_f^{k-1} g]; \quad ad_f^0 g = g.
\]

Given a set of \( k \) smooth vector fields, and \( x \in \mathbb{R}^n \), the subspace of \( \mathbb{R}^n \) defined by \( \Delta (x) = < f_1(x), ..., f_k(x) > \) is called a smooth distribution. A distribution is said to be involutive if \( [f, g](x) \in \Delta (x) \forall f, g \in \Delta (x) \). The involutivity condition holds for all the vectors in \( \Delta (x) \) if and only if it holds for the vectors \( f_1(x), ..., f_k(x) \).

B. Linearization algorithm of control systems

A linear control system with state variables \( x \in \mathbb{R}^n \) and control variables \( u \in \mathbb{R}^m \) is said to be in Brunovsky canonical form if it is expressed as:

\[
\dot{x} = \begin{pmatrix} B_{k_1} & 0_{k_1 \times k_2} & \cdots & 0_{k_1 \times k_m} \\ 0_{k_2 \times k_1} & B_{k_2} & \cdots & 0_{k_2 \times k_m} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{k_m \times k_1} & 0_{k_m \times k_2} & \cdots & B_{k_m} \end{pmatrix} x + \begin{pmatrix} e_{k_1} \\ \vdots \\ e_{k_m} \end{pmatrix} w
\]

where \( k_1 + \cdots + k_m = n, e_k \) is the \( (k_1 + \cdots + k_i) \)-th vector of the canonical basis expressed as a column vector, and
Given a multiple input nonlinear control system

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{1} \]

where \( x = (x_1, \ldots, x_n) \) and \( f, g_i : \mathbb{R}^n \to \mathbb{R}^n \) are vector fields, it is said to be linearizable by static feedback if there exist a change of coordinates \( z(x) \) and new control variables \( w \) with a regular feedback law \( v = \alpha(x) + \beta(x)w \) such that the system is in Brunovsky canonical form in terms of \( z \) and \( w \).

Whenever a system is feedback linearizable, the algorithm to linearize it consists of the following steps:

1. Construct the distributions
\[
D_0 = \langle g_1, \ldots, g_m \rangle \\
D_1 = \langle D_0, ad_f g_1, \ldots, ad_f g_m \rangle \\
\vdots \\
D_k = \langle D_{k-1}, ad_f^k g_1, \ldots, ad_f^k g_m \rangle
\]
where \( k \) is the minimum integer such that \( \dim D_k = n \).

2. Check that all the distributions \( D_i, i = 0, \ldots, k \) are involutive.

3. Define the indices
\[
r_0 = d_0 \\
r_i = d_i - d_{i-1}, \forall i = 1, \ldots, k
\]
where \( d_i = \dim D_i \). Define also
\[
k_j = |\{ r_i | r_i \geq j \}|, \forall j = 1, \ldots, m
\]

4. Find functions \( h_i(x) : \mathbb{R}^n \to \mathbb{R}, 1 \leq i \leq m \) such that \( dh_i(x) \perp D_{k_i-2} \) and such that \( h_i(x) \) is differentially independent from \( h_1(x), \ldots, h_{i-1}(x) \).

5. Finally, the change of variables is given by
\[
\begin{pmatrix}
  h_1 \\
  L_f h_1 \\
  \vdots \\
  L_f^{k_i-1} h_1 \\
  \vdots \\
  h_m \\
  L_f h_m \\
  \vdots \\
  L_f^{k_m-1} h_m
\end{pmatrix}
\]

The condition that \( h_1(x), \ldots, h_m(x) \) are differentially independent is equivalent to the condition that the Jacobian matrix of \( z \) is invertible. The feedback law is given by:
\[
w = \begin{pmatrix}
  L_f^{r_1} h_1 \\
  \vdots \\
  L_f^{r_m} h_m
\end{pmatrix} + \begin{pmatrix}
  L_g 1 h_1 \\
  \vdots \\
  L_g m h_m
\end{pmatrix} v
\]

With this algorithm, the control system in terms of the state variables \( z \) and the control variables is linear and it is in Brunovsky canonical form, with \( m \) blocks of sizes \( k_1, \ldots, k_m \).

### C. Dynamic feedback linearization

Given a nonlinear control system with \( m \) inputs:
\[
\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i
\]

A prolongation of the system is given by:
\[
\begin{cases}
  \dot{z}_1 = f(x) + \sum_{i=1}^{m} g_i(x)z_i^1 \\
  \vdots \\
  \dot{z}_i = f(x) + \sum_{i=1}^{m} g_i(x)z_i^i \\
  \vdots \\
  \dot{z}_{k_i} = z_{k_i} \\
  \dot{z}_{k_i} = v_i
\end{cases}
\]

Whenever a system is not linearizable by static feedback (for instance, if it is driftless, which means that \( f(x) = 0 \)) it may still be linearizable if a suitable prolongation is found in which the new control system is linearizable by static feedback.

Linearization by prolongations is a particular case of linearization by dynamic feedback. Although no sufficient and necessary condition exists to determine if a system is linearizable by dynamic feedback, a finite algorithm is known to determine whether a system is linearizable by prolongations or not.

### III. ROLLING DISK MOTION CONTROL

#### A. State-space model

Consider a thin flat disk rolling without slipping on a plane. The state variables describing the system are the coordinates \((x, y)\) of the contact point of the disk with the plane, the angle \( \theta \) that the disk makes with the horizontal axis and the angle \( \phi \) of a fixed line in the disk with the vertical axis (Figure 1).
FIG. 1. State-space characterization of the motion of a disk rolling without slipping [5]

With this configuration, defining \( \rho \) as the radius of the disk, the state equations of the system are the following:

\[
\begin{aligned}
\dot{x} - \rho \cos \theta \dot{\phi} &= 0 \\
\dot{y} - \rho \sin \theta \dot{\phi} &= 0
\end{aligned}
\]

This control system is characterized by four state variables \( x_1 = x \), \( x_2 = y \), \( x_3 = \phi \), \( x_4 = \dot{\phi} \) and two input variables: \( u_1 = \phi \) and \( u_2 = \dot{\phi} \), which are the rate of turning about the vertical axis and the rate of rolling respectively. This transforms system (3) into an affine non-linear system:

\[
\begin{aligned}
\dot{x}_1 &= \rho \cos x_3 u_1 \\
\dot{x}_2 &= \rho \sin x_3 u_1 \\
\dot{x}_3 &= u_2 \\
\dot{x}_4 &= u_1
\end{aligned}
\]

Since the resulting system is driftless, it is not feedback linearizable. In this example, it is suitable to perform a prolongation of order 2 on the second input \( u_2 \):

\[
\begin{aligned}
x_5 &= u_2 \\
x_6 &= \dot{u}_2 = \dot{x}_5 \\
v_1 &= \dot{x}_6 \\
v_2 &= \ddot{u}_2 = \ddot{x}_6
\end{aligned}
\]

Here, \( x_5 \) and \( x_6 \) are the prolonged state-space variables and \( v_1 \) and \( v_2 \) are the new control inputs. Therefore the complete prolonged system is:

\[
\begin{aligned}
\dot{x}_1 &= \rho \cos x_3 v_1 \\
\dot{x}_2 &= \rho \sin x_3 v_1 \\
\dot{x}_3 &= x_5 \\
\dot{x}_4 &= v_1 \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= v_2
\end{aligned}
\]

**B. Static feedback linearization**

To apply the state-feedback linearization algorithm it is convenient to rewrite the overall system (4) in the usual notation introduced in (1):

\[
\begin{aligned}
\dot{x} &= f(x) + v_1 g_1(x) + v_2 g_2(x) \\
f(x) &= x_5 \frac{\partial}{\partial x_3} + x_6 \frac{\partial}{\partial x_5} \\
g_1(x) &= \rho \cos x_3 \frac{\partial}{\partial x_3} + \rho \sin x_3 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} \\
g_2(x) &= \rho \frac{\partial}{\partial x_6}
\end{aligned}
\]

where \( \frac{\partial}{\partial x_i} \) denotes the \( i \)-th vector in the canonical basis. The steps of the linearization algorithm of control systems can now be applied:

1. Whenever \( x_5 \neq 0 \), the distributions are:

\[
\begin{aligned}
D_0 &= \langle \rho \cos x_3 \frac{\partial}{\partial x_3} + \rho \sin x_3 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5} \rangle \\
D_1 &= \langle - \sin x_3 \frac{\partial}{\partial x_1} + \cos x_3 \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_5} \rangle \\
D_2 &= \mathbb{R}^6
\end{aligned}
\]

2. It can be easily checked that \( D_0 \), \( D_1 \) and \( D_2 \) are involutive and their dimensions are \( d_0 = 2 \), \( d_1 = 4 \) and \( d_2 = 6 \). Therefore the prolonged system is indeed state-feedback linearizable.

3. The indices are \( r_0 = d_0 = 2 \), \( r_1 = d_1 - d_0 = 2 \) and \( r_2 = d_2 - d_1 = 2 \). Therefore \( k_1 = k_2 = 3 \).

4. It can be easily checked that the following functions satisfy \( dh_1 \perp D_1 \) and \( dh_2 \perp D_1 \):

\[
\begin{aligned}
h_1(x) &= x_3 \\
h_2(x) &= \cos x_3 x_1 + \sin x_3 x_2 - \rho x_4
\end{aligned}
\]

The resulting change of variables, which is a diffeomorphism, is given by the following equations:

\[
\begin{aligned}
z_1 &= x_3 \\
z_2 &= x_5 \\
z_3 &= x_6 \\
z_4 &= x_1 \cos x_3 + x_2 \sin x_3 - \rho x_4 \\
z_5 &= -x_5 (x_1 \sin x_3 - x_2 \cos x_3) \\
z_6 &= -x_2^2 (x_1 \cos x_3 + x_2 \sin x_3) \\
&\quad - x_6 (x_1 \sin x_3 - x_2 \cos x_3)
\end{aligned}
\]

Finally, the feedback law for the new control variables \( v_1, v_2 \) can be found using equation (2).

**C. Control design**

Given initial and final conditions (at time \( T \)) for \( x = (x_1, \ldots, x_6) \), which are denoted by \( x_0 \) and \( x_f \), the aim is to design controls \( v_1, v_2 \) such that the dynamic system given in (4) with initial condition \( x(0) = x_0 \) evolves to \( x(T) = x_f \).
It is much easier to work in the linearized system. There, the condition that the trajectories are polynomials \( y_1, y_2 \) of degree \( 2k_1 - 1 \) and \( 2k_2 - 1 \) is imposed. In total, \( 2n \) coefficients have to be determined.

The variables \( z \) can be divided into two blocks of three components each, according to the Brunovsky canonical form: \( z = (\tilde{z}_1, \tilde{z}_2) \). Then, \( \tilde{z}_1 = (y_1, \tilde{y}_1, \ldots, \tilde{y}_i^{(k_i-1)}) \).

Then, it is imposed that \( (y_1(0), \tilde{y}_1(0), \ldots, \tilde{y}_i^{(k_i-1)}(0)) \) and \( (y_1(T), \tilde{y}_1(T), \ldots, \tilde{y}_i^{(k_i-1)}(T)) \) fulfill the starting and final conditions \( z(x_0) \) and \( z(x_f) \), which can be achieved if \( y_i(t) \) is a polynomial of degree \( 2k_i - 1 \).

The controls for the linearized control system are found as:

\[
    w_1(t) = \frac{d^{k_1}}{dt^{k_1}} y_1(t); \quad w_2(t) = \frac{d^{k_2}}{dt^{k_2}} y_2(t)
\]

Finally, the control variables \( v_1, v_2 \) are recovered inverting the feedback law given in (2) and with these controls, the system in (4) is a system of ODEs, which can be solved given an initial condition \( x_0 \) and check if at time \( T \) the system evolves to the desired final state \( x_f \).

**IV. SIMULATIONS**

The system was simulated using Matlab. Computations regarding vector fields were carried out using the Symbolic Math Toolbox [6] and the functions found in [7], and the resulting ODE was numerically solved using the standard Runge-Kutta method implemented in ode45.

The behaviour of the system was then simulated using different initial and final conditions. In Figures 2 and 3, two different examples of the trajectory of the disk in the \( xy \) plane are shown. Fixing \( \rho = 1, T = 1 \), the initial conditions for the examples are:

- \( x_{01} = (0, 0, \frac{\pi}{2}, 0, 1, 2) \);
- \( x_{f1} = (3, 4, \frac{3\pi}{2}, 0, 4, 3) \);
- \( x_{02} = (0, 0, -\pi, 0, 1, 2) \);
- \( x_{f2} = (3, 4, -\frac{\pi}{2}, 0, 4, 3) \);

It can be observed that there are some sharp turns in both trajectories. This is consistent with the motion of a disk: it is not difficult to imagine a disk stopping and turning around at these points. It can also be observed that the disk takes a huge detour to go from \((0, 0)\) to \((3, 4)\), most importantly in the second example, where it reaches coordinates of magnitude 100. This should not be surprising: not only are we fixing the start and the end points in the \( xy \) plane, but also the boundary conditions for \( \phi, \theta, \dot{\phi}, \dot{\theta} \), thus highly constraining the possible trajectories.

A main drawback is that the system does not work for arbitrary initial conditions. In the present model, it was assumed that \( x_5 \neq 0 \) in all points of the trajectory, because otherwise the distribution would not be involutive and the variable change \( z \) would not be invertible. Hence, a trajectory can’t cross the hypersurface \( x_5 = 0 \), as would occur, for instance, if the initial and final values of \( x_5 \) have different signs.

**V. CONCLUSIONS**

Under suitable conditions, it is possible to construct an algorithm to linearize a given nonlinear system. The problem of point to point trajectory generation is then easy to solve, and a solution for the original problem can then be recovered via the inverse change of variables. The above-mentioned algorithm has been applied to the case of a rolling disk, and the Matlab simulations have shown the good behaviour of the system. Even though the trajectories are constrained by the condition \( x_5 \neq 0 \) (which implies that it is not always possible to find a trajectory for arbitrary start and end points), it should be noted that there is no known algorithm to design a controller for a general system.

![FIG. 2. First simulation of the disk trajectory in the \( xy \) plane](image1.png)

![FIG. 3. Second simulation of the disk trajectory in the \( xy \) plane](image2.png)


