Analysis and Applications

Spatial Estimates for Kelvin-Voigt Finite Elasticity with Nonlinear Viscosity: well behaved solutions in space.

RAMON QUINTANILLA

Dep. Matematiques, ESEIAAT, Universitat Politecnica de Catalunya, Colom, 11. Terrassa, Barcelona, 08222, Spain
Ramon.Quintanilla@upc.edu.

GIUSEPPE SACCOMANDI*

Dipartimento di Ingegneria, Università degli Studi di Perugia, Via G. Duranti, Perugia, 06125 Italy
giuseppe.saccomandi@unipg.it

Received (Day Month Year)
Revised (Day Month Year)

We provide some spatial estimates for the nonlinear partial differential equation governing anti-plane motions in a nonlinear viscoelastic theory of Kelvin-Voigt type when the viscosity is a function of the strain rate. The spatial estimates we prove are an alternative of Phragmen-Lindelöf type. These estimates are possible when a precise balance between the elastic and viscoelastic nonlinearities holds.

Keywords: Anti-plane shear; nonlinear visco-elasticity; nonlinear viscosity; spatial estimates.

Mathematics Subject Classification 2000: 22E46, 53C35, 57S20

1. Introduction

One of the simplest models of nonlinear viscoelasticity is based on the classical Kelvin-Voigt model. This model is obtained considering that the Cauchy Stress, $\mathbf{T}$, can be additively split in a hyperelastic part $\mathbf{T}^{ela}$ and a viscous part $\mathbf{T}^{visco}$. As usual for the elastic part, in the incompressible case, we have

$$\mathbf{T}^{ela} = -p\mathbf{I} + 2\frac{\partial \mathcal{W}}{\partial I_1}\mathbf{B} - 2\frac{\partial \mathcal{W}}{\partial I_2}\mathbf{B}^{-1}, \tag{1.1}$$

where $\mathbf{I}$ is the identity tensor, p is the Lagrange multiplier associated with the incompressibility constraint [2], $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green strain tensor, $\mathbf{F}$

*Adjunct professor at the School of Mathematics, Statistics and Applied Mathematics, NUI Galway, University Road Galway, Ireland.

is the gradient of deformation, $I_1 = \operatorname{trace}\mathbf{B}$ and $I_2 = \operatorname{trace}\mathbf{B}^{-1}$, and $\mathcal{W} = \mathcal{W}(I_1, I_2)$ is the strain-energy density function. The viscous part is given by

$$\mathbf{T}^{visco} = \nu\mathbf{D}, \tag{1.2}$$

where $\mathbf{D}$ is the symmetric part of the stretching tensor $\dot{\mathbf{F}}\mathbf{F}^{-1}$ and $\nu \geq 0$ is the viscosity. The viscosity may be a constant, as in the classical Navier-Stokes equations, or a function of the invariants of $\mathbf{B}$ and $\mathbf{D}$ as for example I_1, I_2 and:

$$\operatorname{trace}(\mathbf{D}^2), \quad \operatorname{trace}(\mathbf{BD}), \quad \operatorname{trace}(\mathbf{B}^2\mathbf{D}), \quad \operatorname{trace}(\mathbf{BD}^2), \quad \operatorname{trace}(\mathbf{D}^2\mathbf{B}^2). \tag{1.3}$$

The model introduced here has been considered in many papers. For example, in the quasi-static case general results have been obtained in [3] and in [21] and both contributions are in the case when ν is constant or is a function of the various invariants in (1.3). In the full dynamical case an application of our model has been used in the framework of the propagation of shear waves [5,30]. In nonlinear acoustics a Kelvin-Voigt viscous term with constant viscosity is always introduced to prevent the blow-up of shear waves in computational studies [6].

From a mathematical point of view the case of a viscosity function (i.e. non constant viscosity) is still an open problem. Local existence of solutions has been established in [15,34], but a global existence result is possible only when the viscosity function is bounded from above and below.

In the present paper we restrict our attention to a special subclass of the general nonlinear Kelvin-Voigt constitutive equation. First of all, we consider only generalized neo-Hookean material such that $\mathcal{W} = \mathcal{W}(I_1)$. Second, we consider viscous functions which depend only on the invariant $\operatorname{trace}(\mathbf{D}^2)$. The reason of this choice is to simplify as much as possible the mathematical presentation, remaining in a framework of interest for applications. Indeed, there are many real materials with a non constant viscosity function as for example wheat-flour dough [20], brain matter [31] or hydrogels [4]. A modern overview on the mechanics and rheology viscoelastic materials is provided by [16].

Moreover we consider a shear motion

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + u(X_1; t),$$

or an anti-plane shear motion[a]

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + u(X_1, X_2; t),$$

the balance equations reduce to a single scalar equation in the unknown u. This equation reads

$$\rho_0 u_{,tt} = \left(Qu_{,i} + \nu u_{,it}\right)_{,i}, \tag{1.4}$$

[a]It is well known that for generalized neo-Hookean materials this motion may be sustained [23].

where $i = 1$ for a shear motion, and $i = 1, 2$ for an anti-plane shear motion. Here, we assume a null gradient of pressure along x_3, and we point out that the generalized shear modulus is a function

$$Q = Q(|\nabla u|^2),$$

whereas the viscosity ν may be a constant or a function of any combination of the invariants I_1, I_2 and (1.3). As usual ρ_0 is the constant density.

If the viscosity is constant and the function Q is convex, then equation (1.4) is well posed and this globally in time. This is a classical result obtained in [11] for a one dimensional shear wave and in [8] for the anti-plane shear case. This result can be extended to the case of a non-convex generalized shear modulus [1].

When the viscosity is a function of the invariants, the situation is much more complex and, as we have already pointed out, the finite time blow-up cannot be avoided. A situation that seems to be paradoxical. For a recent up to date and new significant results on such a problem we refer to [33]. Some explicit examples of blow-up in time can be found in [22]

In this framework, it is interesting to understand if the same problem we have time occurs in space. Spatial estimates are important tools to investigate qualitative properties of partial differential equations. For example, these estimates are necessary to ensure that end effects are under control: an important mathematical information for a rigorous design of experimental tests [12]. Clearly if a problem does not enjoy of a global existence in time these estimates are of limited mechanical significance, but a formal theory is possible in any case.

The aim of the present paper is to extend to a non constant viscosity framework the results obtained in [27,26,7] for constant viscosity Kelvin-Voigt nonlinear materials. Our findings are quite interesting because we find that the spatial estimates are possible but only when there is a certain balance among the elastic and the viscoelastic nonlinearities. The nonlinearity of the dissipative part and the nonlinearity of the purely elastic part of the constitutive equations must be in a certain balance. This is another counter-intuitive result in the framework of the nonlinear theory of viscoelasticity: a major challenge in the mathematical theory of continuum mechanics.

2. Setting of the problem

The aim of this paper is to give some spatial estimates for the following problem

$$\rho_0 \hat{u}_{,\hat{t}\hat{t}} = \left[\hat{Q}(|\nabla \hat{u}|^2)\hat{u}_{,i} + \hat{\nu}(|\nabla \hat{u}_{\hat{t}}|^2)\hat{u}_{,i\hat{t}}\right]_{,i}, \tag{2.1}$$

in a semi-infinite strip $\mathcal{R} = [0, \infty) \times [0, l]$ with the boundary conditions

$$\hat{u}(\hat{X}_1, 0, \hat{t}) = \hat{u}(\hat{X}_1, l, \hat{t}) = 0, \quad \hat{X}_1 \geq 0, \ \ \hat{t} > 0 \tag{2.2}$$

$$\hat{u}(0, \hat{X}_2, \hat{t}) = \hat{f}(\hat{X}_2, \hat{t}), \quad \hat{X}_2 \in (0, l), \ \ \hat{t} > 0, \tag{2.3}$$

and the initial conditions

$$\hat{u}(\hat{\mathbf{X}},0)=\hat{u}_{\hat{t}}(\hat{\mathbf{X}},0)=0,\ \ \hat{\mathbf{X}}\in\mathcal{R}. \tag{2.4}$$

In this model the strain-energy density is function only of I_1 and the viscosity ν is function of the strain rate, i.e. of the invariant trace $\mathbf{D}^2$.

The first step is to introduce a set of dimensionless variables:

$$\hat{X}_i=X_i l,\quad \hat{u}=ul,\quad \hat{t}=\varpi t,\quad \hat{Q}=\mu_0 Q,\quad \hat{\nu}=\nu_0\nu.$$

Therefore

$$\varrho u_{,tt}=\left[\upsilon Q\left(|\nabla u|^2\right)u_{,i}+\nu\left(\varpi^{-2}|\nabla u_t|^2\right)u_{,it}\right]_{,i}, \tag{2.5}$$

and

$$\varrho=\frac{\rho_0 l^2}{\varpi\nu_0},\quad \upsilon=\frac{\varpi\mu_0}{\nu_0}.$$

Clearly ϱ is the Reynolds number and υ is an analogue for solid mechanics of the Weissenberg number.

Abusing the notation the domain is the strip $\mathcal{R}=[0,\infty)\times[0,1]$ and the boundary-initial conditions are

$$u(X_1,0,t)=u(X_1,1,t)=0,\quad X_1\geq 0,\ \ t>0,$$

$$u(0,X_2,t)=f(X_2,t),\quad X_2\in(0,1),\ \ t>0, \tag{2.6}$$

$$u(\mathbf{X},0)=\dot{u}(\mathbf{X},0)=0,\ \ \mathbf{X}\in\mathcal{R}.$$

The following hypotheses are imposed on the functions $Q(s^2)$ and $\nu(\dot{s}^2)$:

(i) There exist three positive constants A,B and $p>1$ such that

$$|sQ(s^2)|\leq AW(s^2)^{1/2}+BW(s^2)^{1/p}. \tag{2.7}$$

where $\partial W/\partial s=2Q(s)$ and $W(0)=0$.

(ii) There exist five positive constants D_1,D_2,C_1,C_2 and q such that

$$D_1\dot{s}^2+C_1\dot{s}^q\leq\nu(\dot{s}^2)\dot{s}^2\leq D_2\dot{s}^2+C_2\dot{s}^q. \tag{2.8}$$

It is important to point out that

- The constants A,B,C_1,C_2,D_1 and D_2 are independent of p and q.
- The number p and q must be related in the sense that $q^*=p/(p-1)$ therefore the relationship

$$q^*\leq q<2q^*. \tag{2.9}$$

must be satisfied.

Condition (2.9) links the growth condition for the generalised shear modulus with the viscosity function.

An example of functions satisfying the conditions (i) and (ii) is obtained considering power-law materials

$$\mathcal{W}(I_1) = \frac{\mu}{k(n+1)}\{[1 + k(I_1 - 3)]^{n+1} - 1\},$$

where μ is the infinitesimal shear modulus and k and n are two constitutive parameters. This is a well known strain-energy density function describing strain hardening materials if $n > 0$ and stress softening materials if $0 > n > -1$, for $n = 0$ we recover the neo-Hookean material.

Because we consider the dimensionless version[b] of the strain energy density we have $Q(s^2) = (1 + ks^2)^n$. In the Appendix we give the detailed proof that this material model satisfies conditions (i). The function W should not confused with the dimensionless strain-energy density.

As typical viscosity function let us consider

$$\nu(\dot{s}^2) = (1 + \hat{k}\dot{s}^2)^m$$

where $\hat{k} > 0$ and m are positive constants. The condition (ii) is clearly satisfied with $q = 2(m + 1)$.

Condition (2.9) is satisfied when

$$n \le m \le 2n + 1.$$

Without assumptions (i) and (ii) we are not able to prove the asymptotic behaviour of our solutions but these assumptions are not only a mathematical *caprice*. The mechanical information that these mathematical condition are conveying is that the viscosity has to be not too weak and not too strong with respect the elastic strain strength of the material. This is clear if we are considering a strain-hardening material. In the case of stress softening materials the viscosity in any case must be very *mild*. Just to have an idea for a neo-Hookean material, which in rectilinear and anti-plane shear acts as a linear material [19], it must be $m < 2$. Therefore, the interplay between the strength of the elastic part and the viscous part of the material must be adequately balanced.

We end up warning the reader that to derive our estimates we need to study a nonlinear ordinary differential inequality of the form

$$\frac{\partial G}{\partial z} + \Phi(|G|) \le 0, \tag{2.10}$$

for a function of the solutions $G(z, t)$, where $\Phi(|G|)$ is a suitable function. This kind of differential inequalities has been considered by several authors [13,17,18,26].

[b]For this reason in $Q = Q(s^2)$ and $\nu = \nu(\dot{s}^2)$ the infinitesimal shear modulus and the infinitesimal viscosity are always normalized to 1.

3. Phragmen-Lindelöf Alternatives

In this section we obtain our main results concerning the initial-boundary value problem determined by equation (2.5) with null initial conditions and boundary conditions (2.6) whenever we assume (2.7) and (2.8). These results consists on a Phragmen-Lindelöf alternative for the case $2 < q^* < \infty$ (i.e. $1 < p < 2$) and a similar alternative for the case $1 < q^* < 2$ (and therefore $p > 2$). For the special case $q = q^*$ it is again possible to obtain an exponential alternative.

3.1. *Case* $2 < q^* < \infty$

The analysis starts by considering the function

$$G(z,t) = -\int_0^t \int_0^1 \exp(-\omega\tau)(Qu_{,1} + \nu u_{,1\tau})u_{,\tau} dx_2 d\tau. \tag{3.1}$$

If we apply the divergence theorem and make use of the null initial conditions and the homogeneous boundary conditions on the lateral sides of the strip we obtain that the equality

$$G(z,t) = G(z_0,t) - \int_0^t \int_{z_0}^z \int_0^1 \exp(-\omega\tau)\left(\frac{\varrho\omega u_{,\tau}^2}{2} + \omega W + \nu|\nabla u_{,\tau}|^2\right) da d\tau$$

$$-\frac{1}{2}\int_{z_0}^z \int_0^1 \exp(-\omega t)(\varrho u_{,t}^2 + 2W) da,$$

is satisfied, where $da = dx_1 dx_2$.

Computing

$$\frac{\partial G}{\partial z} = -\int_0^t \int_0^1 \exp(-\omega\tau)\left(\frac{\varrho\omega u_{,\tau}^2}{2} + \omega W + \nu|\nabla u_{,\tau}|^2\right) dx_2 d\tau$$

$$-\frac{1}{2}\int_0^1 \exp(-\omega t)(\varrho u_{,t}^2 + 2W) dx_2,$$

the estimate of the absolute value of the function $G(z,t)$ in terms of the spatial derivative gives

$$|G(z,t)| \leq \int_0^t \int_0^1 \exp(-\omega\tau)(AW^{1/2} + BW^{1/p} + \nu|\nabla u_{,\tau}|)|u_{,\tau}|dx_2 d\tau$$

$$\leq A\left(\int_0^t \int_0^1 \exp(-\omega\tau)W dx_2 d\tau\right)^{1/2} \left(\int_0^t \int_0^1 \exp(-\omega\tau)|u_{,\tau}|^2 dx_2 d\tau\right)^{1/2}$$

$$+ B\left(\int_0^t \int_0^1 \exp(-\omega\tau)W dx_2 d\tau\right)^{1/p} \left(\int_0^t \int_0^1 \exp(-\omega\tau)|u_{,\tau}|^{q^*} dx_2 d\tau\right)^{1/q^*}$$

$$+ C_2\left(\int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^2 dx_2 d\tau\right)^{1/2} \left(\int_0^t \int_0^1 \exp(-\omega\tau)|u_{,\tau}|^2 dx_2 d\tau\right)^{1/2}$$

$$+ D_2\left(\int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^q dx_2 d\tau\right)^{1/p^*} \left(\int_0^t \int_0^1 \exp(-\omega\tau)|u_{,\tau}|^q dx_2 d\tau\right)^{1/q}.$$

Here we use Hölder's inequality and the estimates (2.7) and (2.8) where $p^* = q/(q-1)$.

Poincaré's inequality implies that

$$|G(z,t)| \leq A\left(\int_0^t \int_0^1 \exp(-\omega\tau)W dx_2 d\tau\right)^{1/2} \left(\int_0^t \int_0^1 \exp(-\omega\tau)|u_{,\tau}|^2 dx_2 d\tau\right)^{1/2}$$

$$+ B_1\left(\int_0^t \int_0^1 \exp(-\omega\tau)W dx_2 d\tau\right)^{1/p} \left(\int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^{q^*} dx_2 d\tau\right)^{1/q^*} \tag{3.2}$$

$$+ D_3\left(\int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^2 dx_2 d\tau\right)^{1/2} \left(\int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^2 dx_2 d\tau\right)^{1/2}$$

$$+ C_3\left(\int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^q dx_2 d\tau\right)^{1/p^*} \left(\int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^q dx_2 d\tau\right)^{1/q}.$$

Here B_1, D_3 and C_3 can be easily calculated in terms of the parameters B, C_1 and the Poincaré constant.

If we take into account the inequality

$$\left(\int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^{q^*} dx_2 d\tau\right)^{1/q^*} \leq F(t)\left(\int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^q dx_2 d\tau\right)^{1/q},$$

where

$$F(t) = \left(\omega^{-1}[1-\exp(-\omega t)]\right)^{(q-q^*)/qq^*}, \tag{3.3}$$

we obtain

$$\begin{aligned}
|G(z,t)| \leq{}& A\left(\int_0^t\int_0^1 \exp(-\omega\tau)W dx_2 d\tau\right)^{1/2}\left(\int_0^t\int_0^1 \exp(-\omega\tau)|u_{,\tau}|^2 dx_2 d\tau\right)^{1/2}\\
&+ B_1 F(t)\left(\int_0^t\int_0^1 \exp(-\omega\tau)W dx_2 d\tau\right)^{1/p}\left(\int_0^t\int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^q dx_2 d\tau\right)^{1/q}\\
&+ D_3\left(\int_0^t\int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^2 dx_2 d\tau\right)^{1/2}\left(\int_0^t\int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^2 dx_2 d\tau\right)^{1/2}\\
&+ C_3\left(\int_0^t\int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^q dx_2 d\tau\right)^{1/p^*}\left(\int_0^t\int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^q dx_2 d\tau\right)^{1/q}.
\end{aligned} \tag{3.4}$$

When $q = q^*$ we see that $F(t) \equiv 1$ for every $t > 0$ and

$$\begin{aligned}
|G(z,t)| \leq{}& \left(\frac{A\epsilon_1}{2} + \frac{B_1\epsilon_2^p}{p}\right)\int_0^t\int_0^1 \exp(-\omega\tau)W dx_2 d\tau\\
&+ D_3\int_0^t\int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^2 dx_2 d\tau\\
&+ \frac{A}{2\epsilon_1}\int_0^t\int_0^1 \exp(-\omega\tau)|u_{,\tau}|^2 dx_2 d\tau + \left(\frac{B_1}{q\epsilon_2^q} + C_3\right)\int_0^t\int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^q dx_2 d\tau.
\end{aligned}$$

This means that

$$|G(z,t)| \leq -\lambda_\omega \frac{\partial G(z,t)}{\partial z}, \tag{3.5}$$

where[c]

$$\lambda_\omega = \max\left\{\omega^{-1}\left(\frac{A\epsilon_1}{2} + \frac{B_1\epsilon_2^p}{p}\right), \frac{A}{\epsilon_1\omega\varrho}, C_1^{-1}\left(\frac{B_1}{q\epsilon_2^q} + C_3\right), D_1^{-1}D_3\right\}.$$

From (3.5) we obtain that

$$G(z,t) \leq \lambda_\omega \frac{\partial G(z,t)}{\partial z}, \quad -G(z,t) \leq \lambda_\omega \frac{\partial G(z,t)}{\partial z}.$$

[c]An optimization of the value of λ_ω can be obtained by considering the involved nonlinear equations and selecting the best values for ϵ_i.

The first inequality implies that if there exists a point z_0 such that $G(z_0,t) > 0$, we have

$$G(z,t) \geq G(z_0,t)\exp[\lambda_\omega^{-1}(z-z_0)].$$

On the other hand if such a z_0 does not exists we have $G(z,t) \leq 0$, for every $z \geq 0$, and

$$-G(z,t) \leq -G(0,t)\exp(-\lambda_\omega^{-1}z), \quad z \geq 0.$$

This implies that $G(z,t)$ tends to zero as $z \to \infty$

We have established a Phragmen-Lindelöf alternative of exponential type: either the function $G(z,t)$ growths exponentially for z sufficiently large or the solutions decay exponentially in the form

$$E(z,t) \leq E(0,t)\exp(-\lambda_\omega^{-1}z), \quad z \geq 0, \tag{3.6}$$

where

$$\begin{aligned} E(z,t) = &\int_0^t \int_{R(z)} \exp(-\omega\tau)\left(\frac{\varrho\omega u_{,\tau}^2}{2} + \omega W + \nu|\nabla u_{,\tau}|^2\right) da d\tau \\ &+ \frac{1}{2}\int_{R(z)} \exp(-\omega t)(\varrho u_{,t}^2 + 2W) da, \end{aligned} \tag{3.7}$$

being $R(z) = \{\mathbf{x} \in R, x_1 \geq z\}$.

Therefore we have proved the following

Theorem 3.1. *Let us assume that $q = q^*$ and $1 < p < 2$. Let $u(\mathbf{x},t)$ be a solution of the initial-boundary problem . Then either the solution growths exponentially with z or the estimate is satisfied for the function $E(z,t)$ defined by (3.7).*

Let us consider the case that $q \neq q^*$, but $q < 2q^*$. In this situation we cannot use the previous argument, however we can adapt a variation of this argument which allows us to obtain several estimates. From the estimate (3.4) we obtain that

$$\begin{aligned} |G(z,t)| \leq &\left(\frac{A\epsilon_1}{2} + \frac{B_1 F(t)\epsilon_2^p}{p}\right)\int_0^t \int_0^1 \exp(-\omega\tau) W dx_2 d\tau \\ &+ \frac{A}{2\epsilon_1}\int_0^t \int_0^1 \exp(-\omega\tau)|u_{,\tau}|^2 dx_2 d\tau + \frac{B_1 F(t)}{q^*\epsilon_2^{q^*}}\left(\int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^q dx_2 d\tau\right)^{q^*/q} \\ &+ D_3 \int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^2 dx_2 d\tau + C_3 \int_0^t \int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^q dx_2 d\tau. \end{aligned}$$

This inequality can be written as

$$|G(z,t)| \leq N_1 \left(-\frac{\partial G(z,t)}{\partial z} \right) + N_2 \left(-\frac{\partial G(z,t)}{\partial z} \right)^{\delta}, \tag{3.8}$$

where $\delta = q^*/q$ and N_i are two positive constants which can be obtained in terms of the parameters.

It is worth remarking that $1/2 < \delta < 1$. This last inequality has been studied previously by Horgan and Payne [13]. Again, if there exists a point z_0 such that $G(z_0,t) < 0$, then $G(z,t)$ remains negative for all $z \geq z_0$. Following [13], (p. 134), we obtain the inequality

$$-G(z,t) \leq D_1^* \left[-\frac{\partial G(z,t)}{\partial z} \right] + D_2^* \left[-\frac{\partial G(z,t)}{\partial z} \right]^{1/2}, \tag{3.9}$$

where

$$D_1^* = N_1 + N_2(2-\delta)\delta^{-1}\sigma_1, \;\; D_2^* = 2N_2(\delta^{-1}-1)\delta\sigma_1^{-(2-\delta^{-1})/(2(\delta^{-1}-1))}, \tag{3.10}$$

and σ_1 is an arbitrary positive constant. We have (see [13], p. 135),

$$-G(z,t) \geq D_1^* Q_1(z_0,t) \exp\left(\frac{z-z_0}{D_1^*} \right), \;\; z \geq z_0. \tag{3.11}$$

Here $Q_1(z_0,t)$ is an easily computable quantity in terms of the data of the problem.

When $G(z,t)$ does not satisfy the previous estimate, then $G(z,t)$ must be non negative for all $z \geq 0$. It then follows that (see [13], p. 135),

$$G(z,t) \leq D_3^* \left[-\frac{\partial G(z,t)}{\partial z} \right]^{2\delta} + D_4^* \left[-\frac{G(z,t)}{\partial z} \right]^{\delta}, \tag{3.12}$$

where

$$D_3^* = N_1\delta^{-1}\sigma_2, \;\; D_4^* = N_1(2-\delta^{-1})\sigma_2^{-(\delta^{-1}-1)/(2-\delta^{-1})} + N_2, \tag{3.13}$$

and σ_2 is an arbitrary positive constant. If $z \geq 0$ we obtain (see [13], pp. 135-136)

$$E(z,t) \leq D_4^* \left(\frac{1-\delta}{2D_3^*\delta} [z + Q_2(0,t)] \right)^{-2\delta/(1-\delta)} + D_3^* \left(\frac{1-\delta}{2D_3^*\delta} [z + Q_2(0,t)] \right)^{-4\delta/(1-\delta)}, \tag{3.14}$$

where

$$Q_2(0,t) = \frac{4\delta D_3^*}{2\delta - 1} \left(\left[E(0,t)\frac{1}{D_3^*} + \left(\frac{D_4^*}{2D_3^*} \right)^2 \right]^{1/2} - \frac{D_4^*}{2D_3^*} \right)^{(\delta-1)/\delta}$$

$$- \frac{2\delta D_3^*}{2\delta - 1} \left(\left[E(0,t)\frac{1}{D_3^*} + \left(\frac{D_4^*}{2D_3^*} \right)^2 \right]^{1/2} - \frac{D_4^*}{2D_3^*} \right)^{(2\delta-1)/\delta}.$$

It is worth noting that $Q_2(0,t)$ may be positive or negative, but regardless of the sign, $G(z,t)$ decays at least as fast as $z^{-2\delta/(1-\delta)}$.

Theorem 3.2. *Assume that u is a solution of the initial-boundary-value problem. Then, the function $G(z,t)$ satisfies either the asymptotic condition (3.11) or the energy function $E(z,t)$ satisfies the polynomial spatial decay estimate (3.14).*

3.2. *Case* $1 < q^* < 2$

The analysis starts as in the previous subsection, by defining the function $G(z,t)$ as in as in the previous section. We can obtain the estimate (3.8). Now, we have that $q^* < 2$. Thus, the following estimate:

$$\left(\int_0^t\int_0^1 \exp(-\omega\tau)|u_{,\tau}|^{q^*}dx_2d\tau\right)^{1/q^*} \le F_*(t)\left(\int_0^t\int_0^1 \exp(-\omega\tau)|u_{,\tau}|^2dx_2d\tau\right)^{1/2} \tag{3.15}$$

holds, where

$$F_*(t) = \left(\gamma^{-1}[1-\exp(-\omega t)]\right)^{(2-q^*)/2q^*}. \tag{3.16}$$

We obtain

$$|G(z,t)| \le \left(\frac{A\epsilon_1}{2} + \frac{BF_*(t)\epsilon_2^p}{p}\right)\int_0^t\int_0^1 \exp(-\omega\tau)W dx_2d\tau$$

$$+\frac{A}{2\epsilon_1}\int_0^t\int_0^1 \exp(-\omega\tau)|u_{,\tau}|^2dx_2d\tau + \frac{BF_*(t)}{q^*\epsilon_2^{q^*}}\left(\int_0^t\int_0^1 \exp(-\omega\tau)|u_{,\tau}s|^2dx_2d\tau\right)^{q^*/2}$$

$$+D_3\int_0^t\int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^2dx_2d\tau + C_3\int_0^t\int_0^1 \exp(-\omega\tau)|\nabla u_{,\tau}|^q dx_2d\tau.$$

An estimate can be obtained where $\frac{1}{2} < \delta = q^*/2 < 1$. We can use the arguments used previously to obtain estimates as (3.11) and (3.14) which give the Phragmen-Lindelöf alternative in this case.

4. The Wave Equation

The method used in the previous Section works also when $\nu(\dot{s}^2) \equiv 0$, i.e., when we consider the wave equation

$$\rho u_{,tt} = \left(Q(|\nabla u|^2)u_{,i}\right)_{,i}, \tag{4.1}$$

in the particular case that the assumption (i) holds with $B = 0$[d]. This wave equation has been deeply studied by several authors and we refer to [14] for a review. To the

[d]We note that this assumption and the further arguments are strongly inspired by the results in [9]. However, we here have proposed a family of examples for this kind of assumption and on the other side, in Section 2, new conclusions different from the ones proposed in the cited paper are provided.

equation (4.1) we append the initial conditions (2.6) and the boundary conditions

$$u(\mathbf{X},0)=u_0(\mathbf{X}),\ \ \dot{u}(\mathbf{X},0)=v_0(\mathbf{X}),\ \ \mathbf{X}\in\mathcal{R}. \tag{4.2}$$

The analysis in this case starts by considering the function

$$H(z,t)=-\int_0^t\int_0^1 Qu_{,1}u_{,\tau}dx_2d\tau. \tag{4.3}$$

We have that

$$H(z,t)-H(z_0,t)=-\frac{1}{2}\int_{z_0}^{z}\int_0^1(2W+\rho u_{,t}^2)da+\frac{1}{2}\int_{z_0}^{z}\int_0^1(2W(|\nabla u_0|^2)+\rho v_0^2)da, \tag{4.4}$$

$\forall z\geq z_0$. Direct differentiation in the previous equalities gives

$$\frac{\partial H(z,t)}{\partial z}=-\frac{1}{2}\int_0^1(2W+\rho u_{,t}^2)dx_2+\frac{1}{2}\int_0^1(2W(|\nabla u_0|^2)+\rho v_0^2)dx_2, \tag{4.5}$$

and

$$\frac{\partial H(z,t)}{\partial t}=-\int_0^1 Qu_{,1}u_{,\tau}dx_2. \tag{4.6}$$

The arithmetic-geometric mean inequality implies that

$$\left|\frac{\partial H(z,t)}{\partial t}\right|\leq\frac{1}{2}\int_0^1\left(\epsilon\rho u_{,t}^2+2\frac{A^2}{\epsilon\rho}W\right)dx_2+E_1(z), \tag{4.7}$$

where ϵ is an arbitrary positive constant

$$E_1(z)=\frac{1}{2}\int_0^1\left(2W(|\nabla u_0|^2)+\rho v_0^2\right)dx_2. \tag{4.8}$$

If we select $\epsilon=A\rho^{-1/2}$ we obtain that

$$c^{-1}\frac{\partial H(z,t)}{\partial t}+\frac{\partial H(z,t)}{\partial z}\leq E_1(z) \tag{4.9}$$

and

$$-c^{-1}\frac{\partial H(z,t)}{\partial t}+\frac{\partial H(z,t)}{\partial z}\leq E_1(z) \tag{4.10}$$

where $c=A\rho^{-1/2}$.

Now the inequality (4.9) implies that

$$H(z,c^{-1}(z-z^*))\leq\int_{z^*}^{z}E_1(\xi)d\xi, \tag{4.11}$$

where $z\geq z^*$. In a similar way from (4.10) we see that

$$H(z,c^{-1}(z^{**}-z))\geq-\int_{z}^{z^{**}}E_1(\xi)d\xi, \tag{4.12}$$

when $z \leq z^{**}$. From the previous inequalities and assuming that the total initial energy is finite, i.e.,

$$E^*(0,0) = \frac{1}{2}\int_R (2W(|\nabla u_0|^2) + \rho v_0^2)da < \infty, \tag{4.13}$$

we conclude that

$$\lim_{z\to\infty} H(z,t) = 0, \tag{4.14}$$

for every $t \geq 0$. Therefore we may write

$$H(z,t) = \frac{1}{2}\int_z^\infty \int_0^1 (2W + \rho u_t^2)da - \frac{1}{2}\int_z^\infty \int_0^1 (2W(|\nabla u_0|^2) + \varrho v_0^2)da. \tag{4.15}$$

Inequality (4.11) implies that

$$E^*(z,t) \leq E^*(z^*,0), \tag{4.16}$$

where

$$E^*(z,t) = \frac{1}{2}\int_R (2W + \rho u_t^2)da, \tag{4.17}$$

and where z, z^* and t are related by $t = c^{-1}(z - z^*)$. Now, from (4.12) we see that

$$E^*(z,t) \geq E^*(z^{**},0), \tag{4.18}$$

whenever $t = c^{-1}(z^{**} - z)$.

From the estimates (4.16) and (4.17) we conclude that

$$E^*(z,t) \leq E^*(z^*,t^*), \tag{4.19}$$

where $|t - t^*| \leq c^{-1}|z - z^*|$. We have proved:

Theorem 4.1. *Assume that u is a solution of the initial-boundary-value problem determined by equation* (4.1) *the initial condition (4.2) and the boundary conditions (2.6). Then, the solutions of this problem satisfy the estimate (4.19).*

It is also usual to work with the function

$$\mathcal{E}(z,t) = \int_0^t E^*(z,s)ds. \tag{4.20}$$

When $ct \leq z$, we have the estimate

$$\mathcal{E}(z,t) = \int_0^t E^*(z - cs, 0)ds = c^{-1}\int_{z-ct}^z E^*(\eta,0)d\eta. \tag{4.21}$$

Otherwise, when $ct \geq z$, we have that

$$\mathcal{E}(z,t) = \int_0^{c^{-1}z} E^*(z,s)ds + \int_{c^{-1}z}^t E^*(z,s)ds. \tag{4.22}$$

The first integral can be bounded in the following way

$$\int_0^{c^{-1}z} E^*(z,s)ds \leq c^{-1}\int_0^z E^*(\eta,0)d\eta, \tag{4.23}$$

and the second integral in (4.22) can be estimated as

$$\int_{c^{-1}z}^{t} E^*(z,s)ds \leq \Big(1-\frac{z}{ct}\Big)\,\mathcal{E}(0,t). \tag{4.24}$$

Therefore, we obtain that

$$\mathcal{E}(z,t) \leq c^{-1}\int_0^z E^*(\eta,0)d\eta, + \Big(1-\frac{z}{ct}\Big)\,\mathcal{E}(0,t). \tag{4.25}$$

Hence the following result holds:

Theorem 4.2. *Let u be a solution of the initial-boundary-value problem determined by equation* (4.1)*, the initial condition (4.2) and the boundary conditions (2.6). Then, the function $\mathcal{E}(z,t)$ defined at (4.20) satisfies the estimate (4.21) when $ct \leq z$ and the estimate (4.25) when $ct \geq z$.*

If we assume that the initial conditions are zero, we obtain that $E^*(z,0) = 0$ for every z. In this case the estimate (4.21) implies that

$$\mathcal{E}(z,t) = 0, \tag{4.26}$$

whenever $ct \leq z$ and we conclude that $u(x,t) = 0$ in this region, which is a result of the type of domain of influence. When $ct \geq z$ we obtain that

$$\mathcal{E}(z,t) \leq \Big(1-\frac{z}{ct}\Big)\,\mathcal{E}(0,t), \tag{4.27}$$

which is a very fast rate of decay for the solutions. Therefore, we have obtained that

Theorem 4.3. *Let u be a solution of the initial-boundary-value problem determined by equation* (4.1)*, with null initial condition and the boundary conditions (2.6). Then, the function $\mathcal{E}(z,t)$ vanishes when $ct \leq z$ and the estimate (4.27) holds when $ct \geq z$.*

5. Concluding Remarks

In the Introduction we have discussed details that for materials with non constant viscosity the time-evolution of a BVP may be not well-posed and the lack of global existence in time may be a major problem. On the other hand our results are about spatial estimates and they show that to have *good* estimates we need a certain balance among the elastic and the viscous nonlinearities. The strange fact is that the elastic nonlinearity bounds the viscous nonlinearity (see conditions (i) and (ii) in the setting of the problem). This fact seems to be not in accordance with our experience. Indeed, it is natural to expect that for any nonlinear generalized shear modulus a sufficient strong viscosity function is enough to control the time and spatial behaviour of the solutions but this is not the case.

This status of affair opens an interesting problem from the modelling point of view. The counter-intuitive situations we have pointed out may be, in some sense, a byproduct of a *wrong* model. In amorphous elastomeric materials it is clear that internal dissipation is strongly connected with the mesoscopic structure of the polymeric network. Therefore it is hard to split dissipative and dispersive effects. This situation suggests that a *good* model of viscoelasticity must take into account the internal structure of the material. This approach may help to overcome the various mathematical problem.

In this process the crucial point is the quasi-static limit. In this approximation dispersive effects seems to be not relevant at all [16] and therefore we must be able to introduce mesoscopic information in such a way that dispersive effects will disappear for large times. Clearly, the real challenge is to obtain such a model in the framework of a rigorous axiomatic theory of continuum mechanics and not just adding "ad-hoc" regularising terms.

The situation is also challenging from a mathematical point of view as it is possible to appreciate from [28] and [29], but it seems the only way to produce effective models of dissipative effects in materials subject to large deformations.

Acknowledgments

The work of R. Quintanilla has been supported by the Ministerio de Economia y Competitividad under the research project "Análisis Matemático de Problemas de la Termomecánica" (MTM2016-74934-P), (AEI/FEDER, UE) and Ministerio de Ciencia, Innovación y Universidades under the research project “Análisis matemáico aplicado a la termomecánica” (PID2019-105118GB-I00). The work of G. Saccomandi has been partially supported by GNFM of INDAM and local grants by University of Perugia.

Appendix A. Power-law materials

Lemma Appendix A.1. *Let us consider the function $Q(s^2) = (1 + ks^2)^n$ where k and n are two positive real numbers. There exist two positive constants α, β such that*

$$Q(s^2) \geq \alpha + \beta s^{2n}. \tag{A.1}$$

Proof. We start with the case $1 < n < \infty$. Being

$$(1 + ks^2)^n - 1 - (ks^2)^n \geq 0$$

the lemma is proved with $\alpha = 1$ and $\beta = k^n$.

When $0 < n < 1$ we consider $0 < \alpha < 1$ such that $k^n > \alpha$.

When $s^2 \leq 1$ we see that

$$1 - \alpha - (1-\alpha)s^{2n} \geq 0$$

meanwhile when $s^2 > 1$ we have

$$k^n s^{2n} - \alpha - (k^n - \alpha)s^{2n} \geq 0.$$

Therefore $(1 + ks^2)^n \geq \alpha + \beta s^{2n}$ where $\beta = \max(1-\alpha, k^n - \alpha)$. □

Lemma Appendix A.2. *For the function*

$$W(s^2) = \frac{1}{2(n+1)k}\left((1+ks^2)^{n+1} - 1\right) \tag{A.2}$$

it is

$$W(s^2) \geq \frac{1}{2}\left(\alpha s^2 + \beta s^{2(n+1)}\right), \tag{A.3}$$

where α *and* β *are given in the previous lemma* (Appendix A.1).

Proof. To prove this assertion we note that the function

$$\Omega(s^2) = W(s^2) - \frac{1}{2}\left(\alpha s^2 + \frac{\beta}{n+1}s^{2(n+1)}\right) \tag{A.4}$$

satisfies that $\Omega(0) = 0$ and

$$2\frac{d\Omega}{ds^2}(s^2) = (1+ks^2)^n - \alpha - \beta s^{2n} \geq 0. \tag{A.5}$$

Therefore the assertion is proved.

Now, we see that

$$|sQ(s^2)| \leq \alpha^*|s| + \beta^* s^{2n}. \tag{A.6}$$

Here α^*, β^* are two positive constants which can be easily calculated. We see that condition (i) is clearly satisfied for suitable constants A, B. In this case we have that $q^* = 2n + 2$. □

When $Q(s^2) = (1 + ks^2)^n$ with $-1 < n \leq 0$ the condition (i) is satisfied with $B = 0$. Indeed, let us introduce

$$R(s^2) = \frac{s(1+ks^2)^n}{W(s^2)^{1/2}}. \tag{A.7}$$

Because $n < 0$ we have that

$$\lim_{s\to 0} R(s) = \sqrt{2}, \quad \text{and} \lim_{s\to\infty} R(s) = 0.$$

Therefore we have

$$sQ(s^2) \leq A_1 W(s^2)^{1/2}$$

(here A_1 positive constant) near $s = 0$ and for s large enough. We need to prove a similar estimate in $[a, b] \subset [0, \infty]$. Because the numerator and the never vanishing denominator of (A.7) are two continuous functions defined in a domain which is closed and compact it is always possible to choose a number $A_2 > 0$ such that

$$sQ(s^2) \leq A_2 W(s^2)^{1/2}, \forall s \in [a, b].$$

The maximum among A_1 and A_2 is the number we where searching for.

Appendix B. Estimates in terms of the boundary conditions

The aim of this appendix is to give an upper bound for the amplitude term $E(0, t)$ in terms of the boundary conditions, but assuming that (2.7) and (2.8) hold. To find this bound we pick $\xi(x_1, x_2, t) = f(x_2, t) \exp(-\alpha x_1)$ with $\alpha > 0$. In so doing we clearly have, with uniform convergence,

$$\lim_{x_1 \to 0} \xi(x_1, x_2, t) = f(x_2, t), \qquad \lim_{x_1 \to \infty} \xi(x_1, x_2, t) = 0.$$

Since

$$E(0, t) = -\int_0^t \int_0^1 \exp(-\omega\tau)(Qu_{,1} + \nu u_{,1\tau}) f_{,\tau} dx_2 d\tau,$$

it is possible to compute the following bound

$$E(0, t) = \int_0^t \int_{\mathcal{R}} \exp(-\omega\tau)(\varrho u_{,\tau\tau}\xi_{,\tau} + Qu_{,i}\xi_{,i\tau} + \nu u_{,i\tau})\xi_{,i\tau} da d\tau.$$

By using some cumbersome computations based on the arithmetic-geometric inequality it is possible to obtain

$$E(0, t) \leq \frac{\varrho}{2\alpha\epsilon_1} \exp(-\omega t) \int_0^1 |f_{,t}|^2 dx_2 + \frac{\varrho}{2\alpha\epsilon_2} \int_0^t \int_0^1 \exp(-\omega\tau)|f_{,\tau}|^2 dx_2 d\tau$$

$$+ \frac{\varrho}{2\alpha\epsilon_3} \int_0^t \int_0^1 \exp(-\omega\tau)|f_{,\tau\tau}|^2 dx_2 d\tau + \frac{A}{\alpha\epsilon_4} \int_0^t \int_0^1 \exp(-\omega\tau) \left[\alpha^2|f_{,\tau}|^2 + |f_{,2\tau}|^2\right] dx_2 d\tau$$

$$+ \frac{2B(p-1)^2}{p\alpha\epsilon_5^{p/(p-1)}} \int_0^t \int_0^1 \exp(-\omega\tau) \left[\alpha^{p/(p-1)}|f_{,\tau}|^{p/(p-1)} + |f_{,2\tau}|^{p/(p-1)}\right] dx_2 d\tau$$

$$+ \frac{D_2}{\alpha\epsilon_6} \int_0^t \int_0^1 \exp(-\omega\tau) \left[\alpha^2|f_{,\tau}|^2 + |f_{,2\tau}|^2\right] dx_2 d\tau$$

$$+ \frac{2C_2}{\alpha q^2 \epsilon_7^q} \int_0^t \int_0^1 \exp(-\omega\tau) \left[\alpha^q|f_{,\tau}|^q + |f_{,2\tau}|^q\right] dx_2 d\tau$$

Here ϵ_i $(i = 1, \dots, 7)$ are positive constants such that

$$\epsilon_1 \leq \frac{1}{2}, \quad \omega\epsilon_2 + \epsilon_3 \leq \frac{\omega}{2}, \quad A\frac{\epsilon_2}{2} + B\frac{\epsilon_5^p}{p} \leq \frac{\omega}{2}, \quad D_2\frac{\epsilon_6}{2}D_1 + C_2\frac{(q-1)\epsilon_7^{q/(q-1)}}{C_1 q} \leq \frac{1}{2}.$$

References

[1] G. Andrews and J. M. Ball, Asymptotic behaviour and changes of phase in one-dimensional nonlinear viscoelasticity, *Journal of Differential Equations* **44** (1982) 306–341.
[2] Antman, Stuart S. *Nonlinear Problems of Elasticity* (Springer, New York, 1994).
[3] M. F. Beatty and Z. Zhou, Universal motions for a class of viscoelastic materials of differential type, *Continuum Mechanics and Thermodynamics* **3** (1991) 169–191.
[4] M. H.Chen, L. L. Wang, J. J. Chung, Y. H. Kim, P. Atluri and J. A. Burdick, Methods to assess shear-thinning hydrogels for application as injectable biomaterials. *ACS biomaterials science and engineering* **3** (2017) 3146–3160.
[5] M. Destrade, P. M. Jordan and Giuseppe Saccomandi, Compact travelling waves in viscoelastic solids, *EPL (Europhysics Letters)* **87** (2009) #48001.
[6] M. Destrade, G. Saccomandi and Maurizio Vianello, Proper formulation of viscous dissipation for nonlinear waves in solids, *The Journal of the Acoustical Society of America* **133** (2013) 1255–1259.
[7] J.I. Díaz and R. Quintanilla, Spatial and continuous dependence estimates in linear viscoelasticity, *Jour. Mathematical Anal. Appl.* **273** (2002) 1–16.
[8] H. Engler, Global regular solutions for the dynamic antiplane shear problem in nonlinear viscoelasticity, *Mathematische Zeitschrift* **202** (1989) 251–259.
[9] J. N. Flavin, R.J. Knops and L. E. Payne, Energy bounds in dynamical problems for a semi-infinite elastic beam, in *Elasticity: Mathematical methods and applications* (Ellis-Horwood Chichester, 1989) pp. 101–111.
[10] J. N. Flavin, R. J. Knops and L.E. Payne, Decay estimates for the constrained elastic cylinder of variable cross section, *Quart. Appl. Math.* **XLVII**, (1989) 325–350.
[11] J. M. Greenberg, R. C. MacCamy and V. J. Mizel, On the existence, uniqueness, and stability of solutions of the equation $\sigma'(u_x)u_{xx} + \lambda u_{xtx} = \rho_0 u_{tt}$, *Journal of Mathematics and Mechanics* (1968) 707–728.
[12] C.O. Horgan and J. K. Knowles, Recent developments concerning Saint-Venant's principle, *Advances in applied mechanics* **23** (1983) 179–269.
[13] C. O. Horgan and L. E. Payne, Phragmen-Lindelöf type results for harmonic functions with nonlinear boundary conditions, *Archive for Rational Mechanics and Analysis* **122** (1993) 122–144.
[14] C.O. Horgan and G. Saccomandi, Anti-plane shear deformations for non-Gaussian isotropic, incompressible hyperelastic materials, *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* **457**, (2001)1999–2017.
[15] K. Kuttler and D. Hicks, Initial-boundary value problems for the equation $u_{tt} = (\sigma(u_x))_x + (\alpha(u_x)u_{xt})_x + f$, *Quarterly of applied mathematics* **46** (1988) 393–407.
[16] R. S. Lakes, *Viscoelastic materials* (Cambridge University Press, 2009).
[17] M. C. Leseduarte and R. Quintanilla, Phragmen-Lindelöf alternative for Laplace equation with dynamic boundary conditions, *Journal applied Analysis Computation* **7** (2017) 1323–1335.
[18] Y. Liu, W. Liao and C. Lin, Some alternative results for the nonlinear viscoelasticity equation, *Applicable Analysis*, **89** (2010) 775–787.
[19] R. Mangan, M. Destrade and G. Saccomandi, Strain energy function for isotropic

non-linear elastic incompressible solids with linear finite strain response in shear and torsion, *Extreme Mechanics Letters* **9** (2016) 204–206.

[20] N. Phan-Thien, M. Newberry and R. Tanner, Non-linear oscillatory flow of a soft solid-like viscoelastic material, *Journal of Non-Newtonian Fluid Mechanics* **92** (2000) 67–80.

[21] E. Pucci and G. Saccomandi, On a special class of nonlinear viscoelastic solids, *Mathematics and Mechanics of Solids* **15** (2010) 803–811.

[22] E. Pucci and G. Saccomandi, On the nonlinear theory of viscoelasticity of differential type, *Mathematics and Mechanics of Solids* (2012) 624–630.

[23] E. Pucci and G. Saccomandi, The anti-plane shear problem in nonlinear elasticity revisited, *Journal of Elasticity* **113** (2013) 167–177.

[24] R. Quintanilla, Spatial behaviour for quasilinear parabolic equations in cylinders and cones, *Nonlinear Differential Equations and Applications (NODEA)* **5** (1998) 137–146.

[25] R. Quintanilla, On the asymptotic behaviour of solutions of some nonlinear elliptic and parabolic equations, *Nonlinear Analysis A* **52** (2003) 1275–1293.

[26] R. Quintanilla, Comparison arguments and decay estimates in non-linear viscoelasticity, *International Journal of Non-Linear Mechanics* **39** (2004) 55–61.

[27] R. Quintanilla and G. Saccomandi, Quasistatic anti-plane motion in the simplest theory of nonlinear viscoelasticity. *Nonlinear Analysis: Real World Applications* **9** (2008) 1499–1517.

[28] R. Quintanilla and G. Saccomandi, Spatial behaviour for a fourth-order dispersive equation, *Quarterly of Applied Mathematics* **64** (2006) 547–560.

[29] R. Quintanilla and G. Saccomandi, Phragmen-Lindelöf alternative of exponential type for the solutions of a fourth order dispersive equation, *Rendiconti Lincei-Matematica e Applicazioni* **23** (2012) 105–114.

[30] K. R. Rajagopal and G. Saccomandi, Shear waves in a class of nonlinear viscoelastic solids *The Quarterly Journal of Mechanics and Applied Mathematics* **56** (2003) 311–326.

[31] R. Badar, M. Destrade and M. D. Gilchrist, Mechanical characterization of brain tissue in simple shear at dynamic strain rates, *Journal of the mechanical behavior of biomedical materials* **28** (2013) 71–85.

[32] G. Saccomandi and L. Vergori, Large Time Approximation for Shearing Motions, *SIAM Journal on Applied Mathematics* **76** (2016)1964–1983.

[33] A. B. Stepanov and S. S. Antman, Radially symmetric motions of nonlinearly viscoelastic bodies under live loads *Archive for Rational Mechanics and Analysis* **226** (2017) 1209–1247.

[34] B. Tvedt, Quasilinear equations for viscoelasticity of strain-rate type, *Archive for Rational Mechanics and Analysis* **189** (2008) 237–281.