MASTER'S DEGREE THESIS

Master of Science in Advanced Mathematics and Mathematical Engineering

Title:Second Species Periodic Solutions For the Three Body Problem

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Master thesis

Second Species Periodic Solutions For The Three Body Problem

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Abstract

We are going to explain the construction of second-species periodic solutions for the Restricted Planar Circular 3-Body Problem. These solutions, whose existence had been conjectured by Poincaré, are referred to periodic solutions that travel near singular points.

To do that, we will study two different papers, one written by S.V. Bolotin and R.S. Mackay [1], and the other one written by Jean-Pierre Marco and Laurent Niederman [2]. Although they have much in common, the first one gives a variational approach of the problem (using Lagrangian systems and the Principle of Least Action), while the other one gives a geometrical approach (defining isolated blocks and perturbative methods). We will explain and expand these approaches, to sum up with a briefly comparison between them.

For their study, we will take as a reference the particular case of the Restricted 3-Body Problem corresponding to the Sun, Jupiter and an asteroid, whose singular point will be the collision between these last two bodies.

Resumen

Vamos a explicar la construcción de soluciones periódicas de segunda especie para el Problem Circular Restringido de 3 Cuerpos. Estas soluciones, cuya existencia había conjeturado Poincaré, son soluciones periódicas que pasan cerca de puntos singulares.

Para ello, estudiaremos dos trabajos diferentes, uno llevado a cabo por S.V. Bolotin y R.S. Mackay [1], y otro por Jean-Pierre Marco y Laurent Niederman [2]. Aunque tienen mucho en común, el primero aporta un punto de vista variacional (usando sistemas Lagrangianos y el Principio de Mínima Acción), mientras que el otro aporta un punto de vista geométrico (definiendo bloques aislados y métodos perturbativos). Explicaremos y expandiremos sus aportaciones, para acabar con una breve comparación entre ellos.

Para su estudio, tomaremos como referencia el caso particular del problema restringido de 3 cuerpos correspondiente a el Sol, Júpiter y un asteroide, cuya singularidad será la colisión entre estos dos últimos cuerpos.

Contents

Chapter 1

Introduction

For years the study of the cosmos was a challenge for those who wanted to describe the phenomena present in nature. Since Ptolemy all the astronomical models were based on the pure observation of the stars and planets surrounding us. It was not until the arrival of the first telescope when some of the most important scientists, such as Newton, Kepler or Galileo, were able to get a better view of these phenomena. Due to their research, important laws, such as the Newtonian Gravitational law and the Kepler's laws, came out as the way to describe more precisely the movement of the planets conforming the Solar System.

However, whilst these laws worked well when studying the orbits of one planet around the Sun, they lost accuracy when other objects (such as Asteroids, satellites or other planets), with their correspondent gravitational force, came into play. Although these forces could have been considered negligible (the mass of these objects with respect to the Sun is almost none) they can produce deviations and errors needed to be taken into account.

It was Henry Poincaré who began to give some light in the resolution of this kind of problems. In his work [3], he starts by looking for periodic solutions of problems with 3 bodies, where two of them have negligible masses with respect to the third one (for example, this could be the case when one of the bodies is the Sun, and the other two are planets). Later on, these solutions were named periodic solutions of "first species".

It is also in his work where he expands his results by stating, for example, that the periodic solutions of the general 3-Body problem can be found by perturbing those ones with singularities of the problem before (i.e., singularities corresponding to the collisions with these two particles), and predicting the perturbation of non-Kepler configurations. He named these periodic solutions as solutions of "second species". However, the proof given by Poincaré about the existence of such solutions is confusing and incomplete.

The main goal of this work is to prove, in a more detailed way, the existence of these second species periodic trajectories for the planar restricted circular 3-Body Problem, which we will explain along this work. In order to do that, we will introduce the general N-Body Problem and finally it presents the problem we will work with, the Planar 3-Body Problem, and more concretely, the Restricted Planar 3-Body Probelm. First we will explain the approach made by S.V. Bolotin and R.S. Mackay, which relies on variational techniques [1]. Then, we will explain the approach made by Jean-Pierre Marco and Laurent Niederman [2], which is based on geometrical shadowing techniques. To sum up, we will compare both results in order to get a general view of the proof.

Specifically this goal can be broken down into the following parts:

1. Studying the Kepler equations to understand the 2-Body Problem structure and its solutions. This will help us to get a better view of the topology when we deal with the 3-Body Problem.

2. Explain the model we study, that is the Restricted 3-Body Problem in detail as well as the different reference frames that one can consider for this model.

3. Prove the existence of second species periodic orbits using the variational approach made by S.V. Bolotin and R.S. Mackay.

4. Prove the existence of second species periodic orbits using the geometrical approach made by Jean-Pierre Marco and Laurent Niederman.

5. Compare both results to look for similarities in the reasoning of both approaches, to better understand how one can go from the variational approach to the geometrical one.

This work is structured as follows: Chapter 2 describes in detail the Kepler problem as the simplest scenario, as well as it introduces different reference frames. Chapter 3 introduces the main results of both papers and explains the background to prove them, and ends up with a comparison between both approaches. Chapter 4 explains the proof made in the variational approach given in the first paper [1]. Chapter 5 explains the proof made in the geometrical approach given in the other one [2]. Chapter 6 ends with the conclusions and the future work.

Chapter 2

Context

In this chapter we introduce the N-Body Problem and describe some of its features. We also analyze its simplest setting, the 2-Body Problem. Finally, we describe the Planar Restricted 3-Body Problem, the model we will use for the rest of the work.

2.1 The N-Body Problem

Let us consider N point masses moving in a Newtonian reference system, $\mathbb{R}^3$, with the only force acting on them being their mutual gravitation attraction. For each mass m_i associated with each particle, we define $Q_i = \begin{pmatrix} Q_{i1} \\ Q_{i2} \\ Q_{i3} \end{pmatrix} \in \mathbb{R}^3$ associated with each m_i its position in the space.

The equation to model the forces acting on these particles come from Newton's law of universal gravitation, which can be written as:

$$m_i \ddot{Q}_i = \sum_{i=1}^{N} F_i = \mathcal{G} \sum_{j=1, j\neq i}^{N} m_j m_i \frac{Q_j - Q_i}{||Q_i - Q_j||^3} = \frac{\partial U}{\partial Q_i} \tag{2.1}$$

where $\mathcal{G}$ is the gravitational constant with value $\mathcal{G} = 6.674 \times 10^{-11} \ \frac{m^3}{s^2/kg}$ and U is the self-potential (or the negative of the potential), with expression:

$$U = \sum_{1 \leq i < j \leq N} \frac{\mathcal{G} m_i m_j}{||Q_i - Q_j||} \tag{2.2}$$

Let $Q = (Q_1, \cdots, Q_N) \in \mathbb{R}^{3N}$ and let M be the $3n \times 3n$ diagonal matrix $M = diag(m_1, m_1, m_1, \cdots, m_N, m_N, m_N)$; thus Equation (2.1) is of the form

$$M\ddot{Q} - \frac{\partial U}{\partial Q} = 0 \tag{2.3}$$

Let Δ be the subset of $\mathbb{R}^{3N}$ where $Q_i = Q_j$ for some $i \neq j$, the collision set. It is clear that Equations (2.1) and (2.3) are systems of second order equations in $\mathbb{R}^{3N} \setminus \Delta$, the position space.

Let us pass to the Hamiltonian formalism. We define $P = (P_1, \cdots, P_N) \in \mathbb{R}^{3N}$ by $P = M\dot{Q}$, so $P_i = m_i\dot{Q}_i$ is the momentum of the i^{th} particle. The equations of motion become:

$$\dot{Q}_i = \frac{P_i}{m_i} = \frac{\partial \mathcal{H}}{\partial P_i} \qquad \dot{P}_i = \sum_{i=1}^{N} \frac{\mathcal{G}m_i m_j (Q_j - Q_i)}{||Q_i - Q_j||^3} = -\frac{\partial \mathcal{H}}{\partial Q_i} \tag{2.4}$$

where the Hamiltonian is

$$\mathcal{H}(P, Q) = T(P) + U(Q) \tag{2.5}$$

and T is the kinetic energy:

$$T(P) = \sum_{i=1}^{N} \frac{||P_i||^2}{2m_i} = \frac{1}{2} P^T M^{-1} P \quad \left(= \frac{1}{2} m_i ||\dot{Q}_i||^2 \right) \tag{2.6}$$

The Hamiltonian (2.5) and the Hamiltonian equations (2.4) are defined on the phase space $\mathbb{R}^{6N} \setminus \Delta$.

Constants of Motion

The N-Body Problem is a system of $6N$ first-order equations, so, a complete solution would require $6N - 1$ time-independent first integrals plus one dependent first integral. It is clear that for $N > 2$, it is too optimistic to expect so many global integrals. However, for all N there are ten integrals for the system.

Let

$$L = P_1 + \cdots + P_N$$

be the total linear momentum. From (2.4) it follows that $\dot{L} = 0$, because each term in the sum appears twice with opposite sign. This gives $\ddot{CM} = 0$, where

$$CM = m_1 Q_1 + \cdots + m_N Q_N$$

is the center of mass of the system because $\dot{CM} = L$. Thus the total linear momentum is constant, and the center of mass of the system moves with uniform rectilinear motion. Integrating the center of mass equation gives $CM = L_0 t + C_0$, where L_0 and C_0 are constants of integration. L_0 and C_0 are functions of the initial

conditions, and thus they are integrals of motion. Thus we have six constants of motion or integrals, namely, the three components of L_0 and the three components of C_0.

Let

$$A = Q_1 \times P_1 + \cdots + Q_N \times P_N$$

be the total angular momentum of the system. Then

$$\frac{dA}{dt} = \sum_1^N (\dot{Q}_i \times P_i + Q_i \times \dot{P}_i) = \sum_1^N Q_i \times m_i Q_i + \sum_1^N \sum_1^N \mathcal{G} \frac{m_i m_j Q_i \times (Q_j - Q_i)}{||Q_i - Q_j||^3} = 0$$

The first sum above is zero because $Q_i \times P_i = 0$. In the second sum, one can use that $Q_i \times (Q_j - Q_i) = Q_i \times Q_j$ and then observe that each term in the remaining sum appears twice with opposite sign. Thus the three components of angular momentum are constants of the motion or first integrals too. Finally, the energy $\mathcal{H}$ is also an first integral, so we have found the ten classical integrals of the N-Body Problem.

Note that, since the total angular momentum is constant, we can ensure that if the motion of the particles takes place on a plane, it will remain on the same plane. This fact gives rise to the idea of studying the planar problem, which has 6 integrals of motion instead of 10.

2.1.1 The 2-Body Problem

Before dealing with the more general 3-Body Problem we will look at the 2-Body Problem in detail since it is the only one completely solvable.

First we take a look at the equations

$$m_1 \ddot{Q}_1 = \mathcal{G} m_1 m_2 \frac{(Q_2 - Q_1)}{||Q_2 - Q_1||^3}$$

$$m_2 \ddot{Q}_2 = \mathcal{G} m_1 m_2 \frac{(Q_1 - Q_2)}{||Q_2 - Q_1||^3}$$

Add and subtract these equations to get

$$m_1 \ddot{Q}_1 + m_2 \ddot{Q}_2 = 0, \;\; \ddot{Q}_1 - \ddot{Q}_2 = -\mathcal{G}(m_1 + m_2) \frac{(Q_1 - Q_2)}{||Q_1 - Q_2||^3}$$

or

$$C\ddot{M} = 0, \;\; \ddot{u} = -\mu \frac{u}{||u||^3},$$

where, as we saw before, $CM = m_1Q_1 + m_2Q_2$, $u = Q_1 - Q_2$, and $\mu = \mathcal{G}(m_1 + m_2)$. The first equation integrates to $CM = L_0t + C_0$, where L_0 and C_0 are integration constants, and the second equation is what is called the Kepler problem.

Let us now sketch how the Hamiltonian formalism can be used also to solve this problem, introducing the Jacobi coordinates (g, u, G, v):

$$g = \nu_1 Q_1 + \nu_2 Q_2, \quad G = P_1 + P_2$$
$$u = Q_2 - Q_1, \quad v = -\nu_2 P_1 + \nu_1 P_1,$$

with

$$\nu_1 = \frac{m_1}{m_1 + m_2}, \ \ \nu_2 = \frac{m_2}{m_1 + m_2}, \ \ \nu = m_1 + m_2, \ \ M = \frac{m_1 m_2}{m_1 + m_2}.$$

So g is the center of mass, G is the total linear momentum, u is the position of the particle 2 relative to particle 1, and v is the associated momentum.

By a direct computation one sees that this is a linear symplectic change of variables, so it preserves the Hamiltonian character.

The Hamiltonian (2.5) in the new variables is

$$\mathcal{H}(g, u, G, v) = \frac{||G||^2}{2\nu} + \frac{||v||^2}{2M} - \frac{m_1 m_2}{||u||}$$

and, as well, the equations of motion (2.4) become

$$\dot{g} = \frac{\partial \mathcal{H}}{\partial G} = \frac{G}{\nu}, \quad \dot{G} = -\frac{\partial \mathcal{H}}{\partial g} = 0,$$
$$\dot{u} = \frac{\partial \mathcal{H}}{\partial v} = \frac{v}{M}, \quad \dot{v} = -\frac{\partial \mathcal{H}}{\partial u} = -\frac{m_1 m_2 u}{||u||^3}$$

This implies that total linear momentum G is an integral and the center of mass g moves with constant linear velocity. By taking $g = G = 0$ as initial conditions we are reduced to a problem in the u, v variables alone. The equations then get reduced to

$$\ddot{u} = \frac{\mathcal{G}(m_1 + m_2)u}{||u||^3},$$

which is just the Kepler problem we will discuss in the next section. This means that the motion of one body, e.g. the Earth, when viewed from another, e.g. the Sun, is as if the Sun were a fixed body with mass $m_1 + m_2$ and the Earth were attracted to the Sun by a central force.

2.1.2 The Kepler Problem

Consider a two body problem where the first body is so massive (like the Sun) that its position is fixed to the first approximation and the second body has mass

1. In this case, the equations describing the motion of the second body are

$$\ddot{Q} = -\frac{\mu Q}{||Q||^3}, \tag{2.7}$$

where $Q \in \mathbb{R}^3$ is the position vector of the second body and μ is the constant $\mathcal{G}m$, where $\mathcal{G}$ is the universal gravitational constant and m is the mass of the first body fixed at the origin. In this case, by defining $P = \dot{Q}$, this equation becomes Hamiltonian with

$$\mathcal{H}(Q,P) = \frac{||P||^2}{2} - \frac{\mu}{||Q||} \tag{2.8}$$

and its respective equations of motion described as

$$\dot{Q} = P, \;\; \dot{P} = -\frac{\mu Q}{||Q||^3}. \tag{2.9}$$

Equations (2.7) or Hamiltonian (2.8) defines the Kepler Problem. As we have just seen, the 2-body problem can be reduced to this one.

Let $A = Q \times P$ be the angular momentum, and note

$$\dot{A} = \dot{Q} \times P + Q \times \dot{P} = P \times P + Q \times \left(-\frac{\mu Q}{||Q||^3} \right) = 0 \tag{2.10}$$

so A is constant along the solutions, and the three components of A are first integrals.

We need to state the following Lemma in order to make computations with A.

Lemma 2.1.

$$\frac{d}{dt}\left(\frac{Q}{||Q||}\right) = \frac{(Q \times \dot{Q}) \times Q}{||Q||^3} \quad \left(= \frac{A \times Q}{||Q||^3} \right)$$

Proof.
We get the following results

$$||Q|| = \sqrt{Q_1^2 + Q_2^2 + Q_3^2}$$

so that

$$\frac{d}{dt}||Q|| = \frac{Q \cdot \dot{Q}}{||Q||}$$

and we have the following property:

$$(A \times B) \times C = (C \cdot A)B - (C \cdot B)A$$

Assuming $u = \frac{Q}{||Q||}$ then we have

$$Q = u||Q|| \Longrightarrow \dot{Q} = ||\dot{Q}||u + ||Q||\dot{u} = \frac{Q \cdot \dot{Q}}{||Q||} \cdot \frac{Q}{||Q||} + ||Q||\dot{u}$$

obtaining the result we wanted

$$\frac{d}{dt}\left(\frac{Q}{||Q||}\right) = \dot{u} = \frac{\dot{Q}}{||Q||} - \frac{(Q \cdot \dot{Q})Q}{||Q||^3} = \frac{(Q \cdot Q)\dot{Q} - (Q \cdot \dot{Q})Q}{||Q||^3} = \frac{(Q \times \dot{Q}) \times Q}{||Q||^3}$$

□

If $A = 0$, then

$$\frac{d}{dt}\left(\frac{Q}{||Q||}\right) = \frac{(Q \times \dot{Q}) \times Q}{||Q||^3} = \frac{A \times Q}{||Q||^3} = 0 \tag{2.11}$$

The first equality above is a vector identity, so, if the angular momentum is zero, the motion is collinear. Letting the line of motion be one of the coordinate axes makes the problem a one-degree freedom problem and thus solvable.

However, if $A \neq 0$, then both Q and $P = \dot{Q}$ are orthogonal to A; and so the motion takes place on the plane orthogonal to A, known as the *invariant plane*. In this case, we can change the reference system in such a way that $A = \begin{pmatrix} 0 & 0 & c \end{pmatrix}^T$, so the motion takes place on the orthogonal plane (i.e., in the plane $Q_3 = 0$). Then, the equations of motion on this coordinate plane have the same form as (2.9), but $Q \in \mathbb{R}^2$. In the planar problem only the component of angular momentum perpendicular to the plane is nontrivial; so the problem is reduced to two degrees of freedom with one integral. Such a problem is solvable "up to quadrature". It turns out that the problem is solvable in terms of elementary functions, as we show below.

To do that, we must firstly make a change into polar coordinates, so that we have

$$Q = (Q_1, Q_2, 0) \Longrightarrow Q = (r\cos\theta, r\sin\theta, 0)$$
$$\dot{Q} = (\dot{r}\cos\theta - r\sin\theta\dot{\theta}, \dot{r}\sin\theta + r\cos\theta\dot{\theta}, 0),$$

and we obtain the value of A in these polar coordinates:

$$A = Q \times \dot{Q} = \begin{pmatrix} 0 \\ 0 \\ c = r^2\dot{\theta} \end{pmatrix} \tag{2.12}$$

We also must take into account in order to make the computations later is the following lemma:

Lemma 2.2. *For the value of the angular momentum obtained before, the rate at which the area is swept out by a radius vector is* $\frac{1}{2}r^2\dot{\theta}$.

Proof. We just need to compute the area swept out by a particle

$$Area(S) = \int_S dS \underset{*^1}{=} \int_{\theta_1}^{\theta_2}\int_0^{r(\theta)} r\, dr = \int_{\theta_1}^{\theta_2} \frac{r^2(\theta)}{2}\, d\theta \underset{*^2}{=} \int_{t_1}^{t_2} \frac{r^2\dot{\theta}}{2}\, dt \underset{*^3}{=} \frac{c}{2}(t_2 - t_1) \tag{2.13}$$

$*^1$: Jacobian change from cartesian to polar coordinates.
$*^2$: We can put the angle θ as $\theta = \theta(t)$ such that $d\theta = \dot{\theta}\, dt$.
$*^3$: $A = \begin{pmatrix} 0 & 0 & c = r^2\dot{\theta} \end{pmatrix}^T$. □

Now we are prepared to solve the Kepler problem.
First of all, we multiply the first equation in (2.11) by a factor of $-\mu$ to get

$$-\mu \frac{d}{dt}\left(\frac{Q}{||Q||}\right) = A \times \frac{-\mu Q}{||Q||^3} = A \times \dot{P}$$

Now we integrate this expression with respect of the time (taking into account that A is constant):

$$\frac{-\mu Q}{||Q||} = A \times P + \mu\varepsilon$$

where $\mu\varepsilon$ is a constant 3-dimensional vector. This expression can be written as

$$\mu\left(\varepsilon + \frac{Q}{||Q||}\right) = P \times A \tag{2.14}$$

Then, since $A \cdot Q = 0$ ($A = Q \times \dot{Q}$) and we calculate the scalar product of (2.14) with Q:

$$\mu(\varepsilon \cdot Q + ||Q||) = Q \cdot (P \times A)$$

and multiply by A

$$\mu\left(\varepsilon \cdot A + \frac{Q \cdot A}{||Q||}\right) = (P \times A) \cdot A$$

Using again that $Q \cdot A = 0$, we obtain that $\varepsilon \cdot A = 0$. Thus, if $A \neq 0$, then ε lies in the invariant plane. Otherwise, if $A = 0$, then by (2.14) we have $\varepsilon = -\frac{Q}{||Q||}$ and then ε lies on the line of motion and it has length 1.

Let $A \neq 0$ for the rest of this section. Then we have

$$\mu(\varepsilon \cdot Q + ||Q||) = Q \cdot (P \times A) \underset{definition}{=} det(Q, P, A) = det(A, Q, P)$$
$$= A \cdot (Q \times P) = A \cdot A = ||A||^2 = c^2$$

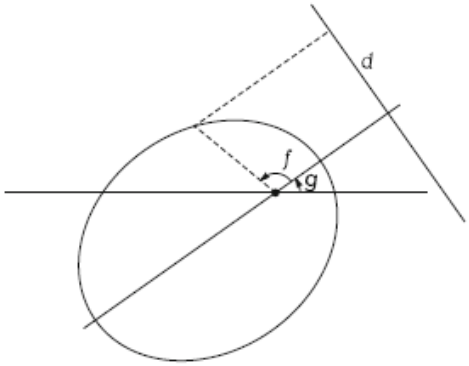


Figure 2.1: The elements of a Kepler motion [4].

so

$$\varepsilon \cdot Q + ||Q|| = \frac{c^2}{\mu} \tag{2.15}$$

If $\varepsilon = 0$, then $||Q|| = \frac{c^2}{\mu}$ is constant. As we saw before, $c = r^2\dot{\theta}$, so $\dot{\theta} = \frac{\mu^2}{c^3}$. Thus when $\varepsilon = 0$, the particle moves on a circle with constant angular velocity.

Now suppose that $\varepsilon \neq 0$, and denote $e = ||\varepsilon||$. Let the plane of motion be illustrated in Figure 2.1. Let r, θ be the polar coordinates of the particle with angle θ measured from the positive Q_1 axis. The angle from the positive Q_1 axis to ε is denoted by g, and the difference of these two angles by $f = \theta - g$. Thus, $\varepsilon \cdot Q = er\cos f$ and Equation (2.15) becomes

$$r = \frac{c^2/\mu}{1 + e\cos f} \tag{2.16}$$

which corresponds with the equation of a conic in polar coordinates. Thus, depending on the value of e, also called the *eccentricity of the conic*, we can obtain different geometries

- $e = 0$: This implies that r is constant, so we have a circle.
- $e < 1$: We would obtain an ellipse.
- $e = 1$: We would obtain a parabola.
- $e > 1$: We would obtain an hyperbola.

It is also important to notice that r is at its minimum when $f = 0$ and so ε points to the point of closest approach. This point is called the *perihelion* if the Sun is at the origin. The angle g is called the *argument of the perihelion* and the angle f is called the *true anomaly*.

Consider the line d illustrated in Figure 2.1 that is at a distance of $\frac{c^2}{\mu e}$ from the origin and perpendicular to ε as illustrated. Equation (2.16) can be rewritten as

$$r = e\left(\frac{c^2}{\mu e} - r\cos f\right)$$

which implies that the distance of the particle from the origin is equal to e times its distance from the line d. This results into Kepler's first law: the particle moves on a conic section of eccentricity e with one focus at the origin.

2.1.3 Description of the elliptic orbits

Before entering in the description of the 3-Body Problem, we are going to describe the most relevant elements of the ellipse, as we are going to work mainly with them along the project.

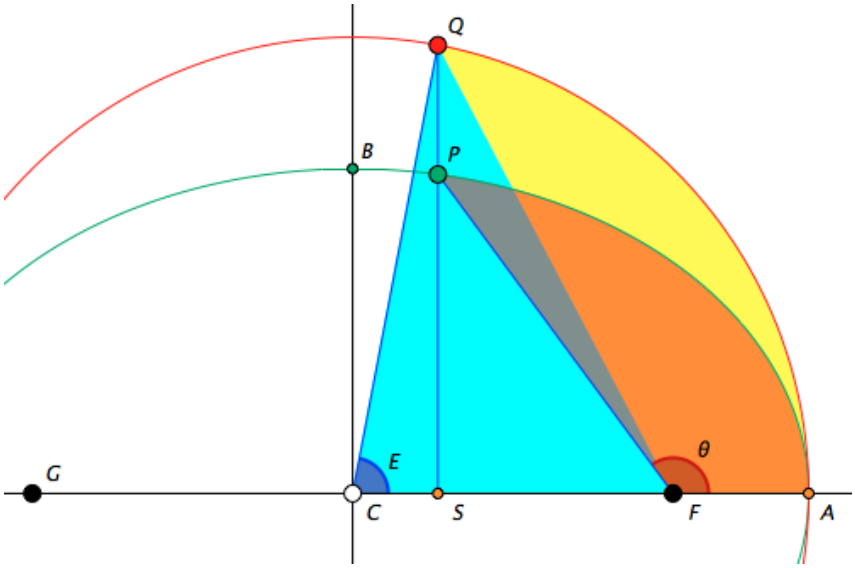


Figure 2.2: Portion of an ellipse and the correspondent circle with center C and radius $\overline{AC}$.

Since it would be an important result for later on, and it allows us to introduce the characteristic elements of the ellipse, we are going to compute the area of a sector of the ellipse from the focus (area in orange in Figure 2.2).

To do this computation, the main idea is to transform the problem of obtaining the area of a sector of the ellipse (in orange) to obtain the one of the corresponding circle (in yellow).

Transforming coordinates

First of all, the polar parametrization of an (axis-aligned) ellipse from its focus is given by the following lemma:

Lemma 2.3. *In polar coordinates (r, θ), the parametrization of the ellipse can be expressed by*

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{2.17}$$

Proof. We are going to consider an ellipse aligned with the axis in such a way that the angle g in Figure 2.1 becomes 0, and then we have that $f = \theta$.

From (2.16), we only need to prove that $\frac{c^2}{\mu} = a(1-e^2)$.

The following picture describes the motion of one particle of mass m around a virtual body of mass M.

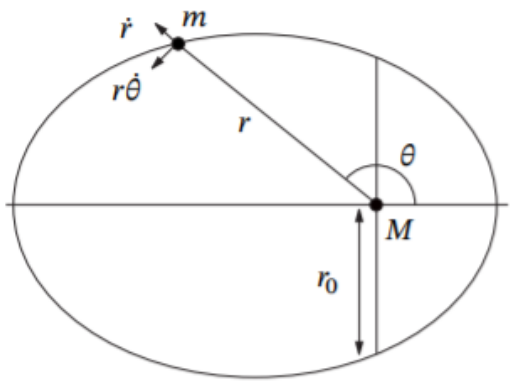


Figure 2.3: Motion of two particles with masses M and m in an elliptic orbit.

The modulus of the angular momentum A is defined, as we saw in (2.12) :

$$c = r^2\dot{\theta}$$

related to the radius r_0 (see Figure 2.3) as (see 2.15):

$$r_0 = \frac{c^2}{\mu}$$

so we only need to see that $r_0 = a(1-e^2)$.

The relation between these terms can be seen in the following picture

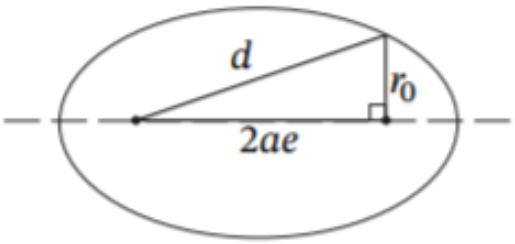


Figure 2.4: Relation between the semi-major axis, the eccentricity and r_0.

So applying Pythagoras and the fact that in an ellipse we have that $d + r_0 = 2a$ and we obtain:

$$(2a - r_0)^2 = 4a^2e^2 + r_0^2$$

so

$$r_0 = a(1 - e^2)$$

proving that $\frac{c^2}{\mu} = a(1 - e^2)$.
So we can write the radius as

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

□

Then, one obtains:

$$\overline{FP} = r(\theta) = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

where a is, as we said before, the semi-major axis and e is the eccentricity. We recall that, as $g = 0$, the true anomaly, denoted by f, is in this case $f = \theta$. Spelled out in coordinates, this is

$$\overline{FP} = \begin{pmatrix} r(\theta)\cos\theta \\ r(\theta)\sin\theta \end{pmatrix}$$

Now we stretch everything in the direction of its minor axis by a factor of

$$\frac{a}{b} = \frac{CA}{CB} = \frac{1}{\sqrt{1 - e^2}},$$

where b is the semi-minor axis. The result will be a circle of radius a. The origin is still the point that used to be focus. The distance between that point and the

center is $\overline{CF} = ae$ (called *linear eccentricity* or *focal distance*). Now we add that to x coordinates to move the origin into the center. Together you now have

$$\overline{CQ} = \begin{pmatrix} r(\theta)\cos\theta + ae \\ \frac{a}{b} r(\theta)\sin\theta \end{pmatrix} = \frac{a}{1+e\cos\theta} \begin{pmatrix} (1-e^2)\cos\theta + e + e^2\cos\theta \\ \sqrt{1-e^2}\sin\theta \end{pmatrix}$$

Eccentric anomaly

After the computation of the vector $\overline{CQ}$, one can obtain now the angle for the point Q against the horizontal axis. This angle, called the *eccentric anomaly* E, satisfies the following relation:

$$\tan E = \frac{\sqrt{1-e^2}\sin\theta}{(1-e^2)\cos\theta + e + e^2\cos\theta}$$

Using the Weierstrass substitution, one can turn this result into the following

$$\begin{aligned} \frac{2\tan\frac{E}{2}}{1-\tan^2\frac{E}{2}} &= \frac{\sqrt{1-e^2}\,2\tan\frac{\theta}{2}}{(1-e^2)(1-\tan^2\frac{\theta}{2}) + e(1+\tan^2\frac{\theta}{2}) + e^2(1-\tan^2\frac{\theta}{2})} \\ &= \frac{2\sqrt{(1+e)(1-e)}\tan\frac{\theta}{2}}{(1+e)-(1-e)\tan^2\frac{\theta}{2}} = \frac{2\sqrt{\frac{1+e}{1-e}}\tan\frac{\theta}{2}}{1-\frac{1-e}{1+e}\tan^2\frac{\theta}{2}}. \end{aligned}$$

Comparing both sides one can see

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}$$

and so we obtain

$$E = 2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

except for the special case when $\theta = \pm\pi$, which yields an infinite tangent but results in $E = \pm\pi$ as well.

Area of circular sector

Now we know that the area of a circular sector is proportional to the angle, so for a circle of radius a this would be $\frac{1}{2}a^2E$. That is the yellow and cyan areas in Figure 2.2, i.e, the sector ACQ.

From center back to focus

But that area is subtended by the center, not by the focus. Therefore one can substract a triangle of base ae and height $a \sin E$ (the cyan triangle $\triangle FCQ$ in Figure 2.2) and we get the area subtended by the focus (AFQ colored yellow):

$$\frac{1}{2}a^2E - \frac{1}{2}a^2e\sin E = \frac{1}{2}a^2(E - e\sin E)$$

Back to the ellipse

This is still an area in the circle. To get back to the original ellipse, we have to undo the scaling, i.e., scale the y direction by $\frac{b}{a}$. This scales areas by the same factor. Thus the area we were seeking for (AFP in red in Figure 2.2) would be

$$\frac{1}{2}ab(E - e\sin E)$$

Mean anomaly

As we vary θ from 0 to 2π, the above area will vary from 0 to $ab\pi$. Since equal areas are swept in equal time, this is closely related to the *mean anomaly*, which sweeps from 0 to 2π in constant time:

$$M = E - e\sin E,$$

so in terms of the mean anomaly, the area will be:

$$\frac{1}{2}abM$$

Period

We want to derive the relationship between the semi-major axis a and the period of the orbit T. To do that, we recall the definition of the area swept out by a particle (2.13)

$$S = \frac{c}{2}(t_2 - t_1)$$

This means that one can obtain the following result

$$\frac{dS}{dt} = \frac{c}{2}$$

By definition, the period T is the time that a particle needs to swept out the complete ellipse. Since the area of the total ellipse is πab, we have

$$S = \frac{1}{2}cT$$

To obtain a relation between the constant c and the semi-major axis a, one can evaluate Equation (2.16) at the perihelion, so $f = 0$ and $r = a(1-e)$, obtaining

$$a(1-e) = \frac{c^2}{1+e}$$

and then

$$a(1-e)^2 = c^2$$

Finally, one can relate the semi-major axis a with the semi-minor axis b as follows

$$b = a\sqrt{1-e^2}$$

So putting everything together, we have

$$2\pi a^2\sqrt{1-e^2} = Tc \Longrightarrow 4\pi^2 a^4(1-e^2) = T^2c^2 \longrightarrow 4\pi^2 a^3 = P^2 \Longrightarrow T = 2\pi a^{\frac{3}{2}}$$

so the relation between the period T and the semi-major axis a is given by:

$$T = 2\pi a^{\frac{3}{2}} \tag{2.18}$$

2.2 Planar 3-Body Problem

Now that we have explained and solved the Kepler Problem, we are ready to describe the model that we will analyze in the forthcoming sections: the *Restricted Circular Planar 3-Body Problem*. To do that, we will start by giving the definition of the more general 3-Body Problem as a particular case of the N-Body Problem we saw before. Then, we will perform and justify the necessary changes in the reference system to go from this general 3-Body scenario to the restricted case.

Let us begin by giving the complete expression of both the equations of motion:

$$\begin{cases} m_1\frac{d^2Q_1}{dt^2} = \mathcal{G}\frac{m_1m_2(Q_2-Q_1)}{||Q_2-Q_1||^3} + \mathcal{G}\frac{m_1m_3(Q_3-Q_1)}{||Q_3-Q_1||^3} \\ m_2\frac{d^2Q_2}{dt^2} = \mathcal{G}\frac{m_2m_1(Q_1-Q_2)}{||Q_1-Q_2||^3} + \mathcal{G}\frac{m_2m_3(Q_3-Q_2)}{||Q_3-Q_2||^3} \\ m_3\frac{d^2Q_3}{dt^2} = \mathcal{G}\frac{m_3m_1(Q_1-Q_3)}{||Q_1-Q_3||^3} + \mathcal{G}\frac{m_3m_2(Q_2-Q_3)}{||Q_2-Q_3||^3} \end{cases} \tag{2.19}$$

and the Hamiltonian (2.5) when $N = 3$:

$$\begin{aligned} \mathcal{H}(Q,P) = &\frac{||P_1||^2}{2m_1} + \frac{||P_2||^2}{2m_2} + \frac{||P_3||^2}{2m_3} \\ &- \mathcal{G}\frac{m_1m_2}{||Q_1-Q_2||} - \mathcal{G}\frac{m_1m_3}{||Q_1-Q_3||} - \mathcal{G}\frac{m_2m_3}{||Q_2-Q_3||} \end{aligned} \tag{2.20}$$

This Hamiltonian equation has 6 degrees of freedom. To reduce it, one can change the reference frame from the Cartesian coordinates to the Jacobi coordinates (see Section 7.2.1) in a similar way as we did in the 2-Body Problem:

$$g_3 = \frac{m_1 Q_1 + m_2 Q_2 + m_3 Q_3}{m_1 + m_2 + m_3}$$
$$u_2 = Q_2 - Q_1$$
$$u_3 = Q_3 - \frac{m_1 Q_1 + m_2 Q_2}{m_1 + m_2}$$

with the associated momenta $G_3 = P_1 + P_2 + P_3$, v_2, v_3.

The Hamiltonian in these new variables $(g_3, u_2, u_3, G_3, v_2, v_3)$ is:

$$\begin{aligned}\mathcal{K}(g_3, u_2, u_3, G_3, v_2, v_3) =& \frac{||G_3||^2}{2\mu_3} + \frac{||v_2||^2}{2M_2} + \frac{||v_3||^2}{2M_3} \\ & - \mathcal{G}\frac{m_1 m_2}{||u_2||} - \mathcal{G}\frac{m_1 m_3}{||u_3 + \alpha_0 u_2||} - \mathcal{G}\frac{m_2 m_3}{||u_3 - \alpha_1 u_2||},\end{aligned}$$

where

$$M_2 = \frac{m_1 m_2}{m_1 + m_3}, \qquad M_3 = \frac{m_3(m_1 + m_2)}{m_1 + m_2 + m_3}$$
$$\alpha_0 = \frac{m_2}{m_1 + m_2}, \qquad \alpha_1 = \frac{m_1}{m_1 + m_2}$$

Notice that $\alpha_0 + \alpha_1 = 1$, and they are introduced so that

$$\begin{aligned}g_2 &= \frac{m_1 Q_1 + m_2 Q_2}{m_1 + m_2} = \frac{m_1 Q_1 + m_2 Q_1 - m_2 Q_1 + m_2 Q_2}{m_1 + m_2} \\ &= Q_1 + \alpha_0 u_2 = Q_2 - \alpha_1 u_2\end{aligned}$$

so

$$||Q_2 - Q_3|| = ||Q_1 - g_2 + g_2 - Q_3|| = ||-u_3 - \alpha_0 u_2|| = ||u_3 + \alpha_0 u_2|| = ||u_3 - \alpha_1 u_2||$$

One can see now that $\mathcal{K}$ does not depend on the variable g_3, and so we can ignore it. This means that $\dot{G_3} = \frac{\partial \mathcal{K}}{\partial g_3} = 0$ (7.29), which implies that G_3 is a constant of motion.

We can take, without lose of generality, $g_3 = G_3 = 0$, and so now $\mathcal{K}$ depends only on (u_2, u_3, v_2, v_3), having 4 degrees of freedom instead of the initial 6.

$$\begin{aligned}\mathcal{K}(u_2, u_3, v_2, v_3) =& \frac{||v_2||^2}{2M_2} + \frac{||v_3||^2}{2M_3} - \mathcal{G}\frac{m_1 m_2}{||u_2||} \\ & - \mathcal{G}\frac{m_1 m_3}{||u_3 + \alpha_0 u_2||} - \mathcal{G}\frac{m_2 m_3}{||u_3 - \alpha_1 u_2||}\end{aligned} \tag{2.21}$$

Even though we reduce the degrees of freedom, we are still far from being able to give a solution of the 3-Body Problem. However, we can still perform more changes in order to keep reducing its complexity.

To do that, we apply now a polar change of coordinates in the following way (see Section 7.2.2):

$$\begin{cases} u_2 = r_1(\cos\theta_1, \sin\theta_1) \\ u_3 = r_2(\cos\theta_2, \sin\theta_2) \end{cases}$$

In order to have a symplectic change, we must also transform the momenta as follows:

$$v_2 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} R_1 \\ \frac{\Theta_1}{r_1} \end{pmatrix}$$
$$v_3 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} R_2 \\ \frac{\Theta_2}{r_2} \end{pmatrix}$$

The Hamiltonian $\mathcal{K}$ in these new variables (r, θ, R, Θ) takes the form (shifting the index $1, 2, 3$ to $0, 1, 2$)

$$\begin{aligned} \tilde{\mathcal{K}}(r, \theta, R, \Theta) =& \frac{1}{2M_1}\left(R_1^2 + \frac{\Theta_1^2}{r_1^2}\right) + \frac{1}{2M_2}\left(R_2^2 + \frac{\Theta_2^2}{r_2^2}\right) - \mathcal{G}\frac{m_0 m_1}{r_1} \\ & - \mathcal{G}\frac{m_0 m_2}{\sqrt{r_2^2 + 2\alpha_0 r_1 r_2 \cos(\theta_2 - \theta_1) + \alpha_0^2 r_1^2}} \\ & - \mathcal{G}\frac{m_1 m_2}{\sqrt{r_2^2 - \alpha_1 r_1 r_2 \cos(\theta_2 - \theta_1) + \alpha_1^2 r_1^2}} \end{aligned}$$

To get rid of the terms $\theta_2 - \theta_1$, we perform a symplectic change in angle variables:

$$\begin{pmatrix} \theta \\ \Theta \end{pmatrix} = \begin{pmatrix} Z & 0 \\ 0 & Z^{-T} \end{pmatrix} \begin{pmatrix} \phi \\ \Phi \end{pmatrix},$$

with $Z = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. Applying these new changes, we obtain what is called the Hamiltonian function in polar coordinates:

$$\begin{aligned} \mathcal{H}(r, \phi, R, \Phi) =& \frac{1}{2M_1}\left(R_1^2 + \frac{(2\Phi_2 - \Phi_1)^2}{r_1^2}\right) + \frac{1}{2M_2}\left(R_2^2 + \frac{(\Phi_1 - \Phi_2)^2}{r_2^2}\right) \\ & - \mathcal{G}\frac{m_0 m_1}{r_1} - \mathcal{G}\frac{m_0 m_2}{\sqrt{r_2^2 + 2\alpha_0 r_1 r_2 \cos\phi_1 + \alpha_0^2 r_1^2}} \\ & - \mathcal{G}\frac{m_1 m_2}{\sqrt{r_2^2 - 2\alpha_1 r_1 r_2 \cos\phi_1 + \alpha_1^2 r_1^2}} \end{aligned} \tag{2.22}$$

Once again, we find that the new Hamiltonian does not depend on ϕ_2, so it can be ignored. Since $\dot{\Phi}_2 = \frac{\partial \bar{\mathcal{K}}}{\partial \phi_2} = 0$, Φ_2 is a constant of motion.

In fact, the previous change shows us that $\Phi_2 = \Theta_1 + \Theta_2$, which is the same as $\Phi_2 = u_2 \times v_2 + u_3 \times v_3$. This is the definition of the total angular momentum, which is preserved as it is one of the 10 constants of motion for the general N-Body Problem (since we are dealing with the planar setting, the angular momentum is just a scalar). Since the total angular momentum is constant, we can ensure that if the motion takes place on a plane, it will remain in the same plane, as it happened in the Kepler problem we solved before.

2.2.1 Equilibrium and Central Configurations

The N-Body Problem is not integrable, so we cannot compute the general solution. However, over the years many special types of solutions have been found using various mathematical techniques. The simplest type of solution one might look for is equilibrium or rest solutions.

Equilibrium Solutions

For (2.1), an equilibrium solution would have to satisfy

$$\frac{\partial U}{\partial Q_i} = 0 \text{ for } i = 1, \ldots, N. \tag{2.23}$$

However, U is homogeneous of degree -1, so by Euler's Theorem on homogeneous functions,

$$\sum_i Q_i \frac{\partial U}{\partial Q_i} = -U. \tag{2.24}$$

Because U is the sum of positive terms, it is positive. If (2.23) were true, then the left side of (2.24) would be zero, which gives a contradiction. Thus there are no equilibrium solutions of the N-Body Problem.

Central Configurations

For a second type of solutions to (2.1), try $Q_i(t) = \phi(t)a_i$, where the $a_i's$ are constant vectors and $\phi(t)$ is a scalar-valued function. Substituting into (2.1) and rearranging yields

$$|\phi|^3 \phi^{-1} \ddot{\phi} m_i a_i = \sum_{j=1, j\neq i}^{N} \mathcal{G} \frac{m_i m_j (a_j - a_i)}{||a_j - a_i||^3} \tag{2.25}$$

Because the right side is constant, the left side must be constant too; let this value be λ. Therefore, (2.25) has a solution if there exists a scalar function $\phi(t)$, a constant λ, and constant vectors a_i such that

$$\ddot{\phi} = -\frac{\lambda\phi}{|\phi|^3}, \tag{2.26}$$

$$-\lambda m_i a_i = \sum_{j=1, j\neq i}^{N} \mathcal{G}\frac{m_i m_j (a_j - a_i)}{||a_j - a_i||^3}, \qquad i = 1, \ldots, N. \tag{2.27}$$

Equation (2.26) is a simple ordinary differential equation (in fact it is the one-dimensional Kepler problem) and so it has many solutions.

Now consider the planar N-Body Problem, where all the vectors lie in $\mathbb{R}^2$. Identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by considering Q_i, P_i complex numbers. Seek a homographic solution of (2.1) by letting $Q_i(t) = \phi(t)a_i$, where the $a_i's$ are constant complex numbers and $\phi(t)$ is a time-dependent complex-valued function. Geometrically, multiplication by a complex number is a rotation followed by a dilation or expansion, i.e., a homography. Thus we seek a solution such that the configuration of the particles is always homographically equivalent to a fixed configuration. Substituting into (2.1) will give us the same equations (2.25), and the same arguments gives Equations (2.26) and (2.27), but now (2.26) is the two-dimensional Kepler problem. That is, if you have a solution of (2.27) where the $a_i's$ are planar, then there is a solution of the N-Body Problem of the form $Q_i = \phi(t)a_i$, where $\phi(t)$ is any solution of the planar Kepler problem, e.g., circular, elliptic, etc.

A configuration of the N particles given by constant vectors $a_1, \ldots, a_N$ satisfying the second equation (2.27) for some λ is called a *central configuration*. In the special case when the $a_i's$ are coplanar, a central configuration is also called a relative equilibrium because, as we show, they become *equilibrium solutions in a rotating coordinate system*. Moreover, central configurations are important in the study of the total collapse of the system because it can be shown that the limiting configuration of a system as it tends to a total collapse is a central configuration [5].

2.3 Restricted 3-Body Problem

Once we gave some insight about the 3-Body Problem, it is time to focus on the restricted case. To do that, we will start by assuming, without loss of generality, that the three bodies are:

- The Sun, whose position will be denoted by Q_S, its momenta by P_S and its mass by m_S.

- Jupiter, whose position will be denoted by Q_J, its momenta by P_J and its mass by m_J.
- An Asteroid, whose position will be denoted by Q_A, its momenta by P_A and its mass by m_A.

We will also assume that the mass of the Asteroid is infinitesimal compared with the other two, and therefore we set $m_A = 0$. Substituting in the equations of motion of the 3-Body Problem (2.19), we obtain the following result:

$$\begin{cases} \frac{d^2Q_S}{dt^2} = \mathcal{G}\frac{m_J(Q_J-Q_S)}{||Q_J-Q_S||^3} \\ \frac{d^2Q_J}{dt^2} = \mathcal{G}\frac{m_S(Q_S-Q_J)}{||Q_S-Q_J||^3} \\ \frac{d^2Q_A}{dt^2} = \mathcal{G}\frac{m_S(Q_S-Q_A)}{||Q_S-Q_A||^3} + \mathcal{G}\frac{m_J(Q_J-Q_A)}{||Q_J-Q_A||^3} \end{cases} \tag{2.28}$$

One can notice after this simplification that the motion of the bodies Q_S and Q_J (called primaries) is not affected by the particle Q_A. This implies that the Q_J and Q_S forms a separated 2-Body Problem, which is solvable as we saw before (see Section 2.1.2).

Once we get the solutions $Q_S(t)$ and $Q_J(t)$ of this separated problem, we can put them in the equation of Q_A and study its motion:

$$\frac{d^2Q_A}{dt^2} = \mathcal{G}\frac{m_S(Q_S(t)-Q_A)}{||Q_S(t)-Q_A||^3} + \mathcal{G}\frac{m_J(Q_J(t)-Q_A)}{||Q_J(t)-Q_A||^3} \tag{2.29}$$

This is what is called the *Restricted 3-Body Problem (R3BP)*. In particular, for the problem we are going to analyse along the paper, we will assume that the motion of Q_A takes place in the plane of rotation of the other two bodies, and so instead of being in the R3BP, we are now in the *Restricted Planar 3-Body Problem (RP3BP)*.

Moreover, we will also assume that the two primaries $Q_S(t)$ and $Q_J(t)$ move on circles, obtaining what is called the *Restricted Planar Circular 3-Body Problem (RPC3BP)*, where usually the masses of the primaries are normalized to $m_S = 1-\mu$ and $m_J = \mu$, where $\mu = \frac{m_J}{m_S+m_J}, m_S \geq m_J$. We also normalize the value of $\mathcal{G}$ to be 1. Substituting all these values in (2.29) we obtain the equation of motion for the massless particle $Q_A = Q = (Q_1, Q_2) \in \mathbb{R}^2$:

$$\frac{d^2Q}{dt^2} = \frac{(1-\mu)(Q_S(t)-Q)}{||Q_S(t)-Q||^3} + \frac{\mu(Q_J(t)-Q)}{||Q_J(t)-Q||^3}, \tag{2.30}$$

where now $Q_S(t)$ and $Q_J(t)$ are the known position of the primaries, which move in a circular orbit:

$$Q_S(t) = -\mu(\cos t, \sin t), \quad Q_J(t) = (1-\mu)(\cos t, \sin t)$$

Denoting $P_A = P = (P_1, P_2) = \frac{dQ}{dt}$ one obtains a system of 4-dimensional non-autonomous differential equations:

$$\begin{cases} \frac{dQ}{dt} = P \\ \frac{dQ}{dt} = \frac{(1-\mu)(Q_S(t)-Q)}{||Q_S(t)-Q||^3} + \frac{\mu(Q_J(t)-Q)}{||Q_J(t)-Q||^3} \end{cases} \tag{2.31}$$

This new system can be written as a Hamiltonian system:

$$\begin{cases} \frac{dQ}{dt} = \frac{\partial \mathcal{H}}{\partial P}(Q, P, t; \mu) \\ \frac{dP}{dt} = -\frac{\partial \mathcal{H}}{\partial Q}(Q, P, t; \mu) \end{cases} \tag{2.32}$$

where $\mathcal{H}$ is the following Hamiltonian function:

$$\begin{aligned} \mathcal{H}(Q, P, t; \mu) = \frac{P_1^2 + P_2^2}{2} - \frac{(1-\mu)}{\sqrt{(Q_1 + \mu\cos t)^2 + (Q_2 + \mu \sin t)^2}} \\ - \frac{\mu}{\sqrt{(Q_1 - (1-\mu)\cos t)^2 + (Q_2 - (1-\mu)\sin t)^2}} \end{aligned} \tag{2.33}$$

The RCP3BP has a first integral, named *Jacobi Constant* and denoted by C, with the following expression:

$$\begin{aligned} C = -2(\mathcal{H} - G) = -(P_1^2 + P_2^2) + 2(Q_1 P_2 - Q_2 P_1) \\ + 2\frac{(1-\mu)}{\sqrt{(Q_1 + \mu\cos t)^2 + (Q_2 + \mu \sin t)^2}} \\ + 2\frac{\mu}{\sqrt{(Q_1 - (1-\mu)\cos t)^2 + (Q_2 - (1-\mu)\sin t)^2}} \end{aligned} \tag{2.34}$$

where G denotes the modulus of the angular momentum in the restricted 3-Body Problem. To understand the behaviour of the body $Q = (Q_1, Q_2)$, now we have to deal with 4 dimensional, non-linear and non-autonomous differential equations.

2.3.1 Planar Restricted 3-Body Problem in rotating coordinates

Since we are working assuming that both the Sun and Jupiter describes a circular motion, we are going to express the RPC3BP in rotating coordinates, as it allows us to fix the positions Q_S and Q_J.

To make this change into the rotating frame, let's consider

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \exp(Jt) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \tag{2.35}$$

be a 2×2 matrix. Now we introduce a set of coordinates that corresponds to put the three bodies in a rotating coordinate frame.

$$x = \exp(Jt)Q, \;\; y = \exp(Jt)P \tag{2.36}$$

Because J is skew-symmetric, $\exp(Jt)$ is orthogonal for all t, so the change of variables is symplectic, with a remainder term of $x^T Jy$, and so the Hamiltonian of the Planar Circular Restricted 3-body problem in rotating coordinates is

$$H(x, y) = \frac{1}{2}||y||^2 - x^T Jy - U(x), \tag{2.37}$$

where $U(x)$ is the self-potential

$$U(x) = \frac{\mu}{d_1} + \frac{1-\mu}{d_2},$$

with d_i the distance from the infinitesimal body to the i^{th} primary, or

$$d_1^2 = (x_1 - 1 + \mu)^2 + x_2^2, \quad d_2^2 = (x_1 + \mu)^2 + x_2^2.$$

The equations of motion can be written now as

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} = y + Jx \\ \dot{y} = -\frac{\partial H}{\partial x} = Jy + \frac{\partial U}{\partial x}. \end{cases} \tag{2.38}$$

The term $x^T Jy$ in the Hamiltonian H reflects the fact that the coordinate system is not an inertial frame, but a rotating one. It gives rise to what is called the *Coriolis force*.

In much of the literature, the equations of motion for the restricted problem are written as a second-order equation in the position variable x. Eliminating y from Equation (2.38) gives

$$\ddot{x} - 2J\dot{x} - x = \frac{\partial U}{\partial x}, \tag{2.39}$$

that corresponds to the Euler-Lagrange given by the Lagrangian associated with the Hamiltonian H. The Hamiltonian H becomes the first integral:

$$F = \frac{1}{2}||\dot{x}||^2 - \frac{1}{2}||x||^2 - U(x) \tag{2.40}$$

Finally, the Jacobi constant C in the rotating frame is described as

$$C = -2F + \mu(1-\mu) \tag{2.41}$$

Substituting H by the previous expression in (2.40), one obtains

$$C = ||x||^2 + 2U(x) + \mu(1-\mu) - ||\dot{x}||^2$$

and one refers to

$$W(x) = ||x||^2 + 2U(x) + \mu(1-\mu) \tag{2.42}$$

as the *Amended Potential* for the Restricted 3-Body Problem.

The amended potential is positive and it tends to infinity as $x \to \infty$ or when x tends to a primary (and therefore to a collision).

Discrete Symmetry

The Hamiltonian of the restricted problem (2.37) has the following symmetry:

$$H(x_1, x_2, y_1, y_2) = H(x_1, -x_2, -y_1, y_2) \tag{2.43}$$

It follows that to any solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ there corresponds another solution $(x_1(-t), -x_2(-t), -y_1(-t), y_2(-t))$. The second is the reflection of the first in the (x_1, y_2)-plane with the time reversed.

An orbit is symmetric if it its own reflection. That means that somewhere along the orbit there is a point such that $x_2 = y_1 = 0$, i.e, at some time it crosses the x_1 axis perpendicularly. If there are two such points the orbit is periodic.

There is another symmetry involving the parameter μ of the problem. A symmetry with respect to the x_2-axis exchanges μ and $(1-\mu)$. One can take advantage of this symmetry and restrict the range of the parameter μ to the interval $(0, \frac{1}{2}]$. Note that for $\mu = \frac{1}{2}$, there is a second symmetry of the phase space; namely

$$H(x_1, x_2, y_1, y_2) = H(-x_1, x_2, y_1, -y_2)$$

This implies that to any solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ there corresponds another solution $(-x_1(-t), x_2(-t), y_1(-t), -y_2(-t))$.

Chapter 3

Main Results

In this chapter, we are going to look for second-species periodic solutions in the RCP3BP (2.33), i.e., periodic orbits that travel close to collision with Jupiter. To do that, we will explain the main results given by S.V. Bolotin and R.S. Mackay [1], where they make a variational approach of the problem, and the results given by Jean-Pierre Marco and Laurent Niederman [2], who make a more geometrical approach.

3.1 Second-species solutions in the Restricted Planar Circular 3-Body Problem

To find the second-species periodic solutions, we are going to explain the results stated and proved by S.V. Bolotin and R.S. Mackay [1], as well as the ones of Jean-Pierre Marco and Laurent Niederman [2].

We will divide the method given by S.V. Bolotin and R.S. Mackay in two parts:

- First of all, they are going to consider the Restricted Planar 3-Body Problem for $\mu = 0$ (which is the same as consider the Kepler's problem between the Asteroid and the Sun). In this model, they will study the collisions given by the Asteroid and Jupiter.

  For each collision, the elliptic orbit given by the Asteroid will be a solution to this Kepler's problem. For each elliptic orbit, they are going to consider the section corresponding to the part between one collision and the following one. Bolotin and Mackay will build then an infinite sequence of such sections, related by the collisions, and they will denote it as a *chain*.

- Finally, from this chain, they will prove the existence and uniqueness of a trajectory of the restricted 3-Body Problem (for a Jacobi Constant C), with

$\mu = \varepsilon$ small enough, that travels close to this chain, generated in such a way that it converges to it as $\varepsilon \to 0$. They will say that this trajectory *shadows* the chain. Therefore, if the chain is periodic, this trajectory will be periodic too, and so they will obtain a second-species periodic solution to the problem. They claim this result in form of the following theorem.

Theorem 3.1. *In the planar circular restricted 3-body problem with masses* $1-\varepsilon$, ε, 0, *for all values of the Jacobi constant* $C \in (-\sqrt{8}, +3)$ *there exists a dense subset* S_C *of rationals in the set* A_C *of allowed frequencies for Kepler ellipses crossing the unit circle, such that for all finite subsets* $T \subset S_C$ *there exists* $\varepsilon_0 > 0$ *such that for any sequence* $\sigma = (\Omega_n = \frac{m_n}{k_n})_{n\in\mathbb{Z}}$ *in* T *and* $0 < \varepsilon < \varepsilon_0$ *there is a unique trajectory of Jacobi Constant* C *near to a chain of collision trajectories formed by transforming ellipses of frequencies* Ω_n *traversed* m_n *times to the rotating frame, and it converges to the chain as* $\varepsilon \to 0$.

Corollary 3.2. *If the chain of collision trajectories is periodic, the shadowing trajectory will be periodic too. This trajectory will be then a second-species periodic solution of the restricted planar* 3*-Body Problem.*

Jean-Pierre Marco and Laurent Niederman give a different proof of the existence of second-species periodic solutions. They will start by considering, as Bolotin and Mackay did, the Kepler's problem corresponding to the restricted 3-Body Problem for $\mu = 0$. In this model, they will take only a part of a particular periodic chain, corresponding to two symmetric sections of elliptic trajectories which collide at one point.

These two trajectories, denoted by

$$\begin{aligned} \varphi_s \colon (-t_c, t_c) &\to \mathbb{C} \\ t &\mapsto \varphi_s(t) \\ \varphi_u \colon (t_c, t_c + l) &\to \mathbb{C} \\ t &\mapsto \varphi_u(t) \end{aligned}$$

where t_c is such that $\varphi_s(-t_c) = \varphi_s(t_c) = \varphi_u(t_c) = \varphi_u(t_c + l) = P$, being P the collision point (Jupiter). They will be two solutions of the Kepler's problem, and therefore they will have an associated momentum, denoted as σ_s and σ_u, which will be bounded by an interval defined as (σ_-, σ_+) (see Section 5.1.1). By symmetry arguments (2.43), Jean-Pierre Marco and Laurent Niederman will be able to ensure that the trajectory of the Asteroid can be reduced to the union of these two trajectories.

From these two solutions, whose union will be denoted by φ, they introduce the following concept:

Definition 3.3. *We will call a family of second-species periodic solution associated to φ_s and φ_u to a family (φ_μ), defined for $\mu > 0$ in a neighborhood of 0, of periodic solutions for every μ of the Hamiltonian H defined in (2.37), such that it converges to φ when $\mu \to 0$.*

If such a family exists, we will say that φ_s and φ_u are generatrices.

These family of solutions (φ_μ) will be, in fact, the second-species solutions we are seeking for. They claim its existence in the following theorem.

Theorem 3.4. *Let $\sigma_1 < \sigma_2$ be two values in the interval (σ_-, σ_+). For every $\eta > 0$, there exists two generatrix solutions φ_s and φ_u of the Hamiltonian H defined in (2.37) for $\mu = 0$, of respective angular momentums a_1 and a_2, such that $|a_1 - \sigma_1| < \eta$ and $|a_2 - \sigma_2| < \eta$.*

Now, before entering in the proofs of both theorems, we are going to explain in detail all the previous results needed to understand them. In these terms, we will be focusing our attention on the two papers we referred before, taking a deeper look into those concepts that could be more confusing.

3.2 S.V. Bolotin and R.S. Mackay Results

First of all, the procedure made by S.V. Bolotin and R.S. Mackay can be divided in four different parts

- To begin with, they will give an introduction to the Hamiltonian and Lagrangian formalism they are going to use along their approach. This is explained in section 3.2.1 of this work.

- Then, they will assume that, for the unperturbed system, there exists a chain as we defined in the previous section. From this assumption, they will come back to the perturbed case, where they will define and prove the existence and uniqueness of a trajectory that *shadows* the chain. This is explained in section 3.2.2 of this work.

- In the third part, they apply the general case study to a concrete scenario, the RCP3BP. This is explained in section 3.2.3 of this work.

- Finally, they will build a sequence of collision trajectories from the Kepler's problem (and therefore, in the unperturbed system) to generate a chain. This is explained in section 3.2.4 of this work.

3.2.1 Introduction

Let $\mathcal{Q}$ be a smooth manifold containing the particle's position as we did in the previous chapter and $\mathcal{P} = \{p_1, \cdots, p_n\}$ a finite set in $\mathcal{Q}$ corresponding to those points where the collision takes place. Let us start by considering the following perturbation of the natural Lagrangian system defined in the configuration space $\mathcal{Q} \setminus \mathcal{P}$, as it will be easier to work with this function for the later results:

$$L_\varepsilon(q, \dot{q}) = L(q, \dot{q}) - \varepsilon V(q), \tag{3.1}$$

where $L(q, \dot{q})$ is the Lagrangian:

$$L(q, \dot{q}) = T(q, \dot{q}) + \langle \omega(q), \dot{q} \rangle - W(q), \tag{3.2}$$

where $\langle \omega(q), \dot{q} \rangle$ is the Coriolis term, $T(q, \dot{q})$ is the kinetic energy, $W(q)$ is the potential energy, and V is the perturbation of this latter potential, which is a function with singularities at the collision set $\mathcal{P}$. We will assume that, in a neighborhood of any point p_i of $\mathcal{P}$, the potential $V(q)$ is defined as

$$V(q) = -\frac{f_i(q)}{\text{dist}(q, p_i)}, \tag{3.3}$$

where f_i is a C^4 function on $U_i \subset \mathcal{Q}$, with $f_i(p_i) > 0$.

Notice that, as these equations do not involve the time explicitly, the kinetic energy T is a quadratic function of the $\dot{q}_i's$

$$T = \frac{1}{2}\langle A(q)\dot{q}, \dot{q} \rangle, \tag{3.4}$$

with $A(q)$ a metric tensor. Then, T can be seen as a Riemannian metric on $\mathcal{Q}$, and so the distance $\text{dist}(q, p_i)$ in (3.3) is defined by means of this new metric.

Lemma 3.5. *With the notation above, the corresponding Hamiltonian function can be expressed as*

$$H_\varepsilon = H + \varepsilon V$$

with

$$H(q, \dot{q}) = T(q, \dot{q}) + W(q)$$

Proof. We can use directly the Legendre Transformation (see Section 7.1.2)

$$H_\varepsilon = \dot{q}\frac{\partial L_\varepsilon}{\partial \dot{q}} - L_\varepsilon$$

So we have

$$
\begin{aligned}
H_\varepsilon &= \dot{q}\left[\frac{\partial}{\partial \dot{q}}\left(L(q,\dot{q}) - \varepsilon V(q)\right)\right] - \left(L(q,\dot{q}) - \varepsilon V(q)\right) = \dot{q}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right) - L(q,\dot{q}) + \varepsilon V(q) \\
&\underset{*}{=} 2T(q,\dot{q}) + \langle \omega(q), \dot{q}\rangle - T(q,\dot{q}) - \langle \omega(q), \dot{q}\rangle + W(q) \\
&= T(q,\dot{q}) + W(q) + \varepsilon V(q) \underset{**}{=} H(q,\dot{q}) + \varepsilon V(q)
\end{aligned}
$$

* Computation of $\frac{\partial L(q,\dot{q})}{\partial \dot{q}}$:

$$
\begin{aligned}
\frac{\partial L(q,\dot{q})}{\partial \dot{q}} &= \frac{\partial}{\partial \dot{q}}\left(T(q,\dot{q}) + \langle \omega(q), \dot{q}\rangle - W(q)\right) = \frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}\langle A(q)\dot{q}, \dot{q}\rangle + \langle \omega(q), \dot{q}\rangle\right) \\
&= \frac{1}{2}\left(\langle A(q), \dot{q}\rangle + \langle A(q)\dot{q}, \mathbb{1}\rangle\right) + \langle \omega(q), \mathbb{1}\rangle.
\end{aligned}
$$

Then

$$
\begin{aligned}
\dot{q}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} &= \frac{1}{2}\left(\langle \dot{q}A(q), \dot{q}\rangle + \langle A(q)\dot{q}, \dot{q}\rangle\right) + \langle \omega(q), \dot{q}\rangle \\
&\underset{\text{Equation 3.4}}{=} 2T(q,\dot{q}) + \langle \omega(q), \dot{q}\rangle
\end{aligned}
$$

** From Equation (2.5) □

Note that there is an abuse of notation here. The Lagrangian function is defined on the tangent bundle $T\mathcal{Q}$ (see Section 7.1.2), so naturally, $\dot{q}$ belongs to the tangent space $T_q\mathcal{Q}$. However, when going through the Legendre Transformation (see Section 7.1.2), the corresponding Hamiltonian H_ε must be defined on the cotangent bundle $T^*\mathcal{Q}$, and so instead of $\dot{q}$, we must be referring to the corresponding element in the cotangent space $p \in T_q^*\mathcal{Q}$ (normally named the momentum, whose relation is given by $p = \frac{\partial L_\varepsilon(q,\dot{q})}{\partial \dot{q}}$), so $H_\varepsilon(q,\dot{q})$ should have been written as $H_\varepsilon(q,p)$. However, we will preserve it as $\dot{q}$ as it is the notation used in [1].

3.2.2 Formulation of the general theorem

We will start by giving a definition to some important concepts that we will use constantly along this chapter.

We fix E such that the domain $\mathcal{D} = \{q \in \mathcal{Q} \mid W(q) < E\}$ contains the set $\mathcal{P}$ and we study the system (L_ε) (3.1) on the energy level $\{H_\varepsilon = E\} \subset T(\mathcal{Q} \setminus \mathcal{P})$.

Definition 3.6. *We say that a trajectory $\gamma_k\colon [0,\tau_k] \to \mathcal{D}$, $k \in K$ (a finite set) of energy E is a collision trajectory if $\gamma_k(0) = p_{\alpha_k}$, $\gamma_k(\tau_k) = p_{\beta_k}$, and $\gamma_k((0,\tau_k)) \subset \mathcal{D} \setminus \mathcal{P}$.*

We suppose that the system (L) (3.2) has nondegenerate (definition to be recalled shortly) collision trajectories. We define now the concept of *chain*.

Definition 3.7. *Given a collision trajectory γ_k, an infinite sequence $(k_i \in K)_{i\in\mathbb{Z}}$ is called a chain if $\gamma_{k_i}(\tau_{k_i}) = \gamma_{k_{i+1}}(0)$ and $\dot{\gamma}_{k_i}(\tau_{k_i}) \neq \pm\dot{\gamma}_{k_{i+1}}(0)$ for all $i \in \mathbb{Z}$.*

Finally, we can define the *shadowing chain*, which will be crucial to understand in order to prove Theorem 3.4.

Definition 3.8. *For each $k \in K$, let $W_k \subset \mathcal{D}$ be a neighbourhood of $\gamma_k([0, \tau_k])$. We say that a trajectory $\gamma\colon \mathbb{R} \to \mathcal{D} \setminus \mathcal{P}$ shadows the chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$ if there exists an increasing sequence $(t_i)_{i\in\mathbb{Z}}$ such that $\gamma([t_i, t_{i+1}]) \subset W_{k_i}$.*

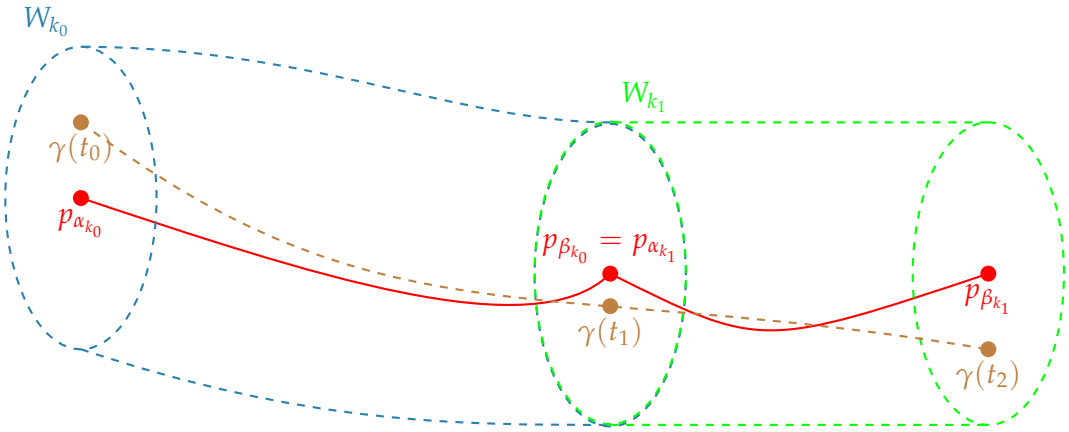


Figure 3.1: Example of a collision trajectory (in red), two neighborhoods W_{k_0} and W_{k_1}, and one shadowing trajectory γ (in brown).

Now we are prepared to state the Theorem that claims the existence of second species solutions, whose proof is the purpose of our work.

Theorem 3.9. *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$ and any chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$:*

- *There exists a trajectory $\gamma\colon \mathbb{R} \to \mathcal{D} \setminus \mathcal{P}$ of energy E for the system (L_ε) (3.1) shadowing the chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$ and it is unique (up to a time shift) if the W_k are chosen small enough.*
- *The orbit γ converges to the chain of collision orbits as $\varepsilon \to 0$: There exists an increasing sequence $(t_i)_{i\in\mathbb{Z}}$ such that*

$$\max_{t_i \le t \le t_{i+1}} dist(\gamma(t), \gamma_{k_i}([0, \tau_k])) \le C\varepsilon$$

where the constant $C > 0$ depends only on the set K of collision orbits.

- *The orbit γ avoids collision by a distance of order ε: there exists a constant $c \in (0, C)$, depending only on K such that*

$$c\varepsilon \leq \min_{t_{i-1} \leq t \leq t_{i+1}} dist(\gamma(t), p_{\alpha_{k_i}})$$

The proof of this Theorem in the Restricted Planar Circular 3-Body Problem, which is the main goal of our work, will be described in Chapter 4. For now let us clarify the definition of *nondegenerate chain*.

Nondegeneracy Chains

We devote this section to the concept of nondegenerate chains, as it appears multiple times along this work. Although in [1], there are up to five equivalent definitions, we will focus only on two of them, as they will be more suitable to use for later computations (see Chapter 4).

- The first definition is variational. Let $\Omega = \Omega(\mathcal{Q}, p_\alpha, p_\beta)$ be the space of $W^{1,2}$ curves $u\colon [0,1] \to \mathcal{Q}$ such that $u(0) = p_\alpha$, $u(1) = p_\beta$. Any point $(u, \tau) \in \Omega \times \mathbb{R}^+$ defines a curve $\gamma\colon [0, \tau] \to \mathcal{Q}$ by $\gamma(t) = u(\frac{t}{\tau})$. Let

$$F(u, \tau) = \int_0^\tau (L(\gamma(t), \dot{\gamma}(t)) + E)\, dt \tag{3.5}$$

  be its action. Then F is a C^2 functional on $\Omega \times \mathbb{R}^+$ and its critical points correspond to trajectories (of the Euler-Lagrange equations associated to the Lagrangian L (3.2)) of energy E connecting p_α with p_β (the fact that u is a curve in $W^{1,2}$ is what makes it possible to use the integration-by-parts theory and to be able to compute the critical points of this functional). The reason why this action integral gives us this information was discussed in the previous chapter (see Section 3.2).

  A collision orbit $\gamma_k\colon [0, \tau_k] \to \mathcal{D}$ is nondegenerate if (u, τ_k), where $u(t) = \gamma_k(\tau_k)$, is a nondegenerate critical point for F (i.e, the second variation is not zero).

- The final definition is the most suitable one for verification in concrete examples: represent the general solution of the system (L) (3.2) as $q = q(\lambda, t)$, where λ is a parameter of dimension 2 $dim\mathcal{Q}$. Then $H(q(\lambda, t), \dot{q}(\lambda, t)) = h(\lambda)$ for some function h (this h can be found explicitly by means of the Legendre Transformation (see Section 7.1.2) .

It is trivial then that collision orbits connecting p_α with p_β with energy $H = E$ correspond to solutions of the system of equations

$$\begin{cases} q(\lambda, 0) = p_\alpha \\ q(\lambda, \tau) = p_\beta \\ h(\lambda) = E \end{cases} \tag{3.6}$$

in the variables λ, τ. The nondegeneracy condition is that the rank of this system at the solution is maximal, i.e., equals 2 $dim\mathcal{Q} + 1$.

3.2.3 Application to the Planar Circular Restricted 3-Body Problem

To get a general idea of the problem we want to solve, let us consider the planar restricted circular 3-body problem (see Section 2.3), where we had two primaries and one body: the Sun, Jupiter and an Asteroid, with normalized masses of $m_S = 1 - \mu$, $m_J = \mu$ and $m_A = 0$ respectively. Moreover, as we are working in a rotating frame, we have the positions of the Sun and Jupiter fixed in such a way that $Q_S = (-\mu, 0)$ and $Q_J = (1 - \mu, 0)$ respectively.

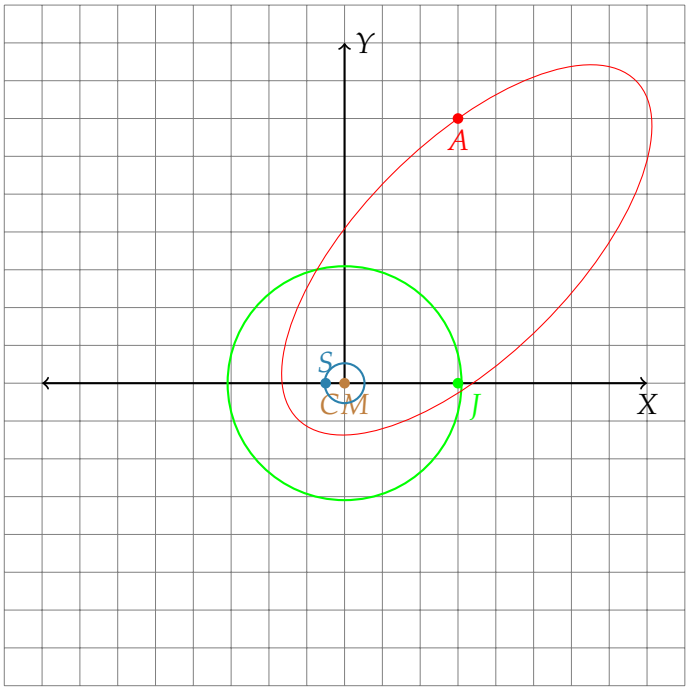


Figure 3.2: Configuration of the Sun (S), Jupiter (J) and Asteroid (A).

As one could see from this representation, the elliptic orbit described by the Asteroid and the circular one described by Jupiter collides at least twice. These

collisions are the main point of our work, so in order to represent and explain them accurately, we need to describe the Hamiltonian in (2.37). However, to make it easier, we are going to perform firstly a classical change of coordinates that involves moving the Sun at the origin, so the translation made from the previous position of the Asteroid (x, y) to the new one, denoted by (q, p), would be

$$\begin{aligned} q_1 = x_1 + \mu \Longleftrightarrow x_1 &= q_1 - \mu \\ x_2 &= q_2 \\ y_1 &= p_1 \\ p_2 = y_2 + \mu \Longleftrightarrow y_2 &= p_2 - \mu \end{aligned} \tag{3.7}$$

It is clear that this is a symplectic change, so, applying it in the Hamiltonian of the rotating Planar Circular Restricted 3-Body Problem in Equation (2.37), we obtain the following Hamiltonian function:

$$\begin{aligned} \tilde{H}(q,p) = \frac{p_1^2 + (p_2 - \mu)^2}{2} + q_2 p_1 - (q_1 - \mu)(p_2 - \mu) \\ - \frac{\mu}{\sqrt{(q_1 - 1)^2 + q_2^2}} - \frac{1-\mu}{\sqrt{q_1^2 + q_2^2}} = \frac{p_1^2 + p_2^2}{2} + q_2 p_1 - q_1 p_2 \\ - \frac{\mu}{\sqrt{(q_1 - 1)^2 + q_2^2}} - \frac{1-\mu}{\sqrt{q_1^2 + q_2^2}} + \mu q_1 - \frac{\mu^2}{2} \end{aligned} \tag{3.8}$$

As we claimed before (see Section 2.3), constants have no meaning in the equation of motion, so we can subtract $-\frac{\mu^2}{2}$, and consider the remaining terms:

$$\tilde{H} = \frac{p_1^2 + p_2^2}{2} + q_2 p_1 - q_1 p_2 - \frac{\mu}{\sqrt{(q_1 - 1)^2 + q_2^2}} - \frac{1-\mu}{\sqrt{q_1^2 + q_2^2}} + \mu q_1$$

By the Hamiltonian equations,

$$\dot{q_1} = \frac{\partial \tilde{H}}{\partial p_1} = p_1 + q_2 \Longrightarrow p_1 = \dot{q_1} - q_2$$

$$\dot{q_2} = \frac{\partial \tilde{H}}{\partial p_2} = p_2 - q_1 \Longrightarrow p_2 = \dot{q_2} - q_1$$

one can express the energy integral in terms of $(q, \dot{q})$ as follows:

$$\begin{aligned} E =& \frac{(\dot{q_1} - p_2)^2 + (\dot{q_2} + q_1)^2}{2} + q_2(\dot{q_1} - q_2) - q_1(\dot{q_2} + q_1) - \frac{\mu}{\sqrt{(q_1 - 1)^2 + q_2^2}} \\ & - \frac{1-\mu}{\sqrt{q_1^2 + q_2^2}} + \mu q_1 = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2) - \frac{1}{2}(q_1^2 + q_2^2) - \frac{\mu}{\sqrt{(q_1 - 1)^2 + q_2^2}} \\ & - \frac{1-\mu}{\sqrt{q_1^2 + q_2^2}} + \mu q_1 = \frac{1}{2}|\dot{q}|^2 - \frac{1}{2}|q|^2 - \frac{\mu}{|q - (1,0)|} - \frac{1-\mu}{|q|} + \mu q_1, \end{aligned}$$

so we obtain

$$E = \frac{1}{2}|\dot{q}|^2 - \frac{1}{2}|q|^2 - \frac{\mu}{|q - (1,0)|} - \frac{1-\mu}{|q|} + \mu q_1, \tag{3.9}$$

which is the similar to the first integral defined in (2.40) taking into account the translation (3.7) performed. However, one can still have the relation $E = -\frac{C}{2}$ (we get rid of the constant $\mu(1-\mu)$ in (2.41) as it does not affect the dynamics), where C is the Jacobi constant defined in Section 2.3.1.

Applying the Legendre transformation (7.24) to the first integral (3.9), we obtain a Lagrangian system of the form (3.1) (considering that $\mu = \varepsilon$), where L and V are given by

$$L(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 + q_1\dot{q_2} - q_2\dot{q_1} + \frac{1}{2}|q|^2 + \frac{1}{|q|} \tag{3.10}$$

and

$$V(q) = \frac{1}{|q|} - \frac{1}{|q - (1,0)|} + q_1 \tag{3.11}$$

Although L has a singularity at $q = (0,0)$, it is not important as we are going to analyze domains that do not contain this singular point. Recall that we are studying collisions with Jupiter which, in these coordinates, correspond to $q = (1,0)$.

We will begin to prove now Theorem 3.1. To do it, we will construct now a set of collision trajectories with the given value of C for the case $\varepsilon = 0$, check their non-degeneracy and construct a non-trivial set of chains of collision that change direction at each collision. These three things are most easily done in the non-rotating frame, to which we shall now revert.

3.2.4 Construction of Collision Orbits

For the case $\mu = \varepsilon = 0$, we will be studying the orbit made by the Asteroid motion with the Sun at one of its focus. In the meantime, Jupiter will be in a circular orbit around the center of mass, as we stated at the beginning. Thus, we

will determine which orbits of the Kepler problem (see Section 2.1) between the Sun and the Asteroid cross the unit circle, as this is a necessary condition for a collision orbit. We restrict attention to the elliptical ones as they will suffice for our purposes.

Denote the semi-major axis of an ellipse by a and its eccentricity by e as we did in Section 2.1.2. Then we can state and prove the following lemma.

Lemma 3.10. *Under the previous notations and definitions (see Section 2.3.1), the Jacobi Constant C (2.34) can be expressed as*

$$C = a^{-1} \pm 2\sqrt{a(1-e^2)} \tag{3.12}$$

Proof. First of all, we express the Hamiltonian of the Kepler's problem between the Sun and the Asteroid, which corresponds to Hamiltonian (2.8) with $\mu = 1$, that in symplectic polar coordinates (see Section 7.2.2) becomes:

$$\mathcal{P}(r, \theta, R, \Theta) = \frac{R^2}{2} + \frac{\Theta^2}{2r^2} - \frac{1}{r}$$

Now we recall the expression of r for the Kepler's problem (2.17):

$$r = \frac{a(1-e^2)}{1 + e\cos f}$$

where a is the semi-major axis of the ellipse, e the eccentricity and f is the eccentric anomaly.

Since $\mathcal{P}$ is a first integral, it will remain the same for any value of r. For this reason, we are going to consider r to be the perihelion, where we know that $f = 0$ (see Section 2.1.2), and so $r = a(1-e)$. Moreover, we can compute R as follows:

$$R = \dot{r} = \frac{a(1-e^2)}{(1+e\cos f)^2} e \sin f \dot{f} = 0$$

Apart from that, we can also compute the modulus of the angular momentum Θ as a function of a and e (see proof of Lemma 2.3):

$$\Theta^2 = a(1-e^2) \tag{3.13}$$

Putting everything together, one can obtain the Hamiltonian $\mathcal{P}$ as a function of a and e:

$$\tilde{\mathcal{P}}(a, e) = \frac{\Theta^2}{2r^2} - \frac{1}{r} = \frac{a(1-e^2)}{2a^2(1-e)^2} - \frac{1}{a(1-e)} = -\frac{1}{2a} \tag{3.14}$$

Once we have related the Hamiltonian with the semi-major axis, we can recall the definition of the Jacobi constant (2.34) in polar coordinates:

$$C = -2(\tilde{\mathcal{P}} - \Theta) \underset{\text{Equation (3.13)}}{=} a^{-1} \pm 2\sqrt{a(1-e^2)}$$

proving the result. □

We will say that the direction for the motion is the same as the rotation of the Sun and Jupiter (prograde motion) when considering the $+$ sign of (3.12), and in the opposite direction (retrograde motion) when considering the $-$ sign.

Thus instead of expressing the elliptic orbits of the Kepler problem as a function of the semi-major axis and the eccentricity, we can give (a^{-1}, C) instead, and deduce $e \in [0,1]$ from Equation (3.12) and obtain

$$e^2 = 1 - \frac{1}{4a}(C - a^{-1})^2 \tag{3.15}$$

The sense of motion is prograde if $C > a^{-1}$, retrograde if $C < a^{-1}$, and the case $C = a^{-1}$ has to be excluded because it gives degenerate ellipses of eccentricity 1 which corresponds to line segments through $\mathcal{O}$.

In Section 2.1.2, we have described the general aspects of the possible elliptic orbits that the Asteroid can take. Now, we will focus only on those ones that cross with the circular motion of Jupiter. Thus, it is clear that these ellipses would be the ones with a perihelion value $r \in (0,1)$ (as the Sun is the center of the circumference of radius 1 of the circular orbit taken by Jupiter). Translating this into Equation (2.17), we obtain that $a(1-e^2)$ must lie between $(1-e)$ and $(1+e)$ (notice that when we are considering the perihelion, we must take $\theta = 0$).

If $a(1-e^2) \in (1-e, 1+e)$, then it is clear that $a^{-1} \in (1-e, 1+e)$. In the variables (a^{-1}, C), and by (3.15) we obtain

$$(C - a^{-1})^2 < 8 - 4a^{-1}$$

which is represented in Figure 3.3:

Note the limiting cases of ellipses tangent of the unit circle ($(C-a^{-1})^2 = 8 - 4a^{-1}$), line segments ($C = a^{-1}$) and parabola ($a^{-1} = 0$).

The angular frequency Ω of the ellipse is given by

$$\Omega = a^{-\frac{3}{2}} \tag{3.16}$$

From the picture it is easy to see then that, given $C \in (-\sqrt{8}, 3)$, we obtain a non-empty set A_C of frequencies of ellipses that cross the unit circle, which consists of one interval for $C \in [2,3)$ or $(-\sqrt{8}, 0]$ and two intervals if $C \in (0,2)$, proving the first statement of the Theorem 3.1.

What is left to prove is that we can obtain a subset of allowed frequencies S_C from which we can obtain a unique trajectory near a chain of collision trajectories. In order to do that, we are going to explain now the strategy performed in [1].

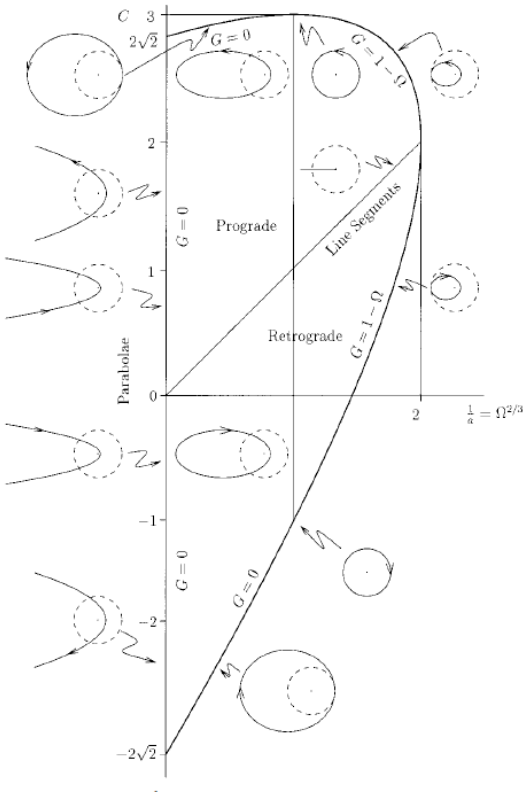


Figure 3.3: The region (a^{-1}, C) for which the Kepler ellipse crosses the unit circle [1].

To obtain a collision orbit of given C, we can choose a so that Ω is rational, say $\Omega = \frac{m}{k}$ in lowest terms, and consider that Jupiter and the Asteroid start at either of the two collision points (as one could see in Figure 3.3, the general behaviour is that the ellipse crosses twice the unit circle), and let Jupiter make k complete revolutions and the Asteroid m complete revolutions (in the direction corresponding to the sign of $C - a^{-1}$). Under these considerations, although it is true that they will collide again at the start point, they might collide earlier at the other intersection point. This would not be a great problem, as it generates a collision orbit, but it is not so easy to see in these cases whether the orbit is degenerate or not.

To avoid this earlier collision, we will determine for each C, a set of rationals (or allowed frequencies) where we can ensure that this behaviour will not take place. Then we shall prove their non-degeneracy, and finally we shall construct chains of these collision orbits for which the direction changes at each collision. We will claim and prove these statements in three separated lemmas.

Lemma 3.11. *For all $C \in (-\sqrt{8}, 3)$ there is a dense set of S_C of rational Ω in the allowed set A_C such that there is no early collision.*

Lemma 3.12. *For $C \in (-\sqrt{8}, 3)$, the collision orbit at $\varepsilon = 0$ (in the model corresponding to the Kepler's problem between the Sun and the Asteroid) in the rotating frame, corresponding to a whole number m of revolutions of an ellipse with rational frequency $\frac{m}{k} \in A_C$ starting and ending at collision with Jupiter, is nondegenerate.*

Lemma 3.13. *Given any sequence $(\Omega_n)_{n\in\mathbb{Z}} \in S_C$ there exists an "orbite à chocs" consisting of ellipses of frequency Ω_n connecting collisions at Jupiter, which leave each collision in neither the same nor the opposite direction as they arrive.*

Now we are going to prove them separately:

- *Proof of Lemma 3.11*

  Given (a^{-1}, C) in the region where the ellipse crosses the unit circle, define $f \in [0, \pi]$ to be the angle between the perihelion and either of the intersections with the unit circle, and $M \in [-\pi, \pi]$ to be 2π times the fraction of the area of the ellipse swept out bu this angle f, signed $+$ for the prograde and $-$ for the retrograde motion (see Figure 3.4)

  Applying Kepler's law of equal areas swept in equal times to both the second and third masses, one sees that there is a collision some time at the other intersection if and only if

  $$2\pi m' - 2M = (2\pi k' - 2f)\Omega$$

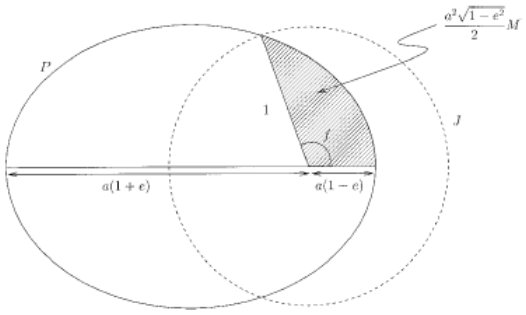


Figure 3.4: Illustration of the angle f and the area defining M [1].

for some non-zero $m', k' \in \mathbb{Z}$. We will explain this result.

Let us suppose that the direction of the motion of both objects is clockwise, and call $t_0 = 0$ the initial time when both Jupiter and the Asteroid are at the first collision point (let us say that it is the upper one), and let them going through the motion. At some time t, both objects collide at the second collision point.

For Jupiter, it is clear that, when it is at the second collision point, the angle swept out will be of $2\pi k' - 2f$, for some $k' \in \mathbb{Z}$, where $2\pi k'$ just represents the number of revolutions that Jupiter made before colliding with the Asteroid. Multiplying by the allowed angular frequency Ω, we will obtain the area swept out by Jupiter before the collision happens.

Now let us consider the Asteroid. Similarly as the Jupiter motion, the area swept out by the Asteroid would be the $2\pi m' - 2M$, for some $m' \in \mathbb{Z}$, where $2\pi m'$ represents the number of revolutions that the Asteroid made before the collision happens, and $2M$ represents the area swept out by the Asteroid if the collision happened at the first revolution.

Finally, using the third Kepler's law, we can ensure an equality between these two areas, obtaining the relation we just saw.

Let

$$G_C = \frac{1}{\pi}(M - f\Omega) \tag{3.17}$$

Then there is collision at the other intersection if and only if $G_C(\Omega) = m' - k'\Omega$. For $\Omega = \frac{m}{k}$ (recall that m and k are the actual revolutions of the Asteroid and Jupiter respectively), this implies that, in order to have another collision, $G_C = \frac{(m'k - k'm)}{k} \in \mathbb{Z} \setminus k$. What is left to do is to find a dense set of rationals where this is false.

Firstly, we notice that the function G_C is analytic on A_C. This is because firstly the angle $f \in [0, \pi]$ is given in terms of C and Ω by

$$e \cos f = \frac{1}{4}(C - a^{-1})^2 - 1 \tag{3.18}$$

which came out just by computation of equations (2.17) and (3.12), as well as knowing that the circular orbit has radius $r = 1$

$$C = a^{-1} \pm 2\sqrt{1 + e \cos f} \Longrightarrow e \cos f = \frac{1}{4}(C - a^{-1})^2 - 1$$

For $\Omega \in A_C$, f avoids 0 and π so it is analytic in Ω.

Secondly, the angle M can be obtained via the "eccentric anomaly" $E \in [-\pi, \pi]$ which is defined by

$$e \cos E = 1 - a^{-1}$$

with sign $+$ for prograde motion, $-$ for retrograde. For $\Omega \in A_C$, E avoids $\{0, \pm\pi\}$ so is analytic on Ω. Then M is given by the Kepler equation

$$M = E - e \sin E$$

so is analytic. As we can obtained $G_C(\Omega)$ via all these formulas, we can ensure that G_C is analytic on A_C.

Suppose for given $C \in (-\sqrt{8}, 3)$ there is a non-empty set open interval $A \subset A_C$ such that for all rationals $\frac{m}{k} \in A$ we have $G_C(\frac{m}{k}) \in \mathbb{Z} \setminus k$. Then choose any $\frac{m_\infty}{k_\infty} \in A$ and a Farey sequence of distinct rationals $\frac{m_n}{k_n} \in A$ tending to $\frac{m_\infty}{k_\infty}$. For each n, there exists $\frac{\tilde{m}_n}{\tilde{k}_n} \in A$ such that $|\tilde{m}_n k_n - m_n \tilde{k}_n| = 1$ (a Farey neighbour). Then for all $j \geq 0$

$$\frac{j m_n + \tilde{m}_n}{j k_n + \tilde{k}_n} \in A$$

just by definition of a Farey sequence and the fact that $\frac{j m_n + \tilde{m}_n}{j k_n + \tilde{k}_n} \in \left[\frac{m_n}{k_n}, \frac{\tilde{m}_n}{\tilde{k}_n}\right]$, which belong to A.

Thus we can ensure that

$$G_C\left(\frac{j m_n + \tilde{m}_n}{j k_n + \tilde{k}_n}\right) \in \mathbb{Z} \setminus (j k_n + \tilde{k}_n)$$

By assumption, for each $\frac{m}{k} \in A$ there exists an integer $n(m, k)$ such that $G_C(m, k) = \frac{n(m,k)}{k}$ (we claimed that $G_C(m, k) \in \frac{\mathbb{Z}}{k}$). Thus for each pair of rationals in A

$$\frac{n(m,k)}{k} - \frac{n(m',k')}{k'} = N\left(\frac{m}{k} - \frac{m'}{k'}\right)$$

This can be rearranged to show that $\frac{n(m,k)-Nm}{k}$ is the same for all rationals $\frac{m}{k} \in A$. To do that, we see that

$$\frac{n(m',k')}{k'} - \frac{m'}{k'}$$

does not depend on k, m, so it has to be the same for all k, m. We denote this difference as $\frac{p}{q}$, so, substituting in the expression we obtained above, we have

$$\frac{n(m,k) - Nm}{k} = \frac{p}{q}$$

Moreover, if we suppose that $\frac{p}{q}$ is in lowest terms, we have that for all $\frac{m}{k} \in A$, $n(m,k) - Nm = p\frac{k}{q}$. If $q > 1$, then there are values of k (those missing some prime factor of q) for which this is impossible (in order to have $p\frac{k}{q} \in \mathbb{Z}$, we need to have $mcd(k,q) = q$ for all k, which is not always true). Thus $q = 1$. Hence,

$$\frac{n(m,k)}{k} = N\frac{m}{k} + p$$

so in general we obtain the following equation

$$G_C(\Omega) = N\Omega + p \tag{3.19}$$

on A. Thus to prove density of frequencies with no early collision in a component of A_C, it suffices to find a contradiction of Equation (3.19) in that component.

From now on the rest of the proof consists on looking for specific counterexamples of this equation for different intervals of C. Even though it is an interesting computation, we will avoid it and assume that it is true as it goes far beyond what we want to achieve with this project. For the complete computation, see [1].

□

- *Proof of Lemma 3.12*

  We will use the third formulation of the nondegeneracy condition to make this proof (see 3.2.2). As we saw before, we can translate the energy into

the Jacobi constant C. Enforcing the collision point p_α to be the starter point of the motion of the Asteroid and Jupiter, we can reduce the dimension of the parameter λ referred in the nondegeneracy formulation to $dim\ \mathcal{Q}$ (in this case $dim\ \mathcal{Q} = 2$). Then, we parametrise the set of planar allowed Kepler ellipses through a given point P (with the Sun S at one focus) by the position $F \in \mathbb{R}^2$ of the second focus. Normalising the distance $SP = 1$, we have

$$a \underset{\text{definition}}{=} \frac{1}{2}(1 + PF)$$

$$e \underset{\text{definition}}{=} \frac{SF}{2a}$$

$$C \underset{\text{Equation (3.12)}}{=} a^{-1} \pm 2\sqrt{a - \frac{SF^2}{4a^2}}$$

The argument of the square root is positive since $e < 1$ in A_C. For $\Omega \in A_C$, the orbit crosses the unit circle transversely at non-zero speed and Jupiter moves along the unit circle at non-zero speed, so the second equation of (3.6) is satisfied nondegenerately for $\Omega = \frac{m}{k}$ (as its derivative with respect to time is non-zero). Thus all we have to check is that the third equation of (3.6) is satisfied nondegenerately too. That is, we have to check that

$$\left.\frac{\partial C}{\partial SF}\right|_a \neq 0$$

Since $e > 0$ in A_C, we have

$$\left.\frac{\partial C}{\partial SF}\right|_a = \pm\frac{SF}{2a\sqrt{a - \frac{SF^2}{4a^2}}} \neq 0$$

□

- *Proof of Lemma 3.13*

  For each $C \in (-\sqrt{8}, 3)$ and $\Omega \in A_C$, there are two possible ellipses leaving it with frequency Ω, and they leave in different and non-opposite directions, as we show below

  Thus at least one of them leaves in a different and non-opposite direction from that in which the Asteroid arrived along the previous collision trajectory.

□

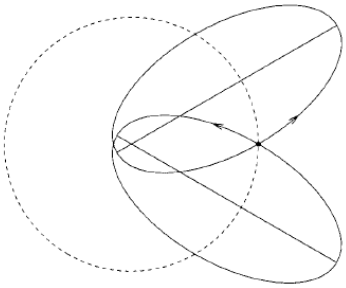


Figure 3.5: Two ellipses that leave the same point of the unit circle with the same values of (a^{-1}, C) [1].

3.3 Jean-Pierre Marco and Laurent Niederman Results

The procedure made by Jean-Pierre Marco and Laurent Niederman can be divided in three parts:

- To begin with, they will give a description of the Hamiltonian of the RCP3BP in different reference systems, as well as studying its Hill's Region. This is explained in section 3.3.1 of this work.
- Then, we will study the unperturbed case, which corresponds to the Kepler problem between the Asteroid and the Sun, in order to determine the levels of energy needed to obtain elliptic solutions.This is explained in section 3.3.2 of this work.
- Finally, we will regularize the perturbed system using a Levi-Civita map to find homoclinic solutions which will generate the second-species solutions. This is explained in section 3.3.3 of this work.

3.3.1 Description and equations of the problem

First, we will recall the equations of motion of this problem (see Section 2.3) in different coordinate frameworks.

Equations in fixed coordinates

We are going to consider the equations of motion in the RPC3BP (2.29) in complex notation:

$$\frac{dP}{dt} = \frac{d^2Q}{dt^2} = -(1-\mu)\frac{Q+\mu e^{it}}{|Q+\mu e^{it}|^3} - \mu\frac{Q-(1-\mu)e^{it}}{|Q-(1-\mu)e^{it}|^3} \tag{3.20}$$

where $Q \in \mathbb{C}$ is the position of the third mass and $P \in \mathbb{C}$ is the velocity (recall that μ is the mass of the Sun, $1-\mu$ is the mass of Jupiter, and the mass of the Asteroid is negligible). Here we abuse the notation and call

$$Q = (Q_1, Q_2) = Q_1 + iQ_2$$

$$P = (P_1, P_2) = P_1 + iP_2$$

Equations in rotating coordinates

As we saw in the previous section, it is easier to work with rotating coordinates instead of fixed ones. Denoting the position and velocity by (ξ, η) respectively in this new reference (note that (ξ, η) are the same that (x, y) in rotating coordinates (see Section 2.3.1) in complex notation, so $\xi = x_1 + ix_2$ and $\eta = y_1 + iy_2$), they must verify

$$Q = \xi e^{it} \qquad P = \eta e^{it} \tag{3.21}$$

Adapting the equations of motion and the corresponding Hamiltonian H in (2.33), we obtain

$$\begin{cases} \frac{d\xi}{dt} = \eta - i\xi \\ \frac{d\eta}{dt} = -i\eta - (1-\mu)\frac{\xi+\mu}{|\xi+\mu|^3} - \mu\frac{\xi-(1-\mu)}{|\xi-(1-\mu)|^3} \end{cases} \tag{3.22}$$

and their corresponding Hamiltonian

$$\begin{aligned} R_\mu(\xi, \eta) =& |\eta|^2 + i(\eta\overline{\xi} - \overline{\eta}\xi) - \frac{2(1-\mu)}{|\xi+\mu|} - \frac{2\mu}{|\xi-(1-\mu)|} \\ &+ 2(1-\mu) + (1-\mu)^2 \end{aligned} \tag{3.23}$$

when $\mathbb{C} \times \mathbb{C}$ is provided with the symplectic form (see Section 7.1.1)

$$\Omega = d\eta \wedge d\overline{\xi} + d\overline{\eta} \wedge d\xi$$

Proof. Consider the function (3.23)

$$\begin{aligned} R_\mu(\xi, \eta) =& |\eta|^2 + i(\eta\overline{\xi} - \overline{\eta}\xi) - \frac{2(1-\mu)}{|\xi+\mu|} - \frac{2\mu}{|\xi-(1-\mu)|} \\ &+ 2(1-\mu) + (1-\mu)^2 \end{aligned}$$

with the symplectic form $\Omega = d\eta \wedge d\overline{\xi} + d\overline{\eta} \wedge d\xi$.

We want to obtain the equations of motion

$$\begin{cases} \dot{\xi} = \eta - i\xi \\ \dot{\eta} = -i\eta - (1-\mu)\frac{\xi+\mu}{|\xi+\mu|^3} - \mu\frac{\xi-(1-\mu)}{|\xi-(1-\mu)|^3} \end{cases}$$

To do that we express η and ξ as follows

$$\xi = a + bi,\ \eta = c + di$$

$$\overline{\xi} = a - bi,\ \overline{\eta} = c - di$$

So the symplectic form becomes

$$\begin{aligned}\Omega &= d\eta \wedge d\overline{\xi} + d\overline{\eta} \wedge d\xi = (dc + idd) \wedge (da - idb) + (dc - idd) \wedge (da + idb) \\ &= (dc \wedge da) - idc \wedge db + idd \wedge da + dd \wedge db + dc \wedge da + idc \wedge db - idd \wedge da + dd \wedge db \\ &= 2(dc \wedge da) + 2(dd \wedge db) = 2(dc \wedge da + dd \wedge db)\end{aligned}$$

From the definition of the symplectic form (see Section 7.1.1), we know that $\Omega(X_R, \cdot) = dR$, so

$$\begin{aligned}&2(X_{R_a} dc - X_{R_c} da + X_{R_b} dd - X_{R_d} db) = (\partial_{a+bi} R) d(a + bi) + (\partial_{c+di} R) d(c + di) \\ &= (\partial_a R) da + (\partial_b R) db + (\partial_c R) dc + (\partial_d R) dd\end{aligned}$$

and so we have the following relations

$$2\dot{a} = \partial_c R \Longrightarrow \dot{a} = \frac{1}{2}\partial_c R$$

$$2\dot{b} = \partial_d R \Longrightarrow \dot{b} = \frac{1}{2}\partial_d R$$

$$-2\dot{c} = \partial_a R \Longrightarrow \dot{c} = -\frac{1}{2}\partial_a R$$

$$-2\dot{d} = \partial_b R \Longrightarrow \dot{d} = -\frac{1}{2}\partial_b R$$

Then

$$\begin{cases}\dot{\xi} = \dot{a} + i\dot{b} = \frac{1}{2}(\partial_c R + i\partial_d R) \\ \dot{\eta} = \dot{c} + i\dot{d} = \frac{1}{2}(-\partial_a R - i\partial_b R)\end{cases}$$

So we only have to express R_μ in these new coordinates and perform the computations:

$$\begin{aligned}R_\mu(\xi, \eta) =& R_\mu(a, b, c, d) = c^2 + d^2 + i\left[(c + di)(a - bi) - (c - di)(a + bi)\right] \\ &- \frac{2(1-\mu)}{\sqrt{(a+\mu)^2 + (b+\mu)^2}} - \frac{2\mu}{\sqrt{(a-(1-\mu))^2 + (b-(1-\mu))^2}} \\ &+ 2(1-\mu) + (1-\mu)^2\end{aligned}$$

So

$$\dot{\xi} = \frac{1}{2}(\partial_c R_\mu + i\partial_d R_\mu) \underset{*}{=} \frac{1}{2}(2c + 2b + 2ai - 2di) = (c + b) + i(a - d)$$

* Computation of $\partial_c R\mu$:

$$\partial_c R\mu = 2c + i(a - bi - (a + bi)) = 2c + i(-2bi) = 2c + 2b$$

* Computation of $i\partial_d R\mu$:

$$i\partial_d R\mu = i[2d + i(i(a - bi) + i(a + bi))] = i[2d + i(2ai)] = 2di - 2ai$$

which coincides with the corresponding equation of motion

$$\dot{\xi} = \eta - i\xi = (c + di) - i(a + bi) = (c + b) + i(a - d)$$

We perform a similar computation for $\dot{\eta}$

$$\begin{aligned}
\dot{\eta} =& -\frac{1}{2}(\partial_a R_\mu + i\partial_b R_\mu) \\
&\underset{*}{=} -\frac{1}{2}\Bigg[-2d + \frac{2(1-\mu)(\mu+a)}{|\xi+\mu|^3} + \frac{2\mu(a-(1-\mu))}{|\xi-(1-\mu)|^3} + i\Bigg(2c + \frac{2(1-\mu)(\mu+b)}{|\xi+\mu|^3} \\
&+ \frac{2\mu(b-(1-\mu))}{|\xi-(1-\mu)|^3}\Bigg)\Bigg] = (d - ic)\frac{1}{|\xi+\mu|^3}((1-\mu)(\mu+a) + i(1-\mu)(\mu+b)) \\
&- \frac{1}{|\xi-(1-\mu)|^3}(\mu(a-(1-\mu)) + i\mu(b-(1-\mu))) \\
&= (d - ic) - \frac{(1-\mu)[(a+bi)+\mu]}{|\xi+\mu|^3} - \frac{\mu[(a+bi)-(1-\mu)]}{|\xi-(1-\mu)|^3} \\
&= -i\eta - \frac{(1-\mu)(\xi+\mu)}{|\xi+\mu|^3} - \frac{\mu(\xi-(1-\mu))}{|\xi-(1-\mu)|^3}
\end{aligned}$$

* Computation of $\partial_a R_\mu$:

$$\begin{aligned}
\partial_a R_\mu &= i[(c + di) - (c - di)] + 2(1-\mu)\frac{\mu+a}{|\xi+\mu|^3} + 2\mu\frac{a-(1-\mu)}{|\xi-(1-\mu)|^3} \\
&= -2d + \frac{2(1-\mu)(\mu+a)}{|\xi+\mu|^3} + \frac{2\mu(a-(1-\mu))}{|\xi-(1-\mu)|^3}
\end{aligned}$$

* Computation of $\partial_b R_\mu$:

$$\begin{aligned}
\partial_b R_\mu &= i[-i(c + di) - i(c - di)] + 2(1-\mu)\frac{\mu+b}{|\xi+\mu|^3} + 2\mu\frac{b-(1-\mu)}{|\xi-(1-\mu)|^3} \\
&= 2c + \frac{2(1-\mu)(\mu+b)}{|\xi+\mu|^3} + \frac{2\mu(b-(1-\mu))}{|\xi-(1-\mu)|^3}
\end{aligned}$$

□

Hill's Region for $\mu \neq 0$

Once we have formulated the Hamiltonian R_μ, we can now fix a level energy h and compute the possible positions and velocities that accomplish this equation, i.e, compute $R_\mu^{-1}(h)$. It can be expressed as

$$|\eta - i\xi|^2 = h + |\xi|^2 + \frac{2(1-\mu)}{|\xi+\mu|} + \frac{2\mu}{|\xi-(1-\mu)|} - 2(1-\mu) - (1-\mu)^2 \geq 0 \quad (3.24)$$

The representations of the Hill's region in rotating coordinates are given in the following pictures, where the h_i are the critical values of R_μ.

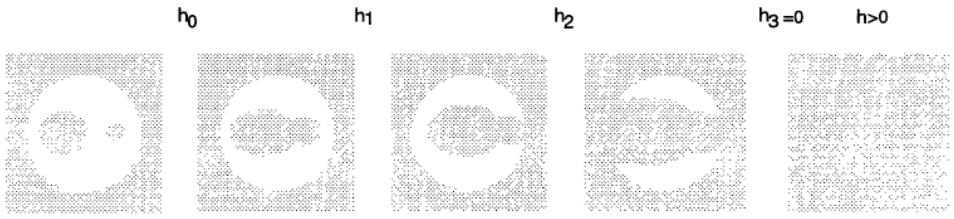


Figure 3.6: Hill's Region [2].

Equations in rotating coordinates centered at P_2

Since one of the goals is to study the collision trajectories with Jupiter (represented as the point P_2), located at $\xi = (1-\mu, 0)$, we are going to make a translation to place Jupiter at the center, as it will be a the (in rotating coordinates) where the collision will take place. This reference frame is, as we said, just a translation of the coordinates (ξ, η) given in the previous rotating reference. Thus we define the new coordinates (M, N) as follows

$$M = \xi - (1-\mu),\ N = \eta - i(1-\mu) \quad (3.25)$$

and the new Hamiltonian is written as

$$\begin{aligned} H_\mu(M,N) =& |N|^2 + i(\overline{M}N - M\overline{N}) - \frac{2\mu}{|M|} \\ & - (1-\mu)\left(\frac{2}{|1+M|} - 2 + (M+\overline{M})\right) \end{aligned} \quad (3.26)$$

for the symplectic form

$$\Omega = dM \wedge d\overline{N} + d\overline{M} \wedge dN \quad (3.27)$$

and the collision with Jupiter happens, in these coordinates, when $M = 0$.

3.3.2 Kepler problem in the rotating frame

In order to work with the Kepler problem between the Asteroid and the Sun, we must consider that the mass of Jupiter is now negligible. When $\mu = 0$, the Hamiltonian can be written as

$$H_0(M, N) = |N|^2 + i(\overline{M}N - M\overline{N}) - \left(\frac{2}{|1+M|} - 2 + (M + \overline{M})\right) \tag{3.28}$$

This would represent the Hamiltonian of the Kepler problem related to the center P_1 (the Sun), in a rotating coordinate framework centered at P_2 (which is now $M = 0$).

It would be easier to study this Hamiltonian in a rotating framework centered on P_1 instead of P_2. This change is as simply as a translation, similar to the one we did before in (3.25). As $\mu = 0$, the point P_1 is now at the center of masses C, so we can use the coordinates (ξ, η) instead. The new Hamiltonian will be

$$R_0(\xi, \eta) = |\eta|^2 + i(\eta\overline{\xi} - \overline{\eta}\xi) - \frac{2}{|\xi|} + 3 \tag{3.29}$$

which is nothing more than substituting $\mu = 0$ (and $(1 - \mu) = 1 - \mu = 1$) in the function $R_\mu(\xi, \eta)$ in (3.23).

Now the following Lemma will give us a relation between this result and the angular momentum, that we will denote by σ, of the Kepler system.

Lemma 3.14. *Preserving the same notation used until now, the angular momentum of the Asteroid in the rotating framework defined above can be expressed as*

$$\sigma = -\frac{i}{2}(\eta\overline{\xi} - \overline{\eta}\xi)$$

Proof. We want to see that the angular momentum $\sigma = -\frac{i}{2}(\eta\overline{\xi} - \overline{\eta}\xi)$. To do that, we express the position ξ and velocity η as follows:

$$\xi = a + bi$$

$$\eta = c + di$$

By definition, the angular momentum is $\sigma = \xi \times \eta = ad - bc$.

So

$$\begin{aligned}\sigma = -\frac{i}{2}(\eta\overline{\xi} - \overline{\eta}\xi) &= -\frac{i}{2}[(c + di)(a - bi) - (c - di)(a + bi)] \\ &= -\frac{i}{2}[dia + adi - cbi - cbi] = ad - bc\end{aligned}$$

□

The Hamiltonian R_0 can be written then in terms of the angular momentum and the physical energy of P_3 in the fixed coordinate system centered at P_1 (which we will denote by F_0) as follows:

$$R_0(\xi, \eta) = F_0(\xi, \eta) - 2\sigma(\xi, \eta) \tag{3.30}$$

with

$$F_0(\xi, \eta) = |\eta|^2 - \frac{2}{|\xi|} + 3 \tag{3.31}$$

The angular momentum σ is a first integral (see Equation 2.10) and F_0 too, so R_0 is completely integrable.

Note that for $\mu = 0$, H_0 is not singular at $M = 0$ (in fact, at the collision point, $H_0(0, N) = |N|^2$). It happens the same for R_0, which is not singular for $\xi = 1$ (at the collision point, $R_0(1, \eta) = |\eta|^2 + i(\eta - \overline{\eta}) + 1$). This result will be useful when we study the collision trajectories.

Topology of the unperturbed problem

Now we are going to study the Hill's region from (3.24) for $\mu = 0$.

For $h < 0$, the region of energy h is the union of a disk centered at P_1 of radius < 1 (without the origin), and the complementary of $\mathbb{C}$ of a disk of radius > 1

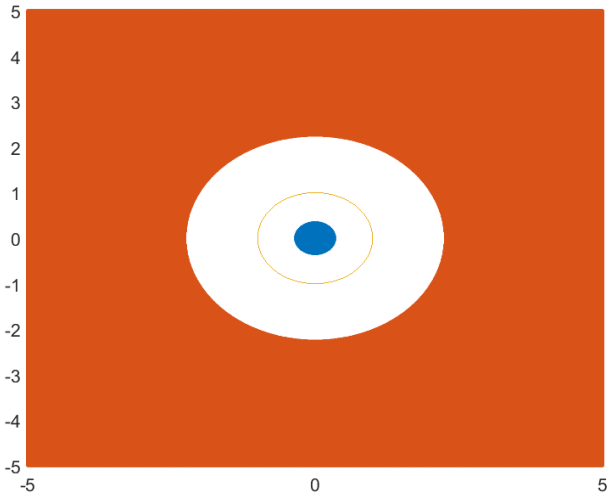


Figure 3.7: Hill's Region for $h < 0$.

For the critical value $h = 0$ the two regions collide under the circle of radius 1

Finally, for $h > 0$, the region is $\mathbb{C}^*$ (see Figure 3.9).

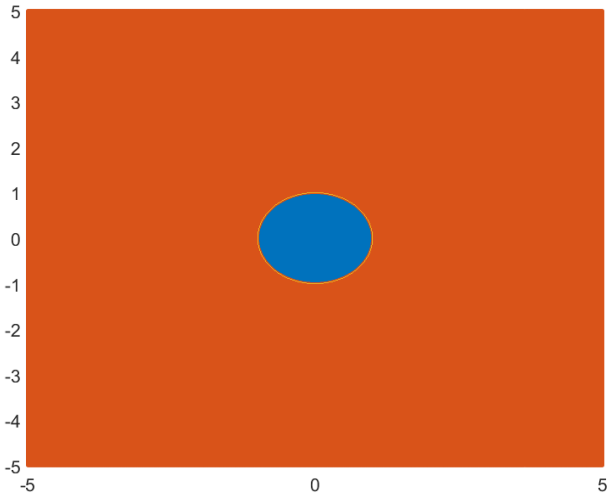


Figure 3.8: Hill's Region for $h = 0$.

Every submanifold of energy $h > 0$ is then diffeomorphic to $\mathbb{C}^* \times \mathbb{T}^1$. (The equation of the submanifold is the one obtained in (3.24), so for a given position of the Hill's Region $\xi \in \mathbb{C}^*$, we obtain the equation of a torus for η, as η initially belong to $\mathbb{C}$).

Structure of the submanifolds for a constant positive energy h

In order to study the structure of the submanifolds for a fixed value of $R_0 = h$ in (3.30), we need to study the different values of the constants of motion F_0 and σ.

For a fixed value F_0, we will say that a solution of R_0 is *elliptic* if $F_0 < 3$, *parabolic* if $F_0 = 3$ and *hyperbolic* if $F_0 > 3$. (i.e. if the trajectory of the solution in fixed coordinates is elliptic, parabolic or hyperbolic respectively). Finally, if the angular momentum $\sigma > 0$ we will say that the solution is *prograde*, and *retrograde* otherwise.

Note the similarities with the analysis done in Section 3.2). There, we will restrict initially to the elliptic domain, so we just consider $F_0 < 3$. However, we said that the motion was prograde if $C > a^{-1}$ and retrograde if $C < a^{-1}$ (being C the Jacobi Constant and a^{-1} the inverse of the semi-major axis), instead of talking about the angular momentum.

But both definitions are similar since Equation (3.12) gives us that, if $C > a^{-1}$, we must consider the positive root of $2\sqrt{a(1-e^2)} > 0$ (because, as seen in Section 2.1.2, in the elliptic orbit we have that $e < 1$), and this term is nothing

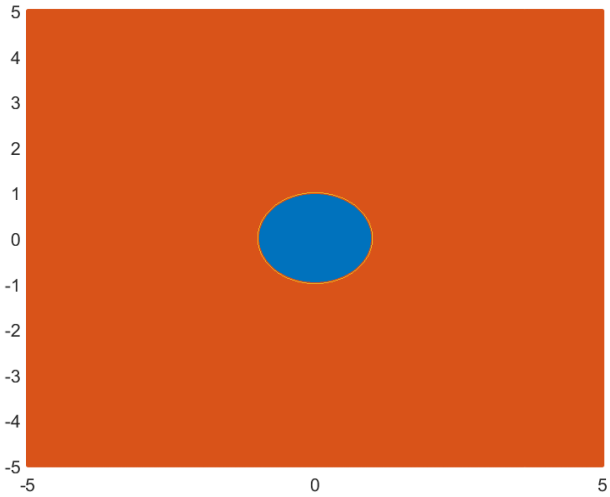


Figure 3.9: Hill's Region for $h > 0$.

more that 2σ, implying directly that $\sigma > 0$. The same similarity can be obtained for the retrograde motion.

We will study now the submanifolds of constant energy $h > 0$, as they will be the ones that we will consider later on (see Chapter 5). Let us fix $h > 0$ and denote by $\mathcal{H} = R_0^{-1}(h)$. $\mathcal{H}$ is the union of

$$\mathcal{H}_+ = \mathcal{H} \cap (F_0^{-1}((3, +\infty))),\ \mathcal{H}_0 = \mathcal{H} \cap F_0^{-1}(3),\ \mathcal{H}_- = \mathcal{H} \cap (F_0^{-1}(-\infty, 3))$$

To do that we state the following lemma:

Lemma 3.15. *All of the submanifolds $\mathcal{H}_+$ and $\mathcal{H}_-$ are diffeomorphic to $\mathbb{C}^* \times \mathbb{T}^1$.*

Proof. It is enough to see that for a fixed value of $\xi \in \mathbb{C}^*$, η lives in a circle.

In complex analysis, the parametrization of a circle centered at $z_0 \in \mathbb{C}$ with radius r is given by

$$|z - z_0|^2 = r^2$$

This result can be also expressed in the following way

$$\begin{aligned} |z - z_0|^2 = r^2 &\iff (z - z_0)(\overline{z} - \overline{z_0}) = z\overline{z} - z\overline{z_0} - z_0\overline{z} + z_0\overline{z_0} \\ &= |z|^2 + |z_0|^2 - (z\overline{z_0} + \overline{z}z_0) = r^2 \end{aligned} \tag{3.32}$$

Now we impose the condition of $\mathcal{H}$, i.e. $R_0(\xi, \eta) = h$, so

$$\begin{aligned} R_0(\xi, \eta) = h &\underset{\text{Eq. (3.29)}}{\iff} |\eta|^2 + i(\eta\overline{\xi} - \overline{\eta}\xi) - \frac{2}{|\xi|} + 3 = h \\ &\underset{i\overline{\xi} = -\overline{i\xi}}{\iff} |\eta|^2 - (\eta(\overline{i\xi}) + \overline{\eta}(i\xi)) = h + \frac{2}{|\xi|} - 3 \end{aligned}$$

To adapt this equation to be as Equation (3.32), we add and substract $|i\xi|^2$, so we obtain:

$$|\eta|^2 + |i\xi|^2 - \eta(\overline{i\xi}) + \overline{\eta}(i\xi)) = h + \frac{2}{|\xi|} - 3 + |i\xi|^2$$

Considering $z = \eta$ and $z_0 = i\xi$, we get

$$|\eta - i\xi|^2 = r^2 \underset{|i\xi|^2=|\xi|^2}{=} h + \frac{2}{|\xi|} - 3 + |\xi|^2 \tag{3.33}$$

So effectively, η lives on a circle with center $i\xi$ and radius $r = \sqrt{h + \frac{2}{|\xi|} - 3 + |\xi|^2}$ □

Fixed h, one can also define the circle $|\eta - i\xi|^2 = r^2$ at the collision point $\xi = 1$, denoted as C_0:

$$C_0 = \{(1, \eta),\ |\eta - i|^2 = h\} \tag{3.34}$$

The same definition can be applied to the coordinates (M, N), obtaining

$$C_0 = \{(0, N),\ |N|^2 = h\} \tag{3.35}$$

From these definitions, we can introduce the concept of singular domain.

Definition 3.16. *We define the singular domain $\mathcal{D}$ of $\mathcal{H}$ as the union of orbits that intersect with C_0, being C_0 the circle of velocities around P_2 under the submanifold $\mathcal{H}$ (i.e, the circle of available velocities for which there is a collision with P_2).*

We will pay more attention to the elliptic singular domain $\mathcal{D}_e \subset \mathcal{D}$, generated by the union of elliptic orbits (in $\mathcal{H}_-$) that intersect with C_0.

Evolution of the structure $\mathcal{H}$ in terms of the energy h

What is left to do is to determine for which values of h we would obtain the elliptic singular domain $\mathcal{D}_e$, as it will be the one we will work with.

Consider C_0 as defined in (3.34). We will denote by η_c those velocities η that live in C_0.

To obtain the elliptic singular domain $\mathcal{D}_e$, we have to intersect this circle C_0 with the submanifold $\mathcal{H}_-$. Such submanifold is described by imposing $R_0(\xi, \eta) = h$ and $F_0(\xi, \eta) < 3$.

On the one hand, from Equation (3.31), we obtain

$$F_0 = F_0(1, \eta_c) = |\eta_c|^2 + 1 \iff \eta_c = \sqrt{F_0 - 1}\, e^{i\theta} \tag{3.36}$$

for some $\theta \in (0, 2\pi)$.

On the other hand, from the definition of the circle C_0 (3.34), we have

$$|\eta_c - i|^2 = h \iff |\eta|^2 + 1 - (\eta\bar{i} + \overline{\eta} i) = h \iff |\eta|^2 + 1 - (-\eta i + \overline{\eta} i) = h$$
$$\underset{\text{Equation(3.36)}}{\iff} h - F_0 = -2\sqrt{F_0 - 1}\sin\theta \iff \sin\theta = \frac{h - F_0}{-2\sqrt{F_0 - 1}}$$
$$\iff \frac{h - F_0}{-2\sqrt{F_0 - 1}} \in (-1, 1)$$

From this result, we can obtain the following inequality:

$$-2\sqrt{F_0 - 1} < h - F_0 < 2\sqrt{F_0 - 1}$$

so we can finally bound h as follows:

$$F_0 - 2\sqrt{F_0 - 1} < h < F_0 + 2\sqrt{F_0 - 1}$$

Since $F_0 < 3$, we can get the upper bound replacing F_0 by 3, so $h < 3 + 2\sqrt{2}$. To obtain the other boundary, one can compute the minimum of the expression $F_0 - 2\sqrt{F_0 - 1}$:

$$(F_0 - 2\sqrt{F_0 - 1})' = 1 - \frac{1}{\sqrt{F_0 - 1}} = 0 \iff F_0 = 2$$

For $F_0 = 2$, $F_0 - 2\sqrt{F_0 - 1} = 0$, so h is bounded by:

$$0 < h < 3 + 2\sqrt{2}$$

so from now on we will suppose that the value of $h \in (0, 3 + 2\sqrt{2})$.

3.3.3 Regularization of the system

Now we will come back to the Hamiltonian system H_μ defined in (3.26). We fix an energy $h > 0$, and we denote by $\mathcal{H}_\mu = H_\mu^{-1}(h)$, and X_{H_μ} the Hamiltonian field associated with H_μ and restricted by $\mathcal{H}_\mu$.

For $M = 0$, the Asteroid is at collision with Jupiter. As we said, there is no problem when we are considering the Kepler's problem between the Asteroid and the Sun (i.e., when we consider $\mu = 0$).

However, for $\mu \neq 0$, we have to consider the Hamiltonian (3.26). In this case, when $M = 0$, there is a singularity. This is a problem when we want to find periodic solutions that goes near the collision. To deal with this singularity, we regularize the Hamiltonian at the collision by performing a change of variables.

Levi-Civita regularization for a fixed energy h

The regularization of the collision with P_2 (Jupiter) is obtained through the application ρ of Levi-Civita, that goes from $\mathbb{C}^* \times \mathbb{C}$ to $\mathbb{C}^* \times \mathbb{C}$, and defined as follows

$$\begin{aligned} \rho \colon \mathbb{C}^* \times \mathbb{C} &\to \mathbb{C}^* \times \mathbb{C} \\ (z, w) &\mapsto \rho(z, w) = \left(\frac{z^2}{h}, \frac{\sqrt{h} w}{\overline{z}} \right) \end{aligned} \tag{3.37}$$

that preserves the symplectic form Ω (3.27) (up to scaling)

$$\begin{aligned} \Omega &= dM \wedge d\overline{N} + d\overline{M} \wedge dN \\ &\implies \rho^*(\Omega) = \frac{2}{\sqrt{h}} (dz \wedge d\overline{w} + d\overline{z} \wedge dw) \end{aligned} \tag{3.38}$$

We also have to take into account also the change in time [4]:

$$4|z|^2 d\tau = dt$$

and so the Hamiltonian (3.26), in the level of energy $H_\mu = h$, is transformed into

$$L_\mu = \frac{|z|^2}{h} (H_\mu - h) \circ \rho$$

in the level of energy $L_\mu = 0$, that has a unique analytic extension in $\mathbb{C}^2$, expressed as

$$L_\mu(z, w) = |w|^2 - |z|^2 + \frac{i|z|^2}{h^{\frac{3}{2}}} (\overline{z} w - z \overline{w}) - 2\mu - \frac{(1-\mu)}{h^3} f(z, \overline{z}, h) \tag{3.39}$$

with

$$f(z, \overline{z}, h) = -|z|^6 + \frac{3}{4} |z|^2 (z^2 + \overline{z}^2)^2 + O_8(z) \tag{3.40}$$

Proof. We will prove the uniqueness and the expression of the analytic expansion L_μ.

Uniqueness

Suppose that there exists two analytic extensions of L_μ in $\mathbb{C}^2$, and denote them by F_1 and F_2. Then, $F_1 - F_2$ is an analytic function that cancels in $\mathbb{C}^* \times \mathbb{C}^2 \subset \mathbb{C}^2 \underset{\mathbb{C}^2 convex}{\implies}$ $F_1 - F_2 = 0$ for every point in $\mathbb{C}^2$, which implies that $F_1 = F_2$ for every point in $\mathbb{C}^2$.

Expression

$$L_\mu(z, w) = \frac{|z|^2}{h} (H_\mu - h) \circ \rho$$

$$(H_\mu - h)(M,N) = |N|^2 - h + i(\overline{M}N - M\overline{N}) - \frac{2\mu}{|M|} - (1-\mu)\left(\frac{2}{|1+M|} - 2 + (M+\overline{M})\right)$$

$$\begin{aligned}(H_\mu - h)\circ\rho =& \left|\frac{\sqrt{h}w}{\overline{z}}\right|^2 - h + i\left(\frac{\overline{z^2}}{h}\frac{\sqrt{h}w}{\overline{z}} - \frac{z^2}{h}\frac{\sqrt{h}\overline{w}}{z}\right) - \frac{2\mu}{\left|\frac{z^2}{h}\right|} \\ & - (1-\mu)\left(\frac{2}{\left|1+\frac{z^2}{h}\right|} - 2 + \frac{z^2+\overline{z^2}}{h}\right) = h\left|\frac{w}{\overline{z}}\right|^2 + \frac{i}{\sqrt{h}}\left(\frac{\overline{z^2}}{\overline{z}}w - \frac{z^2}{z}\overline{w}\right) \\ & - \frac{2\mu}{\left|\frac{z^2}{h}\right|} - (1-\mu)\left(2\left(-1 + \frac{1}{\left|1+\frac{z^2}{h}\right|}\right) + \frac{z^2+\overline{z^2}}{h}\right)\end{aligned}$$

So

$$\begin{aligned}L_\mu(z,w) =& \frac{|z|^2}{h}h\left|\frac{w}{\overline{z}}\right|^2 + \frac{|z|^2 i}{h\sqrt{h}}\left(\frac{\overline{z^2}}{\overline{z}}w - z\overline{w}\right) - 2\mu - \frac{|z|^2}{h}(1-\mu)\left(2\left(-1 + \frac{1}{\left|1+\frac{z^2}{h}\right|}\right)\right. \\ & \left. + \frac{z^2+\overline{z^2}}{h}\right) - |z|^2 = |w|^2 - |z|^2 + \frac{i|z|^2}{h^{\frac{3}{2}}}(\overline{z}w - z\overline{w}) - 2\mu - \frac{(1-\mu)}{h^3}f(z,\overline{z},h) \\ =& w\overline{w} - z\overline{z} + \frac{i}{h^{\frac{3}{2}}}(z\overline{z}^2 w - z^2\overline{z}\overline{w}) - 2\mu - \frac{1-\mu}{h^3}f(z,\overline{z},h)\end{aligned}$$

□

We denote by C_μ the intersection with $\mathcal{L}_\mu = L_\mu^{-1}(0)$ of the plane $z = 0$, i.e., of the circle corresponding to the collision P_2P_3 in regularized coordinates. We choose $L_\mu^{-1}(0)$ because we are working with a fixed energy level h. For $z = 0$, $L_\mu(0,w) = |w|^2 - 2\mu = 0$ is the equation of a circle in the collision of the Asteroid P_3 with P_2 (Jupiter) because $z = 0$ corresponds to the point P_2 in this coordinate reference. In conclusion, we can define C_μ as

$$C_\mu = \{w \in \mathbb{C}, |w|^2 = 2\mu\} \tag{3.41}$$

and we will say then that $(\mathcal{L}_\mu, X_{L_\mu})$ is a regularization of $(\mathcal{H}_\mu, X_{H_\mu})$.

Finally, consider now the circle of velocities C_0 defined in (3.35) (recall that this circle is defined under the Hamiltonian (3.29)). This circle, when going through the Levi-Civita regularization process, collapses to the origin, as one can obtain by replacing $\mu = 0$ at Equation (3.41). For this reason, we are going to make a local study of the dynamics around the origin.

Local study of a fixed point

The Hamiltonian field X_{L_μ} associated with L_μ has, as linear part, the following Hamiltonian $\tilde{L}$:

$$\tilde{L} = |w|^2 - |z|^2$$

whose equations of motion are defined as follows:

$$\begin{cases} \dot{z} = \frac{\partial \tilde{L}}{\partial \overline{w}} = w \\ \dot{w} = -\frac{\partial \tilde{L}}{\partial \overline{z}} = z \end{cases}$$

The corresponding stable and unstable subspaces E_s and E_u are of dimension 2 and have for equations $w = -z$ and $w = z$ respectively.

Proof.

$$X_{L_\mu} = \begin{pmatrix} \dot{z} \\ \dot{w} \\ \dot{\overline{z}} \\ \dot{\overline{w}} \end{pmatrix}$$

It will be enough to study $\dot{z}$ and $\dot{w}$, since $\dot{\overline{z}}$ and $\dot{\overline{w}}$ can be obtained by conjugacy.

$$\begin{pmatrix} \dot{z} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \frac{\partial L_\mu}{\partial \overline{w}} \\ \frac{\partial L_\mu}{\partial \overline{z}} \end{pmatrix} = \begin{pmatrix} w + \frac{i}{h^{\frac{3}{2}}}|z|^2 z \\ z - \frac{i}{h^{\frac{3}{2}}}(2|z|^2 w - z^2 \overline{w}) + \frac{1-\mu}{h^3}\frac{\partial f}{\partial \overline{z}} \end{pmatrix}$$

Then, denoting by $Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we obtain:

$$DX_{L_\mu}(0,0) = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \Longrightarrow P_\lambda = (\lambda^2 - 1)^2 = 0 \Longrightarrow \lambda = \pm 1$$

For E_u:

$$v_u = \mathrm{Ker}(DX_{L_\mu} - I) = \mathrm{Ker} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \Longrightarrow E_u = \{w = z\}$$

Analogically, we obtain $E_s = \{w = -z\}$ □

In a neighbourhood of the origin, the submanifold of energy zero $\mathcal{L}_\mu$ can be expressed as

$$|w|^2 - |z|^2 = 2\mu + O_4(|z|, |w|)$$

- When $\mu \neq 0$, the intersection of $\mathcal{L}_\mu$ with the band B of $\mathbb{C}^2$ of equation $|z|^2 \leq \alpha \quad (\alpha << 1)$ is a closed and complete torus generated by C_μ.

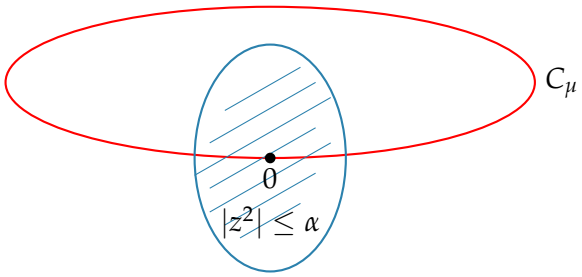


Figure 3.10: The disk $|z^2| \leq \alpha$ and the circle C_μ generates a complet closed torus.

- When $\mu \to 0$, the circle C_μ converges to the origin.
- When $\mu = 0$, the submanifold $\mathcal{L}_0 \cap B$ is homeomorph to a cone with peak at the origin and base a torus of dimension 2 with equation $|z|^2 = |w|^2 = \alpha$.

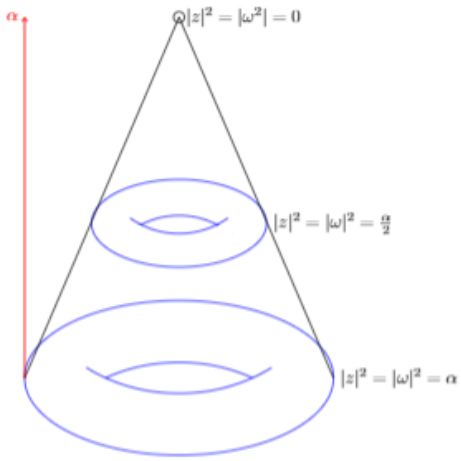


Figure 3.11: Cone with peak at the origin and base a torus of dimension 2 of equation $|z|^2 = |w|^2 = \alpha$.

Finally, we take a look at what happens with the dynamics in the case $\mu = 0$. The circle C_0 (3.35) of the velocities at the point P_2 is sent by ρ to the fixed point O of the system. As a consequence, the cylinder obtained by the transport of C_0 by the flux of the Kepler problem (called singular domain in definition 3.16) is sent by ρ to the union of the 2-dimensional unstable and stable manifolds of the fixed point $(z, w) = 0$.

3.4 Comparison

To make an appropriate comparison between the papers, we are going to split them in two parts, one concerning the solutions to the Kepler's problem between the Sun and the Asteroid when as a result of considering the mass of Jupiter negligible, and the other part related to the arguments made to build the second-species periodic solution. Finally, we will compare separately the way they regularize the collisions.

1. To begin with, let us talk about the reference frame. One could notice that during both approaches, the reference frame used changed constantly, from a fixed coordinate system to the rotating one, the Levi-Civita regularized system, between others. Even though it seems to be a trivial equivalence, it is important as in both works these changes are useful to understand and ease the computations needed to prove more difficult theorems.

   Moreover, one can also see notable differences in the way of treating the problem. The variational approach proves the existence of an orbit that travels near the "homoclinic" solution, which will be by definition a second-specie solution. On the other hand, the geometrical approach proves that the homoclinic solution is the generatrix of a set of second-species solutions.

   However, they are indeed similar. In fact, all resides in the way they defined the concept of "a solution".

   - In the variational we work in the configuration space, and a limit (or homoclinic) solution will be the one that crosses all the singularities of a finite set denoted by $\mathcal{P}$ (see Section 3.2). Thus, one can think of a solution as multiple elliptic orbits taken by the Asteroid around the circular orbit made by Jupiter, each of them produced when the Asteroid and Jupiter collides (see Figure 3.5)
   - In the geometrical approach we work in the phase space, and the generatrix of the second-species solution will be just the union of two of these

ellipses of origin and extreme in the collision point (see Figure 5.1), which in Levi-Civita variables become homoclinic orbits to the origin.

These differences will lead us to solve "different" problems. In the first case we will look for an ellipse that crosses the singularity only once (see Section 3.2.4), while in the other approach we will seek for those solutions that crosses the singularity at least twice (see Section 3.3). Even though they seem to be different, at the end they are the same problem, as the geometrical approach considers a solution to be the union of two of these ellipses, as we said before.

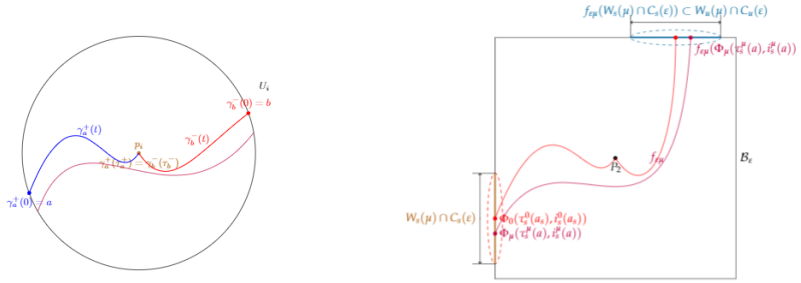
2. Once they have defined their homoclinic solutions as starting point, their arguments differ in the treatment of the problem.

   - The variational approach works in the configuration space and makes an argument based on the action-integral theory and the Lagrangian of the system (see Section 4.1). Based on this, it separates the action in three parts (see Figure 4.4): one corresponding to the action made by the orbit from the boundary of a ball centered at the singularity to the singularity itself; another one corresponding to the action from the singularity to another point of the boundary; and the third one corresponding to the path between this latter boundary point to the boundary of the next singularity. With particular conditions over these points, it can state and prove the existence of a path shadowing the homoclinic solution with an action being the same as we defined with an added little perturbation (see Lemma 4.1). Once it has this result, the only thing that is left to do is to join everything together in a convenient way to generate the shadowing orbit (see Section 4.2).
   - The geometrical approach works in the phase space and builds up a coordinate framework suitable in order to transform a ball centered at the collision into an isolated block, with the particularity that its boundary is divided in two parts: one where the homoclinic solution enters and the other one where it exits (see Section 5.2), just as the ones defined in the analytical approach. Once it has defined such a ball, it only needs to prove that a submanifold generated by the flow of the system applied to a neighbourhood of a point of the homoclinic solution will enter this ball from the corresponding boundary and will exit from the other one via a suitable transition map (see Section 5.3), and the existence of periodic orbits which correspond to second-species solutions will be proved.

To make this comparison more clear, let us take a look at the way they build the second-species periodic solutions in a more restricted scenario.

For the variational approach, suppose that we have a small ball centered at one collision point such as in Section 4.1. Then, one can impose the conditions necessary to be able to use Lemma 4.1 and prove the existence of a unique trajectory ε-close to the singular one for some points of the boundary of such a ball.

However, this idea is pretty similar to the one performed in the geometrical approach, when considering the isolated block around a singularity. Here we stated and proved (in Section 5.3) that a submanifold corresponding to a neighborhood of the homoclinic solution goes through this isolated block via a well-defined transition map, being one part of the boundary the entry point and another part of the boundary the exit point.



(a) Shadowing trajectory given by lemma 4.1 in the variational approach.

(b) Shadowing trajectory result of theorem 5.11 in the geometrical approach.

Figure 3.12: Comparison between both results inside a ball centered at a collision point.

To sum up, we are going to compare the Levi-Civita regularization of both approaches. The first one uses this result to have a well-defined action from which it can state some important results, as it preserves both the Hamiltonian and the integral action (see Section 4.3). The geometrical one uses it instead to be able to have a non-singular Hamiltonian in the perturbed case and prove the existence of periodic orbits in such configuration (see Section 5.3).

Geometrically speaking, the importance of the Levi-Civita map resides on the fact that it allows us to regularize the singularities of the problem (allowing us to use important theorems such as the Implicit Function Theorem to state some important results in a neighborhood of these collision points) in exchange of time

regularization. This means that, while in the regular problem the Asteroid collides with Jupiter at a finite time, in the Levi-Civita case the collision point is a hyperbolic point, so it takes infinite time for the Asteroid to reach it.

As we said before, this change in the time parametrization does not involve neither the Hamiltonian nor the integral actions, which is why it is so useful in the analytic approach (see Section 4.3). On the other hand, it explains why in the geometrical approach there is a need for building a suitable reference framework to work under the idea of the singularity being a hyperbolic point, and where both stable and unstable manifolds are moved in order to become the reference axis to make the computations easier (see Section 5.2).

Chapter 4

Variational Approach

We will start by reviewing in detail the variational approach given by S.V. Bolotin and R.S. Mackay [1]. To do that, we will explain all the concepts treated along the paper, from solving the simplest case scenario of a local ball centered at one singularity, as well as proving Theorem 3.4 and Theorem 3.9, that ensure the existence of a general solution that shadows the homoclinic solution. This solution, if it is periodic, will be the second-species solution we are seeking for.

4.1 A Boundary Value Problem

Suppose we have a small ball U_i centred at $p_i \in \mathcal{P}$(see Section 3.2) and we want to connect two points $a, b \in U_i$. Fix an energy level E with Lagrangian L_ε in (3.1) for $\varepsilon = 0$ (that we will denote by L_0). For any $a \in U_i$ there exists a unique trajectory $\gamma_a^+ : [0, \tau^+(a)] \to U_i$ of energy E connecting a to p_i. Similarly, for any $b \in U_i$, there is a unique trajectory $\gamma_b^- : [\tau^-(b), 0] \to U_i$ of energy E connecting p_i to b (see Figure 4.1).

We denote

$$\begin{aligned} S^+(a) &= \int_0^{\tau^+(a)} (L(\gamma_a^+(t), \dot{\gamma}_a^+(t) + E)dt \\ S^-(b) &= \int_{\tau^-(b)}^0 (L(\gamma_b^-(t), \dot{\gamma}_b^-(t) + E)dt \end{aligned} \tag{4.1}$$

their corresponding actions (see Equation 7.21). These actions can be seen as distances between the points and the singularity respectively (that is, $S^+(a)$ behaves like dist(a, p_i) and $S^-(b)$ like dist(p_i, b), where the distance is defined by the Jacobi metric (7.15)).

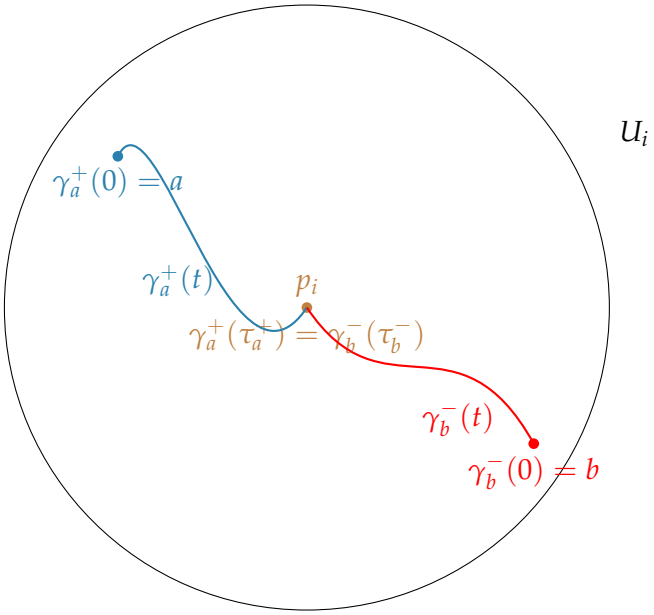


Figure 4.1: Ball U_i centered at p_i with two unique trajectories of energy E denoted by γ_a^+ and γ_b^-.

Let

$$u^+(a) = \dot{\gamma}_a^+(\tau^+(a)), \quad u^-(b) = \dot{\gamma}_b^-(\tau^-(b))$$

be the tangent vectors to γ_a^+, γ_b^- at the collision point $p_i = \gamma_a^+(\tau^+(a)) = \gamma_b^-(\tau^-(b))$. Let $\Sigma_i = \partial U_i$. Fix a small $\delta > 0$ and let

$$X_i = \{(a,b) \in \Sigma_i^2 \mid ||u_+(a) - u_-(b)|| \geq \delta\} \tag{4.2}$$

Equivalently, a pair of points $(a,b) \in \Sigma_i$ belongs to X_i if the solution of the Lagrangian L_0 in (3.1), with energy E connecting a and b does not pass too close to the centre p_i (i.e., these are suitable points from which we can proof the general Theorem 3.9, as they suit up with the definition of *chain* (see Section 3.2)).

Let

$$Y_i = \{(a,b) \in \Sigma_i^2 \mid ||u_+(a) + u_-(b)|| \geq \delta\} \tag{4.3}$$

As well as in X_i, to preserve the definition of chain we need the trajectories to have the property $\dot{\gamma}_a^+(\tau^+(a)) \neq -\dot{\gamma}_b^-(\tau^-(b))$.

Lemma 4.1. *There exists $\varepsilon_0 > 0$ such that:*

- *For any $\varepsilon \in (0, \varepsilon_0]$ and $(a,b) \in X_i$, there exists a unique trajectory $\gamma = \gamma_{a,b}^\varepsilon : [0,\tau] \to U_i$ of energy E for the system (L_ε) (3.1) connecting a to b: $\gamma_{a,b}^\varepsilon(0) = a, \gamma_{a,b}^\varepsilon(\tau) = b$.*

- $\tau = \tau(a,b,\varepsilon)$ *is a* C^2 *function of* $X_i \times (0,\varepsilon_0]$ *and* $\tau(a,b,\varepsilon) \to \tau^+(a) + \tau^-(b)$ *uniformly as* $\varepsilon \to 0$.

- $\gamma^\varepsilon_{a,b}|_{[0,\tau^+(a)} \to \tau^+(a)$ *and* $\gamma^\varepsilon_{a,b}(\cdot+\tau)|_{[\tau_-(b),0]} \to \gamma^-_b$ *uniformly as* $\varepsilon \to 0$. *More precisely, there exists a constant* $C > 0$ *depending only on* δ *such that*

$$\max_{0\le t\le \tau_+(a)} dist(\gamma^\varepsilon_{a,b}(t), \gamma^+_a(t)) \le C\varepsilon$$
$$\max_{\tau+\tau_-(b)\le t\le \tau} dist(\gamma^\varepsilon_{a,b}(t), \gamma^-_b(t-\tau)) \le C\varepsilon$$

- *The action of the trajectory* γ:

$$S(a,b,\varepsilon) = \int_0^\tau (L_\varepsilon(\gamma(t),\dot\gamma(t)) + E)\, dt$$

is a C^2 *function on* $X_i \times (0,\varepsilon_0]$ *and*

$$S(a,b,\varepsilon) = S^+(a) + S^-(b) + \varepsilon s(a,b,\varepsilon) \tag{4.4}$$

where s *is uniformly* C^2 *bounded on* X_i *as* $\varepsilon \to 0$.

- *If, additionally,* $(a,b) \in Y_i$, *then the trajectory* $\gamma^\varepsilon_{a,b}$ *does not pass too close to* p_i:

$$\min_{0\le t\le\tau} dist(\gamma^\varepsilon_{a,b}(t), p_i) \ge c\varepsilon, \quad c > 0 \tag{4.5}$$

For the approach that we are going to make we will suppose that $(a,b) \in Y_i$ for the rest of the paper. The proof of this Lemma is presented in Section 4.3.

4.2 Shadowing Collision Orbits

In this section we prove Theorem 3.9. We will use the same notation as in Section 3.2 and Section 4.1.

For any $k \in K$ a finite set, let α_k and β_k two consecutive collision points, and $x_k \in \Sigma_{\alpha_k}$ and $y_k \in \Sigma_{\beta_k}$ be the intersection points of γ_k with the boundaries of the balls defined around such collision points (see Figure 4.1) and denoted by Σ_{α_k} and Σ_{β_k} respectively. Then $\gamma_k(t) = \gamma^-_{x_k}(t + \tau^-(x_k))$ for $0 \le t \le -\tau^-(x_k)$, $\gamma_k(t) = \gamma^+_{y_k}(t - \tau_k + \tau^+(y_k))$ for $\tau_k - \tau^+(y_k) \le t \le \tau_k$. Without loss of generality we can assume that the points x_k and y_k are not conjugate along γ_k for all k. If not, we change the radius of the balls U_i a little to make the new intersection points non-conjugate.

Once we have described the collision trajectories inside the balls U_{α_k} and U_{β_k}, we want to define a path between them. Let $A_k \subset \Sigma_{\alpha_k}$ be a small neighborhood

of x_k, $B_k \subset \Sigma_{\beta_k}$ a small neighborhood of y_k, and W_k be a small neighborhood of $\gamma([0,\tau_k])$. We may assume that $A_k = W_k \cap \Sigma_{\alpha_k}$ and $B_k = W_k \cap \Sigma_{\beta_k}$.

The non-conjugacy property of the points x_k and y_k along γ_k is equivalent as saying that γ_k is a nondegenerate curve (see Section 3.2). From what we saw in the previous chapter, being nondegenerate means that this curve is a nondegenerate critical point for the action integral (3.5) for a set of curves u. In particular, we can define a subset of this set of curves u, named σ, living in a small neigborhood of W_k with $\sigma(0) = u \in A_k$ and $\sigma(1) = v \in B_k$ (or, up to a time parametrization, we could just say $\sigma(\tau) = v$). This leads to consider the following functional

$$F(\sigma,\tau) = \int_0^\tau (L_\varepsilon(\sigma(t),\dot{\sigma}(t)) + E)\, dt,$$

which gives rise to the idea of using the implicit function theorem. As $F(\gamma_k,\tau) = 0$ and $D_\tau F(\gamma_k,\tau) \neq 0$ (for being a nondenegerate curve), we can ensure by this Theorem that there exists a unique $\sigma^\varepsilon_{uv} = \sigma\colon [0,\tau] \to W_k$ with $\tau = \tau^\varepsilon_{uv}$ (the τ function will describe the time that the curve takes from u to v, and so it is why it has to depend on such points) such that $\sigma(0) = u$, $\sigma(\tau) = v$, and a solution of the Lagrangian (L_ε) (3.1) with energy E, and close to $\gamma_k(t - \tau^-(x_k))$ for $0 \leq t \leq \tau$ (see Figure 4.2). This result can be seen as a consequence of the third definition that we state about the nondegenerate collision curves (see Section 3.2).

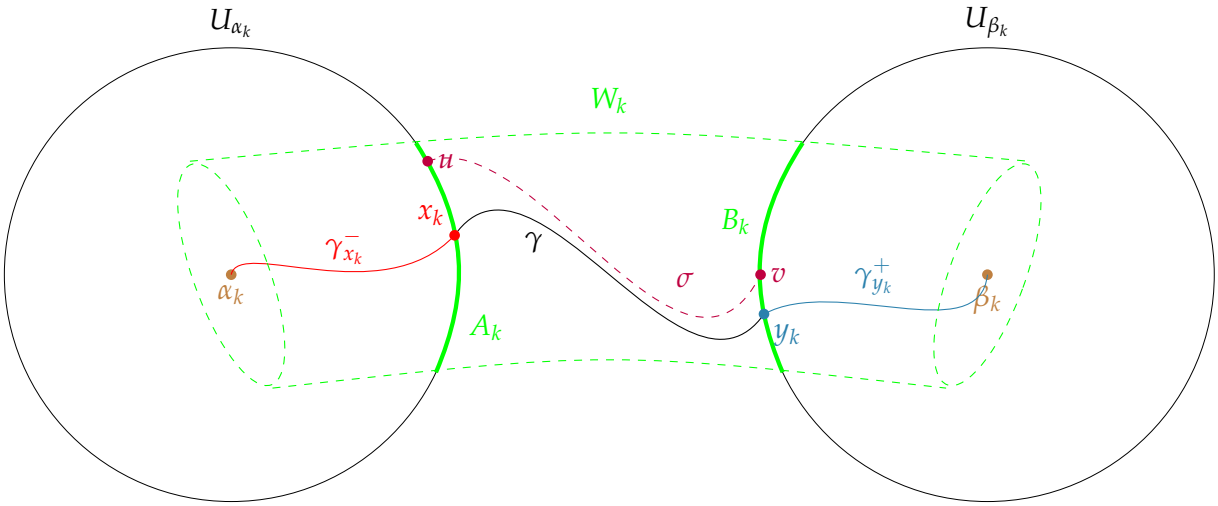


Figure 4.2: Representation of a shadowing orbit σ (in purple), close to the trajectory γ, generated by $\gamma^-_{x_k}$, $\gamma^+_{y_k}$ and the path between them (in black).

Let us consider again this latter functional evaluated at the minimizer σ

$$f_\varepsilon(u,v) = \int_0^\tau (L_\varepsilon(\sigma(t),\dot\sigma(t)) + E)\, dt$$

at $\varepsilon = 0$, and the action integrals S^+ and S^- (4.1) at the collision points α_k and β_k.

Lemma 4.2. *The function $g_k(u,v) = f_0(u,v) + S^-_{\alpha_k}(u) + S^+_{\beta_k}(v)$ on $A_k \times B_k$ has a nondegenerate critical point at $z_k = (x_k, y_k)$.*

Proof. This Lemma follows from the assumption that γ_k is a nondegenerate critical point of the action functional (3.5). Indeed, $g_k(u,v)$ is the action obtained by gluing together the trajectories $\gamma^-_u, \sigma^0_{uv}$ and γ^+_v (with appropriate shift of time parametrisation). Hence g_k is just the action connecting p_α and p_β, with break points u, v, and by definition $z_k = (x_k, y_k)$ must be a noncritical point (as $\gamma_k(\tau^-(x_k)) = x_k$ and $\gamma_k(\tau_k - \tau^+(y_k)) = y_k$.

Another way of understanding this proof is the following: We know that γ_k is a nondegenerate collision orbit for the Lagrangian system (L) (3.2). As a nondegenerate orbit, it is a nondegenerate critical point for the functional (3.5). This action can be divided in three parts: The one corresponding to the ball centered at the first collision point p_{α_k}, whose action is defined by $S^-_{\alpha_k}(u)$, the one corresponding to the path between the two balls, whose action would correspond by definition to $f_0(u,v)$ (as we are considering the non-perturbative case $\varepsilon = 0$), and the one corresponding to the path in the second ball centered at p_{β_k}, whose action is $S^+_{\beta_k}$ (with an appropriate time parametrization). As z_k is the intersection of γ_k with W_k, z_k is then a nondegenerate critical point of the function $g_k(u,v)$. □

Let us choose some coordinates u, v in A_k and B_k. Lemma 4.2 implies that if the neighborhoods A_k, B_k are small enough, there exists $C > 0$ such that

$$||(g_k''(s))^{-1}|| \leq C \text{ for } s \in A_k \times B_k \tag{4.6}$$

where $s = (u,v)$ and $||\cdot||$ is the max norm in $A_k \times B_k$.

This is due to the fact that $z_k = (x_k, y_k)$ is a nondegenerate critical point of g_k on $A_k \times B_k$, which ensures that $g_k''(z_k) \neq 0$ and it is bounded. Thus, we can expand these results on a small neighborhood of z_k, which corresponds to $A_k \times B_k$.

Let $G \subset K^2$ be defined as follows

$$G = \{(k,l) \in K^2 \mid \beta_k = \alpha_l, \dot\gamma_k(\tau_k) \neq \pm\dot\gamma_l(0)\}$$

so p_{α_k} is the first collision point of the trajectory γ_k, and p_{β_k} refers to the second collision point of the trajectory γ_k, which corresponds to the first collision point of the following orbit, denoted by γ_l (of the chain γ_{k_i}), i.e., p_{α_l}.

Taking the neigborhoods A_k, B_k small enough, it can be assumed that for all $(k,l) \in G$, $B_k \times A_l \subset Y_j$, where $j(k,l) = \beta_k = \alpha_l$ and Y_j is defined in (4.3). This can be translated as saying that the shadowing collision orbit, in the path that connects the previous collision orbit γ_k to the following one γ_l, does not pass too close to the collision point p_{α_l} (see Figure 4.3).

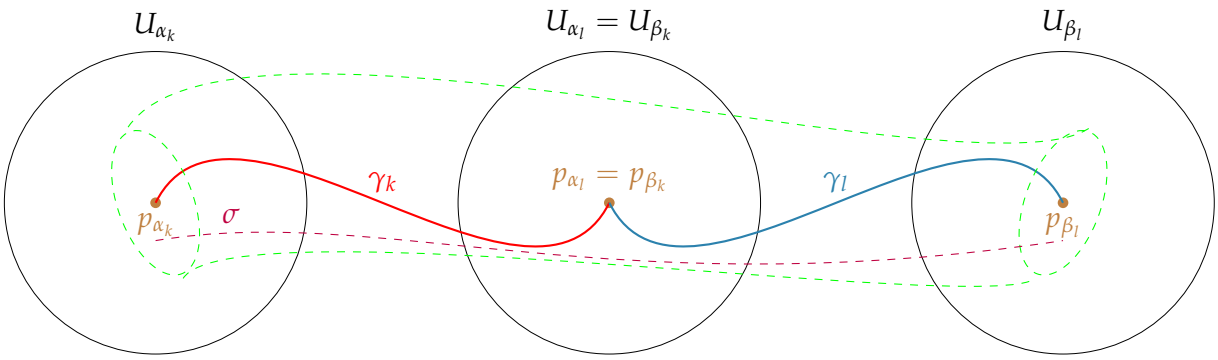


Figure 4.3: Example of a shadowing orbit (denoted by σ) and the collision orbits γ_k and γ_l.

So, until now the results we got are the following:

- First of all, Lemma 4.1 ensures that in a ball centered at one collision point p_i there exists a unique shadowing collision orbit that does not pass too close to the collision point. This result will prove the Theorem 3.9, but only for pieces of orbits which belong to balls centered at the collision points.

- Lemma 4.2 and the results that we saw at the beginning of this chapter ensure us that in the space between two of the previous balls there exists a unique orbit $\sigma_{uv}^{\varepsilon}$ that connects the intersection points u and v (these intersection points will be the ones generated by the intersection of the shadowing collision orbit with the respective balls) that goes close to the collision orbit γ, which is the one generating the chain of collisions.

What is left to do is to connect these two results carefully in order to obtain the shadowing collision orbit that will pass close to the collision chain for every collision point $p_i \in \mathcal{P}$. This shadowing orbit, when imposed a periodicity condition, will be then the second species solution we were seeking for.

To do that, we will state and prove something a little more precise than Theorem 3.9. We will assume that the neighborhoods W_k are small enough.

Theorem 4.3. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and any chain $(k_i \in K)_{i\in\mathbb{Z}}$ of collision orbits there exists a unique (up to a time shift) trajectory $\gamma\colon \mathbb{R} \to (\cup_{k\in K} W_k) \setminus \mathcal{P}$ of energy E for the system (L_ε) (3.1) and a sequence*

$$\cdots < a_i < b_i < a_{i+1} < b_{i+1} < \cdots,$$

such that for all $i \in \mathbb{Z}$:

- $\gamma([a_i, b_i]) \subset W_{k_i}, \gamma(a_i) \in A_{k_i}, \gamma(b_i) \in B_{k_i}$;
- $\gamma([b_i, a_{i+1}]) \subset U_j, \quad j = \beta_{k_i} = \alpha_{k_{i+1}}$

The asymptotic behaviour of this trajectory as $\varepsilon \to 0$ is as follows:

- $b_i - a_i \to \tau_{k_i} - \tau^-(x_{k_i}) - \tau^+(y_{k_i})$ *as* $\varepsilon \to 0$;
- $\gamma(t)$ *is* $O(\varepsilon)$*-close to* $\gamma_{k_i}([\tau^-(x_{k_i}), \tau - \tau^+(y_{k_i})]$ *for* $a_i \le t \le b_i$;
- $\gamma(t) = \gamma^\varepsilon_{q(b_i), q(a_{i+1})}(t - b_i)$ *for all* $t \in [b_i, a_{i+1}]$

The constant ε_0 depends only on the set $\{\gamma_k\}_{k\in K}$ of collision orbits and it is independent of the sequence $(k_i \in K)$. Thus $\gamma(t)$ is $O(\varepsilon)$-close to a chain of collision orbits. Moreover, one can note that Lemma 4.1 gives us directly that the trajectory $\gamma([b_i, a_{i+1}])$ avoids p_j by a distance of order ε (4.5).

Proof. Given a sequence $(k_i)_{i\in\mathbb{Z}}$ with $(k_i, k_{i+1}) \in G$ for all $i \in \mathbb{Z}$, let

$$Y = \prod_{i\in\mathbb{Z}} (A_{k_i} \times B_{k_i})$$

with supremum norm in the chosen charts on A_k and B_k. Choose $\varepsilon_0 > 0$ and let $\varepsilon \in (0, \varepsilon_0]$. Then trajectories of the system (L_ε) (3.1) with energy E near the chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$ correspond to critical points of the functional

$$F_\varepsilon(u, v) = \sum_{i\in\mathbb{Z}} (f_\varepsilon(u_i, v_i) + S(v_i, u_{i+1}, \varepsilon))$$

over sequences $(u, v) = (u_i, v_i)_{i\in\mathbb{Z}} \in Y$. This result can be understood as the sum of the action integral between $v_i \in \partial B_{k_i}$ and $p_{\beta_{k_i}}$, which is $S^-(v_i)$, plus the action integral between $u_{i+1} \in \partial B_{k_i} = \partial A_{k+1}$ and $p_{\beta_{k_i}} = p_{\alpha_{k_{i+1}}}$, which is $S^+(u_{i+1})$. The sum of these two action integrals (plus a remainder term denoted by ε due to the fact that both action integral S^+ and S^-, defined in the previous section, are described under the Lagrangian (L) (3.2), not (L_ε) (3.1)) generates the function $S(v_i, u_{i+1}, \varepsilon)$ described in Lemma 4.1.

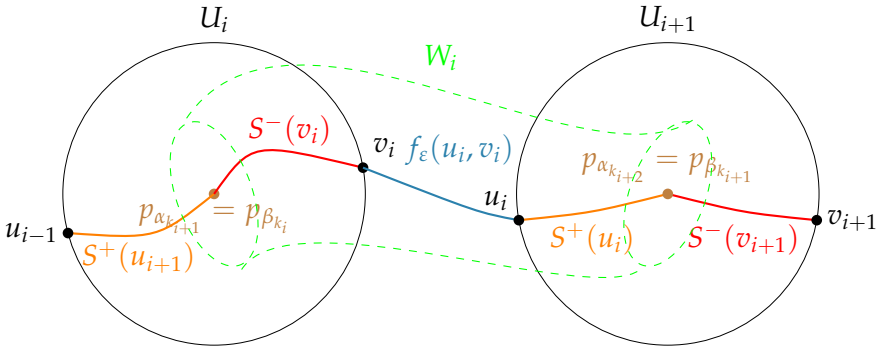


Figure 4.4: Example of the construction of the action integral of the collision trajectory. For the shadowing orbit, we need to add a factor εs (see Lemma 4.1).

Moreover, we have to add the action integral between u_i and v_i, defined at the beginning of the chapter by $f_\varepsilon(u_i, v_i)$. This result, which would be the total action between one element of Y_i, expanded to all Y is what we called $F_\varepsilon(u, v)$ (see Figure 4.4).

Using Lemma 4.2 and Equation (4.4) we can rewrite this total formal action as

$$\begin{aligned} F_\varepsilon(u,v) = \sum_{i\in\mathbb{Z}} (f_\varepsilon(u_i,v_i) + S(v_i,u_{i+1},\varepsilon)) &\underset{\text{Equation (4.4)}}{=} \sum_{i\in\mathbb{Z}} (f_\varepsilon(u_i,v_i) + S^+(u_{i+1}) + S^-(v_i) \\ + \varepsilon s(v_i,u_{i+1},\varepsilon)) &= \sum_{i\in\mathbb{Z}} (f_0(u_{i+1},v_i) + S^+(u_{i+1}) + S^-(v_i) + \varepsilon s(u_{i+1},v_i,\varepsilon)) \\ &\underset{\text{Lemma 4.2}}{=} \sum_{i\in\mathbb{Z}} (g_{k_i}(u_i,v_i) + \varepsilon s(v_i,u_{i+1},\varepsilon)) \end{aligned}$$

As they have to be nondegenerate, trajectories of (L_ε) (3.1) of energy E correspond to nondegenerate zeros (see Section 3.2) of the mapping

$$\begin{aligned} \Phi_\varepsilon = \nabla F_\varepsilon \colon Y &\to Z \\ (u,v) &\mapsto (U,V) \end{aligned}$$

defined by

$$\begin{cases} U_i = \frac{g_{k_i}}{\partial u_i}(u_i,v_i) + \varepsilon \frac{\partial s}{\partial u_i}(v_{i-1},u_i,\varepsilon) \\ V_i = \frac{\partial g_{k_i}}{\partial v_i}(u_i,v_i) + \varepsilon \frac{\partial s}{\partial v_i}(v_i,u_{i+1},\varepsilon) \end{cases} .$$

Since the function s is uniformly C^2 bounded as $\varepsilon \to 0$, and the second derivative matrix of g is uniformly invertible, the chain of collision trajectories is a nondegenerate zero of Φ_0 and we can use the implicit function theorem, which gives us a locally unique continuation for a range of ε independent of the sequence (k_i). This

means that Φ_ε has a unique zero (which is the shadowing collision orbit) provided that

$$\varepsilon_0^{-1} > C \max_{(k,l)\in G} \max_{B_k \times A_k} ||s''||,$$

where C is the constant in (4.6) and s'' is the Hessian of the perturbed action (we need ε_0^{-1} to be greater independently of the sequence, that is why we impose it to be greater than the maximum). □

4.3 Regularisation of Collisions

In this section we prove Lemma 4.1. Before that, let us make some considerations.

Instead of working with the domain $\mathcal{D}$ (see Section 3.2), we can use the following

$$\mathcal{D} = \{q \in \mathcal{Q} \mid W(q) < E\} \Longleftrightarrow \mathcal{D} = \{q \in \mathcal{Q} \mid W(q) - E < 0\},$$

and taking $E = 0$, we have, in particular, that $W(p_i) < 0$ for all $p_i \in \mathcal{P}$. In addition, it can also be assumed that $\omega(p_i) = 0$. If this is not true, let ϕ be a smooth function on $\mathcal{Q}$ such that $\nabla\phi(p_i) = \omega(p_i)$. We can subtract $\langle \nabla\phi(q), \dot{q}\rangle$ from L without changing the Lagrange equations.

Lemma 4.4. *The Hamiltonian corresponding to the Lagrangian L_ε in (3.1) has the form*

$$H_\varepsilon = \frac{1}{2}\left\langle A^{-1}(q)(p - \omega(q)), p - \omega(q)\right\rangle + W(q) + \varepsilon V(q) \tag{4.7}$$

Proof. The Lagrangian $L_\varepsilon(q, \dot{q})$ (3.1) takes the form

$$L_\varepsilon(q, \dot{q}) = \frac{1}{2}\langle A(q)\dot{q}, \dot{q}\rangle + \langle \omega(q), \dot{q}\rangle - W(q) - \varepsilon V(q)$$

We know that

$$\begin{aligned} p = \frac{\partial L_\varepsilon}{\partial \dot{q}} &= \frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}\langle A(q)\dot{q}, \dot{q}\rangle + \langle \omega(q), \dot{q}\rangle - W(q) - \varepsilon V(q)\right) \\ &= \omega(q) + \frac{1}{2}\left(A(q)\dot{q} + A(q)\dot{q}\right) = \omega(q) + A(q)\dot{q} \end{aligned}$$

In order to obtain the associated Hamiltonian, we perform the Legendre Transformation (see Section 7.1.2):

$$H_\varepsilon = \dot{q}\frac{\partial L_\varepsilon}{\partial \dot{q}} - L_\varepsilon = \dot{q}p - L_\varepsilon$$

and so we obtain

$$\begin{aligned}H_\varepsilon(q,p) &= \dot{q}\left(\omega(q)+A(q)\dot{q}\right) - \frac{1}{2}\left\langle A(q)\dot{q},\dot{q}\right\rangle - \left\langle \omega(q),\dot{q}\right\rangle + W(q) + \varepsilon V(q)\\ &\underset{\dot{q}=A^{-1}(q)(p-\omega(q))}{=} \left\langle A^{-1}(q)(p-\omega(q)),p\right\rangle - \frac{1}{2}\left\langle p-\omega(q),A^{-1}(q)(p-\omega(q))\right\rangle\\ &- \left\langle \omega(q),A^{-1}(q)(p-\omega(q))\right\rangle + W(q)+\varepsilon V(q)\\ &= \frac{1}{2}\left\langle A^{-1}(q)(p-\omega(q)),p-\omega(q)\right\rangle + W(q)+\varepsilon V(q)\end{aligned}$$

□

In local conformal coordinates $q \in \mathbb{R}^2 = \mathbb{C}$ near p_k, we have

$$L_\varepsilon(q,\dot{q}) = \frac{1}{2}a(q)|\dot{q}|^2 + i\lambda(q)(q\dot{\overline{q}} - \overline{q}\dot{q}) - W(q) + \frac{\varepsilon f(q)}{|q|}.$$

As a generalization of (3.10) with $a(0) > 0$, $W(0) < 0$, and $f(0) > 0$. Without loss of generality it can be assumed that $a(0) = 1$, $f(0) = \frac{1}{4}$ and $W(0) = -\frac{1}{8}$. If not, we can just scale the variables, change time and rescale ε. The Hamiltonian corresponding to this latter Lagrangian is

$$H_\varepsilon(q,p) = \frac{1}{2}|p - i\lambda(q)q|^2 + W(q) - \varepsilon\frac{f(q)}{|q|}$$

using directly the result of Lemma 4.4.
Let $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the squaring map $h(x) = x^2, x \in \mathbb{C}$.

Lemma 4.5. *There exists a C^4 Hamiltonian*

$$\mathcal{H}(x,y) = \frac{1}{2}(|y|^2 - |x|^2) + O_4(x,y) \tag{4.8}$$

on $\mathbb{R}^4\{x,y\}$ such that for $x \neq 0$ the canonical transformation $g\colon (x,y) \to (q,p)$, $q = h(x), y = h'(x)^T p$ takes trajectories of the system with Hamiltonian $\mathcal{H}$ on the energy level $\mathcal{H} = \varepsilon$ to the trajectories of the system with Hamiltonian H_ε on the energy level $H_\varepsilon = 0$.

Proof. We will use the Mathieu transformation [4]. In this case, we are given a point transformation $q = h(x)$, with $\frac{\partial h}{\partial x} = 2x$ invertible (the inverse would be just $\frac{1}{2x^{-1}}$, which is well-defined as we are supposing that $x \neq 0$.) This point transformation can be extended to a symplectic transformation by defining the function $\tilde{S}(x,p) = h(x)^T p$ and

$$q = \frac{\partial \tilde{S}}{\partial p}(x,p) = h(x), \quad y = \frac{\partial \tilde{S}}{\partial x} = \frac{\partial h}{\partial x}(x)\cdot p = h'(x)\cdot p$$

which gives us directly the result, as we can define from this function the desired g that takes trajectories of the regularized system $\mathcal{H}$ at energy level $\mathcal{H} = \varepsilon$ to the trajectories of the system with Hamiltonian H_ε on the energy level $H_\varepsilon = 0$, by means of the Levi-Civita regularization process [4]. □

The transformation g does not preserve the time parametrisation of the solution, but it preserves the actions.

$$\int_\gamma \langle y, dx \rangle = \int_{g(\gamma)} \langle p, dq \rangle$$

by the definition of Mathieu transformation. So, for trajectories of H_ε with energy 0, we have that the action $\int \langle p, dq \rangle = \int L_\varepsilon$ (this is just the definition of the action of the trajectory $g(\gamma)$), so we have the equivalent action $\int \langle y, dx \rangle$ for $\mathcal{H}$.

Consider now a more general Hamiltonian system on $\mathbb{R}^{2m}$ with a Hamiltonian of the form (4.8). The hyperbolic equilibrium 0 has m-dimensional stable and unstable manifolds $W^\pm_{loc}$. Since $W^\pm_{loc}$ are Lagrangian manifolds (see Section 7.1.1) and project diffeomorphically to $\mathbb{R}^m$, they are defined, by Poincare's Lemma, by generating functions $s^\pm$ on a small ball U with center 0 in $\mathbb{R}^m$ (as they are Lagrangian manifolds, their form is closed; and by Poincare's Lemma they are exact).

$$W^\pm_{loc} = \{(x, y) \mid y = \mp \nabla s^\pm(x), x \in U\} \tag{4.9}$$

By the definition of $W^\pm_{loc}$, for any point $a \in U$ there exists a unique trajectory $\omega_a \colon [0, \infty) \to U$ such that $\lim_{t\to\infty} \omega_a^+(t) = 0$ and $\omega_a^+(0) = a$. Similarly, there exists a unique trajectory $\omega_a^- \colon (-\infty, 0] \to U$ such that $\lim_{t\to-\infty} \omega_a^-(t) = 0$ and $\omega_a^-(0) = a$. Using (4.9) one can compute the actions of these trajectories:

$$\int_{\omega_a^+} \langle y, dx \rangle = \int_a^0 -\nabla s^+(x) dx = s^+(a)$$
$$\int_{\omega_a^-} \langle y, dx \rangle = \int_0^a \nabla s^-(x) dx = s^-(a)$$

For the trajectories $\omega_a^\pm(t)$, let $z_a^\pm(t) \in W^\pm_{loc}$ be the corresponding orbits in the phase space.

Lemma 4.6. *Let $T > 0$ be sufficiently large. Then for any points $a, b \in U$ and $\tau \geq T$:*

- *There exists a unique trajectory of Hamiltonian (4.8)*

$$z(t) = (x(t), y(t)) = f(a, b, \tau, t),$$
$$(\tau, t) \in D_T = \{(\tau, t) \mid \tau \geq T, 0 \leq t \leq \tau\},$$

such that $x(0) = a$ and $x(\tau) = b$.

- *The map f is C^2 on $U^2 \times D_T$ and*

$$f(a,b,\tau,t) = z_a^+(t) + z_b^-(t-\tau) + e^{-\tau}\phi(a,b,\tau,t) \tag{4.10}$$

where ϕ is uniformly C^2 bounded on $D^2 \times D_T$.

- *The action*

$$S(a,b,\tau) = \int_0^\tau \langle y, dx \rangle$$

of the trajectory $z(t)$ is C^2 on $U^2 \times [T,\infty)$ and

$$S(a,b,\tau) = s^+(a) + s^-(b) + e^{-\tau}R(a,b,\tau) + \tau h(a,b,\tau) \tag{4.11}$$

where R is uniformly C^2 bounded as $\tau \to \infty$ and $h(a,b,\tau)$ is the energy of z

This Lemma can be deduced from the strong λ-lemma [7]. However, we will prove the Lemma for the simplest scenario, i.e., considering $\mathbb{R}^2 \times \mathbb{R}^2$ and the linear part of the regularized Hamiltonian, to get an idea on how these results can be obtained.

Proof. As we have a symmetric behaviour between spaces, it will be enough to make the computations considering just $\mathbb{R}^2$, so $x \in \mathbb{R}$ and $y \in \mathbb{R}$. The regularised Hamiltonian function would be then

$$\mathcal{H}(x,y) = \frac{1}{2}(|y|^2 - |x|^2) = \frac{1}{2}(y^2 - x^2) \tag{4.12}$$

with equations of motion

$$\begin{cases} \dot{x} = \frac{\partial \mathcal{H}}{\partial y} = y \\ \dot{y} = -\frac{\partial \mathcal{H}}{\partial x} = x \end{cases}$$

So we have the following system

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = Az$$

with $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One can clearly see that $(0,0)$ is an hyperbolic point (it is the solution of $\dot{z} = 0$) and now we determine its behaviour by computing the eigenvalues

$$P_\lambda = det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Longrightarrow \lambda_1 = 1, \;\; \lambda_2 = -1$$

So $(0,0)$ is a saddle point.

Then, we compute the stable and unstable manifold, denoted by W^+_{loc} and W^-_{loc} respectively.

$$W^+_{loc} = Ker(A+I) = Ker\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \Longrightarrow W^+_{loc} = \langle(1,-1)\rangle$$

$$W^-_{loc} = Ker(A-I) = Ker\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix} \Longrightarrow W^-_{loc} = \langle(1,1)\rangle$$

As one can know from the literature, the general solution of this type of linear ODE systems is

$$z(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

So in our case we would obtain

$$z(t) = c_1 e^t \begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\-1\end{pmatrix} \tag{4.13}$$

so

$$\begin{cases} x(t) = c_1 e^t + c_2 e^{-t} & \in \mathbb{R} \\ y(t) = c_1 e^t - c_2 e^{-t} & \in \mathbb{R} \end{cases}$$

Now we impose the initial conditions:

$$\begin{cases} x(0) = a \Longleftrightarrow c_1 + c_2 = a \\ x(\tau) = b \Longleftrightarrow c_1 e^{\tau} + c_2 e^{-\tau} = b \end{cases} \tag{4.14}$$

By a direct computation, we can deduce the values of the coefficients c_1 and c_2

$$\begin{cases} c_1 = \frac{-b+ae^{-\tau}}{e^{-\tau}-e^{\tau}} \\ c_2 = \frac{b-ae^{\tau}}{e^{-\tau}-e^{\tau}} \end{cases} \tag{4.15}$$

So the solution of the system would be

$$z(t) = \frac{-b+ae^{-\tau}}{e^{-\tau}-e^{\tau}} e^t \begin{pmatrix}1\\1\end{pmatrix} + \frac{b-ae^{\tau}}{e^{-\tau}-e^{\tau}} e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$$

which is unique.

Now we must express this solution as a function $f(a,b,t,\tau) = z^+_a(t) + z^-_b(t-\tau) + e^{-\tau}\phi(a,b,\tau,t)$, with ϕ bounded.

Let us compute then $z^+_a(t)$ ($z^-_b(t-\tau)$ would be similar, changing the initial conditions)

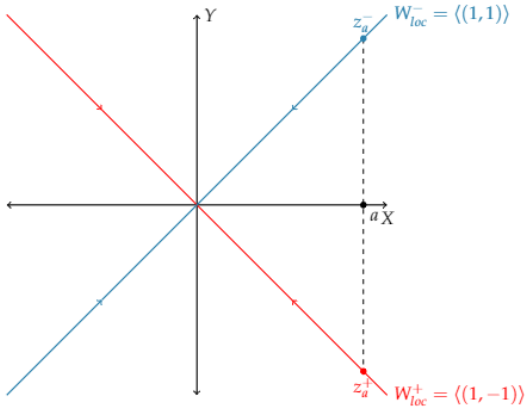


Figure 4.5: Representation of the stable and unstable manifolds for the linear case. As we can see, z_a^+ and z_a^- are just the points of these manifolds whose projection onto the X-axis corresponds to the point a.

z_a^+ is given by the projection of the general solution $z(t)$ into the stable manifold W_{loc}^+ (see Figure 4.5), that is

$$x(0) = a, \;\; y(0) = -a$$

Substituting in (4.13), we obtain the following system of equations

$$\begin{cases} c_1 + c_2 = a \\ c_1 - c_2 = -a \end{cases}$$

which leads to the following expression of $z_a^+(t)$

$$z_a^+(t) = ae^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

that fulfills the required conditions

$$\lim_{t \to +\infty} z_a^+(t) = 0, \;\; x_a^+(0) = a$$

By a similar procedure, we obtain $z_b^-(t-\tau) = be^{t-\tau} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

What is left to do is to compute the function ϕ, which would be the remainder. To do that, we perform the following computations:

We impose the equality

$$z(t) = f(a,b,t,\tau) = z_a^+(t) + z_b^-(t-\tau) + e^{-\tau}\phi(a,b,\tau,t)$$

So

$$\frac{-b+ae^{-\tau}}{e^{-\tau}-e^{\tau}}e^{t}\begin{pmatrix}1\\1\end{pmatrix}+\frac{b-ae^{\tau}}{e^{-\tau}-e^{\tau}}e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}=ae^{-t}\begin{pmatrix}1\\-1\end{pmatrix}+be^{t-\tau}\begin{pmatrix}1\\1\end{pmatrix}+e^{-\tau}\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix}$$

Now we compute ϕ_1

$$\begin{aligned}\phi_1(a,b,t,\tau)&=e^{\tau}\left[e^{t}\left(\frac{-b+ae^{-\tau}}{e^{-\tau}+e^{\tau}}-be^{-\tau}\right)+e^{-t}\left(\frac{b-ae^{\tau}}{e^{-\tau}+e^{\tau}}-a\right)\right]\\&=e^{t}\left(\frac{a-be^{\tau}}{e^{-\tau}-e^{\tau}}-b\right)+e^{-t}\left(\frac{be^{\tau}-ae^{2\tau}}{e^{-\tau}-e^{\tau}}-ae^{\tau}\right)\\&=e^{t}\left(\frac{a-be^{2\tau}}{1-e^{2\tau}}-b\right)+e^{-t}\left(\frac{be^{2\tau}-ae^{\tau}}{1-e^{2\tau}}\right)\end{aligned}$$

As t is bounded, we can suppose that $e^t = C$, so $\lim_{\tau\to+\infty}\phi_1 = -\frac{b}{C} < \infty$, so ϕ_1 is bounded.

A similar computation can be done for ϕ_2, obtaining the following result

$$\phi_2(a,b,t,\tau)=e^{t}\left(\frac{a-be^{2\tau}}{1-e^{2\tau}}-b\right)-e^{-t}\left(\frac{be^{2\tau}-ae^{\tau}}{1-e^{2\tau}}\right),$$

and supposing again that $e^t = C$, $\lim_{\tau\to+\infty}\phi_2 = \frac{b}{C} < \infty$, so ϕ_2 is bounded too.

Finally, we must show that the action

$$S(a,b,\tau)=\int_0^{\tau}\langle y,dx\rangle$$

can be written as

$$S(a,b,\tau)=s^{+}(a)+s^{-}(b)+e^{-\tau}R(a,b,\tau)+\tau h(a,b,\tau)$$

where R is bounded as $\tau\to+\infty$

To do that, we just compute the action using the general solution (4.13)

$$\begin{aligned}S(a,b,\tau)&=\int_0^{\tau}\langle y,dx\rangle=\int_0^{\tau}\langle(c_1e^{t}-c_2e^{-t},c_1e^{t}-c_2e^{-t})\rangle dt=\int_0^{\tau}(c_1e^{t}-c_2e^{-t})^2dt\\&=\int_0^{\tau}c_1^2e^{2t}-2c_1c_2+c_2^2e^{-2t}dt=\left[\frac{c_1^2}{2}e^{2t}-2c_1c_2t-\frac{c_2^2}{2}e^{-2t}\right]_0^{\tau}\\&=\frac{c_1^2}{2}e^{2\tau}-2c_1c_2\tau-\frac{c_2^2}{2}e^{-2\tau}-\frac{c_1^2+c_2^2}{2}\end{aligned}$$

Now, in a similar way as we did before with $z^+(a)$ and $z^-(b)$, we compute $s^+(a)$ and $s^-(b)$:

$$s^+(a) = \int_{\omega^+(a)} \langle y, dx \rangle \underset{z_a^+(t)=(ae^{-t},-ae^{-t})}{=} \int_0^{+\infty} \langle -ae^{-t}, ae^{-t} \rangle dt$$
$$= \int_0^{+\infty} -a^2 e^{-2t} dt = \left[\frac{a^2}{2} e^{-2t} \right]_0^{+\infty} = -\frac{a^2}{2}$$
$$s^-(b) = \int_{\omega^-(b)} \langle y, dx \rangle \underset{z_b^-(t)=(e^{t-\tau}b,e^{t-\tau}b)}{=} \int_{-\infty}^{\tau} \langle e^{t-\tau}b, e^{t-\tau}b \rangle dt$$
$$= \int_{-\infty}^{\tau} b^2 e^{2t-2\tau} dt = \left[\frac{1}{2} b^2 e^{2t-2\tau} \right]_{-\infty}^{\tau} = \frac{b^2}{2}$$

We have to rearrange the result so we obtain the following equality

$$\frac{c_1^2}{2} e^{2\tau} - 2c_1c_2\tau - \frac{c_2^2}{2} e^{-2\tau} - \frac{c_1^2 + c_2^2}{2} = -\frac{a^2}{2} + \frac{b^2}{2} + e^{-\tau} R(a,b,\tau) + \tau h(a,b,\tau)$$

To do that, we use (4.14), so

$$\begin{cases} a = c_1 + c_2 \iff a^2 = c_1^2 + c_2^2 + 2c_1c_2 \\ b = c_1 e^{\tau} + c_2 e^{-\tau} \iff b^2 = c_1^2 e^{2\tau} + 2c_1c_2 + c_2^2 e^{-2\tau} \end{cases}$$

so

$$\frac{b^2 - a^2}{2} = \frac{1}{2} \left(c_1^2 e^{2\tau} + c_2^2 e^{-2\tau} - c_1^2 - c_2^2 \right) = \frac{c_1^2}{2} e^{2\tau} + \frac{c_2^2}{2} e^{-2\tau} - \frac{c_1^2 + c_2^2}{2}.$$

Substituting in the previous equality, we obtain that

$$R(a,b,\tau) = -c_2^2$$

$$h(a,b,\tau) = -2c_1c_2$$

Finally, we must see that $\lim_{\tau \to +\infty} R(a,b,\tau) < \infty$

$$\lim_{\tau \to +\infty} R(a,b,\tau) = \lim_{\tau \to +\infty} -c_2^2 \underset{Equation(4.15)}{=} \lim_{\tau \to +\infty} -\frac{b^2 - 2ae^{\tau} + a^2 e^{2\tau}}{e^{-2\tau} - 2 + e^{2\tau}}$$
$$= \lim_{\tau \to +\infty} -\frac{a^2 e^{4\tau}}{1 + e^{4\tau}} = -a^2 < +\infty$$

□

Lemma 4.7. *The energy $h(a,b,\tau)$ of the trajectory $z(t)$ is a C^2 function on $U^2 \times [T, \infty)$ and has the form*

$$h(a,b,\tau) = e^{-\tau}(h_0(a,b) + h_1(a,b,\tau)) \tag{4.16}$$

where

$$h_0(a,b) = 2\langle v^+(a), v^-(b) \rangle, \quad v^{\pm}(a) = \lim_{t \to \pm\infty} (e^{\pm t} \dot{\omega}_a^{\pm}(t))$$

and $||h_1||_{C^2(U^2\times[\tau,\infty))} \to 0$ *as* $\tau \to \infty$.

Here $v^{\pm}(a)$ *are tangent vectors at* 0 *to the asymptotic trajectories* $\omega_a^{\pm}$*, and* $\langle , \rangle$ *is the Euclidean scalar product.*

We will prove this Lemma for the specific case we have been dealing with until now (i.e., considering the linear part of the Hamiltonian function $\mathcal{H}$ in $\mathbb{R}^2$) and then we will give a more general proof.

Proof. We recall Equation (4.11) so we have

$$h(a,b,\tau) = -2c_1c_2,$$

with $c_1 = \frac{-b+ae^{-\tau}}{e^{-\tau}+e^{\tau}}$ and $c_2 = \frac{b-ae^{\tau}}{e^{-\tau}+e^{\tau}}$.

First of all, we compute $h_0(a,b) = 2\langle v^+(a), v^-(b)\rangle$

$$\begin{aligned} v^+(a) &= \lim_{t\to+\infty}(e^t\dot{\omega}_a^+(t)) \underset{\text{proof of Lemma 4.6}}{=} \lim_{t\to+\infty}(e^t(a\dot{e^{-t}})) \\ &= \lim_{t\to+\infty}(-a) = -a \\ v^-(b) &= \lim_{t\to-\infty}(e^{-t}\dot{\omega}_b(t)) \underset{\text{proof of Lemma 4.6}}{=} \lim_{t\to-\infty}(e^{-t}(e^t b)) \\ &= \lim_{t\to-\infty} b = b \end{aligned}$$

So $h_0(a,b) = -2ab$. Moreover, we have that

$$\begin{aligned} h(a,b,\tau) = -2c_1c_2 &= -2\left[\frac{(-b+ae^{-\tau})(b-ae^{\tau})}{(e^{-\tau}-e^{\tau})^2}\right] \\ &= -2\left(\frac{-b^2+abe^{\tau}+abe^{-\tau}-a^2}{(e^{-\tau}+e^{\tau})^2}\right) \end{aligned}$$

Imposing the equality we want to prove, we can deduce the value of h_1

$$-2c_1c_2 = e^{-\tau}(h_0+h_1) = e^{-\tau}(-2ab+h_1)$$

Then

$$\begin{aligned} h_1(a,b,\tau) &= -2e^{\tau}\left[\frac{-b^2+ab(e^{\tau}+e^{-\tau})-a^2}{(e^{-\tau}-e^{\tau})^2}\right] + 2ab \\ &= -\frac{-2b^2e^{\tau}+2ab(e^{2\tau}+1)-2a^2e^{\tau}+2ab(e^{-\tau}-e^{\tau})^2}{(e^{-\tau}-e^{\tau})^2} \\ &= -\frac{-2b^2e^{\tau}+2abe^{2\tau}+2ab-2a^2e^{\tau}+2abe^{-2\tau}-4ab+2abe^{2\tau}}{(e^{-\tau}-e^{\tau})^2} \\ &= -\frac{e^{\tau}(-2b^2-2a^2)+4abe^{2\tau}+2abe^{-2\tau}+2ab}{(e^{-\tau}-e^{\tau})^2} \\ &= -\frac{e^{3\tau}(-2b^2-2a^2)+4abe^{4\tau}+2ab+2abe^{2\tau}}{e^{4\tau}-e^{2\tau}+1} \end{aligned}$$

We compute the limit to see if it is bounded

$$\lim_{\tau\to\infty} h_1(a,b,\tau) = \lim_{\tau\to\infty} -\frac{e^{3\tau}(-2b^2-2a^2)+4abe^{4\tau}+2ab+2abe^{2\tau}}{e^{4\tau}-e^{2\tau}+1} = -4ab < \infty$$

So h_1 is bounded. □

Proof. First of all we are going to define a symplectic change of coordinates that ensures us that both stable and unstable manifolds are on the axis, so $W^-_{loc} = \{v = 0\}$ and $W^+_{loc} = \{u = 0\}$ for $(u,v) \in \mathbb{R}^{2m}$. This transformation $(x,y) \mapsto (u,v)$ is given by

$$y = \frac{u+v}{\sqrt{2}} + O_3(u,v), \quad x = \frac{u-v}{\sqrt{2}} + O_3(u,v)$$

so we have

$$\begin{aligned}\mathcal{H}(u,v) &= \frac{1}{2}\left(\left|\frac{u+v}{\sqrt{2}} + O_3(u,v)\right|^2 - \left|\frac{u-v}{\sqrt{2}} + O_3(u,v)\right|^2\right) + O_7(u,v)\\ &= \frac{1}{2}\left(\frac{1}{2}(u^2+2u\cdot v+v^2+O(u,v)) - \frac{1}{2}(u^2-2u\cdot v+v^2+O(u,v))\right)\\ &= \langle u,v\rangle(1+O(u,v))\end{aligned}$$

so the equations of motion would be

$$\begin{cases}\dot{u} = \frac{\partial \mathcal{H}}{\partial v} = u + O_2(u)\\ \dot{v} = -\frac{\partial \mathcal{H}}{\partial u} = -v + O_2(v)\end{cases}$$

leading to stable and unstable manifolds W^+_{loc} and W^-_{loc} in a neighborhood of the equilibrium point 0 with equations $W^+_{loc} = \{v = 0\}$ and $W^-_{loc} = \{u = 0\}$ respectively.

The Hamiltonian system on the unstable manifold W^-_{loc} defined by $\dot{u} = u + O_2(u)$ can be transformed [8] to a linear equation $\dot{\xi} = \xi$ by a C^2 change of variables $\xi = f(u)$. Hence the phase flow on W^-_{loc} takes the form

$$g_{-t}(u,0) = (f^{-1}(e^{-t}f(u)),0) = e^{-t}(f(u)+G(u,t),0) \tag{4.17}$$

where $||G||_{C^2} \to 0$ uniformly on W^-_{loc} as $t \to \infty$.

The first equality comes from the following computation

$$\dot{\xi} = \xi \Longrightarrow \xi(t) = e^t\xi_0 \underset{f(u)=\xi}{\Longrightarrow} g_{-t}(u,0) = (f^{-1}(e^{-t}f(u)),0)$$

The second equality comes from the Taylor expansion of f^{-1} at 0, and the fact that $\dot{u} = u + O_2(u)$.

A similar representation holds for the flow on the stable manifold.

$$g_t(0,v) = e^{-t}(0, g(v) + E(v,t)), \tag{4.18}$$

where $||E||_{C^2} \to 0$ uniformly on W^+_{loc} as $t \to \infty$. Note that

$$\begin{cases} \lim_{t\to\infty} e^t g_{-t}(u,0) = \lim_{t\to\infty} e^t(e^{-t}(f(u) + G(u,t),0)) = (f(u),0) \\ \lim_{t\to\infty} e^t g_t(0,v) = \lim_{t\to\infty} e^t(e^{-t}(0,g(v) + E(v,t)) = (0,g(v)) \end{cases} \tag{4.19}$$

One can write now the trajectory $z(t)$ in terms of the phase flows g_t and g_{-t} [9]. Considering $t = \frac{\tau}{2}$ we have

$$z\left(\frac{\tau}{2}\right) = (g_{-\frac{\tau}{2}}(u,0), g_{\frac{\tau}{2}}(0,v)) = (e^{-\frac{\tau}{2}}f(u), e^{-\frac{\tau}{2}}g(v)) + e^{-\frac{\tau}{2}}F(u,v,\tau)$$

Since $\mathcal{H}$ is a first integral, replacing this latter expression into the Hamiltonian in coordinates (u,v) will give us the result we desired

$$h(a,b,\tau) = \mathcal{H}\left(z\left(\frac{\tau}{2}\right)\right) = e^{-\tau}(\langle f(u), g(v)\rangle + h_1(u,v,\tau)),$$

where $||h_1||_{C^2} \to 0$ as $\tau \to \infty$. Now passing to the variables x, y and using (4.19), we obtain Lemma 4.7. □

Finally, we will state the last proposition that will allow us, along with the two lemmas we just have proved, to give the proof of Lemma 4.1 we were seeking for.

Take small $\nu > 0$ and let $B = \{(a,b) \in U^2 \mid h_0(a,b) \geq \nu\}$. Then for $(a,b) \in B$ the function $h_0(a,b)$ is bounded away from zero. If it is non zero, then $h(a,b,\tau)$ has to be monotone in τ for sufficiently large τ. For small $\varepsilon > 0$, solving the equation $h(a,b,\tau) = \varepsilon$ for τ yields a C^2 function $\tau = \tau_\varepsilon(a,b)$. To see how we can obtain this τ function we define $F(a,b,t) = h(a,b,t) - \varepsilon$ and we have

- $F(a,b,\tau) = 0$ by hypothesis.
- $D_\tau F = \frac{\partial}{\partial\tau} h(a,b,\tau) \neq 0$ because h is monotone in τ.

By the Implicit Function Theorem, there exists $\tau = \tau_\varepsilon(a,b)$, C^2 function in B, such that $F(a,b,\tau_\varepsilon(a,b)) = 0$

This result gives rise to the following proposition

Proposition 4.8. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0]$:*

- *For any $(a,b) \in B$, there exists a unique trajectory $z^\varepsilon_{a,b} = (x^\varepsilon_{a,b}, y^\varepsilon_{a,b})\colon [0,\tau] \to U \times \mathbb{R}^m$ of energy ε connecting the points a and b.*
- *The time $\tau = \tau_\varepsilon(a,b)$ is a C^2 function on B and*

$$\tau_\varepsilon(a,b) = -\log\ \varepsilon + \mu(a,b,\varepsilon)$$

where the function μ is uniformly bounded on B as $\varepsilon \to 0$

- *We have*

$$z^\varepsilon_{a,b}(t) = z^+_a(t) + z^-_b(t) + \varepsilon\zeta(a,b,\varepsilon) \tag{4.20}$$

where the function ζ is uniformly C^1 bounded as $\varepsilon \to 0$.

- *The action $f_\varepsilon(a,b) = S(a,b,\tau_\varepsilon(a,b))$ of the trajectory $z^\varepsilon_{a,b}$ is a C^2 function on B and*

$$f_\varepsilon(a,b) = s^+(a) + s^-(b) + \varepsilon r(a,b,\varepsilon) - \varepsilon\log\varepsilon \tag{4.21}$$

where r is uniformly C^2 bounded on B as $\varepsilon \to 0$

Note that from (4.16) and (4.20) we have

$$\min_{0\le t\le\tau} |x^\varepsilon_{a,b}|^2 = 2\varepsilon(|v^+(a)||v^-(b)| - \langle v^+(a), v^-(b)\rangle) + o(\varepsilon) \tag{4.22}$$

Hence $x^\varepsilon_{a,b}(t)$ avoids 0 by a distance of $\sqrt{\varepsilon}$ provided that $v^+(a) \neq v^-(b)$.

Proof.

- We can use directly the previous lemmas and the fact that $h(a,b,\tau) = \varepsilon$ to prove the existence and uniqueness of the trajectory connecting two points of B with energy ε.
- From Lemma 4.7 we know that

$$\varepsilon = h(a,b,\tau) = e^{-\tau}(h_0 + h_1) \underset{\text{both bounded on } B}{\approx} e^{-\tau}\gamma(a,b,\varepsilon)$$

We take logarithms and we obtain

$$\log\ \varepsilon = -\tau + \log\gamma \Longrightarrow \tau = -\log\ \varepsilon + \mu(a,b,\varepsilon)$$

with $\mu(a,b,\varepsilon)$ bounded from the previous results.

- The expression of the new trajectory $z^\varepsilon_{a,b}(t)$ comes from (4.10), replacing τ for $-\log\ \varepsilon + \mu$.

- The expression of the new action $f_\varepsilon(a,b)$ comes from (4.11) replacing τ for $-\log\ \varepsilon+\mu$.
- We will check this minimum result for the linear case (4.12), where we have the explicit form of the trajectory.

  We recall the function that describes the trajectory $z(t)$ in (4.10)

$$x(t)=\frac{(ae^{-\tau}-b)e^{t}+(b-ae^{\tau})e^{-t}}{e^{-\tau}-e^{\tau}}$$

  So

$$\begin{aligned}x^2(t)=&\frac{1}{(e^{-\tau}-e^{\tau})^2}[e^{2t}(ae^{-\tau}-b)^2\\&+e^{-2t}(b-ae^{\tau})^2+2(ae^{-\tau}-b)(b-ae^{\tau})]\end{aligned}\tag{4.23}$$

  Then, we perform the derivative with respect to time t.

$$\begin{aligned}\dot{x^2}(t)=\frac{1}{(e^{-\tau}-e^{\tau})^2}\left[2e^{2t}(ae^{-\tau}-b)^2-2e^{-2t}(b-ae^{\tau})^2\right]=0\\\iff e^{2t}(ae^{-\tau}-b)^2-e^{-2t}(b-ae^{\tau})^2=0\end{aligned}$$

  Which implies

$$e^{4t}=\frac{(b-ae^{\tau})^2}{(ae^{-\tau}-b)^2}\Longrightarrow e^{2t}=\pm\frac{b-ae^{\tau}}{ae^{-\tau}-b},\ \ e^{-2t}=\pm\frac{ae^{-\tau}-b}{b-ae^{\tau}}$$

  Substituting in the original expression (4.23)

$$x^2(t)=\frac{1}{(e^{-\tau}-e^{\tau})^2}\left[4(b-ae^{\tau})(ae^{-\tau}-b)\right]$$

  Using now that $\tau_\varepsilon\approx -log\ \varepsilon$, we have that

$$\min_{0\le t\le\tau}|x^{\varepsilon}_{a,b}(t)|^2=4ab\frac{\varepsilon-2+\frac{1}{\varepsilon}}{\varepsilon^2-2+\frac{1}{\varepsilon^2}}-\frac{4(a-b)^2}{\varepsilon^2-2+\frac{1}{\varepsilon^2}}$$

  which can be rearranged to obtain

$$\min_{0\le t\le\tau}|x^{\varepsilon}_{a,b}(t)|^2=4ab\varepsilon+o(\varepsilon)$$

  but we observe that

$$4ab\varepsilon=2\varepsilon(2ab)=2\varepsilon(|v^+(a)||v^-(b)|-\langle v^+(a),v^-(b)\rangle$$

  due to the fact that $v^+(a)=-a$ and $v^-(b)=b$

□

Now we are ready to prove the Lemma 4.1.

Proof. We saw in the previous chapter that collision solutions $\gamma^+_{h(a)}\colon [0,\tau^+(h(a))] \to U_i$ with p_i for the system (L) (3.2) (3.2) correspond (up to time parametrization; as we saw in Lemma 4.5, the canonical transformation g does not preserve the time) to asymptotic orbits $\omega^+_a\colon [0,+\infty) \to U = h^{-1}(U_i)$ to the equilibrium 0 for the system with Hamiltonian $\mathcal{H}$ on the level set $\mathcal{H} = 0$ (see Section 3.2).

In the conformal coordinates with centre p_i, the map h takes the regularized trajectory $\omega^+_a\colon [0,+\infty) \to U$ to the initial trajectory $\gamma^+_{h(a)}\colon [0,\tau^+(h(a))] \to U_i$ (with changed time parametrization), i.e. $\gamma^+_{h(a)}(t) = h(\omega^+_a(t_+(t)))$, where $t_+\colon [0,+\infty] \to [0,\tau^+(h(a))]$ (note that in the regularized trajectory ω^+_a, corresponding to the stable manifold W^+_{loc}, it takes $+\infty$ time to reach the equilibrium point 0, i.e, $\omega^+_a(+\infty) = 0$). Similarly, $\gamma^-_{h(a)}(t) = h(\omega^-_a(t_-(t))$, where $t_-(t)\colon [-\infty,0] \to [\tau^-(h(a)),0]$.

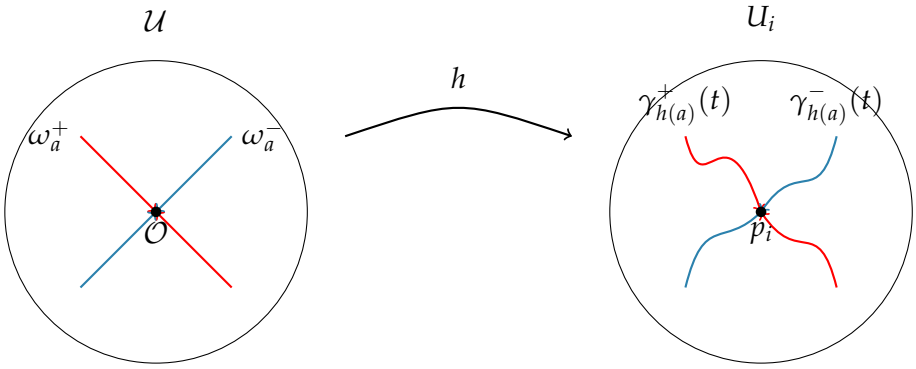


Figure 4.6: The map h takes the regularized trajectory ω_a to the initial trajectory $\gamma_{h(a)}$

Using the definition of the squaring map h, we obtain that for any $a \in U$

$$h(v^{\pm}(a)) = \mp\lambda u^{\pm}(h(a))$$

This result makes sense because if h takes the regularized trajectory $\omega^{\pm}_a$ to the initial collision trajectory $\gamma^{\pm}_{h(a)}$, it will send as well the tangent vectors at 0 of these $\omega^{\pm}_a$ to the tangent vectors at p_i of the collision trajectories $\gamma^{\pm}_{h(a)}$. The fact of being a squaring map is what gives us this linear relation of constant λ between these two vectors. In fact, a direct computation shows that $\lambda = \frac{|v^{\pm}(a)|^2}{2}$.

Moreover, as we saw at the beginning of this section, this map does not change the action, so $S^{\pm}(h(a)) = s^{\pm}(a)$.

For being a squaring map at $\mathbb{C}$, we have also that $h(y) = \lambda h(x), \lambda \geq 0$ is equivalent to $y \perp x$. Hence $u^{+}(h(a)) \neq u^{-}(h(b))$ if, and only if, $v^{+}(a)$ and $v^{-}(b)$ are not orthogonal (this is a condition that we need to impose as we have defined in Section 4.1).

Take two points $\tilde{a}, \tilde{b} \in \Sigma_i$ such that $|u_{+}(\tilde{a}) - u_{-}(\tilde{b}| \geq \delta$ (i.e, $\tilde{a}, \tilde{b} \in X$ as in Lemma 4.1), and

$$h_0(a, b) = 2\langle v^{+}(a), v^{-}(b)\rangle \geq (1 - \mu)$$

provided $(1 - \mu) \geq 0$ sufficiently small, so we are under the conditions of Proposition 4.8. By this proposition, we can connect a to b by a trajectory $z^{\varepsilon}_{a,b} = (x^{\varepsilon}_{a,b}, y^{\varepsilon}_{a,b})$ of energy ε for the system with Hamiltonian (4.8). Under the map h and an appropriate time reparametrization, $h(x^{\varepsilon}_{a,b})$ gives a trajectory of energy 0 for the system (L_{ε}) (3.1) connecting $\tilde{a} = h(a)$ to $\tilde{b} = h(b)$ (using Lemma 4.5). Since the action is invariant under the transformation h, (4.21) implies (4.4) with $r(a, b, \varepsilon) = s(h(a), h(b), \varepsilon)$. Finally, condition (4.5) comes directly from (4.22) and the fact that $|h(x)| = |x|^2$, and the Lemma 4.1 is proved for the 2D case. □

Chapter 5

Geometrical Approach

Once we have reviewed in detail the theory and the results given by S.V. Bolotin and R.S. Mackay [1], it is time to move onto the work made by Jean-Pierre Marco and Laurent Niederman [2]. We will start by giving some complementary definitions and notations apart from those ones defined in Section 3.3, as well as introducing some new concepts such as "generatrix solutions", which will allow us to state the theorems needed to ensure the existence of the second species solutions we are looking for.

5.1 Homoclinic solutions and Second Species solutions

5.1.1 Formulation of the problem

We fix an energy h that ensures the existence of an elliptic singular domain (see Section 3.3), i.e., $h \in (0, 3+2\sqrt{2})$. We make the following considerations

- For $\mu > 0$, we will denote by $\mathcal{P}_\mu$ to the system $(\mathcal{H}_\mu, X_{H_\mu})$ (H_μ defined in (3.26)).

- For $\mu = 0$, $\mathcal{P}_0$ will be the singular system obtained by the restriction of $(\mathcal{H}, X_{H_0})$ (H_0 defined in (3.28)) to the complementary in $\mathcal{H}$ of the circle C_0 (3.35).

- For $\mu \geq 0$, we will denote by $\mathcal{Q}_\mu$ the regularized system $(\mathcal{L}_\mu, X_{L_\mu})$ (L_μ defined in (3.39)).

Homoclinic limit solutions

We consider the Hamiltonian R_0 (3.23):

$$R_0(\xi,\eta) = |\eta|^2 - \frac{2}{|\xi|} + 3 - 2\sigma$$

with a fixed energy $h \in (0, 3+2\sqrt{2})$, and two angular momentums σ_u and σ_s in the interval $(\sigma_-(h), \sigma_+(h))$ correspondent to the elliptic singular domain $\mathcal{D}_e$ defined for this energy value (see Section 3.3). We denote by φ_u and φ_s two solutions of the problem $\mathcal{P}_0$ of respective momentums σ_u and σ_s. Moreover, we will suppose that their domains are consecutive, i.e., of the form $\mathrm{Dom}(\varphi_s) = (-t_c, t_c)$ and $\mathrm{Dom}(\varphi_u) = (t_c, t_c + l)$. Under the configuration space, their trajectories are 2 arcs of ellipses, but in rotating coordinates, they are solution with origin and extreme in the point P_2, as we can see in the following picture [2] (in fact, these are solutions of the problem $\mathcal{P}_0$ with $\xi = 1$ and $\eta_* = \sqrt{h-1+2\sigma_*}\; e^{i\theta_*}$ for some θ_*).

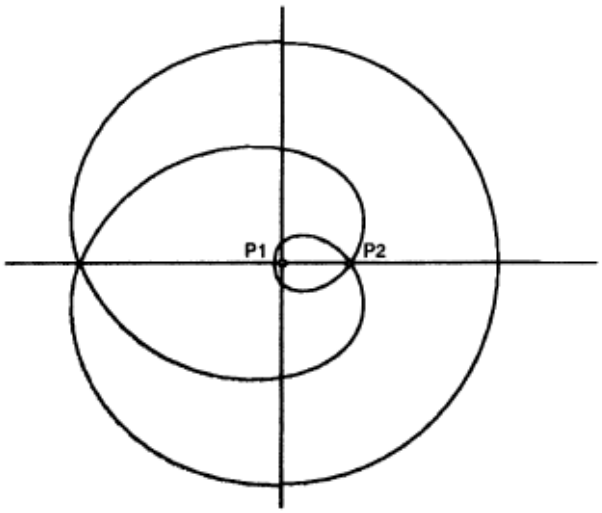


Figure 5.1: Example of two double singular trajectories [2].

Of course, all these solutions can be considered in the coordinate system $M = \xi - (1-\mu)$, $N = \eta - i(1-\mu)$, of Hamiltonian (3.29):

$$H_0(M,N) = |N|^2 + i(\overline{M}N - M\overline{N}) - \left(\frac{2}{|1+M|}\right) - 2 + (M + \overline{M})$$

Let us study now the regularized problem (see Section 3.3.3). To do that, we will recall some results concerning the Levi-Civita map ρ (3.37).

- It duplicates solutions. In fact, this map takes solutions of the regularized domain $\mathbb{C}^* \times \mathbb{C}$ to the regular domain in such a way that $\rho(z) \mapsto z^2$ (we do not need to consider $\rho(\omega)$).

  This means that, if we consider for example the homoclinic solution φ_s, this will correspond to two regularized solutions of the form $z_s = \pm\sqrt{\varphi_s}$.

- It sends the collision point $P_2 = (M = 0, N_*) \mapsto (0, 0)$ (see Section 3.3.3).

Family of second species solutions

We denote by φ the function defined under $\mathrm{Dom}(\varphi_s) \cup \mathrm{Dom}(\varphi_u)$ such that $\varphi_{|\mathrm{Dom}(\varphi_s)} = \varphi_s$ and $\varphi_{|\mathrm{Dom}(\varphi_u)} = \varphi_u$. Under this notation, one can define φ_s and φ_u as the generatrix solutions defined in Section 3.1.

A direct translation of this concept can be made for the correspondent regularized problem $\mathcal{Q}_\mu$. The two families $(\psi_\mu^\pm)$ of solutions associated to the covering converge each one for $\mu \to 0$ to the union of two different homoclinic solutions related with the fixed point O. As we have noticed before, one solution φ_μ corresponds to two solutions of the regularized problem, in this case denoted by $\psi_\mu^\pm$. As φ_μ converges to φ when $\mu \to 0$, $\psi^\pm$ will converge to two solutions $\psi^\pm$ when $\mu \to 0$. These solutions will be different homoclinic solutions related to the fixed point O.

For the rest of the chapter we will determine the analytic conditions needed to ensure that the limit solutions φ_s and φ_u become generatrices. In particular, we will prove the following theorem.

Theorem 5.1. *Let $\sigma_1 < \sigma_2$ be two angular momentums in the interval (σ_-, σ_+). For every $\eta > 0$, there exists two generatrix solutions φ_s and φ_u of the problem $\mathcal{P}_0$, of respective angular momentums m_1 and m_2 such that $|m_1 - \sigma_1| < \eta$ and $|m_2 - \sigma_2| < \eta$*

5.1.2 Existence and density of homoclinic solutions

We will start by proving the existence, in the case $\mu = 0$ and for an energy $h \in (0, 3 + 2\sqrt{2})$, of an infinite number of homoclinic symmetric orbits in the system $\mathcal{Q}_0$.

Double collision ellipses in P_2

We will work here under fixed coordinates $Q = (Q_1, Q_2)$ satisfying (3.20) for $\mu = 0$, without fixing the energy to begin with. The problem will be to find a Kepler elliptic solution describe by P_3 surrounding P_1 (of mass 1), such that

it collides at least two times with P_2, which will be describing a circular orbit of radius 1 with a normalized angular momentum. These solutions will be of interest because two different collisions in the fixed coordinate system are translated into one auto-intersection with the fixed point P_2 in the rotating coordinate system (ξ, η), obtaining something similar to the Figure 5.1.

It will be enough to know the semi-major axis a and the eccentricity e of the solution. The first thing is to take a look at the period of the solution. When there is a rational relation between the periods of P_3 and P_2 (in fact, the period of P_2 would be 2π), the particles P_2 and P_3 have an infinite amount of collisions at the same point. In the rotating frame (see Section 3.3), this behaviour corresponds to periodic solutions of system (3.23). However, they are not interesting for our purposes. That is why we will suppose that $\frac{T}{2\pi}$ is not rational.

So, let us denote by C_- and C_+ the points where the consecutive collisions take place. The origin of times will be the middle value of the interval between collisions, so they happen at times $-t_c$ and $+t_c$ respectively. The points C_- and C_+ are the intersection points between the ellipse and the circular trajectory, and they are symmetric in relation to the semi-major axis of the ellipse. Our reference system will be then (P_1, Q_1, Q_2), where $P_1 = (0, 0)$: (P_1, Q_1) is the major axis of the orbit of P_3 to his apocentre and (P_1, Q_2) is orthogonal to (P_1, X_1) (see Figure 5.2).

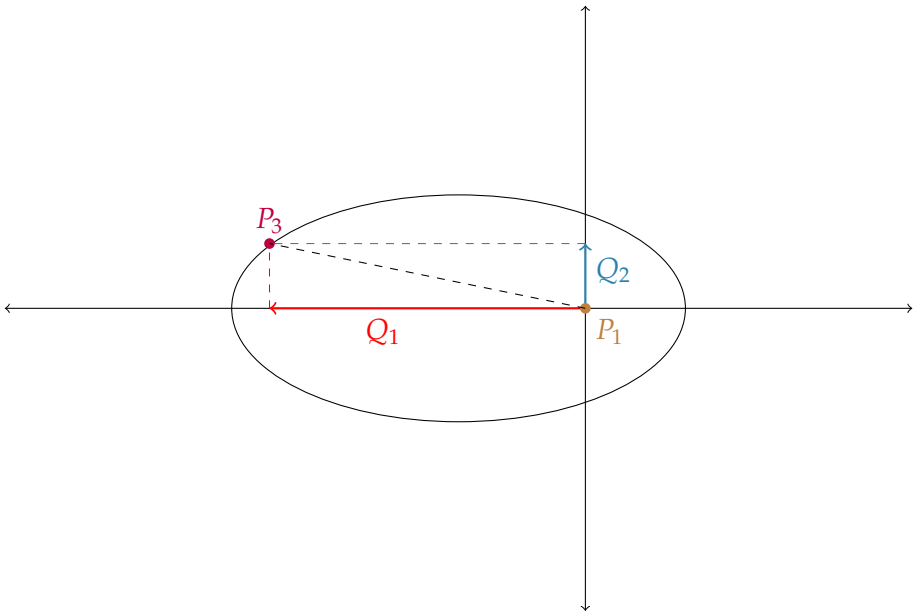


Figure 5.2: Representation of the reference system (P_1, Q_1, Q_2).

Since the particle P_2 has a constant angular momentum, at $t = 0$ it is place at the axis (P_1, Q_1), and by symmetry the particle P_3 it is also placed at (P_1, Q_1) in

the apocentre or pericentre (due to the third Kepler's Law).

If E denotes the eccentric anomaly of P_3 in the fixed reference frame (P_1, X_1, X_2), its trajectory is described by the following equations

$$\begin{cases} Q_1 = a(e - \cos E) \\ Q_2 = -a\sqrt{1-e^2}\sin E \end{cases} \tag{5.1}$$

which are the classical equations that describe the coordinates of a point from the focus (in this case the focus is our reference P_1), and the travel time is given by the Kepler Equation

$$t = a^{\frac{3}{2}}(E - e\sin E) \tag{5.2}$$

Supposing that at $t = t_0$ the point P_3 is at C_-, for an eccentric anomalie value $E_0 = E$, we would obtain

$$\begin{cases} \cos(t_0) = a(e - \cos E_0) \\ \sin(t_0) = -a\sqrt{1-e^2}\sin E_0 \\ t_0 = a^{\frac{3}{2}}(E_0 - e\sin E_0) \end{cases} \tag{5.3}$$

It is clear that we can write the positions Q_1 and Q_2 as the $cos(t_0)$ and $sin(t_0)$ respectively due to the fact that at C_-, the point P_3 can be seen as a point in the circular trajectory of P_2. Moreover, by symmetry we would obtain the equations for C_+ just considering $t = -t_0$ instead.

These equations lead to the following results:

$$\cos(E_0) = \frac{a-1}{ae}, \quad \cos(t_0) = \frac{1-a(1-e^2)}{e} \tag{5.4}$$

and if we note by cos^{-1} the determinations of the angle in $[0, \pi]$, the third equation leads to

$$\begin{aligned} &2\pi(ka^{\frac{3}{2}} - m) + a^{\frac{3}{2}}\left[\cos^{-1}\left(\frac{a-1}{ae}\right) - \sqrt{e^2 - \left(\frac{a-1}{a}\right)^2}\right] \\ &- \cos^{-1}\left(\frac{1-a(1-e^2)}{e}\right) = 0 \end{aligned}$$

where k and m are any two integer numbers. We will prove it.

Proof.

$$\begin{cases} \cos(t_0) = a(e - \cos(E_0)) \\ \sin(t_0) = -a\sqrt{1-e^2}\sin(E_0) \end{cases}$$

So

$$\begin{aligned}1 &= \cos^2(t_0) + \sin^2(t_0) = a^2(e - \cos E_0)^2 + a^2(1-e^2)\sin^2 E_0 \\ &\underset{\sin^2 E_0 = 1-\cos^2 E_0}{=} a^2(1-e^2) - a^2(1-e^2)\cos^2 E_0 + a^2(e^2 - 2e\cos E_0 + \cos^2 E_0) \\ &= a^2 - a^2e^2 - a^2\cos^2 E_0 + a^2e^2\cos E_0 + a^2e^2 - 2a^2e\cos E_0 + a^2\cos^2 E_0 \\ &= a^2 + a^2e^2\cos^2 E_0 - 2a^2e\cos E_0\end{aligned}$$

Then we obtain

$$\cos E_0 = \frac{\frac{2}{e} \pm \sqrt{\frac{4}{e^2} - \frac{4(a^2-1)}{a^2e^2}}}{2} = \frac{\frac{2}{e} \pm \sqrt{\frac{4a^2-4a^2+4}{a^2e^2}}}{2} = \frac{\frac{2}{e} \pm \frac{2}{ae}}{2} = \frac{1}{e} - \frac{1}{ae} = \frac{a-1}{ae}$$

Substituting this expression in the first equation of the system we can deduce $cos(t_0)$

$$\cos(t_0) = a(e - \cos E_0) = ae - a\frac{a-1}{ae} = \frac{ae^2 - a + 1}{e} = \frac{1 - a(1-e^2)}{e}$$

Finally, we are going to obtain the last expression as follows

$$\cos t_0 = \frac{1-a(1-e^2)}{e} \Longrightarrow t_0 = \cos^{-1}\left(\frac{1-a(1-e^2)}{e}\right) + 2m\pi, m \in \mathbb{Z}$$

$$\cos E_0 = \frac{a-1}{ae} \Longrightarrow E_0 = \cos^{-1}\left(\frac{a-1}{ae}\right) + 2k\pi, k \in \mathbb{Z}$$

Then we use Kepler's Equation

$$\begin{aligned}&t_0 = a^{\frac{3}{2}}(E_0 - e\sin(E_0)) \iff \\ &\iff a^{\frac{3}{2}}\left[\cos^{-1}\left(\frac{a-1}{ae}\right) + 2k\pi - \sqrt{e^2 - \left(\frac{a-1}{a}\right)^2}\right] - \cos^{-1}\left(\frac{1-a(1-e^2)}{e}\right) - 2m\pi = 0 \\ &\iff 2\pi(ka^{\frac{3}{2}} - m) + a^{\frac{3}{2}}\left[\cos^{-1}\left(\frac{a-1}{ae}\right) - \sqrt{e^2 - \left(\frac{a-1}{a}\right)^2}\right] - \cos^{-1}\left(\frac{1-a(1-e^2)}{e}\right) = 0\end{aligned}$$

□

First density result

Lemma 5.2. *For an energy $h \in (0, 3+2\sqrt{2})$, the torus of the problem $\mathcal{P}_0$ containing the arcs corresponding to symmetric orbits of double collision generate a submanifold dense on $\mathcal{D}_e$*

Proof. We know that the period of the point P_3 in an ellipse is $T = 2\pi a^{\frac{3}{2}}$ (2.18). As we said before, we are supposing that the relation $\frac{T}{2\pi}$ is not rational, which leads to say that $a^{\frac{3}{2}}$ is not rational. Then, the manifold generated by $(ka^{\frac{3}{2}} - m)_{k,m\in\mathbb{Z}}$ is dense in $\mathbb{R}$. We will introduce the following function

$$G(a,e) = a^{\frac{3}{2}}\left[\cos^{-1}\left(\frac{a-1}{ae}\right) - \sqrt{e^2 - \left(\frac{a-1}{a}\right)^2}\right] - \cos^{-1}\left(\frac{1-a(1-e^2)}{e}\right)$$

So we will have to prove that the torus of parameters (a,e) has a double collision orbit if and only if $G(a,e) = 2\pi(ka^{\frac{3}{2}} - m)$ is true for any $k, m \in \mathbb{Z}$.

We fix first an energy h. The following equation relates the energy h with the given parameters (a,e)

$$h = -\frac{1}{a} + 2\sqrt{a(1-e^2)}$$

which is the same relation as in (2.33). The problem becomes now to find parameters (a',e') in a neighborhood of (a,e), with the same energy h, such that $G(a',e') = 2\pi(ka'^{\frac{3}{2}} - m)$

With a fixed $\delta > 0$. We can find $k, m \in \mathbb{Z}$ such that $|G(a,e) - 2\pi(ka^{\frac{3}{2}} - m)| < \delta$ (this is a general result from Dirichlet's theorem). We can suppose that $|k|$ is arbitrary big, and define the following map:

$$\psi\colon (a',e') \mapsto (h(a',e'), G(a',e') - 2\pi k a'^{\frac{3}{2}})$$

This map is clearly a local diffeomorphism from a neighborhood $\mathcal{O}$ of (a,e) to a neighborhood $\mathcal{U}$ of $\psi(a,e)$, and the values δ and k can be chosen such that $(h, 2\pi m) \in \mathcal{U}$. By the inverse function theorem, the inverse map ψ^{-1} is well-defined, and $\psi^{-1}(h, 2\pi m) = (a',e')$ ends the proof (because we have found a neighborhood of (a,e) such that $h(a',e') = h$ and $G(a',e') - 2\pi k a'^{\frac{3}{2}} = 2\pi m$). □

Corollary 5.3. *For an energy $h \in (0, 3+2\sqrt{2})$, the fixed point of the problem $\mathcal{Q}_0$ has an infinite number of symmetric homoclinic orbits.*

Proof. Every double collision solutions from $\mathcal{P}_0$ is translated by the Levi-Civita map in 2 homoclinic solutions for the problem $\mathcal{Q}_0$. Then they must be infinite, all contained in the submanifold $\mathcal{L}_0$. □

5.2 Construction of second species solutions

Once we have proved the existence of infinite number of homoclinic orbits for the problem $\mathcal{P}_0$ (and in Q_0), we are ready to build up the second species solutions for the problem $\mathcal{P}_\mu$ from the limit solutions φ_s and φ_u.

5.2.1 Definitions and notations

- From now on, p will be the canonical projection of $T^*(\mathbb{C})$ to $\mathbb{C}$ (i.e, it will give us the position). We will preserve the notations and hypothesis from the previous section, in particular we will fix $h \in (0, 3+2\sqrt{2})$.

- For $\mu \geq 0$, we will denote by Φ_μ the flux of the system $\mathcal{P}_\mu$ and ψ_μ the correspondent one in the regularized system $\mathcal{Q}_\mu$ (see Section 5.1.1). For $u \in T^*(\mathbb{C})$ given, and for a subset Δ of $\mathbb{R}$, we denote by $\Phi_\mu(\Delta, u)$ the set of $\Phi_\mu(t, u)$ for $t \in \Delta$ where this expression is well-defined.

- Recall that the Hamiltonian of the problem $\mathcal{Q}_\mu$ is written as (see 3.39)

$$L_\mu(z,w) = |w|^2 - |z|^2 - 2\mu + i\frac{|z|^2}{h^{\frac{3}{2}}}(\bar{z}w - z\overline{w}) - \frac{(1-\mu)}{h^3}f(z,\bar{z},h) \qquad (5.5)$$

  with

$$f(z,\bar{z},h) = -|z|^6 + \frac{3}{4}|z|^2(z^2+\bar{z}^2)^2 + O_8(z)$$

- We will denote by S the symmetry in $\mathbb{C}\times\mathbb{C}$ defined by $S(M,N) = (\overline{M}, -\overline{N})$. For every μ, the Hamiltonian H_μ (in 3.26) is invariant by S. Since $S^*(\Omega) = -\Omega$, for every solution φ of $\mathcal{P}_\mu$, the map $\tilde{\varphi}$ defined by $\tilde{\varphi}(t) = S(\varphi(-t))$ is also a solution of $\mathcal{P}_\mu$. Let us prove it.

Proof. 1. H_μ is invariant by S

$$\begin{aligned}
H_\mu(S(M,N)) =& H_\mu(\overline{M},-\overline{N}) = |-\overline{N}|^2 + i\left(\overline{\overline{M}}(-\overline{N}) - \overline{M}\overline{(-\overline{N})}\right) - \frac{2\mu}{|\overline{\overline{M}}|} \\
&- (1-\mu)\left(\frac{2}{|1+\overline{\overline{M}}|} - 2 + (\overline{M}+\overline{\overline{M}})\right) \\
&= |N|^2 + i(M(-\overline{N}) - \overline{M}(-N)) - \frac{2\mu}{|M|} \\
&- (1-\mu)\left(\frac{2}{|1+M|} - 2 + (\overline{M}+M)\right) \\
&= |N|^2 + i(\overline{M}N - M\overline{N}) - \frac{2\mu}{|M|} \\
&- (1-\mu)\left(\frac{2}{|1+M|} - 2 + (\overline{M}+M)\right) = H_\mu(M,N)
\end{aligned}$$

2. $S^*(\Omega) = -\Omega$

   We recall the definition of the symplectic form Ω (3.27)

$$\Omega = dM \wedge d\overline{N} + d\overline{M} \wedge dN$$

   So

$$\begin{aligned}
S^*(\Omega) &= d\overline{M} \wedge (-d\overline{\overline{N}}) + d\overline{\overline{M}} \wedge (-d\overline{N}) \\
&= -d\overline{M} \wedge dN - dM \wedge d\overline{N} = -\Omega
\end{aligned}$$

□

- A solution φ of $\mathcal{P}_\mu$ will be symmetric if its orbit is invariant by S (i.e, $S(\varphi(t)) = \varphi(t)\ \forall t \in \Delta$).
- For $\mu \geq 0$, we will define

$$\begin{aligned}
\mathcal{I}_\mu &= \{(M,N) \in \mathcal{H}_\mu, Im(M) = Re(N) = 0\} \\
&= \{(M_1,0,0,N_2),\ H_\mu(M_1,0,0,N_2) = h\}.
\end{aligned} \tag{5.6}$$

  $\mathcal{I}_\mu$ will be a submanifold of dimension 1 of $\mathcal{H}_\mu = H_\mu^{-1}(h)$ (initially of dimension 4, but we impose the 2 conditions of $\mathcal{I}_\mu$ plus the one given by fixing the energy h), and invariant by S (due to this latter result, we can think of $\mathcal{I}_\mu$ as a symmetric axis in the configuration space, which will allow us to understand better the operations perform along the paper).
- One solution φ of $\mathcal{P}_\mu$ will be of *orthogonal crossing* when its orbit $\varphi(t)$ intersects with $\mathcal{I}_\mu$ in such a way that the trajectory $p(\varphi(t))$ is orthogonal to the correspondent axis $p \circ \mathcal{I}_\mu = OM_1$ at the point $p \circ (\varphi(t))|_{M_1}$.

5.2.2 Construction of second species solutions

All the construction is based on the following lemma:

Lemma 5.4. *Let ϕ be a solution of $\mathcal{P}_\mu$, for which there exists two instants t_1 and t_2 in Dom(ϕ) such that $\phi(t_1) \in \mathcal{I}_\mu$ and $\phi(t_2) \in \mathcal{I}_\mu$, with $t_1 \neq t_2$. Then ϕ is periodic, of period $T = 2(t_2 - t_1)$, and symmetric, because $S(\phi(T - t)) = \phi(t)$, $\forall t \in \mathbb{R}$.*

Proof.

- Suppose that $\phi(0), \phi(T) \in \mathcal{I}_\mu$. We are going to prove that $\phi(0) = \phi(2T)$.

  First of all, since ϕ is a solution of $\mathcal{P}_\mu$, then $\psi(t) = S(\phi(-t))$ is a solution of $\mathcal{P}_\mu$ too.

  Moreover, $\psi(0) = S(\phi(0)) = \phi(0)$, so for all t, we can ensure that $\psi(t) = \phi(t)$, which implies that $\phi(t) = S(\phi(-t))$.

  Finally, we use the fact that $\phi(T) \in \mathcal{I}_\mu$, which implies that $\phi(T) = S(\phi(T))$. Combining this expression with the one we just proved, we obtain

  $$\phi(-T) = S(\phi(-(-T))) = S(\phi(T))$$

  $$\phi(T) = S(\phi(T))$$

  so $\phi(T) = \phi(-T) \Longrightarrow \forall s,\ \phi(T + s) = \phi(-T + s)$, which leads to

  $$\phi(0) = \phi(2T)$$

  proving that ϕ is $2T$-periodic.

- If $\phi(t_1), \phi(t_2) \in S$, and we denote $\varphi(t) = \phi(t + t_1)$. So $\varphi(0) = \phi(t_1)$ and $\varphi(t_2 - t_1) = \phi(t_2) \in S$.

  From the previous result, we can ensure that $\varphi(t)$ is $2T$-periodic, with $T = t_2 - t_1 \Longrightarrow \phi(t) = \varphi(t - t_1)$ is $2T$-periodic.

□

Now we can describe the method to build up the second species solutions. To begin with, we were given two arcs of elliptic solutions φ_s and φ_u of energy h from the problem $\mathcal{P}_0$, that verifies $\mathrm{Dom}(\varphi_s) = (-t_c, t_c)$, and $\mathrm{Dom}(\varphi_u) = (t_c, t_c + l)$, with the trivial extension of the solutions, i.e., $p \circ \varphi_s(\pm t_c) = p \circ \varphi_u(t_c + l) = P_2$ (see Section 5.1).

One can immediately notice that φ_s and φ_u are symmetric, as they fulfill the conditions of Lemma 5.4, since $\varphi_s(0) \in \mathcal{I}_0$ and $\varphi_u(\tau) \in \mathcal{I}_0$, where $\tau = t_c + \frac{l}{2}$ (see Figure 5.3). We will denote by $A_s = \varphi_s(0)$, $A_u = \varphi_u(\tau)$, $n_s = p \circ A_s$, $n_u = p \circ A_u$ the orthogonal intersection points of the trajectories φ_s and φ_u with the axis OM_1.

Now consider $\mu > 0$ small. We denote by $A_s(\mu)$ and $A_u(\mu)$ the points in $\mathcal{I}_\mu$ such that $p \circ A_s(\mu) = n_s$ and $p \circ A_u(\mu) = n_u$, and their limits when $\mu \to 0$ are A_s and A_u respectively (this means that when we introduce a new mass μ, the coordinates of the position of $A_s(\mu)$ and $A_u(\mu)$ does not change, only the momentum N_s and N_u, in such a way that when $\mu \to 0$, $N_s \to A_{s|N}$ and $N_u \to A_{u|N}$). From the manifold (of dimension 1) $\mathcal{I}_\mu$ we can define then two neighborhoods $I_s(\mu)$, $I_u(\mu)$ of the points $A_s(\mu)$ and $A_u(\mu)$ in $\mathcal{I}_\mu$ which are diffeomorphic to intervals.

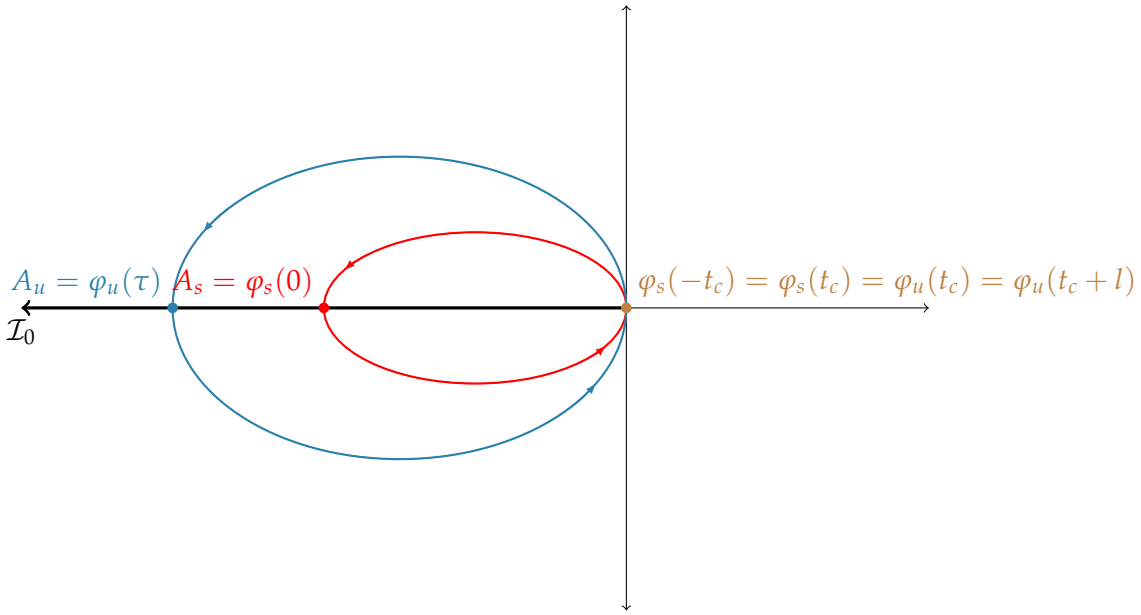


Figure 5.3: Periodic solutions φ_s and φ_u with symmetry at the axis $\mathcal{I}_0$.

For $\varepsilon > 0$ given (detailed later), we will define the following 2-dimensional submanifolds of $\mathcal{H}_\mu$.

$$V_s(\mu) = \Phi_\mu([0, t_c + \varepsilon), I_s(\mu)), \quad V_u(\mu) = \Phi_\mu((-\tau - \varepsilon, 0], I_u(\mu))$$

which are nothing more than the translation of the arcs $I_s(\mu)$ and $I_u(\mu)$ done by the flux Φ_μ. So, in order to prove that φ_s and φ_u are generatrices, it will be enough to prove that $V_s(\mu) \cap V_u(\mu) \neq \emptyset \ \forall \mu > 0$, as it would mean that we can find an orbit for every μ which will be periodic (by the construction we just made we will be able to ensure that the conditions of Lemma 5.4 will be accomplished by this orbit) and when $\mu \to 0$, it will become the union of φ_s and φ_u (see Section 5.1).

Our goal will be then to find the initial solutions φ_s and φ_u for which we can build the manifolds $V_s(\mu)$ and $V_u(\mu)$ in such a way they intersect in $\mathcal{H}_\mu$ for any $\mu \neq 0$.

5.2.3 Local analysis at the neighborhood of the origin

We will suppose given two elliptic solutions φ_s and φ_u for the problem $\mathcal{P}_0$ and all the notation described above. The first step is to diagonalize the linear part of the regularized Hamiltonian L_μ (3.37), in a suitable base, as it is easier to work with only the diagonal part for the computations.

Choice of the coordinate system

As before, the elliptic solutions are extended by continuity to the limits of their domains ($p \circ \varphi_s(\pm t_c) = p \circ \varphi_u(t_c) = p \circ \varphi_u(t_c + l) = P_2$). We will denote from now on $\varphi_s(t_c) = (0, N_s)$ and $\varphi_u(t_c) = (0, N_u)$ in the rotating coordinate system centered at P_2 (see Section 3.3). Then, we can write the following:

$$\begin{aligned} \varphi_s(t) &= (N_s(t - t_c) + O_2(t - t_c), N_s + O_1(t - t_c)) \\ \varphi_u(t) &= (N_u(t - t_c) + O_2(t - t_c), N_u + O_1(t - t_c)) \end{aligned} \tag{5.7}$$

Proof. We are going to see that $\phi_s(t)$ can be written as

$$\varphi_s(t) = (N_s(t - t_c) + O_2(t - t_c), N_s + O_1(t - t_c))$$

around $t = t_c$. The same computation can be done to prove the expression for $\phi_u(t)$.

By Taylor's expansion around t_c we have:

$$\varphi_s(t) = \varphi_s(t_c) + \dot{\varphi}_s(t_c)(t - t_c) + O_2(t - t_c)$$

By definition we know that $\phi_s(t_c) = (0, N_s)$. Moreover, we know that ϕ_s is the solution of H_0 in rotating coordinates around P_2 (3.28), so

$$\dot{\varphi}_s(t_c) = (N_s, -iN_s)$$

Then

$$\begin{aligned} \varphi_s(t) &= (0, N_s) + (N_s, -iN_s)(t - t_c) + O_2(t - t_c) \\ &= (N_s(t - t_c) + O_2(t - t_c), N_s + O_1(t - t_c)) \end{aligned}$$

□

The points $(0, N_s)$ and $(0, N_u)$ are in $\mathcal{H}_0$ (since by definition they correspond to points of φ_s and φ_u, which are solutions of the problem $\mathcal{P}_0$), so $|N_s|^2 = |N_u|^2 = h$ (this is a direct substitution of $H_0(0, N)$ in (3.28)). We will denote by r_s and r_u the determinations of the square roots of N_s and N_u (So $\sqrt{N_s^2} = \sqrt{h} r_s$ and $\sqrt{N_u^2} = \sqrt{h} r_u$). Now we will go through the regularization process by applying the Levi-Civita map (3.37), but before that we need to recall that every solution in $\mathcal{P}_\mu$ is translated into two solutions of the problem $\mathcal{Q}_\mu$, but we can deduce one from the other by the symmetry $z \mapsto -z$.

We will denote then by $\gamma_s = (z_s, \omega_s)$ and $\gamma_u = (z_u, \omega_u)$ the functions such that

$$\rho \circ \gamma_s = \varphi_s \quad \rho \circ \gamma_u = \varphi_u$$

By Taylor's expansion in a neighborhood of t_c, we can express these functions as follows

$$\begin{aligned} z_s(t) &= v_s \sqrt{t_c - t} + O_{\frac{3}{2}}(t_c - t) \\ \omega_s(t) &= -v_s \sqrt{t_c - t} + O_{\frac{3}{2}}(t_c - t) \end{aligned} \tag{5.8}$$

$$\begin{aligned} z_u(t) &= v_u \sqrt{t - t_c} + O_{\frac{3}{2}}(t - t_c) \\ \omega_u(t) &= v_u \sqrt{t - t_c} + O_{\frac{3}{2}}(t - t_c) \end{aligned} \tag{5.9}$$

where

$$v_s = i\sqrt{h} r_s \quad \text{and} \quad v_u = \sqrt{h} r_u \tag{5.10}$$

Before defining the appropriate base, let us state and prove the following result

Lemma 5.5. *If (e_1, e_2, f_1, f_2) are complex numbers verifying*

$$(e_1|f_1) = (e_2|f_2) \neq 0 \qquad (e_1|f_2) = (e_2|f_1) \neq 0 \tag{5.11}$$

where $(z_1|z_2) = Re(z_1\overline{z_2})$, and such that the couple (e_1, f_1) is a $\mathbb{R}$-base of $\mathbb{C}$, the vectors

$$\begin{aligned} S_1 = (e_1, -e_1), \quad S_2 = (e_2, -e_2) \\ U_1 = (f_1, f_1), \quad U_2 = (f_2, f_2) \end{aligned} \tag{5.12}$$

generate a $\mathbb{R}$-base of $\mathbb{C}^2$, symplectic with factor $2(e_1|f_1)$.

Proof. We are going to see that (S_1, S_2, U_1, U_2) is a $\mathbb{C}^2$ symplectic base with factor $2(e_1|f_1)$.

First of all, we introduce the following notation

$$e_1 = a + bi = (a, b)$$

$$e_2 = c + di = (c, d)$$
$$f_1 = e + fi = (e, f)$$
$$f_2 = g + hi = (g, h)$$

now we write the base matrix $M = \begin{pmatrix} S_1 & S_2 & U_1 & U_2 \end{pmatrix}$

$$M = \begin{pmatrix} a & c & e & g \\ b & d & f & h \\ -a & -c & e & g \\ -b & -d & f & h \end{pmatrix}$$

We have to prove that $M^T J M = \alpha J$, $\alpha = 2(e_1|f_1)$ (see Section 7.1.1)

$$JM = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & c & e & g \\ b & d & f & h \\ -a & -c & e & g \\ -b & -d & f & h \end{pmatrix} = \begin{pmatrix} -a & -c & e & g \\ -b & -d & f & h \\ -a & -c & -e & -g \\ -b & -d & -f & -h \end{pmatrix}$$

$$M^T JM = \begin{pmatrix} a & b & -a & -b \\ c & d & -c & -d \\ e & f & e & f \\ g & h & g & h \end{pmatrix} \begin{pmatrix} -a & -c & e & g \\ -b & -d & f & h \\ -a & -c & -e & -g \\ -b & -d & -f & -h \end{pmatrix} = \begin{pmatrix} 0 & 0 & A & B \\ 0 & 0 & C & D \\ -A & -C & 0 & 0 \\ -B & -D & 0 & 0 \end{pmatrix}$$

where $A = 2ae + 2bf$, $B = 2ag + 2bh$, $C = 2ce + 2df$ and $D = 2cg + 2dh$.

Moreover we know

$$(e_1|f_1) = ae + bf$$
$$(e_2|f_2) = cg + dh$$
$$(e_1|f_2) = ag + bh$$
$$(e_2|f_1) = ce + df$$

and by hypothesis we suppose that $(e_1|f_1) = (e_2|f_2) \neq 0$ and $(e_1|f_2) = (e_2|f_1) = 0$.

Substituting we obtain

$$M^T JM = \begin{pmatrix} 0 & 0 & 2(e_1|f_1) & 0 \\ 0 & 0 & 0 & 2(e_2|f_2) \\ -2(e_1|f_1) & 0 & 0 & 0 \\ 0 & -2(e_2|f_2) & 0 & 0 \end{pmatrix} = 2(e_1|f_1)J = \alpha J$$

□

So, if $(\tilde{s}, \tilde{u}) = (\tilde{s}_1, \tilde{s}_2, \tilde{u}_1, \tilde{u}_2)$ are the coordinates of $\mathbb{C}^2$ defined in this base, we can write the Hamiltonian L_μ in this base. The result, denote by l_μ, is the following expression

$$l_\mu(\tilde{s}, \tilde{u}) = 2(\tilde{s}_1\tilde{u}_1 + \tilde{s}_2\tilde{u}_2) + \frac{\mu}{\alpha} + \left(\sum_{1 \leq i,j \leq 2} \tilde{s}_i\tilde{u}_j \right) O_2(\tilde{s}, \tilde{u}) + O_6(\tilde{s}, \tilde{u}) \tag{5.13}$$

(with $\tilde{s} = \tilde{s}_1 e_1 + \tilde{s}_2 e_2$ and $\tilde{u} = \tilde{u}_1 f_1 + \tilde{u}_2 f_2$). From now on, we will choose

$$e_1 = v_s,\ f_1 = v_u,\ e_2 = iv_u,\ f_2 = iv_s \tag{5.14}$$

and we will suppose that condition (5.11) is verified, so v_s and v_u must be chosen such that they are not orthogonal (i.e. N_s and N_u must be non collinear).

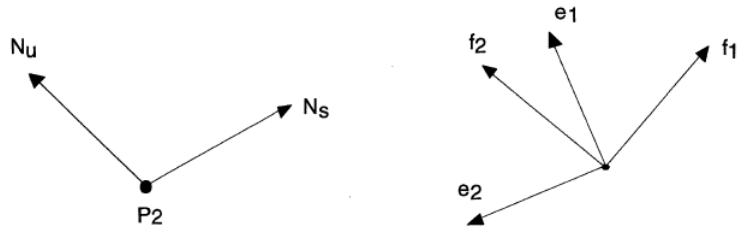


Figure 5.4: Example of admissible velocities [2].

Once we have defined this new base, we can introduce a new change of coordinates $(s, u) = (\tilde{s} + O_5(\tilde{s}, \tilde{u}), \tilde{u} + O_5(\tilde{s}, \tilde{u}))$ in a neighborhood of the origin [10]. The Hamiltonian L_μ in this new coordinates is written by

$$l_\mu(s, u) = 2(s_1u_1 + s_2u_2) + \frac{\mu}{\alpha} \tag{5.15}$$

with $s = s_1e_1 + s_2e_2$ and $u = u_1f_1 + u_2f_2$.

This Hamiltonian $l_\mu(s, u)$ corresponds to the linear part of the Hamiltonian defined in (5.13), in a neighborhood of the origin. The local stable subspace E^s and the local unstable subspace E^u of the Hamiltonian (5.15) are defined as follows

$$E^s_{\text{loc}}(0) = \{(s, u),\ s = 0\}$$
$$E^u_{\text{loc}}(0) = \{(s, u),\ u = 0\}$$

By the Stable Manifold Theorem, we can ensure that there exists stable and unstable submanifolds, denoted by $W^s_{\text{loc}}(0)$ and $W^u_{\text{loc}}(0)$ respectively, described as:

$$W^s_{\text{loc}}(0) = \{(s, u),\ s = 0\}$$

$$W^u_{\text{loc}}(0) = \{(s,u),\ u = 0\}$$

Moreover, we can ensure by Hartman-Grobman Theorem that there exists a continuous conjugation that linearized the vectorial field, in such a way that we can consider the solutions φ_s and φ_u to be taken by the axis Os_1 and Ou_1 respectively.

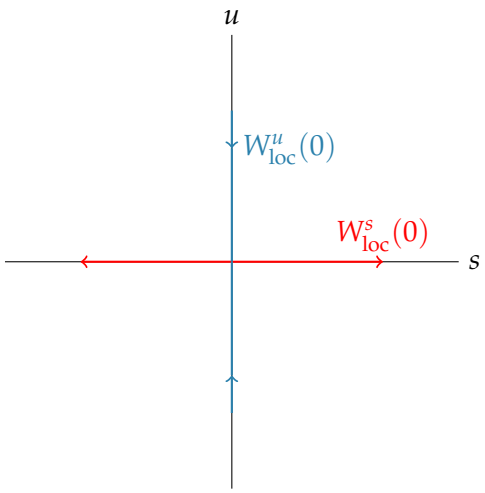


Figure 5.5: Representation of the stable and unstable manifold in coordinates (s,u) in a neighborhood of the origin.

In this way we have been able to define a base for which we obtain a linear regularized Hamiltonian l_μ in a neighborhood of the origin.

Isolated Blocks

We will use the notation in [11]. We define the norm $||(s,u)|| = Sup(|s|,|u|)$, and we denote as $\mathcal{B}_\varepsilon = \mathcal{B}(\varepsilon h^{\frac{3}{4}})$ the ball centered at 0 of radius $\varepsilon h^{\frac{3}{4}}$ in $\mathbb{C}^2$. Its boundary is the intersection of the manifolds $C_s(\varepsilon)$ and $C_u(\varepsilon)$ defined by

$$C_s(\varepsilon) = \{|s| = \varepsilon h^{\frac{3}{4}}; |u| \leq \varepsilon h^{\frac{3}{4}}\} \qquad C_u(\varepsilon) = \{|s| \leq \varepsilon h^{\frac{3}{4}}; |u| = \varepsilon h^{\frac{3}{4}}\}$$

One can verify easily that for ε small enough, $\mathcal{B}(\varepsilon h^{\frac{3}{4}})$ is an isolated block (it comes directly from the definition. For more details, see [11]) for the linearized field X_{L_μ} defined from the Hamiltonian L_μ. Then it is an isolated block from the general field X_{L_μ} too. The entering manifold would be $C_s(\varepsilon)$ (it is the manifold that contains the points of $\partial\mathcal{B}_\varepsilon$ such that there exists a $\delta > 0$ so $\psi_\mu((-\delta,0), C_s(\varepsilon)) \cap \mathcal{B}_\varepsilon \neq \emptyset$), and the exit manifold would be $C_u(\varepsilon)$ (it is the manifold that contains the points of $\partial\mathcal{B}_\varepsilon$ such that there exists a $\delta > 0$ so $\psi_\mu((0,\delta), C_u(\varepsilon)) \cap \mathcal{B}_\varepsilon \neq \emptyset$). It is easier for later on to understand $C_s(\varepsilon)$ and $C_u(\varepsilon)$ as Poincaré section.

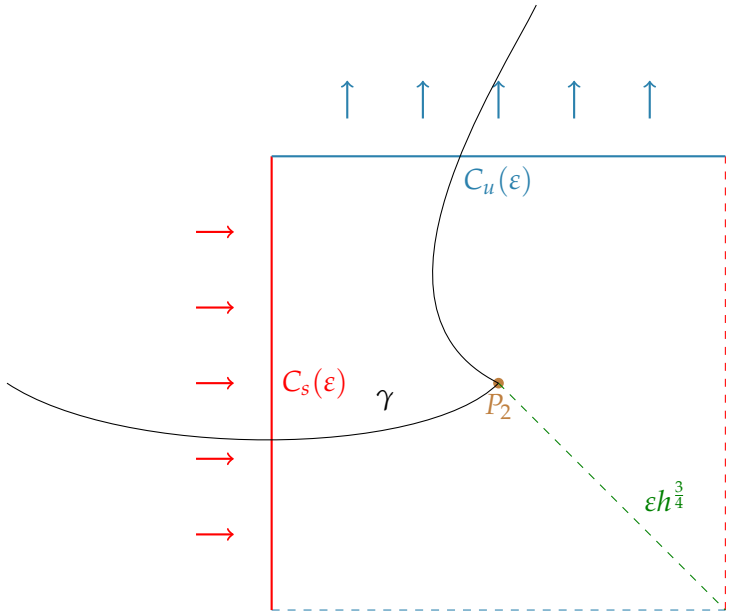


Figure 5.6: Isolated blocks for the ball $\mathcal{B}_\varepsilon$, with the collision trajectory γ.

Choice of coordinates in $C_s(\varepsilon)$ and $C_u(\varepsilon)$

We define the coordinates (x_s, y_s, θ_s) of a point (s, u) in $C_s(\varepsilon)$ by the equalities

$$u = x_s s + y_s(is), \qquad s = \varepsilon h^{\frac{3}{4}} e^{i\theta_s} \tag{5.16}$$

In the same way, we define the coordinates (x_u, y_u, θ_u) of a point (s, u) in $C_u(\varepsilon)$ by the following

$$s = x_u u + y_u(iu), \qquad u = \varepsilon h^{\frac{3}{4}} e^{i\theta_u} \tag{5.17}$$

The explicit expressions of the coordinates (x, y) are given by:

$$\begin{aligned} x_s &= \frac{1}{\varepsilon^2 h^{\frac{3}{2}}}(s_1 u_1 + s_2 u_2) \\ y_s &= \frac{1}{\varepsilon^2 h^{\frac{3}{2}}}(s_1 u_2 - s_2 u_1) \\ x_u &= \frac{1}{\varepsilon^2 h^{\frac{3}{2}}}(s_1 u_1 + s_2 u_2) \\ y_u &= \frac{1}{\varepsilon^2 h^{\frac{3}{2}}}(s_2 u_1 + s_1 u_2) \end{aligned} \tag{5.18}$$

Proof. We recall the definition of u and s

$$\begin{cases} u = x_s s + y_s(is) \\ s = \varepsilon h^{\frac{3}{4}} e^{i\theta_s} \end{cases}$$

So we have

$$u = u_1 + iu_2 = x_s(s_1 + is_2) + y_s(is_1 - s_2)$$

Then

$$u_1 = s_1 x_s - y_s s_2 \Longrightarrow x_s = \frac{u_1 + y_s s_2}{s_1}$$

On the other hand

$$u_2 = s_2 x_s + y_s s_1$$

Substituting x_s from the first expression we got

$$\begin{aligned} u_2 = s_2 \frac{u_1 + y_s s_2}{s_1} + y_s s_1 \Longrightarrow u_2 s_1 = s_2 u_1 + y_s s_2^2 + y_s s_1^2 \\ \Longrightarrow u_2 s_1 - u_1 s_2 = y_s |s|^2 \Longrightarrow y_s = \frac{1}{|s|^2}(u_2 s_1 - u_1 s_2) \\ \underset{|s|^2 = \varepsilon^2 h^{\frac{3}{2}}}{=} \frac{1}{\varepsilon^2 h^{\frac{3}{2}}}(u_2 s_1 - u_1 s_2) \end{aligned}$$

Finally, substituting this latter expression in x_s we have

$$\begin{aligned} x_s = \frac{u_1 + \frac{u_2 s_1 - s_2 u_1}{|s|^2} s_2}{s_1} = \frac{u_1(s_1^2 + s_2^2) + (u_2 s_1 - s_2 u_1) s_2}{s_1 |s|^2} \\ = \frac{u_1 s_1 + u_2 s_2}{|s|^2} = \frac{1}{\varepsilon^2 h^{\frac{3}{2}}}(u_1 s_1 + u_2 s_2) \end{aligned}$$

The computation is similar for x_u, y_u. □

Transition map from $C_s(\varepsilon)$ to $C_u(\varepsilon)$

In order to define this transition map, which will be crucial in the next section, we just focus on the linearized part at the neighborhood of the fixed point $(s, u) = (0, 0)$, which will be associated only on the quadratic part of the Hamiltonian l_μ (5.15) and then independent of μ. It has as expression:

$$\frac{ds}{dt} = -\frac{dl_\mu}{du} = -s, \qquad \frac{du}{dt} = \frac{dl_\mu}{ds} = u$$

It has by flux the expression $\phi(t, (s_0, u_0)) = (e^{-t} s_0, e^t u_0)$. Then, we can define the transition map f_ε associated to this system as follows

$$f_\varepsilon(s, u) = \left(\frac{|u|}{\varepsilon h^{\frac{3}{4}}} s, \frac{\varepsilon h^{\frac{3}{4}}}{|u|} u \right) \tag{5.19}$$

The coordinates (x_u, y_u, θ_u) of $f_\varepsilon(s,u) \in C_u$ are written in terms of the coordinates (x_s, y_s, θ_s) of $(s,u) \in C_s$ as

$$x_u = x_s, \quad y_u = -y_s, \quad e^{i\theta_u} = \frac{u}{|u|} \tag{5.20}$$

and the latest equality is expressed as

$$\begin{cases} \cos(\theta_u) = \frac{x_s \cos\theta_s - y_s \sin\theta_s}{\sqrt{x_s^2+y_s^2}} \\ \sin(\theta_u) = \frac{x_s \sin\theta_s + y_s \cos\theta_s}{\sqrt{x_s^2+y_s^2}} \end{cases} \tag{5.21}$$

Proof.

- $f_\varepsilon(s,u) \in C_u(\varepsilon)$

$$\left| \frac{|u|}{\varepsilon h^{\frac{3}{4}}} s \right| = \frac{|u|}{\varepsilon h^{\frac{3}{4}}} |s| \underset{(s,u)\in C_s(\varepsilon)}{=} |u| \leq \varepsilon h^{\frac{3}{4}}$$

$$\left| \frac{\varepsilon h^{\frac{3}{4}}}{|u|} u \right| = \varepsilon h^{\frac{3}{4}}$$

- We are going to prove the last equality for θ_u. We know that in $C_u(\varepsilon)$, $u = \varepsilon h^{\frac{3}{4}} e^{i\theta_u} \Longrightarrow e^{i\theta_u} = \frac{u}{\varepsilon h^{\frac{3}{4}}} = \frac{u}{|u|}$

$$\begin{aligned} e^{i\theta_u} = \frac{u}{|u|} &\underset{in\ C_s(\varepsilon)}{=} \frac{x_s s + y_s (is)}{|s|\sqrt{x_s^2+y_s^2}} \underset{|s|=\varepsilon h^{\frac{3}{4}}}{=} \frac{x_s e^{i\theta_s} + y_s (i e^{i\theta_s})}{\sqrt{x_s^2+y_s^2}} \\ &= \frac{x_s \cos\theta_s + i x_s \sin\theta_s + i y_s \cos\theta_s - y_s \sin\theta_s}{\sqrt{x_s^2+y_s^2}} \end{aligned}$$

So

$$e^{i\theta_u} = \cos(\theta_u) + i\sin(\theta_u) \Longrightarrow \begin{cases} \cos(\theta_u) = \frac{x_s \cos\theta_s - y_s \sin\theta_s}{\sqrt{x_s^2+y_s^2}} \\ \sin(\theta_u) = \frac{x_s \sin\theta_s + y_s \cos\theta_s}{\sqrt{x_s^2+y_s^2}} \end{cases}$$

□

Once we have defined this transition map (at least in the linearized case), we can use the C^1 approximation Theorem (see [10] for more information), to ensure that the transition map f_ε approaches to the exact transition map $f_{\varepsilon\mu}$ in topology C^1 by order of ε^2.

5.3 Conditions of existence of generatrix solutions

Now that we have set up all the concepts, we are prepared to give, and explain, all the conditions needed to ensure the existence of second species solutions. We will preserve all the notation introduced until now, in particular we want to recall that $\mathrm{Dom}\varphi_s = (-t_c, t_c)$ and $\mathrm{Dom}\varphi_u = (t_c, t_c + l)$.

We will denote by a_s and a_u the semi-major axis of these two elliptic solutions φ_s and φ_u respectively. It is easy to see that, for $\mu \geq 0$, the semi-major axis can be chosen as a parameter in $\mathcal{I}_\mu$ in a neighborhood of $\varphi_s(0)$ and $\varphi_u(0)$. We will denote $i_s^\mu \colon U_s \to \mathcal{I}_\mu$ and $i_u^\mu \to \mathcal{I}_\mu$ to these analytic parametrizations, where U_s and U_u are the neighborhoods of a_s and a_u in $\mathbb{R}$, in such a way that $i_s^0(a_s) = \varphi_s(0)$, and $i_u^0(a_u) = \varphi_u(0)$.

Apart from that, we will denote by $\tilde{\Phi}_\mu$ the regularization (by the Levi-Civita map ρ) of Φ_μ in $\mathcal{L}_\mu$, compatible with γ_s defined in Section 5.2.3, whose difference with ψ_μ in $\mathcal{Q}_\mu$ is just the time parameter (i.e., $\tilde{\Phi}_\mu$ is just the result of applying the map ρ to Φ_μ, while ψ_μ includes also the time change to completely regularize it.

To determine the conditions to ensure the existence of generatrix solutions we will enunciate, and prove, the following lemmas.

Lemma 5.6. *For ε and μ small enough, there exists a real $\delta_s > 0$ such that the relation $\tilde{\Phi}_\mu(t, i_s^\mu(a)) \in C_s(\varepsilon)$ defines an implicit function $\tau_s^{\mu\varepsilon}(a)$, defined in the interval $I_{\delta_s} = (a_s - \delta_s, a_s + \delta_s)$, whose image is a neighborhood of t_c. The map $q_s^{\mu\varepsilon}$ defined by $q_s^{\mu\varepsilon}(a) = \tilde{\Phi}_\mu(\tau_s^{\mu\varepsilon}(a), i_s^\mu(a)$ is an analytic diffeomorphism of I_{δ_s} under its image in $C_s(\varepsilon)$. We have the same result for the section $C_u(\varepsilon)$, where we can define in a similar way δ_u, I_{δ_u} and $q_u^{\mu\varepsilon}$.*

The implicit function $\tau_s^{\mu\varepsilon}$ is the one that determines the time that the flux needs to go from the corresponding point in $\mathcal{I}_\mu$ (which would be $i_s^\mu(a)$) to the Poincaré section $C_s(\varepsilon)$ (in fact, one can use the Implicit Function Theorem because the derivative is non-zero, by the property of transversality given by the definition of Poincaré section).

As we saw in Section 5.2.2), the manifold that describes the flux of this submanifold $\mathcal{I}_\mu$ was $V_s(\mu)$ (in the non regularized case). So, we can say that this intersection point of the flux $\tilde{\varphi}_s$ with $C_s(\varepsilon)$ is equivalent to the intersection of the regularized manifold of $V_s(\mu)$ by the Levi-Civita map (that we will denote by $W_s(\mu)$) with the boundary of the isolated block $\partial\mathcal{B}_\varepsilon(0)$. From now on, the parameter ε is fixed by context, and the functions defined in Lemma 5.6 will be simply $\tau_s^\mu, q_s^\mu, q_u^\mu$.

The first step is the computation of the expansions, at first order, of the functions q_s^0 and q_u^0. In order to do that, we will be focusing on the non-regularized

system (as it is easier to compute). For $t < t_c$, we denote by $h_s(t,a) = \Phi_0(t, i_s^0(a))$, with $a \in I_{\delta_s}$, and for $t > t_c$ and $a \in I_{\delta_u}$, we denote by $h_u(t,a) = \Phi_0(t, i_u^0(a))$.

Lemma 5.7. *The Taylor's expansion at first order at the point (t_c, a_s) of the function h_s is written as*

$$\begin{aligned} M(t,a) &= N_s(t-t_c) + \Delta M_s(a-a_s) + O_2((a-a_s),(t_c-t)) \\ N(t,a) &= N_s + \Delta N_s(a-a_s) + O_2((a-a_s),(t_c-t)) \end{aligned} \tag{5.22}$$

with

$$\begin{aligned} \Delta M_s &= \frac{1}{a_s} + (N_s+i)\left(\frac{1-a_s\sqrt{\sigma_s}}{2a_s^2 e_s}\sin t_c - \frac{3t_c}{2a_s}\right) + \frac{a_s\sigma_s^2-\sigma_s}{2a_s^3 e_s}\left(a_s - i\frac{2e_s\sin t_c}{1-e_s^2}\right)e^{-it_c} \\ \Delta N_s &= -\frac{N_s+i}{2a_s} - \left(\frac{1-a_s\sigma_s}{2a_s^2 e_s}\sin t_c - \frac{3t_c}{2a_s}\right) + i(a_s e_s - \cos t_c)\frac{a_s\sigma_s - 1}{2a_s^3 e_s} \end{aligned} \tag{5.23}$$

where $\sigma_s = \sqrt{a_s(1-e_s^2)}$. The analog relations for h_u are obtained directly replacing the indices s by u.

Proof. We will start working in the fixed coordinate system. In these coordinates, the point $h(t,a)$ is given, as a function depending on the eccentric anomaly, by

$$\begin{aligned} X(t,a) &= a(e+\cos E) + ia(1-e^2)^{\frac{1}{2}}\sin E \\ Y(t,a) &= -\frac{\sin E}{\sqrt{a}(1+e\cos E)} + i\frac{(1-e^2)^{\frac{1}{2}}\cos E}{\sqrt{a}(1+e\cos E)} \end{aligned}$$

with the relation

$$t = a^{\frac{3}{2}}(E + e\sin E)$$

The expression for the position X is a similar expression as the one we have when computing the ellipses with double collision (see Section 5.1.2). In fact, the difference between the two of them resides in the fact that in this case the center is P_2, not P_1. From this expression, one can obtain $Y(t,a)$ as follows:

$$\begin{aligned} Y(t,a) = \frac{dX}{dt} &\underset{*}{=} \frac{1}{a^{\frac{3}{2}}(1+e\cos E)}\frac{dX}{dE} = \frac{1}{a^{\frac{3}{2}}(1+e\cos E)}\left[-a\sin E + ia(1-e^2)^{\frac{1}{2}}\cos E\right] \\ &= -\frac{\sin E}{\sqrt{a}(1+e\cos E)} + i\frac{(1-e^2)^{\frac{1}{2}}\cos E}{\sqrt{a}(1+e\cos E)} \end{aligned}$$

$^*\ t \underset{\text{Kepler Equation}}{=} a^{\frac{3}{2}}(E+e\sin E) \Longrightarrow dt = a^{\frac{3}{2}}(1+e\cos E)dE \Longrightarrow \frac{d}{dt} = \frac{1}{a^{\frac{3}{2}}(1+e\cos E)}\frac{d}{dE}$

The variables are related by the equalities $X(t_c,a_s) = e^{it_c}$, $Y(t_c,a_s) = (N_s+i)e^{it_c}$. It

is just a matter of substituting the point $P_2 = (0,0)$ at the rotating coordinates to the fixed ones and the velocity at the collision in rotating coordinates, which we know is equal to N_s, to the fixed coordinates too (see Section 3.3). Then, we can deduce directly the computation of the computation of the first derivatives, taking into account the relations

$$\frac{de}{da}(t_c, a_s) = \frac{a_s^2(1-e_s^2) - (a_s(1-e_s^2))^{\frac{1}{2}}}{2a_s^3 e_s}$$
$$\frac{\partial E}{\partial a}(t_c, a_s) = \frac{1 - (a_s(1-e_s^2))^{\frac{1}{2}} a_s}{2a_s^{\frac{5}{2}} e_s} \sin t_c - \frac{3t_c}{2a_s^{\frac{3}{2}}}$$

Relations given by the fact that we are fixing an energy h for the system [4]. Finally, one can obtain the results given in the Lemma by performing the change from the coordinates (X, Y) to the rotating ones (M, N) given by $M = Xe^{-it} - 1$, $N = Ye^{-it} - i$. □

Lemma 5.8. *The approximations of the functions q_s^0 and q_u^0 in a neighborhood of a_s and a_u are given, relatively to the coordinates (x_s, y_s, θ_s) and (x_u, y_u, θ_u) in $C_s(\varepsilon)$ and $C_u(\varepsilon)$, by*

$$q_s^0(a_s) = (0,0,0) \quad q_u^0(a_u) = (0,0,0)$$

and

$$\frac{dx_s}{da}(a_s) = O(1)$$
$$\frac{dy_s}{da}(a_s) = -\frac{Im(\Delta M_s \overline{N}_s)}{2h^{\frac{5}{2}}\varepsilon^2} + O(1)$$
$$\frac{d\theta_s}{da}(a_s) = \frac{1}{\varepsilon}\frac{ds_2}{da}(a_s) + O(\varepsilon)$$
$$\frac{dx_u}{da}(a_u) = O(1)$$
$$\frac{dy_u}{da}(a_u) = -\frac{Im(\Delta M_u \overline{N}_u)}{2h^{\frac{5}{2}}\varepsilon^2} + O(1)$$
$$\frac{d\theta_u}{da}(a_u) = \frac{1}{\varepsilon}\frac{du_2}{da}(a_u) + O(\varepsilon)$$

Proof.

1. We denote by $\tau_s(\varepsilon) = \tau_s^0(a_s)$ and $\tau_u(\varepsilon) = \tau_u^0(a_u)$ the functions defined in Lemma 5.6 (this would be referring to the time that the elliptic solutions take to go from the point $\varphi_s(0)$ and $\varphi_u(0)$ to the corresponding point in

$C_s(\varepsilon)$ and $C_u(\varepsilon)$, denoted by $q_s(a_s)$ and $q_u(a_u)$). So $q_s(a_s) = \varphi_s(\tau_s(\varepsilon))$ and $q_u = \varphi_u(\tau_u(\varepsilon))$.

The choice of coordinates (s, u) in section 5.1 in this two points leads to

$$q_s(a_s) = (\varepsilon h^{\frac{3}{4}}, 0, 0, 0), \quad q_u(a_u) = (0, 0, \varepsilon h^{\frac{3}{4}}, 0)$$

deduced from the fact that $q_s(a_s) \in \varphi_s$, which belongs to the axis Os_1 (see Section 5.1) and $q_s(a_s) \in C_s(\varepsilon)$. A similar reasoning works for $q_u(a_u)$.

As we said before, one can think of $C_s(\varepsilon)$ and $C_u(\varepsilon)$ as Poincaré sections. In this way, it is clear that the field must go through $\partial\mathcal{B}_\varepsilon$ transversally (it is a consequence of being a Poincaré section), so the functions $\tau_s(\varepsilon)$ and $\tau_u(\varepsilon)$ must be analytic. The developments (5.10) and the choice of the base (5.12) and (5.14) implies that

$$\varphi_s(t) = (\sqrt{t_c - t} + O_{\frac{3}{2}}(t_c - t), 0, 0, 0)$$

$$\varphi_u(t) = (0, 0, \sqrt{t - t_c} + O_{\frac{3}{2}}(t - t_c), 0)$$

in coordinates (s, u). Since by definition $||\varphi_s(\tau_s)|| = ||\varphi_u(\tau_u)|| = \varepsilon h^{\frac{3}{4}}$ we have that

$$\sqrt{t_c - \tau_s} + O_{\frac{3}{2}}(t_c - \tau_s) = \varepsilon h^{\frac{3}{4}} \Longrightarrow t_c - \tau_s \approx \varepsilon^2 + O_4(\varepsilon)$$

$$\sqrt{\tau_u - t_c} + O_{\frac{3}{2}}(\tau_u - t_c) = \varepsilon h^{\frac{3}{4}} \Longrightarrow \tau_u - t_c \approx \varepsilon^2 + O_4(\varepsilon)$$

2. In order to know the approximations of the functions q_s and q_u, we have to use the functions h_s and h_u defined in Lemma 5.7, and go through all the changes of variables needed to obtain the expressions in coordinates (x_s, y_s, θ_s) and (x_u, y_u, θ_u) (see Section 5.2.3). To begin with, we will evaluate the derivatives of the function h_s under the parameter a

$$\frac{dh_s}{da}(\tau_s, a_s) = \frac{\partial h_s}{\partial a}(\tau_s, a_s) + \frac{\partial h_s}{\partial t}(\tau_s, a_s)\frac{\partial \tau_s}{\partial a}(a_s)$$

It is enough to know the estimations of the partial derivatives of h_s in Levi-

Civita coordinates, i.e.

$$\begin{aligned}
\frac{\partial z}{\partial a}(t,a_s) &= -\frac{1}{2\sqrt{t_c-t}}v_s + O_{\frac{1}{2}}(t_c-t)\\
\frac{\partial \omega}{\partial t}(t,a_s) &= \frac{1}{2\sqrt{t_c-t}}v_s + O_{\frac{1}{2}}(t_c-t)\\
\frac{\partial z}{\partial a}(t,a_s) &= -\frac{\Delta M_s\overline{N_s}}{2h\sqrt{t_c-t}}v_s + O_{\frac{1}{2}}(t_c-t)\\
\frac{\partial \omega}{\partial a}(t,a_s) &= \frac{\overline{\Delta M_s}N_s}{2h\sqrt{t_c-t}}v_s + O_{\frac{1}{2}}(t_c-t)
\end{aligned}$$

where v_s is defined in (5.10). The two first equations are a result of performing the derivative with respect to t of equation (5.8). On the other hand, the last two equations come from a different computation that involves the result of Lemma 5.7.

$$\begin{aligned}
h_s(t,a) \underset{\textit{Lemma 5.7}}{=} &(N_s(t-t_c) + \Delta M_s(a-a_s) + O_2((a-a_s),(t-t_c)),\\
&N_s + \Delta N_s(a-a_s) + O_2((a-a_s),(t-t_c)))
\end{aligned}$$

We know that $h_s(t,a) = \rho(z,\omega) = \left(\frac{z^2}{h}, \frac{\sqrt{h}\omega}{\bar{z}}\right)$, so

- $\frac{z_s^2}{h} = N_s(t-t_c) + \Delta M_s(a-a_s) + O_2((a-a_s),(t-t_c))$

  $$z_s^2 = hNs(t-t_c) + h\Delta M_s(a-a_s) + hO_2((a-a_s),(t-t_c))$$

  Now we perform the derivatives with respect to a and evaluate at (t,a_s)

  $$\begin{aligned}
  2z_s\frac{\partial z_s}{\partial a}(t,a_s) = h\Delta M_s + hO_2(t-t_c) \Longrightarrow \frac{\partial z_s}{\partial a} &= \frac{h\Delta M_s + hO_2(t-t_c)}{2z_s}\\
  &= \frac{h\Delta M_s + O_2(t-t_c)}{2(v_s\sqrt{t_c-t} + O_{\frac{3}{2}}(t-t_c))} \underset{*}{=} -\frac{\Delta M_s\overline{N}_s}{2h\sqrt{t_c-t}}v_s + O_{\frac{1}{2}}(t-t_c)
  \end{aligned}$$

  * Computation of $\frac{h\Delta M_s}{2v_s\sqrt{t_c-t}}$

  $$\begin{aligned}
  \frac{h\Delta M_s}{2v_s\sqrt{t_c-t}} &\underset{|N_s|^2=h}{=} \frac{\Delta M_s\overline{N}_sN_s}{2i\sqrt{h}r_s\sqrt{t_c-t}} \underset{r_s=\sqrt{N_s}}{=} \Delta M_s\overline{N_s}\left(-\frac{1}{2}i\frac{\sqrt{h}}{h}\sqrt{N_s}\frac{1}{\sqrt{t_c-t}}\right)\\
  &= -\frac{\Delta M_s\overline{N}_s}{2h\sqrt{t_c-t}}i\sqrt{h}r_s = -\frac{\Delta M_s\overline{N}_s}{2h\sqrt{t_c-t}}v_s
  \end{aligned}$$

- $\frac{\sqrt{h}\omega_s}{\bar{z}_s} = N_s + \Delta N_s(a-a_s) + O_2((t-t_c),(a-a_s))$

We perform the computations

$$\begin{aligned}\omega_s &= \frac{Ns\overline{z}_s}{\sqrt{h}} + \frac{\Delta N_s \overline{z}_s (a-a_s)}{\sqrt{h}} + O_2((t-t_c),(a-a_s)) \\ &= \overline{z}_s + O(a-a_s)^2 + O_2((t-t_c),(a-a_s))\end{aligned}$$

Now we perform the derivatives with respect to a and evaluate at (t, a_s)

$$\begin{aligned}\frac{\partial \omega_s}{\partial a}(t,a_s) = \frac{\partial \overline{z}}{\partial a}(t,a_s) + O_2(t-t_c) &\Longrightarrow \frac{\partial \omega_s}{\partial a}(t,a_s) = \frac{\partial \overline{z}}{\partial a}(t,a_s) \\ &= -\left(-\frac{\overline{\Delta M_s N_s}}{2h\sqrt{t_c-t}} v_s\right) + O_{\frac{1}{2}}(t-t_c) \\ &= \frac{\overline{\Delta M_s} N_s}{2h\sqrt{t_c-t}} v_s + O_{\frac{1}{2}}(t_c-t)\end{aligned}$$

Rearranging these two latter expressions, one can obtain

$$\begin{aligned}\frac{\partial z}{\partial a}(t,a_s) &= -\frac{1}{2h\sqrt{t_c-t}}[Re(\Delta M_s \overline{N_s})v_s + Im(\Delta M_s \overline{N_s})(iv_s)] + O_{\frac{1}{2}}(t_c-t) \\ \frac{\partial \omega}{\partial a}(t,a_s) &= \frac{-1}{2h\sqrt{t_c-t}}[Re(\Delta M_s \overline{N_s})(-v_s) + Im(\Delta M_s \overline{N_s}(iv_s)] + O_{\frac{1}{2}}(t_c-t)\end{aligned}$$

and so we obtain, in coordinates S_1, S_2, U_1, U_2 (recall that $S_1 = (v_s, -v_s)$ and $U_2 = (iv_s, iv_s)$) the following

$$\left(\frac{\partial z}{\partial t}(t,a_s), \frac{\partial \omega}{\partial t}(t,a_s)\right) = -\frac{1}{2h\sqrt{t_c-t}} S_1 + O_{\frac{1}{2}}(t_c-t)$$

$$\left(\frac{\partial z}{\partial a}(t,a_s), \frac{\partial \omega}{\partial a}(t,a_s)\right) = -\frac{Re(\Delta M_s \overline{N_s})}{2h\sqrt{t_c-t}} S_1 - \frac{Im(\Delta M_s \overline{N_s})}{2h\sqrt{t_c-t}} U_2 + O_{\frac{1}{2}}(t_c-t)$$

Taking into account the linearized change of variables (5.12) and (5.14) and the result we obtained above that ensures that $t_c - \tau_s(\varepsilon) = O_2(\varepsilon)$, we have

$$\begin{aligned}\frac{\partial s_1}{\partial t}(\tau_s, a_s) &= -\frac{1}{2h\varepsilon} + O(\varepsilon) \\ \frac{\partial s_2}{\partial t}(\tau_s, a_s) &= O(\varepsilon) \\ \frac{\partial u_1}{\partial t}(\tau_s, a_s) &= O(\varepsilon) \\ \frac{\partial u_2}{\partial t}(\tau_s, a_s) &= O(\varepsilon)\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial s_1}{\partial a}(\tau_s, a_s) &= -\frac{Re(\Delta M_s \overline{N_s})}{2h\varepsilon} + O(\varepsilon) \\
\frac{\partial s_2}{\partial a}(\tau_s, a_s) &= O(\varepsilon) \\
\frac{\partial u_1}{\partial a}(\tau_s, a_s) &= O(\varepsilon) \\
\frac{\partial u_2}{\partial a}(\tau_s, a_s) &= -\frac{Im(\Delta M_s \overline{N_s})}{2h\varepsilon} + O(\varepsilon)
\end{aligned}$$

What is left to do is to compute the derivative $\frac{d\tau_s}{da}(a_s)$, which is given by

$$\frac{d\tau_s}{da}(a_s) = -\frac{\partial |s|^2}{\partial a}(\tau_s, a_s) \left(\frac{\partial |s|^2}{\partial t}(\tau_s, a_s) \right)^{-1} = -Re(\Delta M_s \overline{N_s}) + O_2(\varepsilon)$$

Let us see why. By definition of $|s|^2$ and τ_s, we know that $|s|^2(\tau_s, a_s) = \varepsilon h^{\frac{3}{4}} = C$. So we can apply the Implicit Function Theorem and obtain the following

$$\begin{aligned}
\frac{\partial |s|^2(\tau_s, a_s)}{\partial a} = 0 \Longrightarrow \frac{\partial |s|^2}{\partial a}(\tau_s, a_s) + \frac{\partial |s|^2}{\partial t}(\tau_s, a_s) \cdot \frac{d\tau_s}{da}(a_s) = 0 \\
\underset{\frac{\partial |s|^2}{\partial t}(\tau_s, a_s) \neq 0}{\Longrightarrow} \frac{d\tau_s}{da}(a_s) = -\frac{\partial |s|^2}{\partial a}(\tau_s, a_s) \left(\frac{\partial |s|^2}{\partial t}(\tau_s, a_s) \right)^{-1}
\end{aligned}$$

Now we substitute

$$\begin{aligned}
-\frac{\partial |s|^2}{\partial a}(\tau_s, a_s) = -\frac{\partial (s_1^2 + s_2^2)}{\partial a}(\tau_s, a_s) &= \left(-2s_1 \frac{\partial s_1}{\partial a} - 2s_2 \frac{\partial s_2}{\partial a} \right)(\tau_s, a_s) \\
&= 2s_1 \left(\frac{Re(\Delta M_s \overline{N_s})}{2h\varepsilon} + O(\varepsilon) \right) + 2s_2 O(\varepsilon)
\end{aligned}$$

On the other hand

$$\frac{\partial |s|^2}{\partial t}(\tau_s, a_s) = \left(2s_1 \frac{\partial s_1}{\partial t} + 2s_2 \frac{\partial s_2}{\partial t} \right)(\tau_s, a_s) = -2s_1 \left(\frac{1}{2h\varepsilon} + O(\varepsilon) \right) + 2s_2 O(\varepsilon)$$

So

$$\frac{d\tau_s}{da}(a_s) = -Re(\Delta M_s \overline{N_s}) + O_2(\varepsilon)$$

Applying the same change of variables as above, we deduce

$$\begin{aligned} \frac{ds_1}{da}(a_s) &= O(\varepsilon) \\ \frac{ds_2}{da}(a_s) &= O(\varepsilon) \\ \frac{du_1}{da}(a_s) &= O(\varepsilon) \\ \frac{du_2}{da}(a_s) &= -\frac{Im(\Delta M_s \overline{N_s})}{2h\varepsilon} + O(\varepsilon) \end{aligned} \tag{5.24}$$

Finally, using (5.18) we obtain the final result in the indicated form. We can make the same reasoning and computations for h_u.

□

The statement of the Theorem of existence of generatrix solutions need a supplementary definition concerning the manifolds $V_s(\varepsilon)$ and $V_u(\varepsilon)$ defined in Section 5.2.2.

Definition 5.9. *We denote by $\tilde{P}_{2s}$ (resp. $\tilde{P}_{2u}$) to the lift given by the Levi-Civita map ρ of the point P_2 in the manifold $V_s(0)$ (resp. $V_u(0)$). We will say that $V_s(0)$ (resp. $V_u(0)$) is p-regular in P_2 when the restriction of p to $V_s(0)$ is a local diffeomorphism at the neighborhood of $\tilde{P}_{2s}$ (reps. $\tilde{P}_{2u}$).*

From this definition, one can apply Lemma 5.7 and Lemma 5.8 to prove the following result:

Lemma 5.10. *$V_s(0)$ (resp. $V_u(0)$) is p-regular in P_2 if and only if $Im(\Delta M_s \overline{N}_s) \neq 0$ (resp. $Im(\Delta M_u \overline{N}_u \neq 0)$).*

Proof. In order to have a local diffeomorphism in a neighborhood of $\tilde{P}_{2s}$, we need to use the Inverse Function Theorem on p in $V_s(0)$ (recall that $V_s(0) = \Phi_0((0, t_c + \varepsilon), I_s(0))$, which can be seen as $h_s((0, t_c + \varepsilon), i_s^0(I_{\delta_s}))$. So, it is equivalent to look at the local diffeomorphism over $V_s(0)$ as it is to look at it over $h((0, t_c + \varepsilon, i_s^0(I_{\delta_s}))$. Then, we can use the results given in both Lemma 5.7 and Lemma 5.8 to ensure that the derivatives $\frac{dx_s}{da}$ and $\frac{dy_s}{da}$ are not zero if and only if $Im(\Delta M_s \overline{N}_s) \neq 0$. The same argument works for $\tilde{P}_{2u}$. □

Now we are ready to state, and proof, the Theorem of existence of generatrix solutions:

Theorem 5.11. *We suppose that the manifolds $V_s(0)$ and $V_u(0)$ are p-regular in P_2, and the velocities N_s and N_u are non-collinear. Then, there exists a real number $\mu_1 > 0$ such that, for every $\mu < \mu_1$, the manifolds $V_s(\mu)$ and $V_u(\mu)$ have a transversal intersection in the manifold $\mathcal{H}_\mu$. The solutions φ_s and φ_u are then generatrices.*

Proof.
It will be enough to prove it under the regularized system $\mathcal{L}_\mu$, and demonstrate the intersection property for the regularized manifolds defined at the beginning of this section, and denoted by $W_s(\mu)$ and $W_u(\mu)$. The proof will be performed in 6 stages.

1. *Domains of definition of the arcs q_s^μ and q_u^μ.*

   We will begin by precising the length of the intervals I_{δ_s} and I_{δ_u} introduced in Lemma 5.6, in order to be able to give uniform estimations later on.

   We denote by C_0 the maximum of the norm of the differential of the Kepler field in non-regularized rotating coordinates (problem $\mathcal{P}_0$) in a big enough neighborhood of the image of φ_s. Then we can state the following inequality:

   $$||\Phi_0(\tau_s^0(a_s), i_s^0(a)) - \Phi_0(\tau_s^0(a_s), i_s^0(a_s))|| \leq e^{C_0\tau_s^0(a_s)}||i_s^0(a) - i_s^0(a_s)||$$

   Proof. We denote $f(t) = \Phi_0(t, i_s^0(a)) - \Phi_0(t, i_s^0(a_s))$, so our goal will be to bound $f(\tau_s^0(a_s))$. In order to do that, we begin with the following result:

   $$f(\tau_s^0(a_s)) = f(0) + \int_0^{\tau_s^0(a_s)} f'(s)\, ds$$

   denoting by X the Kepler field associated with the flux Φ_0, we can rewrite this equality as

   $$\begin{aligned}&\Phi_0(\tau_s^0(a_s), i_s^0(a)) - \Phi_0(\tau_s^0(a_s), i_s^0(a_s))\\ &= (i_s^0(a) - i_s^0(a_s)) + \int_0^{\tau_s^0(a_s)} \left[X(\Phi_0(s, i_s^0(a))) - X(\Phi_0(s, i_s^0(a_s)))\right] ds\end{aligned}$$

   taking norms we can write the following inequality:

   $$\begin{aligned}&||\Phi_0(\tau_s^0(a_s), i_s^0(a)) - \Phi_0(\tau_s^0(a_s), i_s^0(a_s))||\\ &\leq ||(i_s^0(a) - i_s^0(a_s))|| + \left|\left| \int_0^{\tau_s^0(a_s)} \left[X(\Phi_0(s, i_s^0(a))) - X(\Phi_0(s, i_s^0(a_s)))\right]\, ds\right|\right|\\ &\leq ||i_s^0(a) - i_s^0(a_s)|| + \int_0^{\tau_s^0(a_s)} ||X(\Phi_0(s, i_s^0(a))) - X(\Phi_0(s, i_s^0(a_s)))||\, ds\end{aligned}$$

   and now we can apply the IVT (Intermediate Value Theorem) and we obtain:

   $$||X(\Phi_0(s, i_s^0(a))) - X(\Phi_0(s, i_s^0(a_s)))|| \leq \ max||DX||\ ||\Phi_0(s, i_s^0(a)) - \Phi_0(s, i_s^0(a_s))||$$

so we can rewrite the previous expression as follows:

$$||\Phi_0(\tau_s^0(a_s), i_s^0(a)) - \Phi_0(\tau_s^0(a_s), i_s^0(a_s))|| \leq ||i_s^0(a) - i_s^0(a_s)|| + \\ + \int_0^{\tau_s^0(a_s)} \left[max||DX||\ ||\Phi_0(s, i_s^0(a)) - \Phi_0(s, i_s^0(a_s))|| \right] ds$$

Now we can use Gronwall's Lemma and we get the result:

$$\begin{aligned} &||\Phi_0(\tau_s^0(a_s), i_s^0(a)) - \Phi_0(\tau_s^0(a_s), i_s^0(a_s))|| \\ &\leq ||i_s^0(a) - i_s^0(a_s)||\ e^{\int_0^{\tau_s^0(a_s)} max||DX||\ ds} \\ &\leq ||i_s^0(a) - i_s^0(a_s)||\ e^{C_0 \tau_s^0(a_s)} \end{aligned}$$

□

The function i_s^0 is analytic in a neighborhood of a_s and, considering the proper analytic extension of this function under a complex domain containing a_s, we can ensure, by Cauchy, the following result

$$\left|\left| \frac{di_s^0}{da} \right|\right|_{U_s} \leq C_1 ||i_s^0||_{U_s}$$

where $||\cdot||_{U_s}$ is the L^∞ norm in a neighborhood of U_s (introduced at the beginning of this section) and where C_1 is a constant > 0. Then we obtain:

$$||\Phi_0(\tau_s^0(a_s), i_s^0(a)) - \Phi_0(\tau_s^0(a_s), i_s^0(a_s))|| \underset{first\ ineq.}{\leq} e^{C_0 \tau_s^0(a_s)} ||i_s^0(a) - i_s^0(a_s)||$$

$$\underset{IVT}{\leq} e^{C_0 \tau_s^0(a_s)} \left|\left| \frac{di_s^0}{da} \right|\right|_{U_s} |a - a_s| \underset{i_s^0\ analytic}{\leq} C_1 ||i_s^0||_{U_s} |a - a_s| e^{C_0 \tau_s^0(a_s)}$$

As we are supposing that $a \in I_{\delta_s}$ (see Lemma 5.6), then $|a - a_s| \leq \delta_s$. Fixing a value $\delta_s = \varepsilon^{\frac{7}{3}}$ (it will make the following computations easier) we will have directly the following result

$$||\Phi_0(\tau_s^0(a_s), i_s^0(a)) - \Phi_0(\tau_s^0(a_s), i_s^0(a_s))|| = O(\varepsilon^{\frac{7}{3}})$$

We can consider now the image $\rho(C_\varepsilon)$ of the surface of the section by the Levi-Civita transformation, and using the results of Lemma 5.7, we obtain the following equality:

$$\begin{aligned} (M, N)(\Phi_0(\tau_s^0(a_s), i_s^0(a))) &\underset{Lemma\ 5.7}{=} (M((\tau_s^0(a_s), i_s^0(a)), N(\tau_s^0(a_s), i_s^0(a))) \\ &\underset{t_c - \tau_s^0(a_s) = -O_2(\varepsilon)}{=} (-\varepsilon^2 N_s, N_s) + O(\varepsilon^{\frac{7}{3}}) \qquad \forall a \in I_{\delta_s} \end{aligned}$$

So one can observe that $P_2 \notin \Phi_0(\tau_s^0(a_s), i_s^0(I_{\delta_s}))$ (for ε small enough). It is easy to prove then that the map q_s^0 is well-defined in I_{δ_s}, with $|\tau_s^0(a) - \tau_s^0(a_s)| = O(\varepsilon^{\frac{7}{3}})$ (this difference, by IVT, has the same order as $|a - a_s| = O(\varepsilon^{\frac{7}{3}})$). Finally, we can obtain the expression of $q_s^0(a)$ in coordinates (s, u), by applying the results obtained along the proof of Lemma 5.8.

$$(s, u)(q_s^0(a)) = (\varepsilon, 0) + O(\varepsilon^{\frac{4}{3}})$$

The distance of the domain I_{δ_u} can be chosen in the same way, with $\delta_u = \varepsilon^{\frac{7}{3}}$.

2. *Limit development of the arc q_s^0.*

   Once we have obtained the coordinates on (s, u) of $q_s^0(a)$ for an arbitrary semi-mayor axis in a neighborhood of a_s, the second step is to compute the proper estimations of the tangent vectors to the manifold $V_s(\mu)$, for a value of μ small enough. Once we have that, we will be able to deduce an estimation for the derivatives of all the arc $q_s^0(I_{\delta_s})$.

   By analyticity of the field $\mathcal{P}_0$, we can obtain directly the time derivatives:

$$\frac{\partial(M, N)}{\partial t}(\Phi_0(\tau_s^0(a), i_s^0(a))) = (N_s, 0) + O(\varepsilon^{\frac{7}{3}})$$

   That comes from performing the derivatives of M, N with respect to t in Lemma 5.7.

   Then, using the same procedure as before, one can compute the derivatives with respect to t of the transformed coordinates in Levi-Civita.

$$\frac{\partial(s, u)}{\partial t}(q_s^0(a)) = \frac{\partial(s, u)}{\partial t}(q_s^0(a_s)) + O(\varepsilon^{\frac{4}{3}}) \qquad \forall a \in I_{\delta_s}$$

   However, the derivatives with respect to the semi-major axis a need a more detailed study. For this reason, we introduce the following function

$$\begin{aligned} Z_0(t) = &[\Phi_0(t + \tau_s^0(a) - \tau_s^0(a_s), i_s^0(a + h)) - \Phi_0(t + \tau_s^0(a) - \tau_s^0(a_s), i_s^0(a))] - \\ &- [\Phi_0(t, i_s^0(a_s + h)) - \Phi_0(t, i_s^0(a_s))] \end{aligned}$$

   where h is a real number near to 0 and $a \in I_{\delta_s}$. With the previous notation and the analyticity of i_s^0, we have the following result:

$$\left|\left|\frac{dZ_0}{dt}\right|\right| \leq C_0||Z_0|| + |h|O(\varepsilon^{\frac{7}{3}})$$

Proof. We know that Φ_0 is solution of $\frac{d}{dt}\Phi_0 = f(\Phi_0)$, where f is the Kepler field. So, by definition of C_0, we have

$$||Df|| \leq C_0$$

On the other hand we know

$$\begin{aligned}
&\frac{d}{dt}\left[\Phi_0(t,z_1) - \Phi_0(t,z_0)\right] = f(\Phi_0(t,z_1)) - f(\Phi_0(t,z_0)) \\
&= \int_0^1 Df(\Phi_0(t,z_0) + s(\Phi(t,z_1) - \Phi_0(t,z_0))\, ds \cdot [\Phi_0(t,z_1) - \Phi_0(t,z_0]
\end{aligned}$$

So we have that

$$\begin{aligned}
\frac{d}{dt}Z_0(t) = &\int_0^1 Df(\Phi_0(t+\tau_s^0(a) - \tau_s^0(a_s), i_s^0(a))) + s(\Phi_0(t+\tau_s^0(a) - \\
&- \tau_s^0(a_s), i_s^0(a+h)) - \Phi_0(t+\tau_s^0(a) - \tau_s^0(a_s), i_s^0(a)))\, ds \cdot \\
&[\Phi_0(t+\tau_s^0(a) - \tau_s^0(a_s), i_s^0(a+h)) - \Phi_0(t+\tau_s^0(a) - \tau_s^0(a_s), i_s^0(a))] - \\
&- \int_0^1 Df(\Phi_0(t, i_s^0(a))) + s(\Phi_0(t, i_s^0(a+h)) - \Phi_0(t, i_s^0(a)))\, ds \\
&\cdot [\Phi_0(t, i_s(a+h)) - \Phi_0(t, i_s(a))]
\end{aligned}$$

Denoting by $\tilde{t} = t + \tau_s^0(a) - \tau_s^0(a_s)$, we break this equality in two parts

$$\frac{d}{dt}Z_0(t) = A_1 + A_2$$

where

$$\begin{aligned}
A_1 &= \int_0^1 Df(\Phi_0(\tilde{t}, i_s^0(a) + s(\Phi_0(\tilde{t}, i_s^0(a+h)) - \Phi_0(\tilde{t}, i_s^0(a)))\, ds \cdot \\
&\cdot [\Phi_0(\tilde{t}, i_s^0(a+h)) - \Phi_0(\tilde{t}, i_s^0(a)) - (\Phi_0(t, i_s^0(a+h)) - \Phi_0(t, i_s^0(a)))] \\
&= \int_0^1 Df(\Phi_0(\tilde{t}, i_s^0(a) + s(\Phi_0(\tilde{t}, i_s^0(a+h)) - \Phi_0(\tilde{t}, i_s^0(a)))\, ds \cdot Z_0
\end{aligned}$$

As we see, we sustract this last term in A_1 to obtain the integral multiplied

by Z_0. Now we have to add it in A_2 to recover the original expression of A_1

$$\begin{aligned}
A_2 = &- \int_0^1 \left(Df(\Phi_0(t, i_s^0(a))) + s(\Phi_0(t, i_s^0(a+h)) - \Phi_0(t, i_s^0(a))) \right) ds \\
&\times [\Phi_0(t, i_s^0(a+h)) - \Phi_0(t, i_s^0(a))] + \int_0^1 \Big(Df(\Phi_0(\tilde{t}, i_s^0(a)) \\
&+ s(\Phi_0(\tilde{t}, i_s^0(a+h)) - \Phi_0(\tilde{t}, i_s^0(a))) \Big) ds \times (\Phi_0(t, i_s^0(a+h)) - \Phi_0(t, i_s^0(a))) \\
&= \Bigg[\int_0^1 \left(Df(\Phi_0(\tilde{t}, i_s^0(a)) + s(\Phi_0(\tilde{t}, i_s^0(a+h)) - \Phi_0(\tilde{t}, i_s^0(a))) \right) ds \\
&- \int_0^1 \left(Df(\Phi_0(t, i_s^0(a))) + s(\Phi_0(t, i_s^0(a+h)) - \Phi_0(t, i_s^0(a))) \right) ds \Bigg] \\
&\times (\Phi_0(t, i_s^0(a+h)) - \Phi_0(t, i_s^0(a)))
\end{aligned}$$

Now we can bound this terms independently, so we obtain

$$|A_1| \leq C_0 ||Z_0||$$

For A_2, we have to bound each term to obtain $|h| O(\varepsilon^{\frac{7}{3}})$.

On the one hand, using the fact that $|\tilde{t} - t| \lesssim \varepsilon^{\frac{7}{3}}$, the integral factor is bounded by $O(\varepsilon^{\frac{7}{3}})$.

On the other hand, the last factor $|\Phi_0(t, i_s(a+h) - \Phi_0(t, i_s(a))| \lesssim O(|h|)$ by the analyticity of $i_s^0(a)$. Joining both boundaries we obtain the desired inequality. □

Applying Gronwall's Lemma once again we obtain the following inequality

$$||Z_0(\tau_s^0(a_s))|| \leq ||Z_0(0)|| e^{C_0 \tau_s^0(a_s)} + |h| O(\varepsilon^{\frac{7}{3}})$$

Proof. We want to prove the following inequality

$$||Z_0(\tau_s^0(a_s))|| \leq ||Z_0(0)|| e^{C_0 \tau_s^0(a_s)} + |h| O(\varepsilon^{\frac{7}{3}})$$

We know that

$$\begin{aligned}
\int_0^{\tau_s^0(a_s)} \frac{dZ_0}{dt}\, dt = Z_0(\tau_s^0(a_s)) - Z_0(0) &\Longrightarrow Z_0(\tau_s^0(a_s)) \\
&= Z_0(0) + \int_0^{\tau_s^0(a_s)} \frac{dZ_0}{dt}\, dt
\end{aligned}$$

So

$$||Z_0(\tau_s^0(a_s))|| \leq ||Z_0(0)|| + \int_0^{\tau_s^0(a_s)} \left|\left|\frac{dZ_0}{dt}\right|\right| dt$$
$$\underset{\text{Prev. Inequality}}{\leq} ||Z_0(0)|| + C_0 \int_0^{\tau_s^0(a_s)} ||Z_0||\, dt + |h|O(\varepsilon^{\frac{7}{3}})$$

Finally, we have

$$||Z_0(\tau_s^0(a_s))|| \leq ||Z_0(0)|| + C_0 \int_0^{\tau_s^0(a_s)} ||Z_0||\, dt \underset{\text{Gronwall's lemma}}{\Longrightarrow} ||Z_0(\tau_s^0(a_s))||$$
$$\leq ||Z_0(0)||e^{C_0\tau_s^0(a_s)} + |h|O(\varepsilon^{\frac{7}{3}})$$

□

The analyticity of the flux Φ_0 leads us to the following result

$$||Z_0(0)|| \leq C_0 h|\tau_s^0(a) - \tau_s^0(a_s)| + ||[i_s^0(a+h) - i_s^0(a)] - [i_s^0(a_s+h) - i_s^0(a_s)]||$$

being

$$||Z_0(\tau_s^0(a_s))|| = |h|O(\varepsilon^{\frac{7}{3}})$$

which ensures that the limit when $h \to 0$ is

$$\lim_{h\to 0} \frac{Z_0(\tau_s^0(a_s))}{h} = \lim_{h\to 0}\left[\frac{\Phi_0(\tau_s^0(a), i_s^0(a+h)) - \Phi_0(\tau_s^0(a), i_s^0(a))}{h}\right]$$
$$- \lim_{h\to 0}\left[\frac{\Phi_0(\tau_s^0(a_s), i_s^0(a_s+h)) - \Phi_0(\tau_s^0(a_s), i_s^0(a_s))}{h}\right]$$
$$= O(\varepsilon^{\frac{7}{3}})$$

Then, by definition of the derivative in terms of limits, we obtain

$$\frac{\partial(M,N)}{\partial a}(\Phi_0(\tau_s^0(a), i_s^0(a))) - \frac{\partial(M,N)}{\partial a}(\Phi_0(\tau_s^0(a_s), i_s^0(a_s))) = O(\varepsilon^{\frac{7}{3}})$$

which in coordinates (s,u) can be written as

$$\frac{\partial(s,u)}{\partial a}(q_s^0(a)) = \frac{\partial(s,u)}{\partial a}(q_s^0(a_s)) + O(\varepsilon^{\frac{4}{3}})$$

As it does not depend on a, we obtain finally the derivative along the arc q_s^0:

$$\frac{d(s,u)}{da}(q_s^0(a)) = \frac{d(s,u)}{da}(q_s^0(a_s)) + O(\varepsilon^{\frac{4}{3}}) \qquad \forall a \in I_{\delta_s}$$

3. *Distance between the perturbed arcs and the non-perturbed ones.*

   Once we were able to give an expression for q_s^0 along the arc I_{δ_s}, the next step is to introduce a perturbation μ, and compute the distance between q_s^0 and q_s^μ. To do that, we will study the system at the boundary of a disk centered in P_2 of radius ε (note that this disk is different from $\mathcal{B}_\varepsilon$ defined in Section 5.2, where we defined the manifolds $C_s(\varepsilon)$ and $C_u(\varepsilon)$, since the radius has a different value). For that reason, we have to take into account also the distance between this exterior neigborhood and $\rho(C_s(\varepsilon))$.

   The study of the restricted problem in rotating coordinates $\mathcal{P}_\mu$ at the exterior of a disk $\mathcal{D}(P_2, \varepsilon)$ leads to the following distance

   $$||\Phi_\mu(t, i_s^\mu(a)) - \Phi_0(t, i_s^\mu(a))|| = O\left(\frac{\mu}{\varepsilon^2}\right)$$

   since the maximum is computed considering a time interval of the form $[0, t_c - C_2\varepsilon]$ (to ensure it does not collide with the manifold $\mathcal{D}(P_2, \varepsilon)$, as it has to be treated separately).

   This difference comes from the difference between the two correspondent Hamiltonian functions H_0 and H_μ (see Section 3.3). One can see that this difference is of order $O\left(-\frac{\mu}{|M|}\right)$, so between the fluxes Φ_0 and Φ_μ one has to consider the derivative (since this difference is related to the difference between the two vectorial fields) which is, in this time interval, $O\left(\frac{\mu}{|M|^2}\right) = O\left(\frac{\mu}{\varepsilon^2}\right)$.

   Now we denote by $\mathcal{V}_\mu(\varepsilon)$ to the lift of the manifold $\mathcal{H}_\mu$ of the disk $\mathcal{D}(P_2, \varepsilon)$ (i.e., consider $\mathcal{V}_\mu(\varepsilon)$ to be the set of points in $\mathcal{D}(P_2, \varepsilon)$ that are in $\mathcal{H}_\mu$). The travel time (for the problem $\mathcal{P}_\mu$) between the entrance in $\mathcal{V}_\mu(\varepsilon)$ and the arrival to $C_s(\varepsilon)$ is of order ε. So we have

   $$||\Phi_\mu(\tau_s^0(a), i_s^\mu(a)) - \Phi_0(\tau_s^0(a), i_s^\mu(a))|| = O\left(\frac{\mu}{\varepsilon^2}\right) + O\left(\frac{\mu}{\varepsilon^4}\varepsilon\right) = O\left(\frac{\mu}{\varepsilon^3}\right)$$

   In order to understand this result, one has to divide this difference in two parts:

   - The first part corresponds to the difference between the fluxes at a time interval of $[0, t_c - C_2\varepsilon]$, as the previous equality shows. This explains why the first element of the sum is $O\left(\frac{\mu}{\varepsilon^2}\right)$.
   - The second part corresponds to the difference between the fluxes in between $\mathcal{V}_\varepsilon$ and $C_s(\varepsilon)$. However, $C_s(\varepsilon)$ lives in the regularized system,

while $\mathcal{V}_\mu(\varepsilon)$ in the non-regularized one (in other words, we are mixing up both problems $\mathcal{P}_\mu$ and $\mathcal{Q}_\mu$). So, as we know, going through the Levi-Civita map ρ implies that the position M gets squared (in this case it would mean that, instead of having a difference of ε^{-2}, we would have a difference of ε^{-4}). Joining this result with the fact that the travel time between the two manifolds is of order ε, we obtain the second part of the sum.

Now, in the same way as we did in the first step of the proof, we obtain an estimation of the distance in the problem $\mathcal{P}_0$

$$||\Phi_0(\tau_s^0(a), i_s^\mu(a)) - \Phi_0(\tau_s^0(a), i_s^0(a))|| \leq ||i_s^\mu(a) - i_s^0(a)||e^{C_0\tau_s^0(a)}$$

being the function $i_s^\mu - i_s^0$ analytic and verifying $||i_s^\mu - i_s^0||_{U_s} = O(\mu)$. Putting together these two last results, we obtain the following estimation for the distance

$$||\Phi_\mu(\tau_s^0(a), i_s^\mu(a)) - \Phi_0(\tau_s^0(a), i_s^0(a))|| = O\left(\frac{\mu}{\varepsilon^3}\right)$$

4. *Choice of the mass μ as a function of ε.*

Now we will relate the order of the parameters μ and ε, to ease the following computations by working with only one variable. We are going to fix here the value of $\mu = \varepsilon^{\frac{11}{2}}$, that guarantees the following

$$||\Phi_\mu(\tau_s^0(a), i_s^\mu(a)) - \Phi_0(\tau_s^0(a), i_s^0(a))|| = O\left(\frac{\mu}{\varepsilon^3}\right) = O(\varepsilon^{\frac{5}{2}})$$

And now we can use this estimations to compute $M(\Phi_\mu(\tau_s^0(a =, i_s^\mu(a))))$ in terms of the non-perturbed arc in the same way as we did before (using Lemma 5.7)

$$M(\Phi_\mu(\tau_s^0(a), i_s^\mu(a)))| = \varepsilon^2|N_s| + O(\varepsilon^{\frac{7}{3}}) + O(\varepsilon^{\frac{5}{2}}) > 0$$

Expressing this result in coordinates (s, u) we obtain (in the same way as we did when we were computing $(s, u)(q_s^0(a))$ in terms of $(s, u)(q_s^0(a_s))$)

$$(s, u)(q_s^\mu(a)) = (s, u)(q_s^0(a)) + O(\varepsilon^{\frac{3}{2}}) \quad \forall a \in I_{\delta_s} \tag{5.25}$$

with $|\tau_s^\mu(a) - \tau_s^0(a)| = O(\varepsilon^{\frac{5}{2}})$, by the same reasoning as before; the difference of the vectorial fields is of order $O\left(\frac{\mu}{\varepsilon^3}\right) = O(\varepsilon^{\frac{5}{2}})$ (take into account that until now we were computing the difference between $\Phi_\mu(\tau_s^0(a), i_s^\mu(a))$ and $q_s^0(a)$).

In order to have the desired distance between $q_s^\mu(a)$ and $q_s^0(a)$, we should substitute $\tau_s^0(a)$ by $\tau_s^\mu(a)$. That is why with this last estimation one can obtain Equation 5.25).

5. *Development of the perturbed arcs.*

We will use a similar procedure as in paragraph 2. To perform the development of the function q_s^μ, we need to compute the derivatives with respect of the parameters a and t. The time derivatives verify:

$$\frac{\partial(M,N)}{\partial t}(\Phi_\mu(\tau_s^\mu(a), i_s^\mu(a))) = \frac{\partial(M,N)}{\partial t}(\Phi_0(\tau_s^0(a), i_s^0(a))) + O\left(\frac{\mu}{\varepsilon^4}\right)$$

being

$$\frac{\partial(s,u)}{\partial t}(q_s^\mu(a)) \underset{\text{Derivate Equation (5.25)}}{=} \frac{\partial(s,u)}{\partial t}(q_s^0(a)) + O(\varepsilon^{\frac{1}{2}}) \qquad \forall a \in I_{\delta_s}$$

One can obtain these results by a direct computation of all the previous results.

Like before, to estimate the derivatives with respect to the semi-major axis a, we will study the variations of:

$$\begin{aligned} Z_\mu(t) = &[\Phi_\mu(t + \tau_s^\mu(a) - \tau_s^0(a), i_s^\mu(a+h)) - \Phi_\mu(t + \tau_s^\mu(a) - \tau_s^0(a), i_s^\mu(a))] \\ &- [\Phi_0(t, i_s^0(a+h)) - \Phi_0(t, i_s^0(a))] \end{aligned}$$

where h is a real number near 0 and $a \in I_{\delta_s}$. Our objective will be to obtain an estimation for the value $Z_\mu(\tau_s^0(a))$, as we did in paragraph 2

Denote by t_ε the instant when the solution $\Phi_0(t, i_s^0(a))$ intersects with $\partial\mathcal{V}_\mu(\varepsilon)$. Then, for $t \in [0, t_\varepsilon]$ we obtain the following estimation

$$\begin{aligned} \left|\left|\frac{dZ_\mu}{dt}\right|\right| \leq &\left[C_0 + O\left(\frac{\mu}{\varepsilon^3}\right)\right] ||Z_\mu|| + \left[||\Phi_0(\tau_s^\mu(a) - \tau_s^0(a), i_s^\mu(a_s)) - i_s^0(a)||e^{C_0 t_\varepsilon}\right. \\ &\left. + O\left(\frac{\mu}{\varepsilon^3}\right)\right] ||i_s^0(a+h) - i_s^0(a)|| \end{aligned}$$

In order to understand this result, one has to divide the inequality in two parts. The first part corresponds to the interval $t \in [0, t_\varepsilon)$, which has a similar procedure as in b when estimating the value of $\left|\left|\frac{dZ_0}{dt}\right|\right|$. One notes that the derivative of the flux can be estimated in terms of the associated Kepler field, which has a maximum at C_0. So the first sum comes from the estimation of the Kepler field plus the difference between fields $X_\mu - X_0$, that we know it is of order $O\left(\frac{\mu}{\varepsilon^3}\right)$. On the other hand, the second sum comes from considering the limit case when $t = t_\varepsilon$, where we can apply all the previous results.

The choice of μ and the analyticity of i_s^μ and i_s^0 allow us to rewrite this inequality as follows:

$$\left|\left|\frac{dZ_\mu}{dt}\right|\right| \leq [C_0 + O(\varepsilon^{\frac{5}{2}})]||Z_\mu|| + |h|O(\varepsilon^{\frac{5}{2}})$$

The first sum comes from a direct substitution of $\mu = O(\varepsilon^{\frac{11}{2}})$. The second sum comes from the fact that the difference between the i_s parametrization can be reduced to $|h|$ multiplied by a constant and so the remaining term would be $O\left(\frac{\mu}{\varepsilon^3}\right)$, which as we said is equivalent to $O(\varepsilon^{\frac{5}{2}})$.

Then we can apply Gronwall's Lemma as we did in paragraph 2 to obtain the following

$$||Z_\mu(t_\varepsilon)|| \leq ||Z_\mu(0)||e^{C_0 t_\varepsilon} + |h|O(\varepsilon^{\frac{5}{2}})$$

For $t \in [t_\varepsilon, \tau_s^0(a)]$, one has to take into account the distance between $\rho(C_s(\varepsilon))$ and $\mathcal{V}_\mu(\varepsilon)$, as we did in paragraph 3. As we saw before, this consideration is translated to square the ε term, so instead of having $O\left(\frac{\mu}{\varepsilon^3}\right)$, we will obtain $O(\left(\frac{\mu}{\varepsilon^6}\right)$. Moreover, now at the second term of the sum we evaluate the inequality at $\tau_s^0(a)$ instead of t_ε, so the exponential term changes too.

$$\begin{aligned}\left|\left|\frac{dZ_\mu}{dt}\right|\right| \leq & \left[C_0 + O\left(\frac{\mu}{\varepsilon^6}\right)\right]||Z_\mu|| \\ & + \left[||\Phi_0(\tau_s^\mu(a) - \tau_s^0(a), i_s^\mu(a_s)) - i_s^0(a)||e^{C_0\tau_s^0(a)} + O\left(\frac{\mu}{\varepsilon^6}\right)\right] \\ & \times ||i_s^0(a+h) - i_s^0(a)||\end{aligned}$$

Once again, taking into account the choice of the mass μ and the analyticity of i_s^0 and i_s^μ we have

$$\left|\left|\frac{dZ_\mu}{dt}\right|\right| \leq [C_0 + O(\varepsilon^{-\frac{1}{2}})]||Z_\mu|| + |h|O(\varepsilon^{-\frac{1}{2}})$$

The estimation $\tau_s^0(a) - t_\varepsilon = O(\varepsilon)$ saw in paragraph 3 leads to the inequality:

$$\begin{aligned}||Z_\mu(\tau_s^0(a))|| &\leq ||Z_\mu(t_\varepsilon)|| + \left|\left|\frac{dZ_\mu}{dt}\right|\right|(\tau_s^0(a) - t_\varepsilon) \\ &\leq ||Z_\mu(t_\varepsilon)||(1 + O(\varepsilon^{\frac{1}{2}})) + |h|O(\varepsilon^{\frac{1}{2}})\end{aligned}$$

and the estimation of $||Z_\mu(t_\varepsilon)||$ implies that

$$||Z_\mu(\tau_s^0(a))|| \leq C||Z_\mu(0)||(1 + O(\varepsilon^{\frac{1}{2}})) + |h|O(\varepsilon^{\frac{1}{2}})$$

What is left to do is to estimate the value of $||Z_\mu(0)||$, so we substitute in the expression and we obtain

$$\begin{aligned} Z_\mu(0) =&[\Phi_\mu(\tau_s^\mu(a) - \tau_s^0(a), i_s^\mu(a+h)) - \Phi_\mu(\tau_s^\mu(a) - \tau_s^0(a), i_s^\mu(a))] \\ &- [i_s^0(a+h) - i_s^0(a)] \end{aligned}$$

We know that the flux Φ_μ is analytic in a neighborhood of $(0, i_s^\mu(a))$, so we have

$$\begin{aligned} &[\Phi_\mu(\tau_s^\mu(a) - \tau_s^0(a), i_s^\mu(a+h)) - \Phi_\mu(\tau_s^\mu(a) - \tau_s^0(a), i_s^\mu(a))] \\ &- [\Phi_\mu(0, i_s^\mu(a+h) - \Phi_\mu(0, i_s^\mu(a)] = [\Phi_\mu(\tau_s^\mu(a) - \tau_s^0(a), i_s^\mu(a+h)) - \Phi_\mu(\tau_s^\mu(a) \\ &- \tau_s^0(a), i_s^\mu(a))] - [i_s^\mu(a+h) - i_s^\mu(a)] = |h| O(\varepsilon^{\frac{5}{2}}) \end{aligned}$$

and we only have to give an estimation for

$$||[i_s^\mu(a+h) - i_s^\mu(a)] - [i_s^0(a+h) - i_s^0(a)]||$$

The function $i_s^\mu - i_s^0$ is analytic in U_s, so

$$\Big|\Big|\frac{d(i_s^\mu - i_s^0)}{da}\Big|\Big|_{Us} \leq C_3 ||i_s^\mu - i_s^0||_{Us} \underset{\text{Definition}}{=} O(\mu) = O(\varepsilon^{\frac{11}{2}})$$

where C_3 is a constant value, and

$$||[i_s^\mu(a+h) - i_s^\mu(a)] - [i_s^0(a+h) - i_s^0(a)]|| = |h| O(\varepsilon^{\frac{11}{2}})$$

Putting together all the previous results we obtain

$$||Z_\mu(\tau_s^0(a))|| = |h| O(\varepsilon^{\frac{1}{2}})$$

making the limit when $h \to 0$ as we did in paragraph 2, we have

$$\frac{\partial(M,N)}{\partial a}(\Phi_\mu(\tau_s^\mu(a), i_s^\mu(a))) = \frac{\partial(M,N)}{\partial a}(\Phi_0(\tau_s^0(a), i_s^0(a))) + O(\varepsilon^{\frac{1}{2}})$$

which in coordinates (s,u) can be expressed as

$$\frac{\partial(s,u)}{\partial a}(q_s^\mu(a)) = \frac{\partial(s,u)}{\partial a}(q_s^0(a)) + O(\varepsilon^{-\frac{1}{2}})$$

where we deduce finally the uniform estimation along the arc:

$$\frac{d(s,u)}{da}(q_s^\mu(a)) = \frac{d(s,u)}{da}(q_s^0(a)) + O(\varepsilon^{-\frac{1}{2}}) \qquad \forall a \in I_{\delta_s} \tag{5.26}$$

The equivalent estimations of (5.25) and (5.26) at the neighborhood of $i_u^\mu(a_u)$ are also valid with $\mu \geq 0$.

6. *End of the proof.*

Finally, we will introduce on $C_s(\varepsilon)$ and $C_u(\varepsilon)$ two new coordinate systems to define the intersections $\mathcal{L}_\mu \cap C_s(\varepsilon)$ and $\mathcal{L}_\mu \cap C_u(\varepsilon)$. They will be given by the following transformations:

$$\mathcal{T}_\mu^s(x_s, y_s, \theta_s) = (\tilde{x}_s, y_s, \theta_s) \qquad \mathcal{T}_\mu^u(x_u, y_u, \theta_u) = (\tilde{x}_u, y_u, \theta_u)$$

where

$$\tilde{x}_s = \frac{1}{2\varepsilon^2 h^{\frac{3}{2}}} \left(\tilde{l}_\mu^s(x_s, y_s, \theta_s) - \frac{\mu}{\alpha} \right)$$

and

$$\tilde{x}_u = \frac{1}{2\varepsilon^2 h^{\frac{3}{2}}} \left(\tilde{l}_\mu^u(x_u, y_u, \theta_u) - \frac{\mu}{\alpha} \right)$$

$\tilde{l}_\mu^{(s,u)}$ is the restriction of the Hamiltonian l_μ in $C_s(\varepsilon)$ or $C_u(\varepsilon)$, expressed in the correspondent coordinates (x, y, θ). The analyticity of l_μ implies directly the same for the maps $\mathcal{T}_\mu^s$ and $\mathcal{T}_\mu^u$, and the relations (5.13) and (5.18) implies the following equalities in $\partial\mathcal{B}_\varepsilon$

$$\tilde{x}_s = x_s + O(\varepsilon); \qquad \tilde{x}_u = x_u + O(\varepsilon)$$

So the transformations $\mathcal{T}_\mu^s$ and $\mathcal{T}_\mu^u$ are analytic diffeomorphism ε-near to the identity on $\partial\mathcal{B}_\varepsilon$. Then one can define the following:

$$\mathcal{L}_\mu \cap C_s(\varepsilon) = \left\{ (\tilde{x}_s, y_s, \theta_s) \in C_s(\varepsilon) \mid \tilde{x}_s = -\frac{\mu}{2\alpha\varepsilon^2 h^{\frac{3}{2}}} \right\}$$
$$\mathcal{L}_\mu \cap C_u(\varepsilon) = \left\{ (\tilde{x}_u, y_u, \theta_u) \in C_u(\varepsilon) \mid \tilde{x}_u = -\frac{\mu}{2\alpha\varepsilon^2 h^{\frac{3}{2}}} \right\}$$

One can obtain these sets just by replacing $\tilde{l}_\mu^{(s,u)}$ by 0, as $\mathcal{L}_\mu$ corresponds to the points with 0 energy level.

On the other hand, we know that Equation (5.25) for $a = a_s$ implies:

$$(s, u)(q_s^\mu(a_s)) = (\varepsilon, 0) + O(\varepsilon^{\frac{3}{2}})$$

Moreover, relation (5.26) and the hypotheses that $Im(\Delta M_s \overline{N}_s) \neq 0$ implies that $u_2(q_s^\mu(a))$ varies under an interval of length of order $O(\varepsilon^{\frac{4}{3}})$ when a varies

in I_{δ_s}:

$$\frac{du_2}{da}(q_s^\mu(a)) \underset{\text{Equation (5.26)}}{=} \frac{du_2}{da}(q_s^0(a)) + O(\varepsilon^{-\frac{1}{2}})$$
$$\underset{\text{Result Paragraph 2}}{=} \frac{du_2}{da}(q_s^0(a_s)) + O(\varepsilon^{\frac{4}{3}}) + O(\varepsilon^{-\frac{1}{2}})$$
$$\underset{\text{Equation (5.24)}}{=} -\frac{Im(\Delta M_s \overline{N}_s)}{2h\varepsilon} + O(\varepsilon^{\frac{4}{3}})$$

In particular, there exists $a_s^\mu \in I_{\delta_s}$ such that $u_2(q_s^\mu(a_s^\mu)) = 0$. We denote by $m_\mu^{(s)}$ the correspondent point $q_s^\mu(a_s^\mu))$, intersection of the arc $W_s(\mu) \cap C_s(\varepsilon)$ with the plane $u_2 = 0$ (see Lemma 5.6).

Our objective will be to find the representation in coordinates (x_u, y_u, θ_u) of the linear transition map $f_\varepsilon(m_\mu^{(s)})$. Since we have an explicit expression for $\tilde{x}_s$ (and so $x_u = \tilde{x}_s$), we only need to focus on computing y_s and u (because $y_u = -y_s$ and $e^{i\theta_u} = \frac{u}{|u|}$) (see Equation (5.20)).

The coordinates (s, u) of the point $m_\mu^{(s)}$ verify

$$(s_1, s_2, u_1)(m_\mu^{(s)}) = (\varepsilon, 0, 0) + O(\varepsilon^{\frac{4}{3}}) \tag{5.27}$$

It is a point of $C_s(\varepsilon)$, so $|s|$ must have modulus $\varepsilon h^{\frac{3}{4}}$ (We will suppose h to be 1 for convenience).

and

$$\left(\frac{ds_1}{da}, \frac{ds_2}{da}, \frac{du_1}{da}, \frac{du_2}{da}\right)(m_\mu^{(s)}) = \left(0, 0, 0, -\frac{Im(\Delta M_s \overline{N}_s)}{2h\varepsilon}\right) + O(\varepsilon^{-\frac{1}{2}}) \tag{5.28}$$

which is a direct computation from equations (5.24) and (5.26).

Using now Equation (5.18) to transform these expressions in coordinates (x_s, y_s, θ_s), and knowing by hypothesis that $Im(\Delta M_s \overline{N}_s) \neq 0$

$$y_s(m_\mu^{(s)}) = O(\varepsilon^{\frac{2}{3}}) \quad \text{and} \quad \frac{dy_s}{da}(m_\mu^{(s)}) = O(\varepsilon^{-2})$$

On the other hand, relation (5.15) implies:

$$[2s_1(m_\mu^{(s)}) + (s_1(m_\mu^{(s)}) + s_2(m_\mu^{(s)}))O_2(\varepsilon)]u_1(m_\mu(s)) + \frac{\mu}{\alpha} + O_6(\varepsilon) = 0$$

and using (5.27) we obtain the explicit expression for $u_1(m_\mu^{(s)})$

$$u_1(m_\mu^{(s)}) = -\frac{\mu}{2\alpha\varepsilon} + O\left(\frac{\mu}{\varepsilon^{\frac{2}{3}}}\right) = -\frac{\varepsilon^{\frac{9}{2}}}{2\alpha} + O(\varepsilon^{\frac{29}{6}}) \tag{5.29}$$

We can ensure that $\alpha \neq 0$ because we are supposing that N_s and N_u are not collinear.

Then, the coordinates of the linear transition map f_ε applied to the point $m_\mu^{(s)}$, denoted by $(\tilde{x}_u'^{(l)}, y_u'^{(l)}, \theta_u'^{(l)})$ are of the form

$$y_u'^{(l)} = O(\varepsilon^{\frac{2}{3}}) \qquad \theta_u'^{(l)} = 0$$

Moreover, the hypotheses $Im(\Delta M_s \overline{N}_s) \neq 0$ and Equation (5.29) implies.

$$\frac{dy_u'^{(l)}}{da}(f_\varepsilon(m_\mu^{(s)})) = O\left(\frac{1}{\varepsilon^2}\right); \qquad \frac{d\theta_u'^{(l)}}{da}(f_\varepsilon(m_\mu^{(s)})) = O\left(\frac{1}{\varepsilon^{\frac{11}{2}}}\right)$$

Finally, using the approximation Theorem in Section 5.2.3, we can estimate the value of the coordinates of the exact transition map $f_{\varepsilon\mu}$, denoted by $(\tilde{x}_u', y_u', \theta_u')$

$$y_u' = O(\varepsilon^{\frac{2}{3}}) \qquad \theta_u' = O(\varepsilon^2)$$

$$\frac{dy_u'}{da}(f_{\varepsilon\mu}(m_\mu^{(s)})) = O\left(\frac{1}{\varepsilon^2}\right); \qquad \frac{d\theta_u'}{da}(f_{\varepsilon\mu}(m_\mu^{(s)})) = O\left(\frac{1}{\varepsilon^{\frac{11}{2}}}\right)$$

Now we make an analogous study for the arc $W_u(\mu) \cap C_u(\varepsilon)$. For $a \in I_{\delta_u}$, we have:

$$(s_1, s_2, u_1, u_2)(q_u^\mu(a)) = (0, 0, \varepsilon, 0) + O(\varepsilon^{\frac{4}{3}}) \tag{5.30}$$

and

$$\left(\frac{ds_1}{da}, \frac{ds_2}{da}, \frac{du_1}{da}, \frac{du_2}{da}\right)(q_u^\mu(a)) = \left(0, -\frac{Im(\Delta M_u \overline{N}_u)}{2h\varepsilon}, 0, 0\right) + O(\varepsilon^{-\frac{1}{2}}) \tag{5.31}$$

It follows an analogous reasoning as in Equations (5.27) and (5.28) (see Section 5.2.3).

Once again, we can use the same estimation as in Equation (5.25) at a neighborhood of $i_u^\mu(a_u)$, so

$$(s, u)(q_u^\mu(a_u)) = (0, \varepsilon) + O(\varepsilon^{\frac{3}{2}})$$

Moreover, estimation (5.31) and the hypothesis that $Im(\Delta M_u \overline{N}_u) \neq 0$ implies that $s_2(q_u^\mu(a))$ varies on an interval of length of order $O(\varepsilon^{\frac{4}{3}})$ when a varies on I_{δ_u} (the proof is analogous to the one made for $u_2(q_s^\mu(a))$. In particular, there exists $a_u^\mu \in I_{\delta_u}$ such that $s_2(q_u^\mu(a_u^\mu)) = 0$. We will denote by $m_\mu^{(u)}$ the point $q_u^\mu(a_u^\mu)$, which corresponds to the intersection of the arc $W_u(\mu) \cap C_u(\varepsilon)$ with the plane of equation $s_2 = 0$. The coordinates of $m_\mu^{(u)}$ will be denoted as $(\tilde{x}_u'', y_u'', \theta_u'')$.

In order ot know the expression of y_u'' and θ_u'', we use once again Equation (5.18), so we have

$$y_u''(a) = O(\varepsilon^{\frac{2}{3}}) \quad \text{and} \quad \theta_u''(a) = O(\varepsilon^{\frac{1}{3}})$$

and the hypothesis $Im(\Delta M_u \overline{N}_u) \neq 0$ and Lemma 5.8 implies:

$$\frac{dy_u''}{da}(m_\mu^{(u)}) = O\left(\frac{1}{\varepsilon^2}\right); \qquad \frac{d\theta_u''}{da}(m_\mu^{(u)}) = O\left(\frac{1}{\varepsilon^{\frac{3}{2}}}\right)$$

We can choose ε small enough such that the tangent of the arc $f_{\varepsilon\mu}((W_s(\mu) \cap C_s(\varepsilon)))$ at the point $f_{\varepsilon\mu}(m_\mu^{(s)})$ is arbitrary near to the axis θ_u (in variables (s, u), this would be equivalent to be in the axis Ou_1, where the point $m_\mu^{(u)}$ lives), so we can ensure there is a "continuous connection" between the arcs $f_{\varepsilon\mu}((W_s(\mu) \cap C_s(\varepsilon)))$ and $W_u(\mu) \cap C_u(\varepsilon)$, which will have, as a consequence, a transversal intersection (when the semi-major axis vary intervals centered at a_s^μ and a_u^μ).

The convergence of periodic solutions φ_μ can be proved immediately from the previous construction, since $W_s(\mu) \cap W_u(\mu)$ intersect $\partial\mathcal{B}_\varepsilon$ in the points that converge to the intersections of φ_s and φ_u with $\partial\mathcal{B}_\varepsilon$ (with semi-major axis in the domains I_{δ_s} and I_{δ_u} respectively). So the solutions φ_s and φ_u are generatrices.

□

Now we are ready to prove Theorem 5.11.

Proof. We set the solutions φ_s and φ_u of angular momentum σ_s and σ_u respectively. We remark that the function angular momentum, in a neighborhood of the initial conditions (corresponding to φ_s and φ_u) in $\mathcal{I}_0$, is strictly monotone. Lemma 5.2 implies then the density of the initial conditions in $\mathcal{I}_0$ corresponding to double collision solutions (i.e. we have a submanifold dense in the domain elliptic). So it is enough to prove that $Im(\Delta M_s \overline{N}_s)$ and $Im(\Delta M_u \overline{N}_u)$, restricted to $\mathcal{I}_0$, are analytic and non-constants, which is clear at least in a neighborhood of the initial conditions.

Finally, by the principle of isolated zeros, which ensures that all zeroes of an analytic function are isolated, there exist initial conditions arbitrarily close to the initial ones fulfilling these previous conditions, and giving rise to the generatrix solutions by all the previous construction, and the Theorem is proved. □

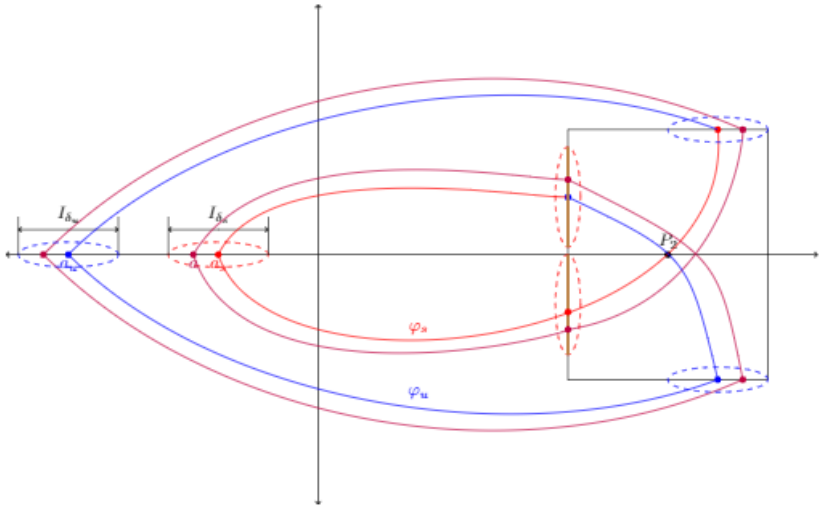


Figure 5.7: Representation of the homoclinic solution (in red and blue) and a second-species solution (in purple) in the reference s,u.

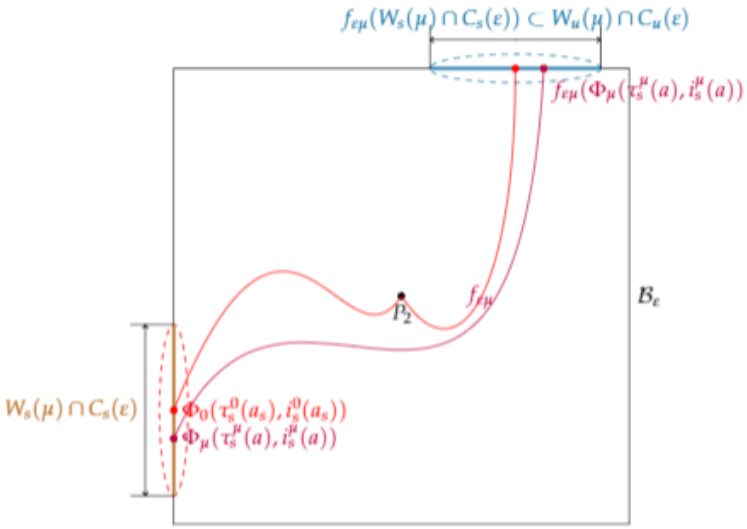


Figure 5.8: A more detailed description of the elements inside the ball $\mathcal{B}_\varepsilon$.

Chapter 6

Conclusions and future work

From the initial requirements (see Chapter 1) to obtain a complete proof of the existence of second species periodic solutions, our work has accomplished:

- Studying the Kepler problem from different points of view in order to better understand the scenario where we were going to work with.
- Introducing and applying dynamical concepts such as Hamiltonian systems, Lagrange equations and symplectic geometry.
- Understanding both geometrical and variational approaches to know how to build second-species periodic solutions.
- Comparing both results to look for similarities in the reasoning of both approaches, to better understand how one can go from the variational approach to the geometrical one.

The variational point of view [1] gives us an explanation on how we can build an "homoclinic" solution going through all the infinitely many singularities that correspond to the collisions of an Asteroid with Jupiter (see Section 3.2). Once we have defined this solution, it leads us through the construction of a true solution (i.e., a solution with a well-defined and definite action integral) shadowing the "homoclinic" one (see Section 4.2).

On the other hand, the geometrical one [2] builds up a suitable environment where one can prove that there exists one homoclinic solution from which one can generate a family of second-species solutions (see Section 5.1). It separates this solution in two parts (one corresponding to the stable manifold and the other one to the unstable one), and introduces different reference frameworks to put as a reference system these two manifolds (see Section 5.2). Finally, it defines two Poincaré

sections in a neighborhood of the singularity and a suitable transition map, from which one can generate the desired second-species solutions (see Section 5.3).

However, as one may have already noticed along the paper, they only work for a two-dimensional configuration space. To expand these results to a more realistic three-dimensional space, S.V. Bolotin and R.S. Mackay [1] use the following methods:

- Instead of using a Levi-Civita regularization map, they use the KS-regularization instead [12].
- They work with a quadratic Hopf map instead of the usual one (see Section 4.3).
- They introduce geodesic coordinates to simplify computations.

Chapter 7

Annex

7.1 Lagrangian & Hamiltonian formalism

In this section we will give some definitions and properties about the Hamiltonian and Lagrangian systems that we will use along the paper. In order to get a complete definition of both points of view, see the work made by Meyer-Offin [4].

7.1.1 Hamiltonian Equations

A Hamiltonian system is a system of $2n$ ordinary differential equations of the form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}(t,q,p),\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}(t,q,p) \tag{7.1}$$

where $H = H(t,q,p)$, called the Hamiltonian, is a smooth real-valued function defined for $(t,p,q) \in \mathcal{O}$, an open set in $\mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^n$. The vectors $q = (q_1, \cdots, q_n)$ and $p = (p_1, \cdots, p_n)$ are traditionally called the position and momentum vectors, respectively, because that is what these variables represent in many mechanical examples.

In general, we can express this system in the following way: Introduce the $2n$ vector z, the $2n \times 2n$ skew-symmetric matrix J, and the gradient by

$$z = \begin{bmatrix} q \\ p \end{bmatrix},\ J = J_n = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},\ \nabla H = \begin{bmatrix} \frac{\partial H}{\partial z_1} \\ \vdots \\ \frac{\partial H}{\partial z_{2n}} \end{bmatrix}$$

where 0 is the $n \times n$ matrix and I is the $n \times n$ identity matrix. In this notation, (7.1) becomes

$$\dot{z} = J\nabla H(t,z) \tag{7.2}$$

In the special case when H is independent of t, the differential equations (7.2) are called autonomous, and the Hamiltonian system is called *conservative*.

A *first integral* for (7.2) is a smooth function $F : \mathcal{O} \longrightarrow \mathbb{R}^1$ which is constant along the solutions (that we denote by ϕ) of (7.2); i.e., $F(\phi(t, t_0, z_0)) = F(z_0)$ is constant. The classical conserved quantities of energy, momentum, etc. are integrals. The level surfaces $F^{-1}(c) \subset \mathbb{R}^{2n}$, where c is a constant, are invariant sets, i.e., they are sets such that if a solution starts in the set, it remains in it.

Lemma 7.1. *If the Hamiltonian system is autonomous, then the associated Hamiltonian function H is a first integral.*

Proof. We begin by expressing how F changes

$$\begin{aligned}\frac{d}{dt}F(\phi(t, t_0, z_0)) &= D_z F(\phi(t, t_0, z_0))\dot{z} + \frac{\partial}{\partial t}F(\phi(t, t_0, z)) \\ &= (\nabla F)^T \circ \phi \cdot J \cdot \nabla H \circ \phi + \frac{\partial}{\partial t}F \circ \phi \\ &= \{F, H\} \circ \phi + \frac{\partial}{\partial t}F \circ \phi\end{aligned}$$

with $\{\cdot, \cdot\}$ being the Poisson bracket. Using the fact that the Hamiltonian system is autonomous (independent of t), we obtain:

$$\frac{d}{dt}(F \circ \phi) = \{F, H\} \circ \phi$$

As we saw before, being a first integral means that F is constant along solutions, so, in order to have a first integral, we must have the following result

$$\{F, H\} \circ \phi = 0$$

As we will see later, the Poisson bracket is an anti-symmetric operator, so $\{H, H\} = 0$, proving the result. □

The most important property of the first integrals is that the solutions lie on the set $F^{-1}(c)$, which is of dimension $2n - 1$. If we were so lucky as to find $2n - 1$ independent integrals, $F_1, \cdots, F_{2n-1}$, then holding all these integrals fixed would define a curve in $\mathbb{R}^{2n}$, the solution curve. This happens in some special cases, for example in the Kepler problem (see Section 2.1.2). We say such a system is *completely integrable*.

The Poisson Bracket

Like the example we saw before, many of the special properties of Hamiltonian systems are formulated in terms of the Poisson bracket operator, so it plays a central role in the theory developed here.

Definition 7.2. *Let H, F and G be smooth functions from $\mathcal{O} \subset \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^1$, and define the Poisson bracket of F and G by*

$$\begin{aligned} \{F, G\} = \nabla F^T J \nabla G &= \frac{\partial F^T}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F^T}{\partial p}\frac{\partial G}{\partial q} \\ &= \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i}(t,p,q)\frac{\partial G}{\partial p_i}(t,q,p) - \frac{\partial F}{\partial p_i}(t,p,q)\frac{\partial G}{\partial q_i}(t,q,p) \right) \end{aligned} \tag{7.3}$$

Clearly $\{F, G\}$ is a smooth map from $\mathcal{O}$ to $\mathbb{R}^1$ as well, and one can easy verify that $\{\cdot, \cdot\}$ is anti-symmetric and bilinear. Moreover, it verifies Jacobi's identity:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0. \tag{7.4}$$

Finally, we will state as a Theorem important results given by the Poisson bracket.

Theorem 7.3. *Let F, G, and H be as above and independent of time t. Then*

1. *F is an integral for (7.2) if and only if $\{F, H\} = 0$.*
2. *H is an integral for (7.2).*
3. *If F and G are integrals for (7.2), then so is $\{F, G\}$.*

Linear Equations

In this theory, as we have noticed, a special role is played by the $2n \times 2n$ matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{7.5}$$

Note that J is orthogonal and skew-symmetric, i.e.,

$$J^{-1} = J^T = -J \tag{7.6}$$

Let z be a coordinate vector in R^{2n}, $\mathbb{I}$ an interval in $\mathbb{R}$, and $S \colon \mathbb{I} \to \mathrm{gl}(2n, \mathbb{R})$ be continuous and symmetric. A linear Hamiltonian system is the system of $2n$ ordinary differential equations

$$\dot{z} = J \nabla H = A(t) z \tag{7.7}$$

where

$$H = H(t,z) = \frac{1}{2} z^T S(t) z, \tag{7.8}$$

so $A(t) = JS(t)$. The Hamiltonian H is then a quadratic form in z with coefficients that are continuous in $t \in \mathbb{I} \subset \mathbb{R}$. If S, and hence H, is independent of t, then H is an integral for (7.7) by Theorem 7.3. This property gives rise to two important definitions

Definition 7.4. *A matrix $A \in gl(2n, \mathbb{F})$ is called Hamiltonian (or sometimes infinitesimally symplectic), if*

$$A^T J + JA = 0 \tag{7.9}$$

We denote the set of such matrices $sp(2n, \mathbb{F})$.

Definition 7.5. *A $2n \times 2n$ matrix T is called symplectic with multiplier μ if*

$$T^T J T = \mu J \tag{7.10}$$

where μ is a nonzero constant. If $\mu = \pm 1$, then T is simply said to be symplectic. The set of all $2n \times 2n$ symplectic matrices is denoted by $Sp(2n, \mathbb{R})$.

Symplectic Manifolds

There is a lot to say in terms of symplectic geometry. However, the purpose of this chapter is to highlight only those theoretical results relevant along this work. For that reason, we will skip some theoretical background needed in order to full comprehend all the results. For more information, see [4].

With that said, we will start by giving a definition of symplectic manifolds, and their relevance to be able to relate differential forms with vector fields.

Definition 7.6. *A symplectic manifold is a pair (M, Ω) where M is a $2n$-dimensional differentiable manifold and Ω is a smooth nondegenerate 2-form on M. Ω is called the symplectic structure or symplectic form.*

The standard example is $\mathbb{R}^{2n}$ with

$$\Omega = \sum_{i=1}^{n} dq_i \wedge dp_i = \sum_{i=1}^{n} dz_i \wedge dz_{n+i} = \frac{1}{2} J_{ij} dz_i dz_j. \tag{7.11}$$

where $\wedge$ refers to the wedge product, and $z = (z_1, \cdots, z_{2n}) = (q_1, \cdots, q_n, p_1, \cdots, p_n)$ are coordinates in R^{2n} (position and momenta). The coefficient matrix of Ω is just J.

In general

$$\Omega = \frac{1}{2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} \omega_{ij}(z) dz_i \wedge dz_j = \frac{1}{2} \omega_{ij} dz_i \wedge dz_j. \tag{7.12}$$

A symplectic structure creates a way to convert a one-form like dH to a vector field like $\dot{z} = J\nabla H$. At each point $p \in M$ the symplectic form defines a nondegenerate bilinear map $\Omega_p \colon T_p^*M \times T_p^*M \to \mathbb{R}^1$, so the map

$$\begin{aligned} \flat \colon T_pM &\to T_p^*M \\ v &\mapsto v^\flat = \Omega_p(v, \cdot) \end{aligned}$$

is a well defined invertible linear map with inverse $\sharp : T_p^*M \to T_pM$

Let $v \in T_pM$ so

$$v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial z_i} = v_i \frac{\partial}{\partial z_i}$$

then

$$v^\flat = \sum_{i=1}^{n} \omega_{ij} v_i dx_j = \omega_{ij} v_i dx_j$$

The form Ω is nondegenerate, so the coefficient matrix has an inverse; let it be ω^{ij} so that $\omega_{ij}\omega^{ij} = \delta_i^j$ the Kronecker delta. Let $w = \sum w_i dx_i$ then

$$w^\sharp = \sum_{i=1}^{n} \omega^{ij} w_i \frac{\partial}{\partial x_j} = \omega^{ij} w_i \frac{\partial}{\partial x_j}$$

Thus, if the Hamiltonian is H, then the Hamiltonian vector field is $dH^\sharp$.

Finally, we will sum up this section by giving an introduction to the Lagrangian submanifolds keeping the notation we were using until now.

Definition 7.7. *Let M be a manifold. A Lagrangian submanifold $L \subset M$ is a submanifold of dimension n such that $\Omega|L = 0$ (recall that Ω is the symplectic form), that is, it is a submanifold of maximal dimension where Ω is zero. Or a Lagrangian submanifold is a submanifold of M such that T_pL is a Lagrangian subspace of T_pM for all $p \in L$ in the sense of Section 7.1.1*

For example, each $T_pN \subset TN$ is Lagrangian. The zero section

$$\begin{aligned} \mathcal{Z} \colon N &\to TN \\ p &\mapsto 0_p \end{aligned}$$

takes N into a Lagrangian submanifold of TN. In coordinates (q, p) where $\Omega = \sum dq^i \wedge dp^i$ the sets where $q = 0$ or where $p = 0$ are Lagrangian submanifolds.

Lemma 7.8. *If A is real Hamiltonian matrix, and all its eigenvalues have nonzero real parts, then there exists Lagrangian linear subspaces L, L^* such that $\mathbb{R}^{2n} = L \oplus L^*$ and all the solutions of $\dot{x} = Ax$ in L (respectively L^*) tend to 0 as $t \to +\infty$ (resp. $t \to -\infty$). These are the stable and unstable sets and they are Lagrangian submanifolds.*

7.1.2 Lagrange Equations

Maupertuis-Jacobi's Principle of Least Action

In 1740, Pierre Louis Moreau de Maupertuis stated that, in analogy with Fermat's Principle of Least Time for light, a particle of mass m under the influence of a force $F = -\nabla U$ (where U is the potential energy) moves along a path which satisfies the *Principle of Least Action*: $\delta S = 0$, where the action integral is defined as

$$S[x] = \int p \cdot dx = \int mv \, ds, \tag{7.13}$$

where $v = \frac{ds}{dt}$ denotes the magnitude of particle velocity, which can be also expressed as

$$v(s) = \sqrt{\frac{2}{m}[E - U(s)]}, \tag{7.14}$$

with the particle's kinetic energy $K = \frac{1}{2}mv^2$ written in terms of its total energy E and its potential energy $U(s)$.

It was Jacobi who, later on, emphasized the connection between Fermat's Principle of Least Time and Maupertuis'Principle of Least Action by introducing a different form of the Principle of Least Action $\delta S = 0$, where Jacobi's action integral is

$$S[x] = \int \sqrt{2m(E - U)} \, ds = 2 \int K \, dt, \tag{7.15}$$

where particle momentum is written as $p = \sqrt{2m(E - U)}$. To obtain the second expression of Jacobi's action integral (7.15), Jacobi made use of the fact that, by introducing a path parameter τ such that $v = \frac{ds}{dt} = \frac{s'}{t'}$ (where a prime here denotes a τ-derivative), we find

$$K = \frac{m(s')^2}{2(t')^2} = E - U$$

so that $2Kt' = s'p$, and the second form of Jacobi's action integral results. Next, Jacobi used the Principle of Least Action to establish the geometric foundation of particle mechanics. Here, the Euler-Jacobi equation resulting from Jacobi's Principle of Least Action is expressed as

$$\frac{d}{ds}\left(\sqrt{E - U}\frac{dx}{ds}\right) = \nabla\sqrt{E - U}$$

Lagrange's Equations from D'Alembert's Principle

The Principle of Virtual Work is one of the oldest principles in Physics that may find its origin in the work of Aristotle on the static equilibrium of levers. It

was finally written in its current form in 1717 by Jean Bernouilli and states that a system composed of N particles is in static equilibrium if the virtual work

$$\delta W = \sum_{i=1}^{N} F_i \cdot \delta x^i = 0 \tag{7.16}$$

for all virtual displacements $(\delta x^1, \cdots, \delta x^N)$ that satisfy physical constraints.

However, it was Jean Le Rond d'Alembert who generalized the Principle of Virtual Work (in 1742) by including the accelerating force $m_i \ddot{x}^i$ in (7.16):

$$\sum_{i=1}^{N} \left(F_i - m_i \frac{d^2 x^i}{dt^2} \right) = 0 \tag{7.17}$$

so that the equations of dynamics could be obtained from this one. Hence, d'Alembert's Principle, states that the work done by all active forces acting in a system is algebraically equal to the work done by all the acceleration forces.

The most historically significant application of d'Alembert's Principle (7.17), however, came from Lagrange, who transformed it into the following expression:

$$\delta K + \delta W = \frac{d}{dt} \left(m \frac{dx}{dt} \cdot \delta x \right) \tag{7.18}$$

where K denotes the Kinetic Energy.

We note that, for a conservative active force derivable from a single potential energy U (i.e., $F = -\nabla U$), the virtual work is $\delta W = -\delta U$, so that time integration of Eq. (7.18) yields an important principle known as *Hamilton's Principle*:

$$\int_{t_1}^{t_2} (\delta K - \delta U) \; dt = \delta \int_{t_1}^{t_2} L \; dt = 0, \tag{7.19}$$

where δx vanishes at $t = t_1$ and t_2 and the function

$$L = K - U \tag{7.20}$$

obtained by subtracting the potential energy U from the kinetic energy K, is known as the *Lagrangian* function of the system.

Hamilton's Principle and Euler-Lagrange Equations

Hamilton's principle (sometimes called The Principle of Least Action) is expressed, as we said before, in terms of a function $L(q, \dot{q}; t)$ known as the Lagrangian, which appears in the action integral

$$S[q] = \int_{t_i}^{t_f} L(q, \dot{q}; t) \; dt \tag{7.21}$$

where the action integral is a functional of the generalized coordinates $q(t)$, providing a path from the initial point $q_i = q(t_i)$ to the final point $q_f = q(t_f)$. The stationarity of the action integral

$$0 = \delta S[q;\delta q] = \left(\frac{d}{d\varepsilon} S[q+\varepsilon\delta q]\right)_{\varepsilon=0} = \int_{t_i}^{t_f} \delta q \cdot \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] dt$$

where the variation δq is assumed to vanish at the integration boundaries ($\delta q_i = 0 = \delta q_f$) yields the *Euler-Lagrange* equation for the generalized coordinate q^j ($j = 1, \cdots, k$):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right) = \frac{\partial L}{\partial q^j} \tag{7.22}$$

Legendre Transformation

The k second-order Euler-Lagrange equations on configuration space $q = (q^1, \cdots, q^k)$ (7.22) can be written as $2k$ first-order differential equations, known as Hamilton's equations, on a $2k$-dimensional phase space with coordinates $z = (q^1, \cdots, q^k; p_1, \cdots, p_k)$, where

$$p_j(q, \dot{q}; t) = \frac{\partial L}{\partial \dot{q}^j}(q, \dot{q}; t) \tag{7.23}$$

defines the j^{th}-component of the canonical momentum. In terms of these new coordinates, the Euler-Lagrange equations (7.22) are transformed into Hamilton's canonical equations (7.1) where the Hamiltonian function $H(q, p; t)$ is defined from the Lagrangian function $L(q, \dot{q}; t)$ by the *Legendre transformation*:

$$H(q, p; t) = p \cdot \dot{q}(q, p, t) - L[q, \dot{q}(q, p, t), t] \tag{7.24}$$

7.2 Coordinate Framework

7.2.1 Jacobi Coordinates

Jacobi coordinates are ideal coordinates for investigations of the N-body problem. Let us see how it is built.

Let $q_i, p_i \in \mathbb{R}^3$ for $i = 1, \cdots, N$, be the coordinates of the N-body problem (see section 2.1). We define a sequence of transformations starting with $g_1 = q_1$ and $\mu_1 = m_1$ and proceed inductively by

$$T_k : \begin{cases} u_k = q_k - g_{k-1} \\ g_k = \left(\frac{1}{\mu_k}\right)(m_k q_k + \mu_{k-1} g_{k-1}) \\ \mu_k = \mu_{k-1} + m_k \end{cases} \tag{7.25}$$

for $k = 2, \cdots, N$. μ_k is the total mass, and g_k is the position vector of the center of mass of the system of particles with indices $1, 2, \cdots, k$. The vector u_k is the position of the k^{th} particle relative to the center of mass of the previous $k-1$ particles (see Figure 7.1).

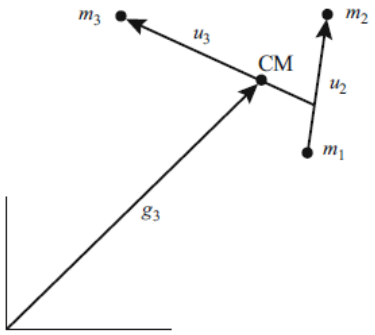


Figure 7.1: Jacobi coordinates for the 3-Body Problem [4].

Consider T_k as a change of coordinates from $g_{k-1}, u_2, \cdots, q_k, \cdots, q_N$ to $g_k, u_2, \cdots, u_k, q_{k+1}, \cdots, q_N$ or simply from g_{k-1}, q_k to g_k, u_{k+1}. The inverse of T_k is

$$T_k^{-1} : \begin{cases} q_k = \left(\frac{\mu_{k-1}}{\mu_k}\right) u_k + g_k, \\ g_{k-1} = \left(\frac{-m_k}{\mu_k}\right) u_k + g_k. \end{cases} \tag{7.26}$$

This is a linear transformation on the q variables only, i.e, on a Lagrangian subspace, so we need to perform the transformation on the p variables in order to have a symplectic transformation. To make the symplectic completion of T_k, define $G_1 = p_1$ and

$$Q_k : \begin{cases} v_k = \left(\frac{\mu_{k-1}}{\mu_k}\right) p_k - \left(\frac{m_k}{\mu_k}\right) G_{k-1}, \\ G_k = p_k + G_{k-1} \end{cases} \tag{7.27}$$

$$Q_k^{-1} : \begin{cases} p_k = v_k + \left(\frac{m_k}{\mu_k}\right) G_k, \\ G_{k-1} = -v_k + \left(\frac{\mu_{k-1}}{\mu_k}\right) G_k \end{cases} \tag{7.28}$$

If we denote the coefficient matrix in (7.25) by A, the coefficient matrices in (7.26), (7.27), and (7.28) are A^{-1}, A^{-T}, and A^T, respectively. Thus, the pair T_k, Q_k is a symplectic change of variables.

These variables satisfy the identities

$$g_{k-1} \times G_{k-1} + q_k \times p_k = g_k \times G_k + u_k \times v_k$$

and

$$\frac{||G_{k-1}||^2}{2\mu_{k-1}} + \frac{||p_k||^2}{2m_k} = \frac{||G_k||^2}{2\mu_k} + \frac{||v_k||^2}{2M_k}$$

where $M_k = m_k \frac{\mu_{k-1}}{\mu_k}$. Additionally, the kinetic energy is

$$K = \sum_{k=1}^{N} \frac{||p_k||^2}{2m_k} = \frac{||G_N||^2}{2\mu_N} + \sum_{k=2}^{N} \frac{||v_k||^2}{2M_k}$$

and the total angular momentum is

$$A = \sum_{k=1}^{N} q_k \times p_k = g_N \times G_N + \sum_{k=2}^{N} u_k \times v_k,$$

where g_N is the center of mass of the system, and G_N is the total linear momentum.

Unfortunately, the formulas for the variables u_k and v_k are not simply expressed in terms of the variables q_k and p_k. Note that

$$u_2 = q_2 - q_1$$

Let $d_{ij} = q_i - q_j$. The Hamiltonian of the N-Body Problem in Jacobi coordinates is

$$\mathcal{H} = \frac{||G_N||^2}{2\mu_N} + \sum_{k=2}^{N} \frac{||v_k||^2}{2M_k} - \sum_{1 \le i < j \le N} \frac{m_i m_j}{||d_{ij}}$$

with equations of motion

$$\begin{cases} \dot{g}_i = -\frac{\partial \mathcal{H}}{\partial G_i} \\ \dot{u}_i = -\frac{\partial \mathcal{H}}{\partial v_i} \\ \dot{G}_i = \frac{\partial \mathcal{H}}{\partial g_i} \\ \dot{v}_i = \frac{\partial \mathcal{H}}{\partial u_i} \end{cases} \tag{7.29}$$

7.2.2 Polar Coordinates

Let x, y be the usual coordinates in the plane and X, Y their conjugate momenta. Suppose we wish to change to polar coordinates (r, θ), in the (x, y)-plane and to extend this point transformation to a symplectic change of variables. Let R, Θ be conjugate to r, θ. Use the generating function $S = Xrcos\theta + Yrsin\theta$ [4], and so

$$
\begin{aligned}
x &= \frac{\partial S}{\partial X} = r\cos\theta, \quad y = \frac{\partial S}{\partial Y} = r\sin\theta, \\
R &= \frac{\partial S}{\partial r} = X\cos\theta + Y\sin\theta = \frac{xX + yY}{r}, \\
\Theta &= \frac{\partial S}{\partial \theta} = -Xr\sin\theta + Yr\cos\theta = xY - yX.
\end{aligned}
\tag{7.30}
$$

If we think of a particle of mass m moving in the plane, then $X = m\dot{x}$ and $Y = m\dot{y}$ are linear momenta in the x and y directions; so, $R = m\dot{r}$ is the linear momentum in the r direction, and $\Theta = mx\dot{y} - my\dot{x} = mr^2\dot{\theta}$ is the angular momentum. The inverse transformation is

$$
X = R\cos\theta - \left(\frac{\Theta}{r}\right)\sin\theta,
$$

$$
Y = R\sin\theta + \left(\frac{\Theta}{r}\right)\cos\theta
$$

Kepler's Problem in Polar Coordinates

The Hamiltonian of the planar Kepler's problem (2.8) in polar coordinates is

$$
\mathcal{P}(r,\theta,R,\Theta) = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} \tag{7.31}
$$

Because $\mathcal{P}$ is independent of θ, is is an ignorable coordinate, so Θ is a first integral. The equations of motion are

$$
\begin{aligned}
\dot{r} &= R, & \dot{\theta} &= \frac{\Theta}{r^2} \\
\dot{R} &= \frac{\Theta^2}{r^3} - \frac{\mu}{r^2}, & \dot{\Theta} &= 0
\end{aligned}
\tag{7.32}
$$

The 3-Body Problem in Jacobi-Polar Coordinates

Now consider the 3-Body Problem in Jacobi coordinates with center of mass at the origin and linear momentum zero; i.e.

$$
H = \frac{||v_2||^2}{2M_2} + \frac{||v_3||^2}{2M_3} - \frac{m_1m_2}{||u_2||} - \frac{m_1m_3}{||u_3 + \alpha_0 u_2||} - \frac{m_2m_3}{||u_3 - \alpha_1 u_2||}
$$

Introduce polar coordinates for u_2 and u_3. That is, let

$$
u_2 = (r_1\cos\theta_1, r_1\sin\theta_1), \qquad u_3 = (r_2\cos\theta_2, r_2\sin\theta_2),
$$

$$v_2 = \left(R_1 \cos\theta_1 - \left(\frac{\Theta_1}{r_1}\right) \sin\theta_1, R_1 \sin\theta_1 + \left(\frac{\Theta_1}{r_1}\right) cos\theta_1 \right),$$

$$v_3 = \left(R_2 \cos\theta_2 - \left(\frac{\Theta_2}{r_2}\right) \sin\theta_2, R_2 \sin\theta_2 + \left(\frac{\Theta_2}{r_2}\right) \cos\theta_2 \right),$$

so the Hamiltonian H becomes (shifting the index $1,2,3$ to $0,1,2$)

$$\begin{aligned} H =& \frac{1}{2M_2}\left\{ R_1^2 + \left(\frac{\Theta_1^2}{r_1^2}\right) \right\} + \frac{1}{2M_3}\left\{ R_2^2 + \left(\frac{\Theta_2^2}{r_2^2}\right) \right\} - \frac{m_0 m_1}{r_1} \\ & - \frac{m_0 m_2}{\sqrt{r_2^2 + \alpha_0^2 r_1^2 - 2\alpha_0 r_1 r_2 \cos(\theta_2 - \theta_1)}} \\ & - \frac{m_1 m_2}{\sqrt{r_2^2 + \alpha_1^2 r_1^2 - 2\alpha_1 r_1 r_2 \cos(\theta_2 - \theta_1)}}. \end{aligned}$$

Note that the Hamiltonian only depends on the difference of the polar angles, $\theta_2 - \theta_1$.

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