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Rigidity of group actions, cotangent lifts and integrable systems

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Abstract

In this master thesis we generalize a theorem by Palais on the rigidity of compact group actions to cotangent lifts. We use this result to prove rigidity for integrable systems on symplectic manifolds including systems with degenerate singularities which are invariant under a torus action. We also prove the b-symplectic analogue of the rigidity results. We illustrate the three basic types of singularities of integrable systems through three models from classical mechanics and we give them as cotangent lifts. Finally we review the focus-focus singularity and the saddle-focus singularity.

Keywords

Rigidity, group actions, cotangent lift, cotangent models, Palais Theorem, symplectic geometry, b-symplectic geometry, integrable systems.

Contents

1. Introduction

Symmetries of mathematical objects, understood as transformations that do not change the shape or the structure of the object, are present in a huge variety of models for physical problems. A set of transformations satisfying certain conditions and endowed with an internal operation is called a *group* and a special category of groups are *Lie groups*, which have the extra structure of a differentiable manifold. The transformation of a manifold by an object of a group is called an *action*, and the classification of actions of Lie groups on smooth manifolds is a principal objective in geometry, since studying how a Lie group acts on a manifold it is possible to derive conclusions on the structure of the manifold.

A principal application of Lie groups theory is the study of dynamical systems, systems of equations that depend on time, which can be seen as continuous transformations of the coordinates of an object in a phase space. Inside the vast family of dynamical systems, the class of *integrable Hamiltonian systems* includes systems which produce a foliation of the phase space by invariant manifolds, and their characterization has been a long-time pursued goal. One of the most important milestones in the field is the Arnold-Liouville-Mineur Theorem, proved independently by Arnold and Liouville in [Arn74] and by Mineur in [Min36]. This result states that, in a regular integrable system, the induced foliation near a compact fibre is a fibration by tori. The theorem also constructs a special set of coordinates, the *action-angle coordinates*, which turn out to be a version of the local coordinates given by the classical Darboux Theorem for symplectic manifolds.

Concerning the stability of the regular tori given by the Arnold-Liouville-Mineur Theorem, the *KAM rigidity theory* states the exact conditions that the joint flows of a system have to satisfy in order to be stable. Namely, KAM theory provides the precise hypothesis that make regular tori survive under small perturbations of the system. With the same idea of stability of transformations under small perturbations, Richard Palais proved in [Pal61a] that two compact Lie group actions on a compact manifold, if they are close enough, are equivalent in the sense that there exists a diffeomorphism conjugating both actions. Other classical stability results for differentiable maps were proved in the 60's and 70's for which stability yields equivalence of close maps (see for instance [Mat68] and [Tho72]).

Symplectic manifolds are the natural setting to test stability ideas as among the classical actions of Lie groups on symplectic manifolds the ones admitting a moment map stand out. These are Hamiltonian actions where the group action can be read off from a mapping $\mu : M \longrightarrow \mathfrak{g}^*$ where $\mathfrak{g}$ is the Lie algebra of the Lie group.

In [Mir07] it was proved that C^2-close symplectic actions on a compact symplectic manifold are equivalent in the sense that not only the actions are conjugated by a diffeomorphism but this diffeomorphism preserves the symplectic form. The proof in [Mir07] (see also [MMZ12]) uses the path method requiring differentiability of degree 2 as the diffeomorphism yielding the equivalence comes from integration of a time-dependent vector field. Generalizations of this result can be easily achieved for

Hamiltonian actions in the symplectic context. In the more general Poisson context technical complications occur due to the lack of a general path method in Poisson geometry and fine Nash-Moser techniques come to the rescue to prove rigidity of Hamiltonian actions of semisimple Lie groups of compact type on Poisson manifolds as proved in [MMZ12]. Those results can be obtained either globally (for compact manifolds) or semilocally (in the neighbourhood of a compact submanifold which is invariant by the group actions).

The lift of Lie group actions to the cotangent bundle, naturally equipped with a canonical symplectic form, provides natural examples of Hamiltonian actions. Non-compactness of cotangent bundles leaves the study of equivalence of actions out of the radar of the compact case and needs to be re-examined with fresh eyes. In this master thesis we analyze the case of cotangent lifted actions where we can easily prove the equivalence of Hamiltonian actions on non-compact manifolds (cotangent bundles) by lifting the diffeomorphism given by Palais (in his Theorem 3.11) from the base. This simple idea allows to reduce the required degree of differentiability by 1 from the case of compact group actions on compact symplectic manifolds. We present a new result on rigidity of lifted actions, which can be thought as an extension of Palais rigidity Theorem to the cotangent lift of an action of a compact group. It has the advantage of being useful at the level of the cotangent bundle, which is a non-compact manifold, in contrast with the compactness required for the manifold in the original Palais Theorem.

Cotangent lifted actions may, a priori, seem a small class of actions to consider. However, this class includes the wide class of regular integrable systems as Kiesenhofer and Miranda proved in [KM17]. There, they show that the action-angle coordinate theorem for integrable systems can be rephrased (see [KM17]) as follows: any integrable system is equivalent in a neighbourhood of a regular torus to the integrable system given by the cotangent lift of translations of this torus to $T^*(\mathbb{T}^n)$.

In this sense group actions turn out to be a useful tool to understand integrable systems. But what happens outside the regular tori? What happens with singularities of integrable systems? As a consequence of the cotangent lift result above and rigidity theorem for cotangent lifts, it follows that integrable systems whose singularities are only of regular and of elliptic type are rigid inside the integrable class. Some of these results can be reproved using normal form theorems for the integrable system and the symplectic form. However, our technique reveals to be useful also when there are no normal forms known for degenerate singularities which are invariant by circle actions. In this direction, we prove a rigidity result for this special class of degenerate singularities of integrable systems.

b-Symplectic geometry is a tool that extends the symplectic structure to manifolds with boundary by considering the boundary as a hypersurface of the double of the manifold and considering vector fields which are tangent to this hypersurface along it. It is then possible to associate a vector bundle (the b-tangent bundle) to model this situation and work with forms as sections of the dual bundle. This setting (see [GMPS15], [GMP11] and [GMP14] for the complete overview) provides a singular model for integrable systems which can be useful for families of physical problems for which symplectic manifolds are not enough to describe them properly. We have adapted some our results on rigidity to the b-symplectic case, proving a b-symplectic analogue of the symplectic Palais Theorem proved by Miranda in [Mir07] and proving a b-cotangent lifted rigidity theorem.

To illustrate the application of the results on rigidity at the level of the cotangent lift to physical problems, we take three simple physical examples of integrable Hamiltonian systems and show that they contain the basic types of non-degenerate singularities (in the Williamson sense [Wil36]). Then, we see that the model based on the cotangent lift of a Lie group action produces these three singularities, that is, we formulate the elliptic, the hyperbolic and the focus-focus singularities as cotangent lifts.

Finally, we discuss two examples of integrable systems with non-degenerate singularities in which the author has worked along the master and which we think that are of interest because they show how the study of non-degenerate singularities is necessary to understand physical problems. The first example is the semitoric integrable system with a focus-focus singularity. The second example is a classical problem in celestial mechanics, the Planar Circular Restricted Three Body Problem (R3BP).

For the semitoric integrable system, we follow the analysis carried by San Vu Ngoc in [VuN03] to conclude that the Lagrangian leaf of a focus-focus singularity is topologically a pinched torus. Then, we study the semi-global invariant of Taylor Series type that Vu Ngoc introduces in his paper and that characterizes the neighbourhood of a focus-focus singular leaf. For the R3BP, we study a special bifurcation, the Hopf Bifurcation. In particular, we study the stability of two fixed points of the system by making an intense use of symplectic transformations and scalings.

Organization of this work

This work is organized as follows. In Section 2 we present all the preliminary contents. We are going to deal with specific objects of differential geometry and integrable systems and, although we suppose the reader is familiar with this concepts, we consider that a Preliminaries section is convenient. In this section we give an overview of basic contents on differential geometry, symplectic and b-symplectic geometry, Lie groups theory and dynamical systems.

In Section 3 we state some of the main results in characterization of regular integrable systems, such as the Arnold-Liouville-Mineur Theorem and the KAM Theorem. We present the Palais Theorem on the equivalence of close compact group actions on compact manifolds, which is the previous step, as well as the motivation, of the new results we present in this master thesis. We also review the recent advances on non-degenerate singularities of integrable Hamiltonian systems.

Section 4 is the core of this master thesis. We state and prove a new theorem about the equivalence of the cotangent lift of close group actions on manifolds. We also prove two new theorems on the equivalence of close integrable systems which are consequence of the first one.

In Section 5 we prove the b-symplectic analogues of the theorems on rigidity of close actions proved in the previous section. We also prove the b-symplectic versions of the results that are needed to prove the Palais Theorem.

In Section 6 we give the definition of three well-known physical models which contain the three basic non-degenerate singularities. Namely, we give the formulation of the harmonic oscillator, which has an elliptic singularity, we give the formulation of the simple pendulum, which has a hyperbolic singularity, and we give the formulation of the spherical pendulum, which has a focus-focus singularity. Then, we give the three singularities as cotangent lifts of group actions.

In Section 7 we give an example of the study of a particular integrable system, the semitoric integrable system with a focus-focus singularity. We detail all the steps that lead to the conclusion that the singular leaf of the focus-focus singularity is topologically a pinched torus and we define the invariant associated to it.

In Section 8 we present another particular integrable system which has a saddle-focus singularity, the Restricted Three Body Problem. We study the the Hopf Bifurcation and we show how the computation of the stability of the equilibrium points of the system is carried via symplectic transformations.

Finally, in Section 9 we summarize the new results obtained and discuss some questions that this master thesis has created and are still open. Among them, there is the natural one of asking if it is possible to relax the hypothesis of the main result that we have proved (Theorem 4.2). It is true for compact group actions and we wonder if it is also true for actions not of a compact group but of a locally compact group, provided that the actions are proper. We give the precise conjecture and we explain why we could not prove it yet. We also discuss other questions that this work has raised and that we hope we can answer in the future, concerning for example other formulations of the cotangent singular models or the application of the cotangent lift to geometric quantization.

Papers related to this master thesis

There are two papers written by the author and professor Eva Miranda which have been produced along with this master thesis. One is based on the new results of section 4 concerning rigidity of lifted actions and it is on arXiv[1] [MM20]. The other one contains the cotangent lift models for non-degenerate singularities of section 6 and is in pre-print.

[1] "Rigidity of cotangent lifts and integrable systems", [arXiv:2006.12477 math.SG]

2. Preliminaries

Most of the definitions and results in this section will be well-known by a reader with notions of differential geometry and Lie groups theory. Nevertheless, a review of them will be useful to establish the notation that will be used throughout all the sections.

2.1 Differential geometry

We start defining the essential concepts in differential geometry.

Definition 2.1. A *smooth manifold* is a two-countable Hausdorff topological space M such that, for every $x \in M$, there exists $\varphi : U \to \mathbb{R}^k$, where $U \subset M$ is an open neighbourhood of x (with the induced euclidean topology) and φ is a local diffeomorphism.

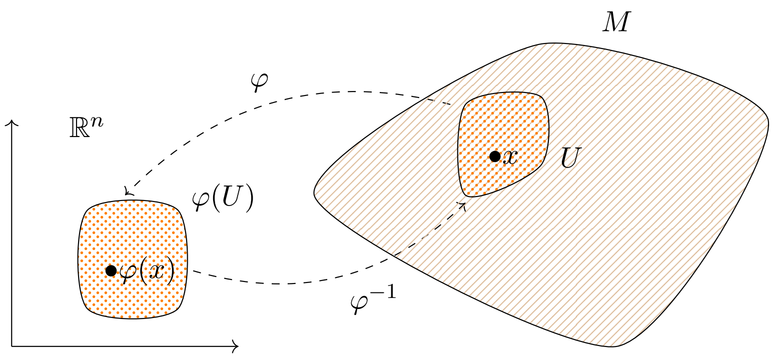

Figure 1: A local chart in a smooth manifold M is a smooth map to $\mathbb{R}^{\dim M}$.

Example 2.2. The circle $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$ is a smooth manifold of dimension 1 which can be equipped with the following charts. Since any point $z \in S^1$ can be written as $z = e^{2i\pi c}$ for a unique $c \in [0, 1)$, for any given $c \in [0, 1)$ the map

$$\begin{array}{rccc} \nu_z : & \mathbb{R} & \longrightarrow & S^1 \\ & t & \longmapsto & e^{2i\pi t} \end{array}$$

restricted to the interval $I_c = (c-1/2, c+1/2)$, namely $\mu_z = \nu_z|_{I_c}$ is a diffeomorphism from I_c to $S^1 \setminus \{-e^{2i\pi c}\}$, which is a neighbourhood of $z = e^{2i\pi c} \in S^1$. Then, $\varphi_z := \mu_z^{-1}$ is a chart of S^1 near z.

Definition 2.3. A *curve* $\gamma(t)$ on a smooth manifold M is a differentiable map from $I \subset \mathbb{R}$ to M.

Definition 2.4. The *tangent space* of the manifold M at the point $x \in M$ is $T_x(M) := \operatorname{Im} d\phi_0$, where $\phi_0(0) = \varphi_0^{-1}(0) = x$. The set of all tangent vectors on M at x is denoted by T_xM.

The *tangent bundle* of M, denoted by TM, is defined as the disjoint union of all the sets of tangent vectors, i.e.:

$$TM := \bigsqcup_{x \in M} T_x M.$$

It is equipped with the canonical projection

$$\begin{aligned} \pi : TM &\longrightarrow M \\ (x, v) &\longmapsto x \end{aligned}$$

With the same setting, the *cotangent bundle* of M, denoted by T^*M, is defined as the vector bundle over M which is dual to the tangent bundle TM. It is also equipped with the canonical projection

$$\begin{aligned} \pi : T^*M &\longrightarrow M \\ (x, p) &\longmapsto x \end{aligned}$$

Definition 2.5. The *linear tangent mapping* of a map $f : M \to N$ at $x = \gamma(0)$ (for a curve γ on M), denoted by $(df)_x$, is defined as follows. If $\gamma'(0)$ is the tangent vector to the curve $\gamma(t) \in M$, $(df)_x : T_x(M) \to T_{f(x)}N$ assigns to it the tangent vector to the curve $f(\gamma(t)) \in N$ at $t = 0$. This definition makes the following diagram commute:

$$\begin{array}{ccc} TM & \xrightarrow{\;df\;} & TN \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{\;f\;} & N \end{array}$$

where π_M and π_N are the natural projections on M and N respectively.

Definition 2.6. A *vector field* X over a manifold M is a derivation. That is, it is a $\mathbb{R}$-linear map $X : \mathcal{C}^\infty(M, \mathbb{R}) \to \mathcal{C}^\infty(M, \mathbb{R})$ such that it satisfies the *Leibniz rule*, i.e. $X(f, g) = fX(g) + X(f)g$. More explicitly, $X(f)$ is a function whose value at a point x is the directional derivative of f at x in the direction $X(x)$. A *vector* X_x at a point $x \in M$ satisfies $X_x(f, g) = f(x)X_x(g) + X_x(f)g(x)$. The set of all vector fields over a manifold is denoted by $\mathfrak{X}(M)$.

Remark 2.7. The linear tangent mapping defined before is also called the *differential of f at $x \in M$*. For a smooth vector field $X \in \mathfrak{X}(M)$, the differential acts on X exactly as $(df_x)(X) = X(f)$. It is more intuitive, though, to think that it is the directional derivative of f with respect to the vector field X. Notice that df_x is an element of $(T_xM)^*$, the dual space of T_xM.

Definition 2.8. If X is a smooth vector field on M and df_x is a linear tangent mapping, the application of df_x to X, denoted by f_*X, is called the *pushforward.*

Definition 2.9. An *integral curve* γ of a vector field $X \in \mathfrak{X}(M)$ is a curve $\gamma : I \to M$ such that
$$\frac{d\gamma}{dt}(t) = X(\gamma(t)) \quad \forall t \in I.$$

Remark 2.10. Locally, integral curves always exist. Take coordinates $(U_\alpha, \varphi_\alpha = (x_1, \ldots, x_n))$ and $X = \sum_{i=1}^n X^i \partial/\partial x_i$. Then, the equality writes as

$$\sum_{i=1}^{n} \gamma'^i(t)\frac{\partial}{\partial x_i} = \sum_{i=1}^{n} X^i(\gamma(t))\frac{\partial}{\partial x_i} \quad \Longleftrightarrow \quad \gamma'^i(t) = X^i(\gamma(t)), \quad \forall i = 1, \ldots, n,$$

which is a system of ODE's whose solution exists locally by Theorem of Existence of ODE's.

Definition 2.11. Assume M is a compact manifold. Then, the *flow* ϕ of a vector field $X \in \mathfrak{X}(M)$ is given by

$$\begin{array}{rccc} \phi: & M \times \mathbb{R} & \longrightarrow & M \\ & (x,t) & \longmapsto & \gamma_x^X(t) \end{array}$$

where $\gamma_x^X(t)$ is an integral curve of X passing through x.

It is immediate to check, from the definition, that any flow ϕ satisfies the following properties:

- $\phi(x, 0) = x$.
- $\forall t \in \mathbb{R}$, $\widetilde{\phi}_t(x) := \phi(x,t)$ is a diffeomorphism.
- $\phi(\phi(x,s),t) = \phi(x, s+t)$.

Definition 2.12. Given two vector fields $X, Y \in \mathfrak{X}(M)$, the *Lie bracket*[2] between X and Y is defined as the vector field $[X,Y]$ that assigns to each $f \in \mathcal{C}^\infty(M)$ the vector field given by
$$[X,Y](f) = X(Y(f)) - Y(X(f)).$$

Definition 2.13. A *differential* r*-form* (or a differential form of degree r, or, simply, an r-form) α at a point x on a smooth manifold M is an element $\alpha_x \in \wedge^r(T_xM)^*$, where $\wedge^r$ is the *wedge product* of r dual vector spaces. The space of all r-forms on M is denoted by $\Omega^r(M)$.

Remark 2.14. The wedge product of r-forms satisfies $\alpha \wedge \beta = (-1)^{|\alpha||\beta|}\beta \wedge \alpha$, where $|\alpha|, |\beta|$ are the degrees of the forms α and β respectively.

[2]This definition of the Lie bracket is compatible with the definition given in the context of Lie groups in Definition 2.51.

Definition 2.15. Let $X, X_2, \dots, X_r \in \mathfrak{X}$ be smooth vector fields and $\alpha \in \Omega^r(M)$ an r-form. The *interior product* $\iota_X \alpha$ of α with X is a $r-1$-form that is defined as

$$\iota_X \alpha(X_2, \dots, X_r) = \alpha(X, X_2, \dots, X_r).$$

It is also called the *contraction* between α and X.

Definition 2.16. Let α be a differential r-form on a smooth manifold M. The *exterior derivative* of α is the differential $(r+1)$-form $d\alpha$ defined in the following way. If $X_0, X_1, \dots, X_r$ are smooth vector fields defined on M, then

$$\begin{aligned} d\alpha(X_0, \dots, X_r) = &\sum_i (-1)^i X_i(\alpha(X_0, \dots, \hat{X}_i, \dots, X_r)) + \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j \dots, X_r), \end{aligned}$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields and $\hat{X}_i$ denotes the omission of the element X_i.

Remark 2.17. The exterior derivative satisfies the following properties:

1. If $U \subset M$ is open, then $\alpha|_U = \beta|_U \Rightarrow d\alpha|_U = d\beta|_U$.
2. $d^2 = 0$.
3. $d(f dx_1 \wedge \cdots \wedge dx_n) = df \wedge dx_1 \wedge \cdots \wedge dx_n$.

Remark 2.18. When a differential form $\alpha \in \Omega^r(M)$ satisfies $d\alpha = 0$, it is called a *closed form*. If $\alpha = d\beta$ for some $\beta \in \Omega^{r-1}(M)$, α is called an *exact form*.

The generalization of the Lie bracket between vector fields is the *Lie derivative*.

In the more general definition, the Lie derivative $\mathcal{L}_X R$ evaluates the change of a tensor field R along the flow of a particular vector field X on a smooth manifold M. We list the three most used Lie derivatives:

- The Lie derivative of a scalar function $f \in \mathcal{C}(M)$ with respect to a vector field X is $\mathcal{L}_X f = X(f)$, the directional derivative of f with respect to the vector field.
- The Lie derivative of a vector field $Y \in \mathfrak{X}(M)$ with respect to a vector field X is $\mathcal{L}_X Y = [X, Y]$, the Lie bracket.
- The Lie derivative of a r-form $\alpha \in \Omega^r(M)$ with respect to a vector field X is $\mathcal{L}_X \alpha = \iota_X d\alpha + d\iota_X \alpha$, an equality which is known as *Cartan's magic formula*.

Definition 2.19. Let $\varphi : M \to N$ be a smooth map and let $f : N \to \mathbb{R}$ be a smooth function. The *pullback* of f by φ is a smooth map defined by

$$(\varphi^* f)(x) := f(\varphi(x)), \qquad x \in M.$$

Let $\alpha \in \Omega^r(N)$ be a differential r-form on N. Let $X_1, \dots, X_r \in \mathfrak{X}(M)$ be smooth vector fields on M. The pullback of α by φ is a differential r-form defined by

$$(\varphi^* \alpha)_x(X_1, \dots, X_r) := \alpha_{\varphi(x)}(d\varphi_x(X_1), \dots, d\varphi_x(X_r)), \qquad x \in M.$$

We give some results on integration of differential distributions, which will be important to understand foliations later.

Lemma 2.20. *Let $X, Y \in \mathfrak{X}(M)$. Then, $[X,Y] = 0 \iff XY - YX = 0 \iff \phi_s^X \circ \phi_t^Y = \phi_t^Y \circ \phi_s^X$, where ϕ_r^Z is the flow of the vector field Z at time r.*

Proposition 2.21. *Let X, Y be two vector fields on a manifold M such that $[X,Y] = 0$. Then, there exists a smooth map $F : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \longrightarrow M$ such that:*

1. $F(0,0) = p$.
2. $dF_{(s,t)}\big|_p \left(\frac{\partial}{\partial s}\right) = X(F(s,t))\big|_p$.
3. $dF_{(s,t)}\big|_p \left(\frac{\partial}{\partial t}\right) = Y(F(s,t))\big|_p$.

Proof. Take $F = \phi_s^X \circ \phi_t^Y(p)$, which satisfies $\phi_s^X \circ \phi_t^Y = \phi_t^Y \circ \phi_s^X$ by Lemma 2.20. Then:

1. $F(0,0) = p$.
2. $dF_{(s,t)}\big|_p \left(\frac{\partial}{\partial s}\right) = \frac{d}{ds}\left(\phi_s^X\right)\left(\phi_s^X \circ \phi_t^Y\right)(p) = X(F(s,t))$.
3. $dF_{(s,t)}\big|_p \left(\frac{\partial}{\partial t}\right) = \frac{d}{dt}\left(\phi_t^Y\right)\left(\phi_t^Y \circ \phi_s^X\right)(p) = Y(F(s,t))$.

□

Definition 2.22. The image of the map $F = \phi_s^X \circ \phi_t^Y$ is called an *integral surface.*

Definition 2.23. A *differential distribution* D of rank k is the object that satisfies the following properties:

- For every $p \in M$, $D_p \leq T_pM$, i.e. D_p is a subspace of dimension k.
- For every $p \in M$, there exists a neighbourhood $U \subset M$ of p and $X_1, \ldots, X_k \in \mathfrak{X}(U)$ such that if $q \in U$, then $\langle X_1(q), \cdots, X_l(q)\rangle = D_q$.

Example 2.24. In $M = R^n$, $D = \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\rangle$ with $k \leq n$ is a distribution of rank k. In fact, it is R^k at every point.

Example 2.25. In $M = R^3$, $D = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\rangle$ is a distribution of rank 2.

Remark 2.26. A distribution of rank 1 is a vector field.

Definition 2.27. Let D be a differential distribution of rank k on a manifold M. Then, D is *integrable* if there exists a k-dimensional embedded submanifold $S \subset M$ such that for all $p \in S$, $T_pS = D_p$.

The distribution of Example 2.24 is integrable, while the distribution of Example 2.25 is not.

Definition 2.28. A distribution D on a manifold M is *involutive* if for every pair (X, Y) of vector fields locally defined on an open neighbourhood U and for every $p \in U$, $[X, Y]_p \in D_p$.

Theorem 2.29 (Frobenius Theorem)**.** *A distribution D is integrable if and only if it is involutive.*

Definition 2.30. A *leaf* of an integrable distribution is a maximal integrable submanifold.

Remark 2.31. In the definition of a leaf Λ of an integrable distribution, *maximal* means that if there exists an integrable submanifold S such that $\Lambda \cap S \neq 0$, then $S \subset \Lambda$.

Remark 2.32. The leaf Λ_p at a point p in a manifold with an integrable distribution can be defined as the set:

$$\Lambda_p = \{q \in M \mid p \text{ can reach } q \text{ following paths in the integrable submanifold}\}.$$

2.2 Complex manifolds

In Section 6 we are going to deal with complex manifolds. We give its basic notions and the construction of the tangent and cotangent bundles of a complex manifold.

Definition 2.33. Let D be an open subset of $\mathbb{C}^n$ and let $f : D \longrightarrow \mathbb{C}$ be a complex-valued function on D. We say that f is *holomorphic* in D if each point $p \in D$ has an open neighborhood U such that f writes as

$$f(z) = \sum_{k_1,\ldots,k_n=0}^{\infty} c_{k_1,\ldots,k_n}(z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n}$$

for all $z \in U$.

Remark 2.34. For a map $f : D \subset \mathbb{C}^n \longrightarrow E \subset \mathbb{C}^m$, we say it is holomorphic if the component functions $f_1, \ldots, f_m : D \longrightarrow \mathbb{C}$ are holomorphic.

Proposition 2.35. *If $f : D \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ is holomorphic, then it satisfies the Cauchy-Riemann equations. I.e., if $f = g + ih$ with $g, h : D \longrightarrow \mathbb{R}$ and $z_i = x_i + iy_i$ with $x_i, y_i \in \mathbb{R}$, then, for all $i = 1, \ldots, n$:*

$$\frac{\partial g}{\partial x_i} = \frac{\partial h}{\partial y_i} \quad \textit{and} \quad \frac{\partial g}{\partial y_i} = -\frac{\partial h}{\partial x_i}. \tag{2.1}$$

Definition 2.36. An *holomorphic atlas* $\mathcal{A} = \{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ is a covering of a topological space X by open subsets $U_\alpha \subset X_{\alpha \in A}$, together with a set of homeomorphisms $\phi_\alpha : U_\alpha \longrightarrow D_\alpha \mathbb{C}^n$ such that the functions $g_{\alpha,\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta)$ are holomorphic. Each $\phi_\alpha : U_\alpha \longrightarrow D_i$ is called a *coordinate chart.* The atlas is called *maximal* if whenever (U, z) is a local complex chart and (U, z) is compatible with $(U_\alpha, z_\alpha) \, \forall \alpha \in A$, $(U, z) \in \mathcal{A}$.

Definition 2.37. A *complex analytical structure* on a topological manifold of real dimension $2n$ is a maximal holomorphic atlas.

Definition 2.38. A *complex manifold* is a topological manifold together with a complex analytical structure.

Remark 2.39. Given a complex manifold X of complex dimension n with atlas $\mathcal{A} = \{(U_\alpha, z_\alpha)\}_{\alpha \in A}$, one can always consider the underlying real differentiable manifold X_0 of real dimension $2n$ with the coordinates inherited from the complex structure on X.

Definition 2.40. A *complex structure* on a real vector space V is a $\mathbb{R}$-linear endomorphism J such that $J \circ J = -Id$.

Example 2.41. The product by i on $\mathbb{C}^n$:

$$\cdot i : \mathbb{C}^n \longrightarrow \mathbb{C}^n \tag{2.2}$$

$$(\dots, x_i + iy_i, \dots) \longmapsto (\dots, ix_i - y_i, \dots) \tag{2.3}$$

induces the following complex structure J on $\mathbb{C}^n \cong \mathbb{R}^{2n}$:

$$J : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n} \tag{2.4}$$

$$(\dots, x_i, \dots, y_i, \dots) \longmapsto (\dots, -y_i, \dots, x_i, \dots) \tag{2.5}$$

And the following diagram commutes

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\cdot i} & \mathbb{C}^n \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{R}^{2n} & \xrightarrow{J} & \mathbb{R}^{2n} \end{array}$$

In matrix form, the linear endomorphism J writes as

$$J : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n} \tag{2.6}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \bar{J} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \tag{2.7}$$

with $\bar{J} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, which has eigenvalues $\pm i$ with multiplicity n and has the following corresponding eigenspaces:

$$E_{+i} = \left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ -i \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -i \end{pmatrix} \right\rangle \quad \text{and} \quad E_{-i} = \left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ i \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ i \end{pmatrix} \right\rangle \tag{2.8}$$

Notice that $\dim_{\mathbb{C}} E_{+i} = \dim_{\mathbb{C}} E_{-i} = n$ and that $E_{+i} \oplus E_{-i} = \mathbb{C}^{2n}$.

Consider a complex manifold X of complex dimension n. It is not obvious which is the tangent space to X at a point for the next reason. Suppose that X_0 is its underlying real differential manifold of dimension $2n$. For any $a = (x_1, \dots, x_n, y_1, \dots, y_n) \in U \subset X_0$, the tangent space of X_0 at a is defined as $T_a X_0 := \operatorname{Im} d\phi_0$, where $\phi_0(0) = \varphi_0^{-1}(0) = x$ and φ_0 is a chart of X_0 centered at 0. The canonical basis of $T_a X_0 \cong \mathbb{R}^{2n}$ is

$$\{\frac{\partial}{\partial x_1}\bigg|_a, \dots, \frac{\partial}{\partial x_n}\bigg|_a, \frac{\partial}{\partial y_1}\bigg|_a, \dots, \frac{\partial}{\partial y_n}\bigg|_a\}.$$

The complexification of $T_a X_0$, i.e., $(T_a X_0)^{\mathbb{C}} = T_a X_0 \otimes_{\mathbb{R}} \mathbb{C} = T_a X_0 \oplus i T_a X_0$, has then real dimension $4n$, twice the dimension of X. In consequence, it is natural to take as the tangent space of X at a only a part of $(T_a X_0)^{\mathbb{C}}$, in order to deal with something similar to the tangent space for real differential manifolds.

The complex structure $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ of Example 2.41 allows us to split $(T_a X_0)^{\mathbb{C}}$ into the sum of two convenient subspaces. By identification of $T_a X_0$ with R^{2n}, J induces the map J_a

$$J_a : \begin{cases} \frac{\partial}{\partial x_i}\Big|_a \longmapsto \frac{\partial}{\partial y_i}\Big|_a \\ \frac{\partial}{\partial y_i}\Big|_a \longmapsto -\frac{\partial}{\partial x_i}\Big|_a \end{cases},$$

which is equivalent to J on $T_a X_0$.

Exactly as in Example 2.41, by C-linear extension of J_a on $(T_a X_0)^{\mathbb{C}}$ we obtain the following eigenspaces

$T_a^{1,0} X := \{w \in (T_a X_0)^{\mathbb{C}} \mid J_a w = iw\}$, the *holomorphic tangent space* and,
$T_a^{0,1} X := \{w \in (T_a X_0)^{\mathbb{C}} \mid J_a w = -iw\}$, the *anti-holomorphic tangent space.*

for which we can take, respectively, the basis

$$\frac{\partial}{\partial z_i}\bigg|_a = \frac{1}{2}\left(\frac{\partial}{\partial x_i}\bigg|_a - i\frac{\partial}{\partial y_i}\bigg|_a\right), \quad \frac{\partial}{\partial \bar{z}_i}\bigg|_a = \frac{1}{2}\left(\frac{\partial}{\partial x_i}\bigg|_a + i\frac{\partial}{\partial y_i}\bigg|_a\right), \quad i = 1, \dots, n$$

Notice that $(T_a X_0)^{\mathbb{C}} = T_a^{1,0} X \oplus T_a^{0,1} X$, and that $\dim_{\mathbb{C}} T_a^{1,0} X = \dim_{\mathbb{C}} T_a^{0,1} X = n$.

Lemma 2.42. *Let $f : D \subset \mathbb{C}^n \to \mathbb{C}$ and suppose $f = g + ih$ with $g, h : D \longrightarrow \mathbb{R}$. Then, the Cauchy-Riemann Equations (2.1) are equivalent to $\frac{\partial f}{\partial \bar{z}_i} = 0 \; \forall i = 1, \ldots, n$.*

Proof. For all $i = 1, \ldots, n$:

$$\frac{\partial}{\partial \bar{z}_i} f = \frac{\partial}{\partial \bar{z}_i}(g + ih) = \frac{1}{2}\left(\frac{\partial}{\partial x_i}(g + ih) + i\frac{\partial}{\partial y_i}(g + ih)\right) \tag{2.9}$$

$$= \frac{1}{2}\left(\left(\frac{\partial}{\partial x_i}(g) - \frac{\partial}{\partial y_i}(h)\right) + i\left(\frac{\partial}{\partial x_i}(h) + \frac{\partial}{\partial y_i}(g)\right)\right). \tag{2.10}$$

Hence,

$$\frac{\partial}{\partial \bar{z}_i} f = 0 \quad \Longleftrightarrow \quad \frac{\partial g}{\partial x_i} = \frac{\partial h}{\partial y_i}, \frac{\partial g}{\partial y_i} = -\frac{\partial h}{\partial x_i}. \tag{2.11}$$

Each partial derivative is evaluated at a, which we omit for simplicity. □

Remark 2.43. If f is an holomorphic function, $\frac{\partial f}{\partial \bar{z}_i} = 0$ for any i. Then, if $w \in T_a^{0,1}X$, $w(f) = 0$. Since we usually want to work with holomorphic functions, it makes sense to take $T_a^{1,0}X$ as the tangent space of X at a.

Definition 2.44. Let X be a complex manifold and take $a \in X$. We define the *holomorphic tangent space* or, simply, the *tangent space* of X at a as $T_a^{1,0}X$.

Definition 2.45. The *holomorphic tangent bundle* or, simply, the *tangent bundle* of a complex manifold X is defined as $TX = \bigsqcup_{a \in X} T_a^{1,0}X = \{(a, w) \mid a \in X, w \in T_a^{1,0}X\}$, and it is equipped with the natural projection $\pi : (a, w) \mapsto a : TX \to X$.

The cotangent space at a point in a complex manifold is defined as the dual space to the tangent space, analogously to what it is done in the real differential case.

Definition 2.46. Let a be a point in a complex manifold X and consider $T_a^{1,0}X$, the tangent space of X at a. The dual space of $T_a^{1,0}X$, i.e. $(T_a^{1,0}X)^*$ is the *holomorphic cotangent space* or, simply, the *cotangent space* of X at a.

Although the cotangent space is already well-defined, we are going to prove that we can also define it from the cotangent space of a real differentiable manifold in the same way we constructed the tangent space, and that the two definitions are equivalent.

Take again X_0, the underlying real differential manifold of dimension $2n$ corresponding to X. For any $a = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in U \subset X_0$, the cotangent space of X_0 at a is defined as $T_a^*X_0 := \mathrm{Hom}_{\mathbb{R}}(T_aX_0, \mathbb{R})$, where T_aX_0 is the tangent space of X at a. The canonical basis of $T_a^*X_0 \cong \mathbb{R}^{2n}$ is

$$\{(dx_1)_a, \ldots, (dx_n)_a, (dy_1)_a, \ldots, (dy_n)_a\}.$$

The complexification of $T_a^*X_0$ is $(T_a^*X_0)^{\mathbb{C}} = T_a^*X_0 \otimes_{\mathbb{R}} \mathbb{C} = T_a^*X_0 \oplus iT_a^*X_0$. Again, it has real dimension $4n$ and we want to take as the cotangent space of X at a only a subspace of $(T_a^*X_0)^{\mathbb{C}}$ of dimension 2.

The splitting of $(T_a^*X_0)^{\mathbb{C}}$ into the sum of two subspaces is analogous to the one applied for the complexified tangent space. If we consider $(dz_i)_a = (dx_i)_a + i(dy_i)_a$ and $(d\bar{z}_i)_a = (dx_i)_a - i(dy_i)_a$ we can define:

$(T_a^*X)^{1,0} := \langle (dz_1)_a, \ldots, (dz_n)_a \rangle$, the *holomorphic cotangent space*, and
$(T_a^*X)^{0,1} := \langle d(\bar{z}_1)_a, \ldots, (d\bar{z}_n)_a \rangle$, the *anti-holomorphic cotangent space.*

It is clear that $(T_a^*X_0)^{\mathbb{C}} = (T_a^*X)^{1,0} \oplus (T_a^*X)^{0,1}$ and that

$$dz_i\left(\frac{\partial}{\partial z_j}\right) = \delta_{ij},\ d\bar{z}_i\left(\frac{\partial}{\partial \bar{z}_j}\right) = \delta_{ij},\ d\bar{z}_i\left(\frac{\partial}{\partial z_j}\right) = dz_i\left(\frac{\partial}{\partial \bar{z}_j}\right) = 0. \tag{2.12}$$

where we omit the evaluation at a. Then, we have proved the following.

Proposition 2.47. *Let $a \in X$ a point in a complex manifold. Then, the cotangent space $(T_a^{1,0}X)^*$ of X at a coincides with $(T_a^*X)^{1,0}$.*

If $f : D \subset \mathbb{C}^n \longrightarrow \mathbb{C}$, we can write

$$(df)_a = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(dx_i)_a + \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(a)(dy_i)_a,$$

or, in the new basis,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a)(dz_i)_a + \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i}(a)(d\bar{z}_i)_a) = \partial f + \bar{\partial} f.$$

Since any holomorphic function f satisfies $\bar{\partial} f$ and we want $(df)_a$ to be an element of the cotangent space, it becomes clear why we chose $(T_a^*X)^{1,0} = \langle (dz_1)_a, \ldots, (dz_n)_a \rangle$ as the cotangent space of X at a.

We end this section with the most basic example of a complex manifold.

Example 2.48. Take $X = \mathbb{C}$, a complex manifold with coordinates $z = x + iy$. The tangent space at $m \in \mathbb{C}$ is:

$$T_m^{1,0}\mathbb{C} = \left\langle \frac{\partial}{\partial z}\bigg|_m \right\rangle = \left\langle \frac{\partial}{\partial x}\bigg|_m - i\frac{\partial}{\partial y}\bigg|_m \right\rangle.$$

So, if $w \in T_m^{1,0}\mathbb{C}$, then:

$$w = (a+ib)(\frac{\partial}{\partial x}\bigg|_m - i\frac{\partial}{\partial y}\bigg|_m) = \left(a\frac{\partial}{\partial x}\bigg|_m + b\frac{\partial}{\partial y}\bigg|_m\right) + i\left(b\frac{\partial}{\partial x}\bigg|_m - a\frac{\partial}{\partial y}\bigg|_m\right).$$

The cotangent space at m is:

$$(T_m^*\mathbb{C})^{1,0} = \langle (dz)_m \rangle = \langle (dx)_m + i(dy)_m \rangle.$$

So, if $\omega \in (T_m^*\mathbb{C})^{1,0}$, then:

$$\omega = (a+ib)\left((dx)_m + i(dy)_m\right) = \left(a(dx)_m - b(dy)_m\right) + i\left(b(dx)_m + a(dy)_m\right).$$

2.3 Lie groups theory

In this section, we introduce the basics of Lie groups and Lie group actions, one of the most important tools in differential geometry and mathematical physics.

Definition 2.49. A finite-dimensional smooth manifold G is called a *Lie group* if it has a group structure and if its group operations

$$\mu : (x, y) \mapsto x \cdot y : G \times G \to G,$$

$$\iota : x \mapsto x^{-1} : G \to G,$$

i.e. the group product and the group inversion, are smooth.

Some examples of Lie groups are $\mathbb{R}$ and $\mathbb{C}$ equipped with the addition operation and $\mathbf{GL}(n, \mathbb{R})$ and $\mathbf{GL}(n, \mathbb{C})$ equipped with the matrix product. A classical set of Lie groups is the family of matrix Lie groups.

Definition 2.50. A *matrix Lie group* is a closed subgroup G of $\mathbf{GL}(n, \mathbb{C})$.

The most studied Lie groups are, in fact, matrix Lie groups. Some examples of matrix Lie groups are:

- $SL(n, \mathbb{R}) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid \det A = 1\}$,
- $O(n, \mathbb{R}) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid A^t A = I\}$,
- $SO(n, \mathbb{R}) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid A^t A = I, \det A = 1\}$,
- $U(n) = \{A \in \mathbf{GL}(n, \mathbb{C}) \mid A^* A = I\}$,
- $SU(n) = \{A \in \mathbf{GL}(n, \mathbb{C}) \mid A^* A = I, \det A = 1\}$
- $Sp(2n, \mathbb{R}) = \{A \in \mathbf{GL}(2n, \mathbb{R}) \mid A^t J A = J\}$, with J a nonsingular skew-symmetric matrix (the group of symplectic matrices).

Associated to any Lie group, there is a *Lie bracket* and a *Lie algebra*. For any G Lie group, $T_e G$, the tangent space of the Lie group G at the identity element e, is denoted by $\mathfrak{g}$. By definition, $\mathfrak{g}$ is a real vector space. Equipped with the addition, it is a commutative Lie group of the same dimension as G.

Definition 2.51. Let V be a vector space. The bilinear map $[\cdot, \cdot] : V \times V \to V$ that satisfies $[X, X] = 0$ and $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ (the *Jacobi identity*) for any $X, Y, Z \in V$ is called the *Lie bracket*.

Definition 2.52. Let G be a Lie group. Then, the tangent space of G at the identity, $\mathfrak{g} = T_e G$, provided with the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, is the *Lie algebra* associated to G.

Remark 2.53. Lie algebra $\mathfrak{g}$ of a group G can be thought of the linearization of G near the identity element.

Example 2.54 (Computation of the Lie algebra of $SL(2,\mathbb{R})$). To compute of $\mathfrak{sl}(2,\mathbb{R})$, the Lie algebra of $SL(2,\mathbb{R})$, suppose $X \in \mathfrak{sl}(2,\mathbb{R})$. Then, X can be represented by the tangent vector at 0 of the path $\gamma : (-\varepsilon,\varepsilon) \subset \mathbb{R} \to SL(2,\mathbb{R})$ such that $\gamma(0) = e_{SL(2,\mathbb{R})} = I$. That is, $X = \gamma'(0)$.

Then, $\gamma(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$.

Since $\gamma(0) = I$, $a(0) = 1$, $b(0) = 0$, $c(0) = 0$, $d(0) = 1$. Since $SL(2,\mathbb{R}) = \{A \in \mathbf{GL}(2,\mathbb{R}) \mid \det A = 1\}$ and $\gamma(t) \in SL(2,\mathbb{R})$, $a(t)d(t) - b(t)c(t) = 1$ for all $t \in (-\varepsilon,\varepsilon)$. Deriving this expression, it follows that $a'(t)d(t) + a(t)d'(t) - b'(t)c(t) - b(t)c'(t) = 0$. Evaluating at $t = 0$, the following holds: $a'(0) + d'(0) = 0$ (equivalently, the trace of $\gamma'(0)$ is 0). Then,

$$X = \gamma'(0) = \begin{pmatrix} a'(0) & b'(0) \\ c'(0) & -a'(0) \end{pmatrix}.$$

We conclude that $\mathfrak{sl}(2,\mathbb{R}) = \{A \in \mathbf{GL}(2,\mathbb{R}) \mid tr(A) = 0\}$.

Some examples of Lie algebras are:

- $\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathbf{GL}(n,\mathbb{R}) \mid tr(A) = 0\}$ is the Lie algebra of $SL(n,\mathbb{R})$,
- $\mathfrak{o}(n,\mathbb{R}) = \{A \in \mathbf{GL}(n,\mathbb{R}) \mid A^t + A = 0\}$ is the Lie algebra of $O(n,\mathbb{R})$,
- $\mathfrak{u}(n) = \{A \in \mathbf{GL}(n,\mathbb{C}) \mid A^* + A = 0\}$ is the Lie algebra of $U(n)$.

For all these Lie algebras, as well as for any other Lie algebra associated to a matrix Lie group, the Lie bracket of two elements X, Y of the algebra takes the form $[X,Y] = XY - YX$.

Taking into account that a Lie group is, in particular, a smooth manifold, it makes sense to consider the derivative of a Lie group homomorphism. The following result shows that a homomorphism of Lie groups induces a homomorphism of Lie algebras.

Proposition 2.55. *Let $\phi : G \to H$ be a Lie group homomorphism which is differentiable at e_G. Then, the differential $d\phi_{e_G} : T_{e_G}G = \mathfrak{g} \to T_{e_H}H = \mathfrak{h}$ is a Lie algebra homomorphism, i.e, a linear map that satisfies $d\phi([X,Y]_{\mathfrak{g}})_{e_G} = [d\phi(X)_{e_G}, d\phi(y)_{e_G}]_{\mathfrak{h}}$ for all $X, Y \in \mathfrak{g}$.*

Proof. Take $x \in G$ and consider the map $\mathbf{Ad}_x : y \mapsto xyx^{-1} : G \to G$, the *$x$-conjugation* map. Its derivative at e_G is $\mathrm{Ad}_x := d(\mathbf{Ad}_x)_{e_G} : \mathfrak{g} \to \mathfrak{g}$, the *$x$-adjoint* map.

Then, the map $\mathrm{Ad} : x \mapsto \mathrm{Ad}_x : G \to \mathbf{GL}(\mathfrak{g})$, where $\mathbf{GL}(\mathfrak{g})$ is the Lie group of all bijective linear maps on $\mathfrak{g}$, is a homomorphism of groups. Its derivative at e_G is $\mathrm{ad} := d(\mathrm{Ad})_{e_G} : \mathfrak{g} \to \mathbf{L}(\mathfrak{g},\mathfrak{g})$, where $\mathbf{L}(\mathfrak{g},\mathfrak{g})$ is the vector space of all linear mappings: $\mathfrak{g} \to \mathfrak{g}$.

The Lie bracket can be now reformulated in the following way:

$$[X, Y] := (\mathrm{ad}X)(Y) \quad \forall X, Y \in \mathfrak{g},$$

which is a definition equivalent to 2.52.

Since ϕ is a homomorphism, it is clear that:

$$\phi((\mathbf{Ad}_x)(y)) = \phi(xyx^{-1}) = \phi(x)\phi(y)\phi(x)^{-1} = ((\mathbf{Ad}_{\phi(x)})(\phi(y)).$$

Differentiating on both sides with respect to y at $y = e_G$ and taking the direction $Y \in \mathfrak{g}$:

$$d\phi(\mathrm{Ad}_x(Y))_{e_G} = (\mathrm{Ad}_{\phi(x)})(d\phi(Y)_{e_G}).$$

Differentiating on both sides with respect to x at $x = e_G$ and taking the direction $X \in \mathfrak{g}$:

$$d\phi((\mathrm{ad}X)(Y))_{e_G} = (\mathrm{ad}(d\phi(X)_{e_G}))(d\phi(Y)_{e_G}),$$

which is equivalent to

$$d\phi([X, Y]_{\mathfrak{g}})_{e_G} = [d\phi(X)_{e_G}, d\phi(y)_{e_G}]_{\mathfrak{h}}.$$

□

What is interesting about Lie algebra $\mathfrak{g}$ and the Lie Bracket is that they make it possible to recover the local structure of the Lie group G. Then, it is common that the study of a group G is done through the study of $\mathfrak{g}$, which is linear.

A principal connection between the Lie algebra $\mathfrak{g}$ and the Lie group G is the *exponential map*. Before introducing it, it is necessary to talk about left-invariance and right-invariance of vector fields.

For any element x of any group G, it is possible to define the left and right multiplications by x, respectively, by

$$\mathrm{L}_x : y \mapsto xy : G \to G,$$

$$\mathrm{R}_x : y \mapsto yx : G \to G.$$

Definition 2.56. A vector field X defined on a Lie group G is called *left invariant* if, for all $x, y \in G$,

$$X(\mathrm{L}_x y) = T_y(\mathrm{L}_x X(y)),$$

and it is called *right invariant* if, for all $x, y \in G$,

$$X(\mathrm{R}_x y) = T_y(\mathrm{R}_x X(y)).$$

In the particular case of taking $y = e_G$, the two conditions in 2.56 define two special left and right invariant vector fields, X^{L} and X^{R}, which satisfy, respectively, $X^{\mathrm{L}}(x) = T_e(\mathrm{L}_x X(e))$ and $X^{\mathrm{L}}(x) = T_e(\mathrm{R}_x X(e))$.

Definition 2.57. The *exponential map* from $\mathfrak{g}$ into G is defined as

$$exp : X \mapsto h(1) : \mathfrak{g} \to G,$$

where $h_X : (\mathbb{R}, +) \to (G, \cdot)$ is the differentiable homomorphism such that $\frac{dh_X}{dt}(0) = X$.

With the definition of the exponential map, Proposition 2.55 can be extended to the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\phi_{e_G}} & \mathfrak{h} \\ {\scriptstyle \exp_G}\downarrow & & \downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow{\phi} & H \end{array}$$

The following theorem proves that the exponential map is well-defined.

Theorem 2.58. *Given $X \in \mathfrak{g}$, there is a unique homomorphism $h_X : (\mathbb{R}, +) \to (G, \cdot)$ which is differentiable at $t = 0$ and its differential there ($\frac{dh_X}{dt}(0)$) is equal to X. It is exactly the integral curve of X^{L} and also of X^{R} that starts at $e \in G$, and it is complete, in the sense that it is defined for all $t \in \mathbb{R}$.*

Proof. Consider $X \in \mathfrak{g}$ and ϕ^t, the flow of $X^{\mathrm{L}}(x) = T_e(\mathrm{L}_x X(e))$, which is left invariant. It satisfies $\phi^t = \mathrm{R}_{(\phi^t(1))}$ [DK99]. Then, it is clear that $\phi^{t+s}(1) = \phi^s(\phi^t(1)) = \phi^t(1)\cdot\phi^s(1)$. This relation shows that, given t, if $\phi^t(1)$ is defined, then, for some $\varepsilon > 0$, $\phi^s(1)$ is defined for $s \in (t-\varepsilon, t+\varepsilon)$. This implies that ϕ^t is a differentiable homomorphism defined for all $t \in \mathbb{R}$, and we can denote it by $h_X : t \mapsto \phi^t(1) : (\mathbb{R}, +) \to (G, \cdot)$. An analogous reasoning shows that the flow of $X^{\mathrm{R}}(x) = T_e(\mathrm{L}_x X(e))$ defines the same homomorphism h_X.

Now, take $h = h_X : (\mathbb{R}, +) \to (G, \cdot)$ such that h is differentiable at $t = 0$ and its differential there is equal to X. Since h is a diffeomorphism around 0, it is possible to differentiate the expression:

$$h(s)h(t) = h(s+t) = h(t+s) = h(t)h(s)$$

respect to s and evaluate at $s = 0$.

From the leftmost equality, one obtains that:

$$\frac{dh}{dt}(t) = T_e\mathrm{R}_{(h(t))}(X) = X^{\mathrm{R}}(h(t))),$$

indicating that $h(t)$ is an integral curve of X^{R}. Since $h(0) = 1$, h is uniquely determined by X.

From the rightmost equality, one concludes that $h(t)$ is an integral curve of X^{L}, uniquely determined by X. □

The exponential map is a smooth map and locally it is a diffeomorphism. It has many other "nice" properties that are not of use in this work but can be found on [DK99] in great detail.

We introduce now Lie group actions.

Definition 2.59. Let G be a Lie group and let M be a smooth manifold. A *Lie group action* of G on M is a smooth collection of smooth maps $\rho_g : M \to M$ with $g \in G$ that satisfies $\rho_{g \cdot h} = \rho_g \circ \rho_h$ and $\rho_e = id$, where e is the identity element of G. The associated map

$$\rho : (g, m) \mapsto g \cdot m : G \times M \to M,$$

is smooth.

Example 2.60. Let G be a Lie group and consider the action α of G on itself defined by left multiplication:

$$\rho : (g_1, g_2) \mapsto g_1 \cdot g_2 : G \times G \to G.$$

It is clear that it defines a Lie group action because:

1. The induced mapping $\rho_{g_1} : g_2 \mapsto g_1 \cdot g_2 : G \to G$ is a diffeomorphism for any $g_1, g_2 \in G$.

2. The induced mapping satisfies $\rho_{g_1 \cdot g_2}(g_3) = (g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) = \rho_{g_1} \circ \rho_{g_2}(g_3)$ for any $g_1, g_2, g_3 \in G$.

Example 2.61. Let M be a compact manifold and $X \in \mathfrak{X}(M)$ (X is a vector field defined on M). Then, the flow (for definition of flow, see Section 2.7) of X at any point $x \in M$, $\phi : (t, x) \mapsto \phi_t(x) : \mathbb{R} \times M \to M$ defines a Lie group action because:

1. $\phi_t(x)$ is defined for all $t \in \mathbb{R}$, because M is compact.

2. $\phi_t(x)$ is a diffeomorphism.

3. $\phi_{t+s}(x) = \phi_t \circ \phi_s(x)$.

Example 2.62. Take $G = \mathbf{GL}(n, \mathbb{R})$ and $M = \mathbb{R}^n$. Then

$$\begin{aligned} \rho : \mathbf{GL}(n, \mathbb{R}) \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (A, x) &\longmapsto A \cdot x \end{aligned}$$

defines a Lie group action because, for any $A, B \in (n, \mathbb{R})$:

1. ρ_A is a differentiable map and its inverse is $\rho_{A^{-1}}$, which is also differentiable.

2. $\rho_{AB}(x) = (AB)x = A(Bx) = \rho_A \circ \rho_B(x)$.

Example 2.63. Take $G = S^1$ and $M = S^2$, the unit sphere on $\mathbb{R}^3$. Consider cylindrical coordinates (r, θ, z). The map

$$\begin{aligned} \rho : S^1 \times S^2 &\longrightarrow S^2 \\ (t, (r, \theta, z)) &\longmapsto (r, \theta + t, z) \end{aligned}$$

is a Lie group action. The reader can check that, for a fixed $t \in S^1$, ρ_t is a diffeomorphism and satisfies the composition property.

When dealing with Hamiltonian dynamical systems, it is usual to consider non-compact Lie groups. Then, it is useful to have a less strong condition than compactness for Lie group actions.

Definition 2.64. A *Lie group action* $\rho : G \times M \to M$ is *proper* if the action map

$$\rho : (g, m) \mapsto (g \cdot m, m) : G \times M \to M \times M,$$

is proper, in the sense that the preimage of any compact set is a compact set.

In the following list there are some properties of proper Lie group actions, and many others can be found on [GGK02].

- If the Lie group G is compact, the action on the manifold M is proper.
- The restriction of a proper action to any closed subgroup $H \subset G$ is a proper action on M.
- The restriction of a proper action to any invariant subset $U \subset M$ is a proper action U.
- If the action is proper, the *evaluation* map

  $$ev_m : G \to M, \qquad g \mapsto g \cdot m$$

  is proper $\forall m \in M$.

Like for any group action, it is possible to define *stabilizers* and *orbits* of the points in the manifold.

Definition 2.65. Let $\rho : G \times M \to M$ be a Lie group action. For each $m \in M$, the *stabilizer* or *isotropy group* of m is

$$G_m = \{g \in G \mid g \cdot m = m\} \subset G.$$

The stabilizer G_m is always a Lie group and, if the action is proper, G_m is compact. If $G_m = \{e\}, \forall m \in M$, the action is called *free* action, while it is called *locally free* action if all the G_m's are discrete.

Definition 2.66. Let $\rho : G \times M \to M$ be a Lie group action. For each $m \in M$, the *orbit* of m is

$$G \cdot m = \{g \cdot m \mid g \in G\} \subset M.$$

The manifold M can be partitioned in orbits, as the evaluation map $ev_m : g \mapsto g \cdot m$ induces a bijection between the quotient G/G_m and the orbit $G \cdot m$. The quotient set M/G is precisely the set of orbits in which M decomposes.

Example 2.67. Take $G = S^1$ and $M = S^2$ and consider the action ρ in Example 2.63. The diffeomorphism ρ_t can be interpreted as the planar rotation of S^2 of angle t around the z-axis. Then, it is clear that M/G, the set of orbits, is the interval $[-1, 1]$.

Example 2.68. Like in Example 2.61, take M a compact manifold and $X \in \mathfrak{X}(M)$. The action of $\mathbb{R}$ on M given by the flows of X induces a natural partition of M into orbits of the form $\gamma_x = \{(\phi^t(x)) \mid t \in T\}$. That is the reason for the definition of orbit of a dynamical system in Section 2.7.

Lemma 2.69. *Let $\rho : G \times M \to M$ be a proper Lie group action. Then, every orbit $G \cdot m$ is a closed subset of M.*

Proof. A proper map between smooth manifolds is closed. Then, as the action is proper, the evaluation map $ev_m : g \mapsto g \cdot m$ is proper and closed. Hence, every orbit is closed. $\square$

Definition 2.70. Let X be a topological space. A *σ-algebra* on X is a family $\mathcal{M}$ of subsets of X such that $\emptyset \in \mathcal{M}$, for every $S \in \mathcal{M}$, $X \setminus S \in \mathcal{M}$ and if $S_1, S_2, \dots \in \mathcal{M}$ is a countable family, then $\cup_{i=1}^{\infty} S_i \in \mathcal{M}$. The smallest of such σ-algebras that contains all the open subsets of X is called *Borel σ-algebra* on X.

Definition 2.71. Let X be a topological space with a σ-algebra $\mathcal{M}$. A *measure* μ on X is a mapping $\mu : \mathcal{M} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that, for every $S_1, S_2, \dots \in \mathcal{M}$ countable family of pairwise disjoint elements,

$$\mu(\cup_{i=1}^{\infty} S_i) = \sum_{i=1}^{\infty} \mu(S_i),$$

where we assume that $a + \infty = \infty$ for any $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. If $\mathcal{M}$ is a Borelσ-algebra, μ is called a *Borel measure.*

If X is a locally compact Hausdorff space, and μ a Borel measure on X, we say that μ is *regular* if, for every element S of the Borel σ-algebra on X, the following two conditions are satisfied:

$$\mu(S) = \inf\{\mu(U) \mid A \subset U, U \, \text{open}\},$$

$$\mu(S) = \sup\{\mu(K) \mid K \subset A, K \, \text{compact}\}.$$

Definition 2.72. Let G be a locally compact group. The *left Haar measure* on G is a regular Borel measure μ on G that is finite on compact subsets of the Borel algebra and that is left invariant, i.e., $\mu(g \cdot S) = \mu(S)$ for every $g \in G$ and every element S of the Borel σ-algebra of G.

The *right Haar measure* can be defined in an analogous way. In every compact Lie group G, every left invariant measure is also a right invariant measure, and vice versa [DK99]. Although it is not a direct result, and we will not prove it (a proof can be found on [DK99]), the Haar measure exists on every locally compact Lie group.

Theorem 2.73. *Let G be a locally compact group. Then, there exists a left (and right) Haar measure μ on G which is unique up to a scalar multiple. Moreover, if G is compact, there is a unique Haar measure such that the integration over all G is 1 (i.e. $\int_G 1 d\mu = 1$, and it is called the normalized Haar measure on G (or, simply, the Haar measure on G).*

The definition of the Haar measure is quite technical but, for instance, the Haar measure on $\mathbb{R}^n$ is the usual Lebesgue measure. This way, equipping a Lie group G with this measure and using the general theory of Lebesgue integration, it is possible to define the integral of all Borel measurable functions f on G.

Example 2.74. Take $G = (\mathbb{R} \setminus \{0\}, \cdot)$. Then, a Haar measure μ is

$$\mu(S) = \int_S \frac{1}{|x|} dx,$$

for any Borel subset S of G. If, for instance, $S = [a, b]$, with $0 < a < b$,

$$\mu(S) \int_a^b \frac{1}{|x|} dx = \log(b/a).$$

Now, if the interval S is multiplied by an element $g \in G$, the measure of gS is

$$\mu(gS) = \log((gb)/(ga)) = \log((b)/(a)) = \mu(S).$$

So, indeed, μ is left invariant (it is also right invariant).

Example 2.75. If $G = \mathbf{GL}(n, \mathbb{R})$ a Haar measure for any for any Borel subset S of G is

$$\mu(S) = \int_S \frac{1}{|det(X)|^n} dX,$$

where dX is the Lebesgue measure on $\mathbb{R}^{n^2}$.

The uniqueness of the Haar measure stated on Theorem 2.73, when the Lie group is compact, makes it possible to apply averaging arguments.

Definition 2.76. Let G be a compact Lie group. Then, the integration of any map $f \in \mathcal{C}(G)$ over all the elements of G with respect to the Haar measure μ, i.e.:

$$f \mapsto \int_G f \, d\mu : \mathcal{C}(G) \to \mathbb{R},$$

is called *averaging* over G.

We are going to use averaging, in particular, in the proof of Theorem 5.5.

For many real Lie groups it is possible to consider their complexification, such as we did with differential manifolds. We have the following definition of the complexification of a compact Lie group.

Definition 2.77. Let K be a compact Lie group. An *analytic complexification* of K is a complex analytic group G together with a Lie group homomorphism $i : K \longrightarrow G$ such that, if $f : K \longrightarrow H$ is another Lie group homomorphism into a complex analytic group H, then there exists a unique analytic homomorphism $F : G \longrightarrow H$ such that $f = F \circ i$.

In the same way, we can consider the complexification of a Lie algebra, which is easier to define because it is only the complexification of a real vector space. To get from a real Lie algebra representation to a complex one, we extend the action of real scalars to complex scalars. In the case of real matrices, complexification is essentially allowing complex coefficients and using the same rules for multiplying matrices as before.

Definition 2.78. The *complexification* $V^{\mathbb{C}}$ of a real vector space V is the space of pairs (v_1, v_2) of elements of V with product by $a + ib \in \mathbb{C}$ given by

$$(a + ib)(v_1, v_2) = (av_1 - bv_2, av_2 + bv_1)$$

This definition makes it possible to think of the complexification of V as $V^{\mathbb{C}} = V + iV$. Now, for any real Lie algebra $\mathfrak{g}$, the complexification $\mathfrak{g}^{\mathbb{C}}$ is the set of pairs of elements (X, Y) of $\mathfrak{g}$, with the usual rule for the product by complex scalars, which can be thought of as $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$.

The Lie bracket on $\mathfrak{g}$ extends in a natural way to a Lie bracket on $\mathfrak{g}^{\mathbb{C}}$ by:

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2] - [Y_1, Y_2], [X_1, Y_2] + [Y_1, X_2]),$$

which can be thought as the following computation:

$$[X_1 + iY_1, X_2 + iY_2] = [X_1, X_2] - [Y_1, Y_2] + i([X_1, Y_2] + [Y_1, X_2])$$

Example 2.79. The Lie group $G = \mathbf{GL}(n, \mathbb{R})$ has the Lie algebra $\mathfrak{gl}(n, \mathbb{R}) \subset \mathfrak{gl}(n, \mathbb{C})$ of real $n \times n$ matrices. Its complexification is nothing else than $\mathfrak{gl}(n, \mathbb{C})$, since $\mathfrak{gl}(n, \mathbb{R})^{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{R}) + i\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{gl}(n, \mathbb{C})$.

Example 2.80. The Lie group $U(n)$ has the Lie algebra $\mathfrak{u}(n) \subset \mathfrak{gl}(n, \mathbb{C})$ of anti-Hermitian matrices. Since the product of the anti-Hermitian matrices by i gives the Hermitian matrices, the complexification $\mathfrak{u}(n)^{\mathbb{C}}$ of $\mathfrak{u}(n)$ is exactly $\mathfrak{gl}(n, \mathbb{C})$.

Remark 2.81. With these two examples, one can see that different Lie algebras can have the same complexification.

Example 2.82. The Lie groups $O(n,\mathbb{R})$ and $SO(n,\mathbb{R})$ have the same Lie algebra, since $SO(n,\mathbb{R})$ is the connected component of $O(n,\mathbb{R})$ that contains the identity. The complexification of the Lie algebra $\mathfrak{so}(n,\mathbb{R})$ of the real anti-symmetric matrices is naturally the Lie algebra of the complex anti-symmetric matrices $\mathfrak{so}(n,\mathbb{C}) \subset \mathfrak{gl}(n,\mathbb{C})$, since $\mathfrak{so}(n,\mathbb{R})^{\mathbb{C}} = \mathfrak{so}(n,\mathbb{R}) + i\mathfrak{so}(n,\mathbb{R}) = \mathfrak{so}(n,\mathbb{C})$.

The topology of the simple orthogonal group over the complex numbers is quite simple. As well as $SO(n,\mathbb{R})$, $SO(n,\mathbb{C})$ is a connected Lie group, since any element can be joined by a path to the identity. The elements of $SO(n,\mathbb{C})$ can be thought as rotations and can be identified in a hyperbolic basis with the invertible elements of $\mathbb{C}$, i.e., with $\mathbb{C}\setminus\{0\}$. The topology of this set can be, at its turn, identified to the Cartesian product $S \times \mathbb{R}$.

2.4 Symplectic geometry

Symplectic geometry is a branch of differential geometry which is fundamental for the formulation of Hamiltonian mechanics, for geometric quantization, and for considering many other problems. It is also the main setting in which the new results of this work actually belong.

Definition 2.83. Given an even dimensional manifold M^{2n}, we say a smooth 2-form ω is a *symplectic form* if ω is closed ($d\omega = 0$) and non-degenerate ($\forall \alpha \in \Omega^1(M), \exists! X \in \mathfrak{X}(M)$ that solves $\iota_X \omega = \alpha$).

Definition 2.84. A *symplectic manifold* is a pair (M,ω) such that M is a differential manifold and ω is a closed non-degenerate 2-form on M.

Remark 2.85. The condition of non-degeneracy of ω stated in the definition is equivalent to the condition that ω is a volume form, which means that $\omega^n := \omega\wedge\cdots\wedge\omega \neq 0$, where n is half the dimension of the manifold M.

Example 2.86. Consider $\mathbb{R}^{2n}$ with the standard coordinates $x_1,\ldots,x_n,y_1,\ldots,y_n$. The standard 2-form

$$\omega_{st} = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is a symplectic form because $d\omega_{st} = 0$ and $\omega_{st}^n = n!\cdot dx_1\wedge dy_1\wedge\cdots\wedge dx_n\wedge dy_n \neq 0$.

Example 2.87. Let Σ be an orientable surface and take $\omega \in \Omega^2(M)$ a volume form in Σ. Then, ω is closed because is a form of degree 2, the maximal degree on the manifold. It is also clear that ω is non degenerate, as $\omega^n = \omega^1 \neq 0$ because the dimension of Σ is 2. Then, (Σ,ω) is a symplectic manifold.

Definition 2.88. A diffeomorphism $\varphi : (M_1,\omega_1) \to (M_2,\omega_2)$ between symplectic manifolds is called a *symplectomorphism* if $\varphi^*\omega_2 = \omega_1$.

When we study integrable Hamiltonian systems it is quite common to apply changes of coordinates. It is essential to preserve the symplectic structure of the underlying manifold along these changes, and this can be achieved if the transformations are symplectic.

Definition 2.89. A *symplectic transformation* of a symplectic vector space (V^{2n}, ω) is a linear transformation $L : V \longrightarrow V$ which preserves ω:

$$\omega(Lu, Lv) = \omega(u, v).$$

If ω is written as a matrix Ω in some fixed basis of V, and L is written as a matrix M, the cpreserving condition is equivalent to the condition that M is a symplectic matrix:

$$M^{\mathrm{T}}\Omega M = \Omega.$$

The following result basically states that every symplectic manifold is locally equivalent to $(\mathbb{R}^{2n}, \omega_{st})$, which is a big result in symplectic geometry.

Theorem 2.90 (Darboux)**.** *Let (M^{2n}, ω) be a symplectic manifold. Then, for all $p \in M$, there exists a local coordinate system $(x_1, y_1, \ldots, x_n, y_n)$ such that*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

in a neighbourhood U of p.

Proof. We use the exponential map. Let p be a point in M and

$$\begin{array}{rccc} \exp_p : & U \subset T_pM & \longrightarrow & V \subset M \\ & u & \longmapsto & \gamma_u(1) \end{array}$$

is a local diffeomorphism. Locally, (M, ω) is symplectic, so T_pM is symplectic with symplectic form $\widetilde{\omega}_1 = \exp^* \omega$. We express $\widetilde{\omega}_1$ in a symplectic base $\{e_i, f_i\}_{i=1}^n$ and we get $\widetilde{\omega}_0$, which satisfies:

- $\widetilde{\omega}_0(e_i, f_j) = \delta_{ij}$,
- $\widetilde{\omega}_0(e_i, e_j) = 0$,
- $\widetilde{\omega}_0(f_i, f_j) = 0$.

In this symplectic base, if J is the matrix $\begin{pmatrix} 0 & -In \\ In & 0 \end{pmatrix}$, then $\widetilde{\omega}_0(u, v) = u^T \cdot J \cdot v$. Now, we define $\omega_0 := (\exp^{-1})^* \widetilde{\omega}_0$ and we apply the *Moser's trick*. We define the path

$$\omega_t = (1 - t)\omega_0 + t\omega_1 \tag{2.13}$$

which is a path of closed forms, since $d\omega_t = (1-t)d\omega_0 + td\omega_1 = 0$ because $d\omega_0 = d\omega_1 = 0$. The forms ω_t are locally non-degenerated, because $\det(\omega_{ij}(p)) \neq 0$ and, since det is a continuous map, there exists a neighbourhood U of p such that $\det(\omega_{ij}(q)) \neq 0$ for any $q \in U$.

By Poincaré Lemma, $\omega_0 - \omega_1 = d\beta$ (because ω_1 and ω_0 are closed and d is linear). So $\omega_0 - \omega_1$ is closed and, then, locally exact. Then, there exists a vector field X_t such that $\iota_{X_t}\omega_t = -\beta$. Let φ_t be the flow of X_t and let us prove that $(\varphi_t)^*\omega_t = \omega_0 = (\varphi_1)^*\omega_1$:

$$\frac{d}{dt}\left(\varphi_t^*\omega_t\right) = \varphi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) = \tag{2.14}$$

$$= \varphi_t^*\left(-\omega_0 + \omega_1 + d\iota_{X_t}\omega_t + \iota_{X_t}d\omega_t\right) = \tag{2.15}$$

$$= \varphi_t^*\left(d\beta + d(-\beta) + 0\right) = \tag{2.16}$$

$$= \varphi_t^*(0) = 0 \tag{2.17}$$

Then, $\varphi_t^*\omega_t$ is constant and, since $\varphi_0^*\omega_0 = 0$, we have that $(\varphi_t)^*\omega_t = \omega_0$. □

Definition 2.91. Let $H \in \mathcal{C}^\infty$ be a smooth function on a symplectic manifold (M, ω) (in Physics, a Hamiltonian, the function of total energy). The *Hamiltonian vector field* X_H associated to H is defined as the only solution of $\iota_{X_H}\omega = -dH$.

Example 2.92. Take $(\mathbb{R}^{2n}, \omega = \sum dx_i \wedge dy_i)$. Let us write the flow of X_H:

$$X_H = \sum_{i=1}^n X_H^{x_i}\frac{\partial}{\partial x_i} + \sum_{i=1}^n X_H^{y_i}\frac{\partial}{\partial y_i} \tag{2.18}$$

So, on the one hand:

$$\iota_{X_H}\omega = \omega(X_H, \cdot) = \sum_{i=1}^n X_H^{x_i}dy_i + \sum_{i=1}^n -X_H^{y_i}dx_i \tag{2.19}$$

And, on the other hand:

$$-dH = \sum_{i=1}^n -\frac{\partial H}{\partial x_i}dx_i + \sum_{i=1}^n -\frac{\partial H}{\partial y_i}dy_i \tag{2.20}$$

This leads to the Hamiltonian equations:

$$\begin{cases} X_H^{x_i}dy_i = \dot{x}_i = -\frac{\partial H}{\partial y_i}dy_i \\ X_H^{y_i}dx_i = \dot{y}_i = \frac{\partial H}{\partial x_i}dx_i \end{cases} \tag{2.21}$$

Definition 2.93. The *Poisson bracket* associated to a symplectic manifold (M, ω) is the operator defined by

$$\begin{aligned} \{\cdot,\cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}(M)^\infty &\longrightarrow \mathcal{C}^\infty(M) \\ (f, g) &\longmapsto \{f, g\} = \omega(X_f, X_g), \end{aligned}$$

where X_f and X_g are the solutions of $\iota_{X_f}\omega = -df$ and $\iota_{X_g}\omega = -dg$, respectively.

Remark 2.94. The Poisson bracket satisfies, for all $f, g, h \in \mathcal{C}^\infty(M)$:

1. $\{f, g\} = -\{g, f\}$ (skew-symmetry).

2. $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{f, h\}\} = 0$ (Jacobi's identity).

3. $\{f, gh\} = \{f, g\}h + g\{f, h\}$ (Leibniz rule).

Theorem 2.95. *Let $H \in \mathcal{C}^\infty$ be a Hamiltonian and X_H the corresponding Hamiltonian vector field. Then, $X_H(H) = 0$*

Proof. Consider the Poisson bracket $\{\cdot, \cdot\}$ defined by

$$\{f, g\} = \omega(X_f, X_g) = -\omega(X_g, X_f) = -\{g, f\} = X_f(g) \tag{2.22}$$

where X_f is defined by $\iota_{X_f}\omega = -df$ and analogously for X_g. Then, by skew-symmetry of $\{\cdot, \cdot\}$, $X_H(H) = 0$. □

Definition 2.96. Let (M, ω) be a symplectic manifold. A vector field $X \in \mathfrak{X}(M)$ is called *symplectic vector field* if it preserves ω, i.e., if $\mathcal{L}_X\omega = 0$, or $d\iota_X\omega = 0$.

Remark 2.97. We will denote by $\mathfrak{X}^{\text{Symp}}(M) = \{X \in \mathfrak{X}(M) \mid d\iota_X\omega = 0\}$ the set of symplectic vector fields on M and by $\mathfrak{X}^{\text{Ham}}(M) = \{X \in \mathfrak{X}(M) \mid \iota_X\omega = -d\beta\}$ the set of symplectic vector fields on M.

Lemma 2.98. *$\mathfrak{X}^{Ham}(M) \subset \mathfrak{X}^{Symp}(M)$.*

Proof. Take $X \in \mathfrak{X}^{\text{Ham}}(M)$. Then:

$$\mathcal{L}_{X_f}\omega = d\iota_{X_f}\omega + \iota_{X_f}d\omega = d(-df) + 0 = -d^2f = 0 \tag{2.23}$$

□

Lemma 2.99. $\mathcal{H}^1(M) = {\mathfrak{X}^{Symp}(M)}\big/{\mathfrak{X}^{Ham}(M)}$

Example 2.100. Take $(\mathbb{R}^{2n}, \omega = dx \wedge dy)$. $X = \frac{\partial}{\partial x_1}$ is Hamiltonian because $\iota_X\omega = dy_1$. Hence, it is symplectic.

Example 2.101. Take $(S^2, \omega = dh \wedge d\theta)$. $X = \frac{\partial}{\partial\theta}$ is Hamiltonian because $\iota_X\omega = -dh$. Hence, it is symplectic.

Example 2.102. Take $(\mathbb{R}^{2n}, \omega = d\theta_1 \wedge \theta_2)$. $X = \frac{\partial}{\partial\theta_1}$ is not Hamiltonian because $\iota_X\omega = d\theta_2$ and θ_2 is not a global function. It is symplectic.

Lemma 2.103. *Let $f, H \in \mathcal{C}^\infty(M)$. Then,*

$$\{f, H\} = 0 \iff f \text{ is constant along the flow of } X_H \tag{2.24}$$

Proof. $\{f, H\} = -X_H(f) = -\frac{d}{dt}(f \circ \gamma)(t)\big|_{t=0} = 0$ □

Definition 2.104. Suppose $f, H \in \mathcal{C}^\infty(M)$ satisfy $\{f, H\} = 0$. Then, f is called an *integral of motion of H*.

Definition 2.105. Let (M^{2n}, ω) be a symplectic manifold. A Hamiltonian system $(M, \omega, H \in \mathcal{C}^\infty(M))$ is called *completely integrable* if $\exists f_1, \ldots, f_n \in \mathcal{C}^\infty(M)$ such that:

1. $\{f_i, f_j\} = 0$ for all $i, j = 1, \ldots, n$
2. $df_1 \wedge \cdots \wedge df_n \neq 0\, a.e.$

Example 2.106. Any Hamiltonian system defined on a surface, $(\Sigma^2, \omega, H \in \mathcal{C}^\infty(M))$, is completely integrable.

Definition 2.107. Let G be a Lie group and (M, ω) a symplectic manifold. A group action $\rho : G \longrightarrow \mathrm{Diff}(M)$ is a *symplectic action* if $\rho : G \longrightarrow \mathrm{Symp}(M) \subset \mathrm{Diff}(M)$, where $\mathrm{Symp}(M)$ is the set of symplectomorphisms on M.

Example 2.108. Take $(\mathbb{R}^{2n}, \omega_{st})$ and $X = \partial/\partial x_1$. The flow of X defines a symplectic action $\psi : \mathbb{R} \longrightarrow \mathrm{Symp}(\mathbb{R}^{2n}, \omega_{st})$ which is the following:

$$\psi(t)(x_1, y_1, \ldots, x_n, y_n) = (x_1 + t, y_1, \ldots, x_n, y_n). \tag{2.25}$$

We give the definition of a Hamiltonian action and the moment map, which are the absolute key concepts in the link between symplectic geometry and integrable systems.

Definition 2.109. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Consider also $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. Suppose $\psi : G \to \mathrm{Diff}(M)$ is an action on a symplectic manifold (M, ω). It is called a *Hamiltonian action* if there exists a map $\mu : M \to \mathfrak{g}^*$ which satisfies:

- For each $X \in \mathfrak{g}$, $d\mu^X = \iota_{X^\#}\omega$, i.e., μ^X is a Hamiltonian function for the vector field $X^\#$, where
  - $\mu^X : p \longmapsto \langle \mu(p), X \rangle : M \longrightarrow \mathbb{R}$ is the component of μ along X,
  - $X^\#$ is the vector field on M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subset G$.
- The map μ is equivariant with respect to the given action ψ on M and the coadjoint action: $\mu \circ \psi_g = \mathrm{Ad}^*_g \circ \mu$, for all $g \in G$.

Then, (M, ω, G, μ) is called a *Hamiltonian G-space* and μ is called the *moment map*.

2.5 b-symplectic geometry

When dealing with physical models it is usual to encounter singularities in the phase space. A natural formulation for these type of singularities at the level of the manifold is the b-geometry and, in the symplectic context, the b-symplectic geometry. We give some definitions and results on b-symplectic manifolds that will be necessary later. The proofs of these results are contained in [GMP11], [GMP14] and [GMPS15].

Definition 2.110. A pair (M, Z), where M is a manifold and Z a (not necessarily connected) hypersurface in M is called a *b-manifold*.

Definition 2.111. A map $f : (M_1, Z_1) \longrightarrow (M_2, Z_2)$ between b-manifolds is called a *b-map* if f is transverse to Z_2 and $Z_1 = f^{-1}(Z_2)$.

Definition 2.112. We call an action ρ of a Lie group G on a b-manifold (M, Z) a *b-action* if for every $g \in G$, the induced diffeomorphism ρ_g is a b-map on (M, Z).

Differential forms with singularities can be introduced formally for b-Poisson manifolds. The idea is that it is possible to extend the symplectic structure from $M \backslash Z$ to the whole manifold M. This singular form will be called a *b-symplectic* form on M.

Definition 2.113. A *b-vector field* on a b-manifold (M, Z) is a vector field which is tangent to Z at every point $p \in Z$.

If x is a local defining function for Z on an open set $U \subset M$ and $(x, y_1, \dots, y_{N-1})$ is a chart on U, then the set of b-vector fields on U is a free $C^\infty(M)$-module with basis

$$(x\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_N}).$$

There exists a vector bundle associated to this module. This vector bundle is called the *b-tangent bundle* and denote it ${}^bTM$. The *b-cotangent bundle* ${}^bT^*M$ of M is defined to be the vector bundle dual to ${}^bTM$.

For each $k > 0$, let ${}^b\Omega^k(M)$ denote the space of *b-de Rham k-forms*, i.e., sections of the vector bundle $\Lambda^k({}^bT^*M)$. The usual space of de Rham k-forms sits inside this space in a natural way; for f a defining function of Z every b-de Rham k-form can be written as

$$\omega = \alpha \wedge \frac{df}{f} + \beta, \text{ with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M). \tag{2.26}$$

The decomposition given by (2.26) enables us to extend the exterior d operator to ${}^b\Omega(M)$ by setting

$$d\omega = d\alpha \wedge \frac{df}{f} + d\beta.$$

The right hand side is well defined and agrees with the usual exterior d operator on $M \setminus Z$ and also extends smoothly over M as a section of $\Lambda^{k+1}({}^bT^*M)$. Note that $d^2 = 0$, which allows us to define the complex of b-forms, the b-de Rham complex. The cohomology associated to this complex is called *b-cohomology* and it is denoted by ${}^bH^*(M)$.

A special class of closed 2-forms of this complex are b-symplectic forms as defined in [GMP14].

Definition 2.114. Let (M^{2n}, Z) be a b-manifold and $\omega \in {}^b\Omega^2(M)$ a closed b-form. We say that ω is *b-symplectic* if ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*M)$ for all $p \in M$. We call (M, Z, ω) a *b-symplectic manifold*.

Definition 2.115. The set of *b-functions* ${}^bC^\infty(M)$ consists of functions with values in $\mathbb{R} \cup \{\infty\}$ of the form

$$c\log|f| + g,$$

where $c \in \mathbb{R}$, where f is a defining function for Z and g is a smooth function. The differential operator d is defined as: $d(c\log|f| + g) := \frac{c\,df}{f} + dg \in {}^b\Omega^1(M)$, where dg is the standard de Rham derivative.

The Lie derivative of b-forms is defined via the Cartan formula:

$$\mathcal{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega) \in^b \Omega^k(M), \tag{2.27}$$

where $\omega \in^b \Omega^k(M)$ and X is a b-vector field.

Finally, the following theorem shows how the b-cohomology is related to De Rham cohomology:

Theorem 2.116 (Mazzeo-Melrose)**.** *The b-cohomology groups of M^{2n} satisfy*

$${}^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

2.6 The cotangent lift

The cotangent bundle of a smooth manifold can be naturally equipped with a symplectic structure in the following way. Let M be a differential manifold and consider its cotangent bundle T^*M. There is an intrinsic canonical linear form λ on T^*M defined pointwise by

$$\langle \lambda_p, v\rangle = \langle p, d\pi_p v\rangle, \qquad p = (m, \xi) \in T^*M, v \in T_p(T^*M),$$

where $d\pi_p : T_p(T^*M) \longrightarrow T_mM$ is the differential of the canonical projection at p. In local coordinates (q_i, p_i), the form is written as $\lambda = \sum_i p_i\, dq_i$ and is called the *Liouville 1-form*. Its differential $\omega = d\lambda = \sum_i dp_i \wedge dq_i$ is a symplectic form on T^*M.

Definition 2.117. Let $\rho : G \times M \longrightarrow M$ be a group action of a Lie group G on a smooth manifold M. For each $g \in G$, there is an induced diffeomorphism $\rho_g : M \longrightarrow M$. The *cotangent lift of* ρ_g, denoted by $\hat{\rho}_g$, is the diffeomorphism on T^*M given by

$$\hat{\rho}_g(q, p) := (\rho_g(q), ((d\rho_g)_q^*)^{-1}(p)), \qquad (q, p) \in T^*M$$

which makes the following diagram commute:

$$\begin{array}{ccc} T^*M & \xrightarrow{\hat{\rho}_g} & T^*M \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{\rho_g} & M \end{array}$$

Lemma 2.118. *The induced diffeomorphism $\hat{\rho}_g$ preserves the form λ and, hence, preserves the symplectic form ω.*

Proof. We will prove that, in general, that given a diffeomorphism $\rho : M \longrightarrow M$, its cotangent lift preserves the canonical form λ. At a point $p = (m, \xi) \in T^*M$, we have:

$$\begin{aligned}\lambda_p &= (d\pi)^*_p \xi = \\ &= (d\pi)^*_p (d\rho)^*_m \left((d\rho)^*_m\right)^{-1} \xi = \\ &= (d(\rho \circ \pi))^*_p \left((d\rho)^*_m\right)^{-1} \xi = \\ &= (d(\pi \circ \hat{\rho}))^*_p \left((d\rho)^*_m\right)^{-1} \xi = \\ &= (d\hat{\rho})^*_p (d\pi)^*_{\hat{\rho}(p)} \left((d\rho)^*_m\right)^{-1} \xi = \\ &= (d\hat{\rho})^*_p \lambda_{\hat{\rho}(p)},\end{aligned}$$

where we used the definitions of the Liouville 1-form and the cotangent lift and the fact that $\rho \circ \pi = \pi \circ \hat{\rho}$. Then, the canonical 1-form is preserved by $\hat{\rho}$.

As a consequence:

$$\hat{\rho}^*(\omega) = \hat{\rho}^*(d\lambda) = d(\hat{\rho}^*\lambda) = d\lambda = \omega.$$

So, the cotangent lift $\hat{\rho}_g$ preserves the Liouville form and the symplectic form of T^*M. □

Remark 2.119. The cotangent lift of a Lie group G on a manifold M, which is an action on (T^*M, ω_{T^*M}), is automatically Hamiltonian (see for instance [GS84b]). This makes the cotangent lift a natural and powerful tool for the formulation of integrable systems, specially in the context of mechanics.

The cotangent lift can also be defined in a b-manifold. In this case there are two different 1-forms that provide a symplectic structure on the b-cotangent bundle, namely, the canonical b-1-form and the twisted b-1-form. Each of this forms produces the same b-symplectic form, but the b-cotangent lift in each of the cases is different.

First, we define the canonical b-1-form:

Definition 2.120. For (M, Z) a b-manifold, we define a *b-form* λ on ${}^bT^*M$ via

$$\langle \lambda_p, v \rangle := \langle p, (\pi_p)_*(v) \rangle, \tag{2.28}$$

where $v \in {}^bT({}^bT^*M)$ and $p \in {}^bT^*M$. The negative differential

$$\omega = -d\lambda$$

is the *canonical b-symplectic form* on ${}^bT^*M$. Here, we view ${}^bT^*M$ as a b-manifold with hypersurface $\pi^{-1}(Z)$ where

$$\pi : {}^bT^*M \to M$$

is the canonical projection. Choosing a local set of coordinates $x_1, \dots, x_n$ on M, where x_1 is a defining function for Z we have a corresponding chart

$$(x_1, \dots, x_n, p_1, \dots, p_n)$$

on T^*M, given by identifying the $2n$-tuple above with the b-cotangent vector

$$p_1 \frac{dx_1}{x_1} + \sum_{i=2}^{n} p_i dx_i \in^b T_x^* M.$$

In these coordinates

$$\lambda = p_1 \frac{dx_1}{x_1} + \sum_{i=2}^{n} p_i dx_i \in^b T^*({}^bT^*M).$$

Then, we define the canonical b-cotangent lift.

Definition 2.121. Assume that (M, Z) is an n-dimensional b-manifold. Consider the b-cotangent bundle ${}^bT^*M$ endowed with the canonical b-symplectic structure obtained naturally if we use the intrinsic definition of the Liouville one-form in the b-setting. Moreover, assume that the action of G on M preserves the hypersurface Z, i.e. ρ_g is a b-map for all $g \in G$. Then the lift of ρ to an action on ${}^bT^*M$ is well-defined:

$$\hat{\rho} : G \times^b T^*M \to^b T^*M : (g, p) \mapsto \rho^*_{g^{-1}}(p).$$

We call this action on ${}^bT^*M$, endowed with the canonical b-symplectic structure, the *canonical b-cotangent lift.*

Proposition 2.122. *The canonical b-cotangent lift is Hamiltonian with equivariant moment map given by:*

$$\mu : {}^bT^*M \to \mathfrak{g}^*, \quad \langle \mu(p), X \rangle := \langle \lambda_p, X^\# |_p \rangle = \langle p, X^\# |_{\pi(p)} \rangle, \tag{2.29}$$

where $p \in {}^bT^*M$, $X \in \mathfrak{g}$, $X^\#$ is the fundamental vector field of X under the action on ${}^bT^*M$ and the function $\langle \lambda, X^\# \rangle$ is smooth because $X^\#$ is a b-vector field.

Now, we define the twisted b-1-form.

Definition 2.123. Let $T^*\mathbb{T}^n$ be endowed with the standard coordinates (θ, a), $\theta \in \mathbb{T}^n$, $a \in \mathbb{R}^n$ and consider again the action on $T^*\mathbb{T}^n$ induced by lifting translations of the torus $\mathbb{T}^n$. Define the following non-smooth one-form away from the hypersurface $Z = \{a_1 = 0\}$:

$$\lambda_{tw,c} \log |a_1| d\theta_1 + \sum_{i=2}^{n} a_i d\theta_i.$$

Then, the form $\omega := -d\lambda_{tw,c}$ is a b-symplectic form on $T^*\mathbb{T}^n$, called the *twisted b-symplectic form* on $T^*\mathbb{T}^n$. In coordinates:

$$\omega_{tw,c} := \frac{c}{a_1} d\theta_1 \wedge da_1 + \sum_{i=2}^{n} d\theta_i \wedge da_i. \tag{2.30}$$

We call the lift together with the b-symplectic form (2.30) the *twisted b-cotangent lift* with modular period c on the cotangent space of a torus.

In a more general setting, the twisted cotangent lift is defined as follows. Consider a $(n-1)$-dimensional manifold N and let λ_N be the standard Liouville one-form on T^*N. Endow the product $T^*(S^1 \times N) \cong T^*S^1 \times T^*N$ with the product structure $\lambda := (\lambda_{tw,c}, \lambda_N)$ (defined for $a \neq 0$). The form $\omega = -d\lambda$ is a b-symplectic structure with critical hypersurface given by $a = 0$.

Suppose K is a Lie group acting on N and consider the component-wise action of $G := S^1 \times K$ on $M := S^1 \times N$, where S^1 acts on itself by rotations. Lift this action to T^*M as described before. This construction, with T^*M endowed with the b-symplectic form ω, is called the *twisted b-contangent lift*.

If $(x_1, \dots, x_{n-1})$ is a chart on N and $(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})$ is the associated chart on T^*N, λ has the following local expression:

$$\lambda = \log|a| d\theta + \sum_{i=1}^{n-1} y_i dx_i. \tag{2.31}$$

This action is, again, Hamiltonian with moment map given by the contraction of the fundamental vector fields with λ:

Proposition 2.124. *The twisted b-cotangent lift on $M = S^1 \times N$ is Hamiltonian with equivariant moment map μ given by*

$$\langle \mu(p), X \rangle := \langle \lambda_p, X^{\#}|_p \rangle, \tag{2.32}$$

where $X^{\#}$ is the fundamental vector field of X under the action on T^*M.

2.7 Integrable systems

The study of evolution in time of physical, chemical and other kinds of systems leads to the mathematical formulation of the time dependence of the different variables involved. The concept of *dynamical system* formalizes this problem. In this section we restate and complement some definitions on integrable Hamiltonian systems that we already gave and we provide some physical examples, which at the end are the motivating problems.

Definition 2.125. A *dynamical system* is a tuple $(M, T, (\phi^t)_{t \in T})$ such that T is $\mathbb{Z}$ or $\mathbb{R}$, M is a non-empty set and $(\phi^t)_{t \in T}$ is a family of functions from M to M that satisfy:

- $\phi^0(x) = x, \quad \forall x \in M$
- $\phi^{s+t}(x) = \phi^s(g^t(x)), \quad \forall x \in M, \forall t, s \in T.$

In most of the cases, and also in this work, M is a smooth manifold and $\phi^t(x)$ a x-continuous function. The function $\phi(x)$, for a fixed t, is called the *flow* through x, while the set $\gamma_x = \{(\phi^t(x)) \mid t \in T\}$ is called the *orbit* through x. If T is taken to be $\mathbb{Z}$, the dynamical system is called *discrete*, while it is called *continuous* if T is taken to be $\mathbb{R}$.

A special family of continuous dynamical systems are *Hamiltonian systems*, which are defined from a system of ordinary differential equations and a smooth function. The following definition comes from the field of dynamical systems.

Definition 2.126. A *Hamiltonian system* is the system of ordinary differential equations:

- $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}(t, q, p)$
- $\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}(t, q, p)$

for $i = 1, \dots, n$, where $H : U \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a smooth function.

The theorem of existence and uniqueness from the theory of differential equations states that, for any $(t_0, p_0, q_0) \in U$, there is a unique solution $\phi(t; t_0, p_0, q_0)$ of the Hamiltonian system 2.126 that is defined in a neighbourhood of t_0 and that satisfies $\phi(t_0; t_0, p_0, q_0) = (p_0, q_0)$. When H does not depend on t, the system is called *autonomous.*

Example 2.127. The central force problem given by $\ddot{q} = k(|q|) \cdot q$, with $q : \mathbb{R} \to \mathbb{R}$ is a classical physical problem that can be formulated as an autonomous Hamiltonian system. By defining $p := dq/dt$, the second order equation of the problem is equivalent to the system:

$$\frac{dq}{dt} = p, \quad \frac{dp}{dt} = k(|q|) \cdot q,$$

and it is easy to check that Hamiltonian

$$H = \frac{1}{2}p^2 - \int_{s=|q|} k(s)ds$$

gives raise to exactly this system.

A generalized definition of Hamiltonian system, in the sense that it can be defined on a general symplectic manifold, is the following.

Definition 2.128. Let (M, ω) be a symplectic manifold. Let $H \in C^\infty(M)$ be a smooth function. Then, the *Hamiltonian system* is given by the vector field X_H that solves of the equation

$$\iota_{X_H}\omega = -dH.$$

Example 2.129. In the problem of magnetic a particle moving in a magnetic field, consider only the spin motion, i.e. the rotation of the spin vector s of the particle. As the spin vector can point to any direction on $\mathbb{R}^3$ but its magnitude is fixed, the problem is naturally set in S^2. S^2 is a surface and can be equipped with the form $\omega = ds_z \wedge d\theta$, where θ is the polar angle and s_z is the z component of the spin vector s. Then, since ω is proportional to the area form of the sphere, (S^2, ω) is a symplectic manifold, as we saw in 2.87.

If the magnetic field of magnitude B is taken in the z-direction, the Hamiltonian H of the particle is defined as $H = s_z B$. Then, the vector field X_H associated to this Hamiltonian is the solution to

$$\iota_{X_H}(ds_z \wedge d\theta) = -d(s_z B),$$

which is

$$X_H = B\frac{\partial}{\partial\theta}.$$

Then, the flows of the Hamiltonian field are circles of S^2 that lay in planes that are normal to the z direction. So, the spin vector of the particle rotates around the magnetic field.

It is easy to see that, in a Hamiltonian system, the Hamiltonian H is a conserved quantity, in the sense that it is constant along the solutions of the ode system[3]. Any smooth function that has this property is called an *integral* of the system. The existence of such functions in a given dynamical system is a well-known and difficult problem, and they are essential in the definition of one subcategory of Hamiltonian systems called *completely integrable systems* or simply *integrable systems.*

Definition 2.130. Let M be a symplectic manifold of dimension $2n$. Let H be a Hamiltonian defined on M. The Hamiltonian system given by H is said to be completely integrable if there exists a function $F : M \to (f_1, \dots, f_n) \in \mathbb{R}^n$ such that $df_1, \dots, df_n$ are independent a.e., the components of F are in involution with respect to the symplectic Poisson bracket ($\{f_i, f_j\} = 0$ for all i, j) and $\{f_i, H\} = 0$ for all i. Such F is called the *moment map* and the function H may be one of the f_i's.

Remark 2.131. A Hamiltonian system of degree $n = 1$ is always integrable and the Hamiltonian function H is the natural first integral.

Example 2.132. The harmonic oscillator is probably the most classical physical example of a Hamiltonian system which is integrable. It is defined from the differential equation

$$\ddot{q} + \omega^2 q = 0,$$

with ω a constant. The associated Hamiltonian equations are:

$$\frac{dq}{dt} = \omega p, \quad \frac{dp}{dt} = -\omega q,$$

where p has been defined as $\omega p = dq/dt$. A first integral of the system, which already makes the system integrable, is $H(q,p) = \frac{\omega}{2}(q^2 + p^2)$.

[3]It is constant along the solutions because, by definition $dH/dt = dq/dt \cdot dp/dt - dp/dt \cdot dq/dt = 0$.

Definition 2.133. An integrable system on M is given by a smooth map $f = (f_1, \dots, f_n) : M \to \mathbb{R}^n$ such that $\{f_i, f_j\} = 0$ for all $1 \leq f_i, f_j \leq n$ and $\operatorname{rank} f = n$ almost everywhere. If the Hamiltonian vector fields provided by each f_i (i.e. the X_i satisfying $\iota_{X_i}\omega = -df_i$), which have commuting flows $\phi_t^1, \dots, \phi_t^n$, are complete, then the system induces an $\mathbb{R}^n$ action on M, called the *joint flow*:

$$\rho : \mathbb{R}^n \times M \longrightarrow M \tag{2.33}$$

$$(t_1, \dots, t_n, p) \longmapsto \phi_t^1 \circ \cdots \circ \phi_t^n(p) \tag{2.34}$$

Remark 2.134. The action ρ of the additive group $(\mathbb{R}^n, +)$ preserves the level sets $M_c = f^{-1}(c)$ for all $c \in \mathbb{R}^n$.

If f is an integrable system with compact connected fibers, then the Arnold-Liouville-Mineur Theorem (see Theorem 3.3 in the next section) tells us that, whenever c is a regular point of f, these fibers M_c are diffeomorphic to $\mathbb{R}^n/L_c$, where $L_c \subset \mathbb{R}^n$ is the *period lattice*, so M_c are called the Liouville tori.

3. Stability and equivalence of actions and integrable systems

Stability is a general concept which, in geometry and dynamical systems, refers to persistence of a certain properties or structures when something else changes. We call that something is *stable* if it remains essentially the same after a small perturbation. We can use it to talk of stable mappings, stable algebraic structures, stable initial positions of integrable systems... For properties of mappings, for instance, the precise definition is the following.

Definition 3.1. A property of a map $f : X \to Y$ is called *stable* if whenever f_0 possesses the property and $f_t : X \to Y$ is an homotopy of f_0, then, for some $\varepsilon > 0$, the property is satisfied for each f_t if $t \in [0, \varepsilon]$.

In the study of group actions, a similar concept is used, *rigidity*, if for any group action on a manifold, all nearby actions are equivalent to it.

Definition 3.2. Let a Lie group G act smoothly on a manifold M and let $\rho : G \times M \longrightarrow M$ denote this action. The action ρ is *rigid* if for every smooth one-parameter family of actions ρ_t of G on M there exists a one-parameter family of diffeomorphisms $h_t : M \longrightarrow M$ which conjugate ρ to ρ_t for all t in a small interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$. We say that two actions which are conjugated via a diffeomorphism are *equivalent*.

When studying integrable systems, one gives a lot of importance to the fibrations and the orbits of the system and whether they are stable under certain conditions or under small perturbations. In this section we review briefly some important known theorems on stability of integrable systems and we expose the basic results concerning non-degenerate singularities of integrable systems. We believe that this is a necessary background that will help the reader to understand the new results in the next section and that it will also illustrate our motivation to prove them.

3.1 The Arnold-Liouville-Mineur Theorem

The most important classical result in determining the topology of fibrations is the *Arnold-Liouville-Mineur* theorem. It states that, in the case of a regular integrable system, the compact fibers have a tori fibrated neighbourhood. The same theorem carries the definition of a special set of coordinates, the action-angle coordinates.

Consider an integrable system defined on a symplectic manifold (M, ω) of dimension $2n$ and given by a proper moment map $F = (f_1, \dots, f_n)$. The flows of the vector fields X_i associated[4] to the f_i can be thought as a Lie group action of $\mathbb{R}^n$ on M as in Example 2.61. In this way, the orbits of the action given by the momentum map F define a foliation by invariant leaves. These leaves are identified with the level sets of F as $\Lambda_c := F^{-1}(c), c \in \mathbb{R}^n$, and are, in general, *Lagrangian submanifolds* of M,

[4]A vector field X_i is associated to a function f_i if it satisfies $\iota_{X_i}\omega = -df_i$.

which means that are of dimension n, half the dimension of M. From this point on, we assume that F is a proper map.

If $df_1 \wedge \cdots \wedge df_n \neq 0$ at a point $m \in M$, a theorem of Darboux-Carathéodory states that it is possible to take local coordinates (x, ξ), in such a way that the foliation Λ_c for c in a neighbourhood of $F(m)$ is given by $\xi = c$, so the same functions $f_1, \dots, f_n$ are the momentum coordinates, and therefore the x variables are the local position coordinates. The Arnold-Liouville-Mineur Theorem says much more.

Theorem 3.3 (Arnold-Liouville-Mineur)**.** *Let (M, ω) be a symplectic manifold and $F : M \to \mathbb{R}^n$ a proper moment map. Suppose that the components $f_1 \dots, f_n$ of F are in involution with respect to the Poisson bracket and that $df_1 \wedge \cdots \wedge df_n \neq 0$ almost everywhere. Assume that $c \in \mathbb{R}^n$ is a regular value of F and that the leaf $\Lambda_c = F^{-1}(c)$ is connected.*

Then, in a neighbourhood $\Omega(\Lambda_c)$ of the leaf Λ_c, there exists a diffeomorphism $\phi : \Omega(\Lambda_c) \to D^n \times \mathbb{T}^n$ such that:

1. *$\phi(\Lambda_c) = \{0\} \times \mathbb{T}^n$.*
2. *$\phi^*(\sum_{i=1}^n d\mu_i \wedge d\beta_i) = \omega$, where $\mu_1, \dots, \mu_n$ are coordinates on D^n and $\beta_1, \dots, \beta_n$ are coordinates on $\mathbb{T}^n$.*
3. *The moment map F is only function of the coordinates $\phi^*(\mu_i)$.*

The theorem says that the $\Omega(\Lambda_c)$ is symplectomorphic to a neighbourhood of the zero section of $T^*(\mathbb{T}^n)$, the cotangent bundle of the n-torus $\mathbb{T}^n$. In particular, Λ_c and the closer leaves are called *Liouville tori.*

The theorem also gives raise to a set of coordinates $p_i = \phi^*(\mu_i), \theta_i = \phi^*(\beta_i)$, called *action coordinates* and *angle coordinates*, respectively. It is a result by Mineur that the p_i coordinates can also be computed equivalently in the following way:

$$p_i(x) = \int_{\Gamma_i(x)} \lambda, \qquad x \in M,$$

where λ is the Liouville 1-form[5] and $\Gamma_i(x)$ is a closed curve that lies on the torus containing x.

In action-angle coordinates, the moment map F writes as $F = (p_1, \dots, p_n)$, meaning that the action coordinates are the normal form of the set of first integrals of the system. Action-angle coordinates give also the normal Darboux form for the symplectic structure of the fibration, which is

$$\omega = \sum_{i=1}^{n} dp_i \wedge d\theta_i.$$

[5]Recall that, if λ is the Liouville 1-form, $-d\lambda = \omega$.

Remark 3.4. Dynamics in the compact leaf Λ_c is basically reduced to two types. If the Hamiltonian H is the component f_1 of F, the Hamiltonian system induced by $\hat{H} = H \circ \phi^{-1}(p_1, \dots, p_n, \theta_1, \dots, \theta_n)$ is

$$\frac{d\theta}{dt} = \partial_p \hat{H}(p, \theta) = \partial_p \hat{H}(p),$$
$$\frac{dp}{dt} = -\partial_\theta \hat{H}(p, \theta) = 0,$$

as F does not depend on θ. The equations of the system make it clear that the each flow will be confined in an invariant toric leaf with p constant and θ varying on the torus as a function of p. This is the reason for calling p the action and θ the angle.

Explicitly, the flows are given by

$$\begin{aligned} \varphi : \mathbb{R} \times D^n \times \mathbb{T}^n &\to D^n \times \mathbb{T}^n \\ (t, p, \theta) &\mapsto (p, \theta + \partial_p \hat{H}(p)). \end{aligned}$$

Although completely integrable systems such as the ones that considers the Arnold-Liouville-Mineur Theorem are rare to find when dealing with physical problems, it is more typical to encounter *close-to-integrable* systems. These are Hamiltonian systems of the form

$$\frac{d\theta}{dt} = \partial_p H_0(p) + \varepsilon \partial_p H_1(p, \theta),$$
$$\frac{dp}{dt} = -\varepsilon \partial_\theta H_1(p, \theta),$$

that are integrable systems with first integrals p_i if $\varepsilon = 0$ and close-to-integrable if $0 < \varepsilon << 1$.

Example 3.5. Consider the manifold $(M = \mathbb{R} \times \mathbb{T}, \omega = dp \wedge d\theta)$ and the Hamiltonian $H = p^2/2 + \varepsilon \cos\theta$ for $p \in \mathbb{R}, \theta \in \mathbb{T}, 0 < \varepsilon << 1$. The equations of the system are

$$\frac{d\theta}{dt} = p, \tag{3.1}$$
$$\frac{dp}{dt} = \varepsilon \sin\theta, \tag{3.2}$$

and its solutions lie on the level sets $H = h$ of the first integral $p = \pm\sqrt{2(h - \varepsilon\cos\theta}$.

In the case $\varepsilon = 0$, the Hamiltonian H is simply $H = p^2/2$ and the orbits are given by $\{(p, \theta) \mid p \neq 0 \text{ constant}\}$ and $\{(p, \theta) \mid p = 0, \theta \,\text{constant}\}$.

The perturbation of the system by adding the term $\varepsilon \cos\theta$ to H changes the shape of the orbits of the type $\{(p, \theta) \mid p \neq 0 \text{ constant}\}$ but they are essentially the same kind of S^1 orbits. So, these invariant tori "survive", as we can see in Figure 2. On the contrary, the orbits of the type $\{(p, \theta) \mid p = 0, \theta \,\text{constant}\}$, which are fixed points, break down and there appear new closed orbits which are not deformable to the original ones.

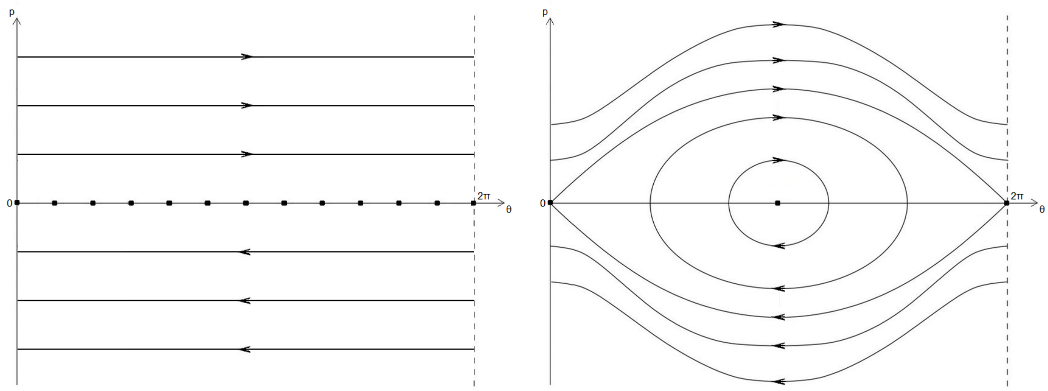

Figure 2: Phase space and some orbits of the simple pendulum system in Example 3.2 for $\varepsilon = 0$ and $\varepsilon > 0$, respectively.

In this example we see that if we add a perturbation to an integrable system, some of the tori given by the Arnold-Liouville-Mineur Theorem may still exist but others may appear or disappear. To the study of this tori persistence is devoted the KAM theory.

3.2 KAM Theory

Generalizing the setting of in Example 3.2, consider a Hamiltonian $H(\theta, p) = H_0(p) + \varepsilon H_1(\theta, p)$, for $(\theta, p) \in (\mathbb{T}^n \times V \subset R^n, \omega = \sum_{i=1}^n d\theta_i \wedge dp_i)$. The corresponding Hamiltonian system is

$$\frac{d\theta}{dt} = \partial_p H_0(p) + \varepsilon \partial_p H_1(\theta, p),$$
$$\frac{dp}{dt} = -\varepsilon \partial_\theta H_1(\theta, p).$$

For $\varepsilon = 0$, the flow of the Hamiltonian system at $p \neq 0$ is

$$\varphi_t(p_1, \ldots, p_n, \theta_1, \ldots, \theta_n) = (p_1, \ldots, p_n, \theta_1 + \nu_1 t, \ldots, \theta_n + \nu_n t).$$

We define the *frequency vector* $\nu := (\nu_1, \ldots, \nu_n, 1) \in \mathbb{R}^{n+1}$.

If the frequency vector ν satisfies $\nu \cdot k = (\nu_1, \ldots, \nu_n, 1) \cdot (k_1, \ldots, k_n, 1) = 0$ for some $k \in \mathbb{Z}^{n+1}$, it is called a *resonant* frequency. If there is no $k \in \mathbb{Z}^{n+1}$ such that $\nu \cdot k = 0$, ν is called a *non-resonant* frequency.

If ν is resonant, any flow in the Hamiltonian system eventually closes and, hence, is a periodic orbit . On the other hand, if ν is non-resonant, the flow fills $\mathbb{T}^n$ densely and it is called a *quasi-periodic* orbit[KMS16].

Definition 3.6. A vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is called *Diophantine* when $\exists \gamma > 0, \tau \geq 0$ such that

$$|a \cdot k| \geq \frac{\gamma}{||k||^\tau} \qquad \forall k \in \mathbb{Z} \setminus \{0\},$$

where $||k|| = \sum_{i=1}^{n} |k_i|$.

In some sense, the Diophantine or *strongly non-resonant* vectors are the vectors in $\mathbb{R}^n$ that are further from any rational vector (i.e. a vector of $\mathbb{Q}^n$).

The Diophantine nature of the frequency vector of the flows in an unperturbed integrable Hamiltonian system strongly affects the persistence of the Liouville tori when the system is slightly perturbed. The KAM theorem, named after Kolmogorov, Arnold and Moser, states the exact conditions for tori persistence.

Theorem 3.7. *Let $H : (\theta, p) \mapsto H(\theta, p) : \mathbb{T}^n \times D^n \to \mathbb{R}$ be an analytic function that only depends on p, i.e. $H : (\theta, p) = h(p)$. Consider the frequency vector $\nu(p) := \partial h(p)/\partial p \in \mathbb{R}^n$. If the frequency vector $\nu(p_0)$ of $p_0 \in D^n$ is Diophantine and it is non-degenerated (i.e. $\det(D_p^2 h(p_0)) \neq 0$), then, the Liouville torus $\mathbb{T}^n \times \{p_0\}$ given by the LAM theorem persists under perturbations of the Hamiltonian H of the form*

$$H_\varepsilon(\theta, p) = H(p) + \varepsilon H_1(\theta, p),$$

with H_1 analytic, if $\varepsilon > 0$ is sufficiently small.

Besides, the flow φ_t of the new perturbed system on the surviving torus $\mathcal{T}$ is conjugated to the linear flow on $\mathbb{T}^n$ with frequency vector $\nu(p_0)$. I.e., it exists a diffeomorphism $\phi : \mathbb{T}^N \to \mathcal{T}$ such that

$$\phi^{-1} \circ \varphi_t \circ \phi(\theta_0) = \theta_0 + \nu(p_0)t.$$

The theorem states that tori corresponding to strongly non-resonant frequencies survive under sufficiently small perturbations, and it is, in fact, a direct consequence of a theorem proved by Kolmogorov in 1954.

Theorem 3.8. *Let $H : \mathbb{T}^n \times D_r^n \to \mathbb{R}$ be in the Kolmogorov normal form, i.e., in the form:*

$$H(\theta, p) = a + \nu \cdot p + Q(\theta, p),$$

for some $a \in \mathbb{R}$, $\nu \in \mathbb{R}^n$ and $Q : \mathbb{T}^n \times D^n \to \mathbb{R}$ a function which is quadratic in p. Assume that H is non-degenerate in the sense that:

$$\det \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \partial_p^2 Q(\theta, 0) d\theta \neq 0.$$

Consider the perturbed Hamiltonian $H_\varepsilon = H + \varepsilon H_1$, with $H_1 : \mathbb{T}^n \times D_r^n \to \mathbb{R}$ analyitic. Then, there exists an ε_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$, there is a diffeomorphism

$$\phi : \mathbb{T}^n \times D_{r^*}^n \to \mathbb{T}^n \times D_r^n,$$

for some $0 < r^* < r$, which transforms H_ε into Kolmogorov normal form:

$$(H_\varepsilon \circ \phi)(\theta, p) = a^* + \nu \cdot p + Q^*(\theta, p),$$

in such a way that $||\phi - \mathrm{id}||_{sup}, |a^* - a|$ and $||Q - Q^*||_{sup}$ are of order ε. The diffeomorphism ϕ is ε-close to the identity, in the sense that there exists a constant K such that $||\phi - \mathrm{id}||_{sup} < K\varepsilon$.

Following the ideas in KAM theory to see which objects persist under perturbation, we want to study the rigidity of Lie group actions.

3.3 Stability, rigidity and Palais Theorem

Following the notation of Palais in [Pal60], we define C^k-*close actions.*

Definition 3.9. Let $f, g : M \to N$ be two smooth maps between smooth manifolds of dimension m and n, respectively. Suppose that $(x_1, \dots, x_m)$ is a coordinate system for $U \subset M$ compact and $(y_1, \dots, y_n)$ is a coordinate system for $V \subset N$. Suppose that $f(U) \subset V$ and $g(U) \subset V$. Then, f and g are C^k-*close maps*, for $k \geq 0$, if there exists an $\varepsilon > 0$ such that $|y_i \circ f(p) - y_i \circ g(p)| < \varepsilon$ for $p \in U$ and $i = 1, \dots, n$ and

$$\Big|\frac{\partial^r (y_i \circ f)}{\partial x_{j1} \cdots \partial x_{jr}}(p) - \frac{\partial^r (y_i \circ f)}{\partial x_{j1} \cdots \partial x_{jr}}(p)\Big| < \varepsilon,$$

for $p \in U$, $r \leq k$, $i = 1, \dots, n$ and $j_\alpha = 1, \dots, m$.

For Lie group actions the definition of closeness is the natural one, considering that the source space is the product of two smooth manifolds, hence a smooth manifold.

Definition 3.10. Two Lie group actions $\rho_1, \rho_2 : G \times M \to M$ are C^k-*close actions* if, for $g_1, g_2 \in G$, ${\rho_1}_{g_1}, {\rho_2}_{g_2}$ are C^k-close maps.

Richard Palais already proved in [Pal61a] an important rigidity result, the existence of a diffeomorphism that conjugates C^1-close actions of a compact Lie group on a compact manifold.

Theorem 3.11 (Palais)**.** *Let G be a compact Lie group and M a compact manifold. Let $\rho_1, \rho_2 : G \times M \longrightarrow M$ be two actions which are C^1-close. Then, there exists a diffeomorphism φ of class C^1 that conjugates ρ_1 and ρ_2, making them equivalent. This diffeomorphism belongs to the arc-connected component of the identity.*

In the case of the manifold being symplectic, Palais Theorem was extended to the following Theorem to obtain that the diffeomorphism conjugating the two close symplectic actions is a symplectomorphism. This was proved by Miranda (see [Mir07] or [MMZ12]).

Theorem 3.12 (Miranda)**.** *Let G be a compact Lie group and (M, ω) a compact symplectic manifold. Let $\rho_1, \rho_2 : G \times M \longrightarrow M$ be two symplectic actions which are C^2-close. Then, there exists a symplectomorphism φ that conjugates ρ_1 and ρ_2, making them equivalent.*

In the proof of Theorem 3.12, the diffeomorphism given by Palais Theorem is used, together with the Moser path method and a De Rham homotopy operator, to prove that the symplectic structure is equivariantly preserved.

3.4 Non-degenerate singularities in integrable Hamiltonian systems

A Hamiltonian system is completely integrable if it is defined by n first integrals in involution with respect to the Poisson bracket. Completely integrable Hamiltonian systems are closely related to Lagrangian foliations through the following result.

Proposition 3.13. *Let $f_1, \dots, f_n$ be n functions such that $\{f_i, f_j\} = 0, \forall i, j$. Suppose that $d_p f_1 \wedge \dots \wedge d_p f_n \neq 0$ at a point $p \in M$. Then, the distribution generated by the Hamiltonian vector fields $\mathcal{D} = \langle X_{f_1}, \dots, X_{f_n} \rangle$ is involutive and the leaf through p is a Lagrangian submanifold.*

The dynamics of an integrable system $F = (f_1, \dots, f_n)$ is explained by the Arnold-Liouville-Mineur Theorem (see Theorem 3.3) at the regular points, namely, at the points of the manifold where the differential $dF = (df_1, \dots, df_n)$ is not singular. This theorem was restated by Kiesenhofer and Miranda in [KM17] revealing that at a semilocal level the regular leaves are equivalent to a completely toric cotangent lift model.

Theorem 3.14. *Let $F = (f_1, \dots, f_n)$ be an integrable system on a symplectic manifold (M, ω). Then, semilocally around a regular Liouville torus, the system is equivalent to the cotangent model $(T^*\mathbb{T}^n)_{can}$ restricted to a neighbourhood of the zero section $(T^*\mathbb{T}^n)_0$ of $T^*\mathbb{T}^n$.*

At the singular points, the degeneracy of dF determines in general how difficult is to understand the dynamics, and for the case of non-degenerate singular points there are powerful results. The following definitions give the precise details of these concepts.

Definition 3.15. A point $p \in M^{2n}$ is a *singular point* of an integrable Hamiltonian system given by $F = (f_1, \dots, f_n)$ if the rank of $dF = (df_1, \dots, df_n)$ at p is less than n. The singular point p has *rank* k and *corank* of $n - k$ if $\text{rank}(dF)_p = \text{rank}\left((df_1)_p, \dots, (df_n)_p\right) = k$.

Definition 3.16. Let $\mathfrak{g}$ be a Lie algebra. A *Cartan subalgebra* $\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}$ that is self-normalizing, i.e., if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, then $Y \in \mathfrak{h}$. If $\mathfrak{g}$ is finite-dimensional and semisimple over an algebraically closed field of characteristic zero, a Cartan subalgebra is a maximal abelian subalgebra (a subalgebra consisting of semisimple elements).

Definition 3.17. Let (M^{2n}, ω) be a symplectic manifold with an integrable Hamiltonian system of n independent and commuting first integrals $f_1, \dots, f_n$. Consider a singular point $p \in M$ of rank 0, i.e. $(df_i)_p = 0$ for all i. It is called a *non-degenerate singular point* if the operators $\omega^{-1} d^2 f_1, \dots, \omega^{-1} d^2 f_n$ form a Cartan subalgebra in the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{sp}(T_p M, \omega)$.

Remark 3.18. The operators $\omega^{-1} d^2 f_i$, where df_i is the Hessian of f_i, associate a function to the Hessian by visualizing the Hessian as a quadratic form $H(u, v)$ and taking the symplectic dual of the function obtained. A good reference for details of the algebraic construction of the Cartan subalgebra is [BF04].

The classification of non-degenerate critical points of the moment map in the real case was obtained by Williamson [Wil36]. In the complex case, all the Cartan subalgebras are conjugate and hence there is only one model for non-degenerate critical points of the moment map.

Theorem 3.19 (Williamson)**.** *For any Cartan subalgebra $\mathcal{C}$ of $\mathfrak{sp}(2n, \mathbb{R})$, there exists a symplectic system of coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{2n}$ and a basis $f_1, \ldots, f_n$ of $\mathcal{C}$ such that each of the quadratic polynomials f_i is one of the following:*

$$
\begin{aligned}
& f_i = x_i^2 + y_i^2 && \text{for } \ 1 \le i \le k_e \\
& f_i = x_i y_i && \text{for } \ k_e + 1 \le i \le k_e + k_h \\
& \begin{cases} f_i = x_i y_{i+1} - x_{i+1} y_i \\ f_{i+1} = x_i y_i + x_{i+1} y_{i+1} \end{cases} && \text{for } \ i = k_e + k_h + 2j - 1, \ 1 \le j \le k_f
\end{aligned}
$$

The three types are called elliptic, hyperbolic and focus-focus, respectively.

Remark 3.20. Notice that the focus-focus components always go by pairs. Because of theorem 3.19, the triple (k_e, k_h, k_f) at a singular point it is an invariant. It is also an invariant of the orbit of the integrable system through the point [Zun96].

If p is a non-degenerate singularity of the moment map F, it is characterized by four integer numbers, the rank k of the singularity and the triple (k_e, k_h, k_f). By the way they are defined, they satisfy $k + k_e + k_h + 2k_f = n$, where n is the number of degrees of freedom of the integrable system.

The following is a result of Eliasson [Eli90] and Miranda and Zung ([Mir03], [Mir14], [MZ04]).

Theorem 3.21 (Smooth local linearization)**.** *Given an smooth integrable Hamiltonian system with n degrees of freedom on a symplectic manifold (M^{2n}, ω), the Liouville foliation in a neighborhood of a non-degenerate singular point of rank k and Williamson type (k_e, k_h, k_f) is locally symplectomorphic to the model Liouville foliation, which is the foliation defined by the basis functions of Theorem 3.19 plus "coordinate functions" $f_i = x_i$ for $i = k_e + k_h + 2j + 1$ to n.*

Remark 3.22. The theorem states the existence of a semilocal symplectomorphism between foliations with a non degenerate singularity of rank k and the same parameters (k_e, k_h, k_f). One could think that functions are also preserved via a symplectomorphism, but it is not possible to guarantee this statement when $h_k \neq 0$ as one can add up analytically flat terms on different connected components (see counterexample in [Mir03]). In general one needs more information about the topology of the leaf to conclude (see Figure 3).

Remark 3.23. Because of Theorem 3.21, if one considers the Taylor expansions of $F = (f_1, \ldots, f_n)$ at the non-degenerate singular point in a canonical coordinate system and removes all terms except for linear and quadratic, the functions obtained remain commuting and define a Liouville foliation that can be considered as the *linearization* of the initial foliation $\mathcal{F}$ given by $f_1, \ldots, f_n$, to which it is symplectomorphic.

The description of non-degenerate singularities at the semilocal level is completed with the following two results.

Theorem 3.24 (Model in a covering). *The manifold can be represented, locally at a non-degenerate singularity of rank k and Williamson type (k_e, k_h, k_f), as the direct product*

$$M^{reg} \times \overset{k}{\cdots} \times M^{reg} \times M^{ell} \times \overset{k_e}{\cdots} \times M^{ell} \times M^{hyp} \times \overset{k_h}{\cdots} \times M^{hyp} \times M^{foc} \times \overset{k_f}{\cdots} \times M^{foc}$$

Where:

- *M^{reg} is a "regular block", given by*

$$f = x,$$

- *M^{ell} is an "elliptic block", representing the elliptic singularity given by*

$$f = x^2 + y^2,$$

- *M^{hyp} is an "hyperbolic block", representing the hyperbolic singularity given by*

$$f = xy,$$

- *M^{foc} is a "focus-focus block", representing the focus-focus singularity given by*

$$\begin{cases} f_1 = x_1 y_2 - x_2 y_1 \\ f_2 = x_1 y_1 + x_2 y_2 \end{cases}.$$

For the first three types of blocks the symplectic form is $\omega = dx \wedge dy$, while for the focus-focus block it is $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

In the case of a smooth system (defined by a smooth moment map), a similar result was proved and described by Miranda and Zung in [MZ04]. It summarizes some previously results proved independently and fixes the case where there are hyperbolic components ($k_h \neq 0$), because in this case the result is slightly different and it has to be taken the semidirect product in the decomposition. As opposite to the case where there are only elliptic and focus-focus singularities, in which the base of the fibration of the neighbourhood is an open disk, if there are hyperbolic components the topology of the fiber can become complicated. The reason is essentially that for the smooth case a level set of the form $\{x_i y_i = \varepsilon\}$ is not connected but consists of two components.

Theorem 3.25 (Miranda-Zung). *Let $V = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ with coordinates $(p_1, ..., p_k)$ for D^k, $(q_1(mod 1), ..., q_k(mod 1))$ for $\mathbb{T}^k$, and $(x_1, y_1, ..., x_{n-k}, y_{n-k})$ for $D^{2(n-k)}$ be a symplectic manifold with the standard symplectic form $\sum dp_i \wedge dq_i + \sum dx_j \wedge dy_j$. Let F be the moment map corresponding to a singularity of rank k with Williamson type (k_e, k_h, k_f). There exists a finite group Γ, a linear system* *on the symplectic manifold V/Γ and a smooth Lagrangian-fibration-preserving symplectomorphism ϕ from a neighborhood of O into V/Γ, which sends O to the torus $\{p_i = x_i = y_i = 0\}$. The smooth symplectomorphism ϕ can be chosen so that via ϕ, the system-preserving action of a compact group G near O becomes a linear system-preserving action of G on V/Γ. If the moment map F is real analytic and the action of G near O is analytic, then the symplectomorphism ϕ can also be chosen to be real analytic. If the system depends smoothly (resp., analytically) on a local parameter (i.e. we have a local family of systems), then ϕ can also be chosen to depend smoothly (resp., analytically) on that parameter.*

In this case, the so-called *twisted hyperbolic* component can arise (see Figure 3), and the group of all linear moment maps preserving symplectomorphisms of the linear direct model of Williamson type (k_e, k_h, k_f) is isomorphic to

$$\mathbb{T}^k \times \mathbb{T}^{k_e} \times (\mathbb{R} \times \mathbb{Z}/2\mathbb{Z})^{k_h} \times (\mathbb{R} \times \mathbb{T}^1)^{k_f}.$$

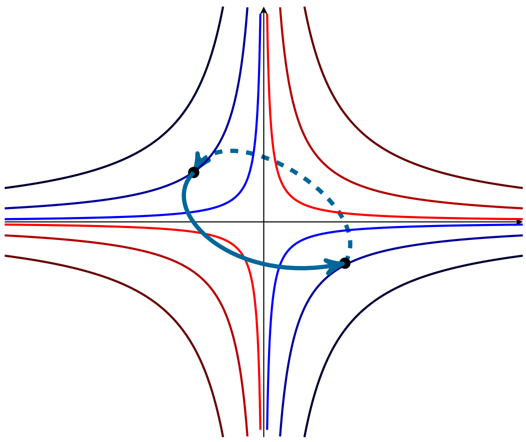

Figure 3: In the neighbourhood of an orbit of rank 1 and Williamson type $(0, 1, 0)$, the return map corresponding to the flow of circle action can give rise to two different behaviours. After one turn, the point can return to itself or it can return to its "opposite" branch (twisted hyperbolic case), and this defines a $\mathbb{Z}/2\mathbb{Z}$ action. The twisted hyperbolic case is described in this picture.

To end this section, we recall a related result which highlights the importance of considering the symplectomorphism at the level of the Lagrangian fibration induced by the Hamiltonian vector fields of the integrable system. Assume that (M, ω) is a symplectic manifold with a non-degenerate singularity of Williamson type (k_e, k_h, k_f). Assume that the foliation $\mathcal{F}$ at the singularity is the linear foliation defined by $\mathcal{F} = \langle X_1, \dots, X_n \rangle$, where the vector fields X_i are the linear Hamiltonian vector fields corresponding to the basis functions of Theorem 3.19. Namely, X_i are the vector fields induced by $\iota_{X_i}\omega = -df_i$, that is:

- $X_i = -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}$ for elliptic components,

- $X_i = -x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}$ for hyperbolic components,
- $X_i = -x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i} - x_{i+1}\frac{\partial}{\partial x_{i+1}} + y_{i+1}\frac{\partial}{\partial y_{i+1}}$ and
  $X_{i+1} = x_{i+1}\frac{\partial}{\partial x_i} + y_{i+1}\frac{\partial}{\partial y_i} - x_i\frac{\partial}{\partial x_{i+1}} - y_i\frac{\partial}{\partial y_{i+1}}$ for focus-focus components.

Then, the following theorem holds.

Theorem 3.26. *[Mir03] Let ω be a symplectic form defined in a neighbourhood of the singularity at p for which the foliation $\mathcal{F}$ is Lagrangian. Then, there exists a local diffeomorphism $\phi : (U, p) \longrightarrow (\phi(U), p)$ such that ϕ preserves the foliation and $\phi^*(\sum_i dx_i \wedge dy_i) = \omega$, where x_i, y_i are local coordinates on $(\phi(U), p)$.*

For completely elliptic singularities (of rank 0 and Williamson type $(k_e, 0, 0)$) Theorem 3.26 was proved by Eliasson [Eli90]. When $h_e \neq 0$, the foliation given by the hyperbolic components is preserved but the components of the moment map are not necessarily preserved (for more details see [Mir03]).

4. New results on rigidity of close lifted actions and applications

4.1 Equivalence of close lifted actions

We state and prove some results on symplectic equivalence of lifted close actions of a compact group on a compact manifold. We start proving a proposition on the equivalence at the level of cotangent lift given equivalence at the base. It is clear that if two symplectic actions are close, so are their fundamental vector fields. In Proposition 4.1 we prove that if two actions are C^1-equivalent, so are their cotangent lifts, and we define explicitly the diffeomorphism that conjugates them. With the same idea, and since any cotangent lifted action is Hamiltonian, we prove that if two actions are C^1-equivalent, then the moment maps induced by their cotangent lifts are also equivalent.

Proposition 4.1. *Let G be a Lie group and let M be a smooth manifold. Let $\rho_1, \rho_2 : G \times M \longrightarrow M$ be two actions which are C^1-equivalent via a conjugation through a diffeomorphism φ. Let $\hat{\rho}_1, \hat{\rho}_2$ be the cotangent lifts of ρ_1, ρ_2, respectively. Then, $\hat{\rho}_1$ and $\hat{\rho}_2$ are C^1-equivalent via the conjugation through $\hat{\varphi}$. The moment maps induced by $\hat{\rho}_1, \hat{\rho}_2$, denoted respectively by μ_1, μ_2, are equivalent via the conjugation through $\hat{\varphi}$ as $\mu_2 = \mu_1 \circ \hat{\varphi}$.*

Proof. Assume $\rho_1, \rho_2 : G \times M \longrightarrow M$ are two C^1-equivalent Lie group actions. Let φ be the C^1-diffeomorphism conjugating the two actions, i.e, let φ be a diffeomorphism such that $\rho_1 \circ \varphi = \varphi \circ \rho_2$. Differentiating both sides, the following equality is obtained:

$$d\rho_{1,\varphi(q)} \circ d\varphi_q = d\varphi_{\rho_2(q)} \circ d\rho_{2,q}.$$

Transposing and inverting the latter equality on both sides, one arrives to the following relation:

$$((d\rho_{1,\varphi(q)})^*)^{-1} \circ ((d\varphi_q)^*)^{-1}(p) = ((d\varphi_{\rho_2(q)})^*)^{-1} \circ ((d\rho_{2,q})^*)^{-1}(p),$$

which shows that $((d\varphi)^*)^{-1}$ is exactly the conjugation between $((d\rho_{1,\varphi(q)})^*)^{-1}$ and $((d\rho_{2,q})^*)^{-1}$.

We define now $\hat{\varphi}(q,p) := (\varphi(q), ((d\varphi_q)^*)^{-1}(p))$, which is a diffeomorphism and can be thought as the *cotangent lift of* φ. Consider the cotangent lift of the actions ρ_1 and ρ_2, i.e. $\hat{\rho}_1$ and $\hat{\rho}_2$. By definition, $\hat{\rho}_i(q,p) = (\rho_i(q), ((d\rho_{i,q})^*)^{-1}(p))$. Then, it is clear that $\hat{\rho}_1 \circ \hat{\varphi} = \hat{\varphi} \circ \hat{\rho}_2$, and we conclude that the cotangent lifts of the actions are equivalent on the cotangent bundle via conjugation by $\hat{\varphi}$, which is precisely the cotangent lift of the diffeomorphism φ that conjugates ρ_1 and ρ_2 on the base.

The cotangent lift of the action $\hat{\rho}_i$ is a Hamiltonian action with moment map $\mu_i : T^*M \longmapsto \mathfrak{g}^*$ given by

$$\langle \mu_i(p), X \rangle := \langle \lambda_p, X^{\#}|_p \rangle = \langle p, X^{\#}|_{\pi(p)} \rangle,$$

where $p \in T^*M, X \in \mathfrak{g}$, $X^\#$ is the fundamental vector field of X generated by the $\hat{\rho}_i$ action and λ is the Liouville 1-form on T^*M.

The diffeomorphism $\hat{\varphi}$ is defined by

$$\hat{\varphi}(q,p) := (\varphi(q), ((d\varphi_q)^*)^{-1}(p))$$

and satisfies $\hat{\rho}_1 \circ \hat{\varphi} = \hat{\varphi} \circ \hat{\rho}_2$. On the other hand, by Lemma 2.118 the Liouville one-form is invariant under the lifted actions, i.e. $\hat{\rho}_i^* \lambda = \lambda$ for $i = 1, 2$, and it is also invariant under the diffeomorphism $\hat{\varphi}$ by Lemma 2.118.

Through the following computation:

$$\begin{aligned}
\langle \mu_2(p), X\rangle &= \langle \lambda_p, X^\#|_p\rangle = \\
&= \langle \lambda_p, \frac{d}{dt}\left(\hat{\rho}_2(\exp(-tX), p)\right)|_{t=0}\rangle = \\
&= \langle \lambda_p, \frac{d}{dt}\left(\hat{\varphi}^{-1}(\hat{\rho}_1(\exp(-tX), \hat{\varphi}(p)))\right)|_{t=0}\rangle = \\
&= \langle \lambda_{\hat{\varphi}(p)}, \frac{d}{dt}\left(\hat{\rho}_1(\exp(-tX), \hat{\varphi}(p))\right)|_{t=0}\rangle = \\
&= \langle \lambda_{\hat{\varphi}(p)}, X^\#|_{\hat{\varphi}(p)}\rangle = \\
&= \langle \mu_1(\hat{\varphi}(p)), X\rangle = \langle \mu_1 \circ \hat{\varphi}(p), X\rangle,
\end{aligned}$$

where we have used that $\hat{\varphi}^{-1} \circ \hat{\rho}_1 \circ \hat{\varphi} = \hat{\rho}_2$. Observe that the fundamental vector fields and the actions are $\hat{\varphi}$-related. If one of the fundamental vector fields is Hamiltonian in the ξ direction (the one given by μ_1), so is the second (the one given by $\mu_1 \circ \hat{\varphi}$). We conclude that the moment maps are equivalent. □

Now we prove a theorem that can be thought as the cotangent lifted version of Theorem 3.12.

Theorem 4.2. *Let G be a compact Lie group and M a compact smooth manifold. Let $\rho_1, \rho_2 : G \times M \longrightarrow M$ be two actions which are C^1-close. Let $\hat{\rho}_1, \hat{\rho}_2 : G \times (T^*M, \omega) \longrightarrow (T^*M, \omega)$ be the cotangent lifts of ρ_1, ρ_2, respectively. Then, there exists a symplectomorphism that conjugates $\hat{\rho}_1$ and $\hat{\rho}_2$, thus making them equivalent.*

Remark 4.3. Notice that the actions have to be C^1-close. Compared with the symplectic version of Palais rigidity Theorem (Theorem 3.12), where they have to be C^2-close, one degree of differentiability is gained here.

Proof. Let G be a compact Lie group and M a compact smooth manifold. Let $\rho_1, \rho_2 : G \times M \longrightarrow M$ be two actions and assume that they are C^1-close. By Theorem 3.11, there exists a diffeomorphism φ that conjugates ρ_1 and ρ_2.

Consider $\hat{\rho}_1, \hat{\rho}_2 : G \times (T^*M, \omega) \longrightarrow (T^*M, \omega)$, the cotangent lifts of ρ_1 and ρ_2, respectively. By Proposition 4.1, the diffeomorphism $\hat{\varphi}$ conjugates $\hat{\rho}_1$ and $\hat{\rho}_2$. To prove that the actions $\hat{\rho}_1$ and $\hat{\rho}_2$ are not only equivalent, but symplectically equivalent, we need to check that $\hat{\varphi}$ preserves the symplectic form. By Lemma 2.118, it preserves the canonical 1-form λ of T^*M and, hence, it preserves the symplectic form ω. □

4.2 Application to integrable systems with non-degenerate singularities

Results of the previous section, namely shows a natural way of applying the result of rigidity of the lifted actions to the category of Hamiltonian systems. Theorem 4.2 guarantees, for instance, that the compact orbits of two C^1-close integrable systems on a symplectic manifold are equivalent at the level of the cotangent lift.

An immediate corollary of Palais rigidity Theorem is the following. Consider two integrable systems in a compact symplectic manifold (M, ω) given by $F = (f_1, \dots, f_n)$ and $\hat{F} = (\hat{f}_1, \dots, \hat{f}_n)$, respectively. Let $X_1, \dots, X_n$ and $\hat{X}_1, \dots, \hat{X}_n$ be the corresponding associated vector fields (those induced by $\iota_{X_i}\omega = -df_i$). If, for each $i = 1, \dots, n$, the flow ψ_i of X_i is close to the flow $\hat{\psi}_i$ of $\hat{X}_i$, and all of them are actions of a compact group (case of toric manifolds), then the two integrable systems are equivalent, i.e., it exists a diffeomorphism φ that conjugates F and $\hat{F}$. This equivalence can even be pictured in terms of the Delzant theorem looking at the corresponding Delzant polytopes [Del88].

In the same direction, a straightforward consequence of Theorem 4.2 at the semilocal level in a neighbourhood of a compact orbit is the following.

Theorem 4.4. *Let $F = (f_1, \dots, f_n) : (M^{2n}, \omega) \to \mathbb{R}^n$ and $\hat{F} = (\hat{f}_1, \dots, \hat{f}_n) : (M^{2n}, \omega) \to \mathbb{R}^n$ be two smooth maps defining two integrable systems. Suppose that the singularities of F and $\hat{F}$ are non-degenerate and a combination of only regular and elliptic components (with compact orbits) i.e., that each singularity of rank $k \neq n$ has Williamson type $(n-k, 0, 0)$. Assume that, for all $1 \leq i \leq n$, f_i and $\hat{f}_i$ are C^2-close. Then, for each $c \in Im(F) \subset \mathbb{R}^n$:*

1. *there exists $\hat{c} \in Im(\hat{F}) \subset \mathbb{R}^n$ that is close to c, and*
2. *there exists a symplectomorphism ϕ_c that makes the neighbourhoods of the leaves $\Lambda_c = F^{-1}(c)$ and $\hat{\Lambda}_{\hat{c}} = \hat{F}^{-1}(\hat{c})$ equivalent. Namely, there exists ϕ_c defined in a neighbourhood of Λ_c such that $\phi_c \circ F = \hat{F} \circ \phi_c$ and $\phi_c^*(\omega) = \omega$.*

Remark 4.5. Observe that for elliptic and regular components the connected components of the leaves equal the orbits.

Proof. By closeness between F and $\hat{F}$, for each $c \in \text{Im}(F) \subset \mathbb{R}^n$ there exists $\hat{c} \in \text{Im}(\hat{F}) \subset \mathbb{R}^n$ that is close to c and such that $\hat{c}$ is a singular value of $\hat{F}$ if and only if c is a singular value of F. Closeness between F and $\hat{F}$ (together with non-degeneracy) guarantees that the number of elliptic components at the singularity $x \in F^{-1}(c)$ is the same as the number of elliptic components at $y \in \hat{F}^{-1}(\hat{c})$.

Now, in view of Theorem 3.24, and since in this case the singularities are the product of only regular and elliptic type, if we prove the existence of the symplectomorphism for the case of a regular value and for the case of a complete elliptic singularity we will be finished.

If c is a regular value of F, by the Arnold-Liouville-Mineur Theorem the neighbourhood of the leaf Λ_c is diffeomorphic to the cotangent bundle of the Liouville torus. The same applies to the the neighbourhood of the leaf $\hat{\Lambda}_{\hat{c}}$. The action on $T^*\mathbb{T}^n$ is the cotangent lift of a compact torus action and then, by Theorem 4.2, there exists a symplectomorphism ϕ_c conjugating F and $\hat{F}$ on the respective leaf neighbourhoods.

Now suppose c is a non-degenerate singular value of F and $x \in F^{-1}(c)$ is a completely elliptic singularity. Consider the action given by the joint flow, which in this case is locally free and has a unique fixed point, the singularity x . By means of the joint flow we identify the action as a torus action (see [MZ04]) and we can apply Theorem 3.12 to obtain rigidity between a neighbourhood of Λ_c and $\hat{\Lambda}_{\hat{c}}$. □

Remark 4.6. In the case of a regular point, another way of proving symplectic rigidity is using the normal form of the moment map, since there is only one local model, which is the one given by the Arnold-Liouville-Mineur Theorem.

Remark 4.7. We do not require that the Williamson type of the non-degenerate singularities of F and $\hat{F}$ is the same, only that they both are combination of regular and elliptic type (in both cases the orbits coincide with the leafs). Notice that if $\hat{F}$ is close enough to F, the elliptic components of a singularity of F will remain elliptic in the associated singularity of $\hat{F}$, and the regular components can not become neither hyperbolic nor focus-focus, so compactness of actions and, hence, rigidity, is guaranteed without having to impose the same Williamson type.

These consequences do not go beyond results that are already known concerning rigidity of integrable systems. In fact, they can be considered special cases of the Arnold-Lioville-Mineur Theorem, since it gives a unique normal form for neighbourhoods of regular points of integrable systems. Nevertheless, Theorem 4.2 can be used in the same context of integrable systems to prove a slightly more ambitious result.

4.3 Application to S^1-invariant degenerate singularities

Consider the following example of a very simple integrable system.

Example 4.8. Let $f = (x^2+y^2)^k$, with $k \geq 2$, be the moment map of an the integrable system in $(\mathbb{R}^2, \omega_{st} = dx \wedge dy)$. It is a completely solvable system, it has an isolated degenerate singularity at the origin, the flows of the Hamiltonian vector field lie in concentric circles, and the singularity is a stable center. Since it is a degenerate singularity, we can not apply directly normal form theorems. Nevertheless, we know that, the system is invariant with respect to the S^1 action and therefore we can use another system (which is non-degenerate) associated to the circle action for which there exists a normal form, which in fact is $x^2 + y^2$ and corresponds to an elliptic singularity.

In order to conclude we need a normal form result for circle actions. We first recall the general symplectic slice theorem and then apply it in the case of a fixed point of a circle action.

Theorem 4.9 (Guillemin-Sternberg [GS84a], Marle [Mar85]). *Let (M, ω, G) be a symplectic manifold together with a Hamiltonian group action. Let z be a point in M such that $\mathcal{O}_z$ is contained in the zero level set of the momentum map. Denote G_z the isotropy group and $\mathcal{O}_z$ the orbit of z. There is a G-equivariant symplectomorphism from a neighbourhood of the zero section of the bundle $T^*G \times_{G_z} V_z$ equipped with the above symplectic model to a neighbourhood of the orbit $\mathcal{O}_z$.*

Recall from Bochner's linearization theorem that in a neighbourhood of a fixed point of an action we can always linearize the group action.

Applying the theorem 4.9 above to the circle action case with a fixed point and applying Bochner's theorem we obtain the classical Marle-Guillemin-Sternberg which gives a local normal form for the moment map of circle actions in a neighbourhood of a fixed point of the action.

Theorem 4.10 ([Mar85, Mar84, GS84a]). *Let (M^{2n}, ω) be a symmplectic manifold endowed with an S^1-Hamiltonian action and let p be a fixed point for this action. Then there exist local coordinates $(x_1, y_1, \dots, x_n, y_n)$ such that $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ and $\mu(x) = \sum_{i=1}^n c_i(x_i^2 + y_i^2)$.*

Remark 4.11. The constants c_j can be interpreted as *weights* of the circle action.

The last conclusion of the example is summarized the following Lemma, which is an easy consequence of the Guillemin-Marle-Sternberg Theorem.

Lemma 4.12. *Consider a 2-dimensional integrable system which has an S^1-invariant degenerate singularity. Then, locally it is function of the quadratic normal form of elliptic type.*

Proof. By Guillemin-Marle-Sternberg Theorem 4.10, the moment map of an S^1-action with a fixed point is a sum of squares in its normal form. Since it is a 2-dimensional system and because of Noether's theorem, one can take coordinates x, y in a neighbourhood of the singularity in such a way that the moment map can be written as

$$f = \phi(x^2 + y^2).$$

□

Consider now $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) and with the standard symplectic form $\omega_{st} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Consider the three following Hamiltonian functions:

$$F = (f_1, f_2) = \left(x_1^2 + y_1^2, x_2^2 + y_2^2\right) \tag{4.1}$$

$$G = (g_1, g_2) = \left((x_1^2 + y_1^2)^2, x_2^2 + y_2^2\right) \tag{4.2}$$

$$H = (h_1, h_2) = \left((x_1^2 + y_1^2)^2, (x_2^2 + y_2^2)^2\right) \tag{4.3}$$

The three integrable systems have an isolated singularity at the origin, but only in the system given by F it is non-degenerate. By the way, this system is the model of the two uncoupled harmonic oscillators and its level sets are 2-dimensional invariant tori. The same level sets appear in the other two systems, but Theorem 4.4 can be directly applied to state rigidity only in the first system, since it has a single non-degenerate singularity of elliptic type (it is already in normal form), while in the others the singularity is degenerate. Observing the system given by G, though, one can see that the second component produces an invariant S^1 action. This S^1 symmetry allows for a symplectic reduction of the system, making it decrease from a 4-dimensional to a 2-dimensional. In this new system, the singularity is still degenerate, but following the idea in Example 4.8 one can find its non-degenerate elliptic normal form. Then, Theorem 4.4 can be applied to obtain rigidity of the reduced system, understanding rigidity as equivalence of close systems. It is not difficult to see that, then, the original system is also rigid.

In view of this procedure, we have the following result.

Theorem 4.13. *Consider an integrable system in a symplectic manifold (M, ω) given by $F = (f_1, \ldots, f_n)$. Suppose that if $p \in M$ is a singularity of F, it is isolated, there are no other singularities in its F-level set, and it is:*

- *either non-degenerate of regular or elliptic type, or*
- *degenerate of the following type: $f_1, \ldots, f_{n-1}$ have a non-degenerate singularity of elliptic type at p, f_n has a degenerate singularity at p and f_n is S^1-invariant.*

Then the system is rigid at the neighbourhood of each compact leaf $\Lambda_c = F^{-1}(c) \subset M$.

Proof. In all the regular leaves or in the leaves containing non-degenerate singularities, Theorem 4.4 already gives rigidity. At any singular leaf containing a degenerate singularity, there exist $(n-1)$ S^1-invariant actions that commute so we can perform a series of $(n-1)$ symplectic reductions successively to reduce the system to a 2-dimensional system, which has a degenerate singularity corresponding to the singularity of f_n. At this point, the moment map of the reduced integrable system still gives an S^1-invariant action which has a moment map $\overline{f_n}$ and because of Theorem 4.10 the function $\overline{f_n}$ can be put in the quadratic normal form corresponding to the elliptic singularity. Then, again by Theorem 4.4, the system associated to $\overline{f_n}$ is rigid at the neighbourhood of the leaf. Because of by Lemma 4.12 the function f_n is a smooth function of $f_n = H(\overline{f_n})$ of $\overline{f_n}$ and thus rigidity also holds for f_n and by reconstruction from the initial integrable system $(f_1, \ldots, f_n)$ in a neighbourhood of a compact leaf. □

Theorem 4.13 states semiglobal rigidity in the very particular case of systems with degenerate singularities that are non-degenerate in $(n-1)$ components of the moment map and have an S^1-invariant action in the degenerate component.

5. The b-symplectic version

It is a natural question to ask if the same results obtained for the standard cotangent lift in the symplectic setting are also true in the b-symplectic case. Since rigidity of the cotangent lift is proved using Palais Theorem, in order to prove rigidity in the b-symplectic setting one has to use a b version of Palais Theorem. At its turn, the proof of Palais Theorem uses the classical Mostow-Palais Theorem on equivariant embeddings, so one has too prove also its b version. In this section, we first state the b version of the Mostow-Palais and the Palais Theorems. Then, we give the proofs of the b-symplectic Palais and the twisted b-cotangent lifted Palais Theorems, which are all new results.

5.1 The b-Mostow-Palais and the b-Palais Theorems

We want to prove the b-versions of the Mostow-Palais embedding Theorem and the Palais Rigidity Theorem. We give a proof which is similar to the proof of Palais Rigidity Theorem that can be found on [GGK02].

We start proving the following Lemma.

Lemma 5.1. *Representation b-functions are $\mathcal{C}^1$-dense in $\mathcal{C}^\infty(M, Z)$.*

Proof. Representation b-functions on G, with respect to the left or right action of G on itself, are uniformly dense in $\mathcal{C}^1(G)$. This follows from the Peter-Weyl Theorem adapted for b-functions. Namely, the matrix coefficients of a finite-dimensional representation T of G span the finite-dimensional representation $T^* \otimes T$ in $\mathcal{C}^\infty(G)$. Then, any linear combination of matrix coefficients is a representation b-function on G. On the other hand, by same the Peter-Weyl Theorem, such linear combinations are uniformly dense in $\mathcal{C}^\infty(G)$.

To prove the lemma, we will show that the *convolution* of a function f on M and a representation function u on G, is a representation function on M. More specifically, for $u \in \mathcal{C}^\infty(G)$ and $f \in \mathcal{C}^\infty(M, Z)$, we set:

$$f_u(x) = \int_G u(g) f(g^{-1}x) dg \tag{5.1}$$

where dg is the normalized Haar measure on the compact group G. And now we see that f_u is a b-representation function whenever u is a b-representation function. Indeed, for every $h \in G$, we have

$$hf_u = f_{hu} \tag{5.2}$$

where $hu(g) := u(h^{-1}g)$. This can be checked through the following calculation:

$$(hf_u(x)) = f_u(h^{-1}x) = \int_G u(g) f(g^{-1}h^{-1}x) dg = \int_G u(h^{-1}g) f(g^{-1}x) dg = f_{hu}(x). \tag{5.3}$$

For a fixed f, the mapping $u \mapsto f_u$ is linear and G-equivariant (as $hf_u = f_{hu}$). Hence, $V(f_u)$ is the image of $V(u)$ and, as a consequence, $V(f_u)$ is finite-dimensional if $V(u)$ is finite-dimensional.

To finish the proof, observe that every function f can be arbitrarily close $\mathcal{C}^1$-approximated by $b$functions of the form f_v with $v \in \mathcal{C}^\infty(G)$. It suffices to take as v a bump function on G which is supported near $e \in G$ and has unit integral. Uniformly approximating v by representation functions u, we obtain a $\mathcal{C}^1$-approximation of f by representation b-functions f_u. $\square$

Before stating the b-Mostow Palais Embedding Theorem, We recall that a G-action on M induces a linear G-representation on $\mathcal{C}^\infty(M, Z)$, where $g \in G$ acts by sending $f \in \mathcal{C}^\infty(M, Z)$ to the b-function $(gf)(x) = f(g^{-1}x)$. Denote by $V(f)$ the span in $\mathcal{C}^\infty(M, Z)$ of the orbit $G \cdot f$. f is said to be a *representation b-function* if $V(f)$ is finite-dimensional.

Theorem 5.2 (b-Mostow-Palais embedding Theorem)**.** *Let a compact Lie group G act on a compact b-manifold (M, Z) via b-maps. Then, there exists an equivariant embedding of (M, Z) into a linear representation of G on a finite-dimensional pair (V, H) of vector spaces where H has codimension 1 in V.*

Proof. The evaluation maps $\delta_x : f \mapsto f(x)$, for $x \in (M, Z)$, give rise to an equivariant injection $x \mapsto \delta_x$ of M, Z into the dual space to $\mathcal{C}^\infty(M, Z)$. For every $f \in \mathcal{C}^\infty(M, Z)$, the space $V(f)$ is naturally a G-representation, and the evaluation map gives rise to an equivariant mapping $(M, Z) \to V(f)^*$. By Whitney's Theorem, every manifold can be smoothly embedded in $\mathbb{R}^m$ for some m, and a C^1 deformation of an embedding remains an embedding Then, since representation b-functions are C^1-dense in $\mathcal{C}^\infty(M, Z)$ (by Lemma 5.1), there exists an embedding $(M, Z) \to \mathbb{R}^m$ whose components $f_1, \ldots, f_m$ are representation b-functions.

Therefore, we obtain an equivariant evaluation map from (M, Z) to the direct sum $V = V(f_1)^* \oplus \cdots \oplus V(f_m)^*$. This map is an embedding because its composition with a suitable linear mapping $V \to \mathbb{R}^m$ is the original embedding $(f_1, \ldots, f_m)$. And if the G action on M is effective, so is the G action on V. Then, G embeds in $GL(N)$, where $N = \dim V$. The action becomes orthogonal by taking any inner product on V and averaging with respect to the G-action. $\square$

Now, before proving the b-Palais Rigidity Theorem, we need the following Proposition.

Proposition 5.3. *Let G be a compact Lie group and V a finite-dimensional vector space. Let ρ_0 be a linear representation of G in V. Then, for every representation ρ which is sufficiently C^0-close to ρ_0, there exists an automorphism A of V which intertwines ρ_0 and ρ, i.e., such that $\rho_0 = A \circ \rho \circ A^{-1}$. Besides, A can be chosen to depend smoothly on ρ and ρ_0.*

Proof. For each representation ρ of G in V, define a linear map $A : V \to V$ by

$$A(x) = \int_G \rho_0(g^{-1})\rho(g)(x)dg, \quad x \in V.$$

Then, for any $h \in G$,

$$A(\rho(h)x) = \int_G \rho_0(g^{-1})\rho(g)\rho(h)(x)dg = \int_G \rho_0(g^{-1})\rho(gh)(x)dg.$$

The change of variable $g \mapsto gh^{-1}$ turns the integral into

$$\int_G \rho_0(hg^{-1})\rho(g)dg = \rho_0(h)\int_G \rho_0(g^{-1})\rho(g)dg = \rho_0(h)A(x).$$

This shows that $A \circ \rho = \rho_0 \circ A$. When A is invertible, the proof is finished. Because A depends continuously on ρ and is equal to the identity map when $\rho = \rho_0$, the map A is invertible if ρ is sufficiently close to ρ_0. □

Theorem 5.4 (b-Palais Theorem)**.** *Let ρ be a b-action of a compact group G on a compact b-manifold (M, Z). For every b-action ρ_1 of G on (M, Z) which is sufficiently C^1-close to ρ, there exists a diffeomorphism $\phi : (M, Z) \to (M, Z)$ which is a b-map and which conjugates the actions: $\rho_1 = \phi\rho\phi^{-1}$. Also, it belongs to the connected component of the identity map.*

Proof. We start applying Theorem 5.2 to both ρ and ρ_1. We obtain two representations of G, say $\bar{\rho}$ and $\bar{\rho}_1$, on vector spaces V and V_1, respectively, and equivariant embeddings $M \to V$ for ρ and $(M, Z) \to V_1$ for ρ_1. We can find a linear isomorphism $V_1 \to V$ after which the embeddings become C^1-close and $\bar{\rho}$ and $\bar{\rho}_1$ become close, making it possible to identify $V = V_1$.

By Proposition 5.3, there exists a linear mapping $A : V \to V$, close to the identity, which sends $\bar{\rho}_1$ to $\bar{\rho}$ ($\bar{\rho} = A \circ \bar{\rho}_1 \circ A^{-1}$). Thus, we assume that the representations are equal and the embedding ψ_1 for ρ_1 is still C^1-close to the embedding ψ for ρ.

The image of ψ_1 lies in a small tubular neighborhood of the image of ψ, which we identify with (M, Z). Let us fix a G-invariant metric on V. The composition ϕ of ψ_1 with the orthogonal projection from the tubular neighborhood to (M, Z) is clearly G-equivariant: $(M, Z, \rho_1) \to (M, Z, \rho)$ and it is a b-map. Since ψ_1 is C^1-close to ψ, this composition is a diffeomorphism. □

5.2 The b-symplectic and the b-cotangent lifted Palais Theorems

We prove now the b-symplectic version of Palais Theorem, which is the b-symplectic analogue of Theorem 3.12.

Theorem 5.5. *Let G be a compact Lie group and (M, Z, ω) a compact smooth b-symplectic manifold. Let $\rho_1, \rho_2 : G \times M \longrightarrow (M, Z, \omega)$ be two b-actions which are C^2-close. Then, there exists a b-symplectomorphism that conjugates ρ_1 and ρ_2, making them equivalent.*

Proof. Let G be a compact Lie group and (M, Z, ω) a compact smooth manifold. Let $\rho_1, \rho_2 : G \times M \longrightarrow M$ be two b-actions and assume that they are C^2-close. By Theorem 5.4, there exists a diffeomorphism φ that conjugates ρ_1 and ρ_2 and such that it is a b-map.

Set $\omega_0 = \omega$ and $\omega_1 = \varphi^*(\omega_0)$ and consider the linear path of b-symplectic structures

$$\omega_t = t\omega_1 + (1-t)\omega_0, \quad t \in [0,1],$$

which is a path of b-symplectic structures since ω_0 and ω_1 are close. We want to see that this path, which takes ω_0 to ω_1, is invariant respect to the action ρ_1.

By the Theorem 5.4 the b-diffeomorphism φ belongs to the arc-connected component of the identity, making it possible to construct an homotopy φ_t from $\varphi_0 := id$ to $\varphi_1 := \varphi$.

Then, we define a b-De Rham homotopy operator Q following the recipe given in [GS77] by Guillemin-Sternberg (see also [CdS01]) which states the following. Suppose that ω_t is a smooth family of b-k-forms and that φ_t represents a one-parameter family of local diffeomorphisms such that $\varphi_t = id$ and $d\varphi_t/dt = X_t \circ \varphi_t$, i.e., φ_t is the flow of the b-vector field X_t. Then,

$$\frac{d}{dt}(\varphi_t^*\omega_t) = \varphi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}\right). \tag{5.4}$$

Fixing $\omega_t = \omega$ in Equation 5.4 and integrating over $t \in [0,1]$, we obtain:

$$\varphi_1^*\omega - \varphi_0^*\omega = \int_0^1 \varphi_t^*(\mathcal{L}_{X_t}\omega_t)dt. \tag{5.5}$$

Applying Cartan's formula, we get to the following equality:

$$\varphi_1^*\omega - \varphi_0^*\omega = \int_0^1 \varphi_t^*(\iota_{X_t}d\omega + d\iota_{X_t}\omega)dt = \tag{5.6}$$

$$= \int_0^1 \varphi_t^*(\iota_{X_t}d\omega)dt + d\int_0^1 \varphi_t^*(\iota_{X_t}\omega)dt. \tag{5.7}$$

Now, using Equation 5.7, we define the following b-De Rham operator Q:

$$Q(\omega) = \int_0^1 \varphi_t^*(\iota_{X_t}\omega)dt,$$

where the b-vector field X_t is defined by the isotopy φ_t. Equation 5.7 applied to $\omega = \omega_0$ tells that:

$$\omega_1 - \omega_0 = Q(d\omega) + dQ(\omega). \tag{5.8}$$

Since ω is a b-symplectic form, $Q(d\omega) = Q(0) = 0$ and:

$$\omega_1 - \omega_0 = dQ(\omega),$$

which proves that ω_0 and ω_1 belong to the same cohomology class and explicitly shows that $\omega_1 - \omega_0 = d\alpha$ for the b-1-form $\alpha = Q(\omega)$.

Now, let X_t be the b-vector field that satisfies

$$\iota_{X_t}\omega_t = -\alpha. \tag{5.9}$$

Notice that X_t is a b-vector field for any t, since α is a b-1-form and ω_t is a b-2-form for any t. Then, X_t will preserve (M, Z). Consider the averaged vector field of X_t with respect to a Haar measure $d\mu$ on G:

$$X_t^G := \int_G \rho_1(g)_*(X_t) d\mu. \tag{5.10}$$

Since the b-diffeomorphism φ conjugates the actions ρ_1 and ρ_2, which both preserve the initial b-symplectic form ω_0, the path of b-symplectic forms ω_t is invariant under ρ_1. Then, the b-vector field X_t^G satisfies the equation

$$i_{X_t^G}\omega_t = -\int_G \rho_1(g)^*(\alpha) d\mu,$$

which can be considered an averaging of Equation 5.9. Then, the invariant b-1-form defined by $\alpha_G = \int_G \rho_1(g)^*(\alpha) d\mu$ satisfies $\omega_1 - \omega_0 = d\alpha_G$ because the path ω_t is invariant under ρ_1.

Finally, consider the equation

$$X_t^G(\phi_t^G) = \frac{\partial \phi_t^G}{\partial t}.$$

At this point, there is a loss of one degree of differentiability with respect to the degree of differentiability of φ, but the existence of ϕ_t^G for all $t \in [0, 1]$ is clear, because the manifold is compact and by 5.4 the conjugating b-diffeomorphism φ is of class $\mathcal{C}^2$.

Then, the flow ϕ_t^G commutes with the action of G given by ρ_1 and satisfies $\phi_t^{G*}(\omega_t) = \omega_0$ for all $t \in [0, 1]$. In particular, at $t = 1$, we have that ϕ_1^G takes ω_1 to ω_0 in an equivariant way. □

We finally prove twisted b-cotangent lift version of the Palais Theorem, the b analogue of 4.2. The twisted b-cotangent lift (see Definition 2.123) is appropriate here because when we want to eventually apply this result to integrable systems, these would be b-integrable Hamiltonian systems defined by b-functions. And the logarithm appearing on the local expression of the twisted b-1-form, i.e.:

$$\lambda_{tw} = \log|y_1| dx_1 + \sum_{i=2}^{n} y_i dx_i,$$

is compatible with the logarithm term of a b-function. On the contrary, the canonical b-cotangent lift is based on the singularity of the type $1/x$ appearing in the canonical b-1-form, and it does not produce a proper b-integrable Hamiltonian system.

Proposition 5.6. *Let G be a compact Lie group and (M, Z) a compact smooth b-manifold. Let $\rho_1, \rho_2 : G \times (M, Z) \longrightarrow (M, Z)$ be two b-actions which are C^1-close. Let $\hat{\rho}_1, \hat{\rho}_2 : G \times ({}^bT^*M, \omega) \longrightarrow ({}^bT^*M, \omega)$ be the twisted b-cotangent lifts of ρ_1, ρ_2, respectively. Then, there exists a b-symplectomorphism that conjugates $\hat{\rho}_1$ and $\hat{\rho}_2$, making them equivalent.*

Before proving this proposition, we prove that if two b-actions are C^1-equivalent, so are their twisted b-cotangent lifts and so are the induced moment maps.

Proposition 5.7. *Let G be a Lie group and let (M, Z) be a smooth manifold. Let $\rho_1, \rho_2 : G \times (M, Z) \longrightarrow (M, Z)$ be two b-actions which are C^1-equivalent via a conjugation through a diffeomorphism φ which is a b-map. Let $\hat{\rho}_1, \hat{\rho}_2$ be the twisted b-cotangent lifts of ρ_1, ρ_2, respectively. Then, $\hat{\rho}_1$ and $\hat{\rho}_2$ are C^1-equivalent via the conjugation through the b-map $\hat{\varphi}$. The moment maps induced by $\hat{\rho}_1, \hat{\rho}_2$, denoted respectively by μ_1, μ_2, are equivalent via the conjugation through $\hat{\varphi}$.*

Proof. Assume $\rho_1, \rho_2 : G \times (M, Z) \longrightarrow (M, Z)$ are two C^1-equivalent Lie group b-actions. Let φ be the C^1-b-diffeomorphism conjugating the two actions, i.e, let φ be a b-diffeomorphism such that $\rho_1 \circ \varphi = \varphi \circ \rho_2$.

Define $\hat{\varphi}(q, p) := (\varphi(q), ((d\varphi_q)^*)^{-1}(p))$, which is a b-diffeomorphism and can be thought as the twisted *b-cotangent lift of φ*. Consider the twisted b-cotangent lift of the actions ρ_1 and ρ_2, i.e. $\hat{\rho}_1$ and $\hat{\rho}_2$. By definition, $\hat{\rho}_i(q, p) = (\rho_i(q), ((d\rho_{i,q})^*)^{-1}(p))$. Then, by the same computations of the proof of Proposition 4.1 we deduce that $\hat{\rho}_1 \circ \hat{\varphi} = \hat{\varphi} \circ \hat{\rho}_2$, and we conclude that the twisted b-cotangent lifts of the actions are equivalent on the cotangent bundle via conjugation by $\hat{\varphi}$, which is precisely the twisted b-cotangent lift of the diffeomorphism φ that conjugates ρ_1 and ρ_2 on the base.

Also, analogously to what was computed on Proposition 4.1, the moment maps induced by the twisted b-cotangent lifts of ρ_1 and ρ_2 are equivalent. □

Proof of Proposition 5.6. Let G be a compact Lie group and (M, Z) a compact smooth b-manifold. Let $\rho_1, \rho_2 : G \times (M, Z) \longrightarrow (M, Z)$ be two b-actions and assume that they are C^1-close. By Theorem 5.4, there exists a diffeomorphism φ that conjugates ρ_1 and ρ_2 and is a b-map.

Consider $\hat{\rho}_1, \hat{\rho}_2 : G \times ({}^bT^*M, \omega) \longrightarrow ({}^bT^*M, \omega)$, the twisted b-cotangent lifts of ρ_1 and ρ_2, respectively. By Proposition 5.7, the diffeomorphism $\hat{\varphi}$ conjugates $\hat{\rho}_1$ and $\hat{\rho}_2$ and it is also a b-map. To prove that the actions $\hat{\rho}_1$ and $\hat{\rho}_2$ are not only equivalent, but b-symplectically equivalent, we need to check that $\hat{\varphi}$ preserves the b-symplectic form. It preserves the canonical b-1-form λ of ${}^bT^*M$ and, hence, it preserves the b-symplectic form ω. □

6. Cotangent models for integrable systems

Integrable Hamiltonian systems with non-degenerate singularities are really common in Mechanics problems, and one does not have to go to complicated models to already find the three basic types of non-degenerate singularities (in the Williamson sense, see Theorem 3.19) described in Section 3.4. In the classical examples of the harmonic oscillator, the simple pendulum and the spherical pendulum, there appear the elliptic singularity, the hyperbolic singularity and the focus-focus singularity, respectively.

On the other hand, the three basic singularities can be formulated (in the elliptic case only formally) as the cotangent lift of a Lie group action, which shows how cotangent models are a useful tool when dealing with integrable systems.

In this section, we give the mathematical description of the physical models illustrating the three types of non-degenerate singularities. Then, we give the formulation of the elliptic, hyperbolic and focus-focus singularities as cotangent lifts.

6.1 The harmonic oscillator

Consider an ideal one-dimensional oscillating system consisting of a mass m connected to a rigid foundation by way of a spring of stiffness constant k, as in Figure 4, with no friction of any kind and, hence, with no loss of mechanical energy. The Hamiltonian of the system is the sum of the kinetic and the elastic potential energies. In terms of the natural coordinates of the phase space of the system $(\mathbb{R}^2, \omega = dx \wedge dv)$, which are the position x and the velocity v of the mass, it writes as:

$$\hat{H}(x, v) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \tag{6.1}$$

Applying the following symplectic transformation:

$$\begin{cases} x = q \cdot \frac{1}{\sqrt[4]{k/m}} \\ v = p \cdot \sqrt[4]{k/m} \end{cases}, \tag{6.2}$$

the symplectic manifold is now $(\mathbb{R}^2, \omega = dq \wedge dp)$ and the Hamiltonian becomes:

$$H(p, q) = \frac{1}{2}\sqrt{mk}\left(p^2 + q^2\right). \tag{6.3}$$

Dropping the physical constants m and k, this Hamiltonian is exactly the normal form of the moment map of a one-dimensional system with an elliptic singularity at the origin, the unique equilibrium point of the system.

6.2 The simple pendulum

The simple pendulum is another of the basic models in classical mechanics. The most natural approximation to its formulation is the Newtonian setting, where we

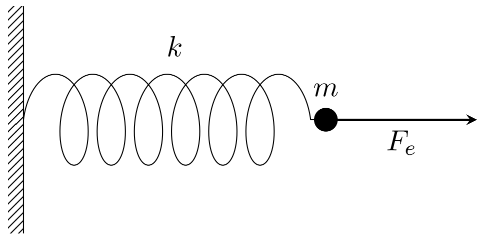

Figure 4: The harmonic oscillator.

consider the forces and acting in the system formed by a mass m attached to an end of a rigid massless rod of length l which has the other end fixed, as in Figure 5. It is assumed that the mass moves in the vertical plane formed by the vertical direction and the initial position and, since the rod has fixed length, the natural coordinate is the angle $\theta \in [0, 2\pi)]$ with respect to the lower vertical equilibrium position.

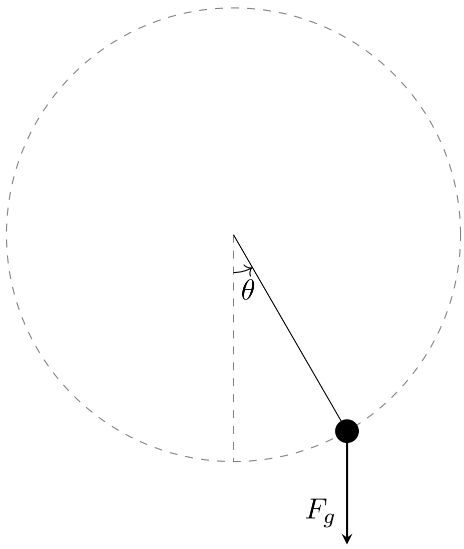

Figure 5: The simple pendulum.

Newton's law states that the acceleration of the mass in the direction of motion, which is always perpendicular to the direction of the rod, is proportional to the total force in this direction of motion. Since the only force in this direction is the component of the gravity force, Newton's law reduces to:

$$ma_\perp = F_\perp. \tag{6.4}$$

Taking into account that the acceleration is related to the angular coordinate through $a_\perp(\theta) = l\frac{\partial^2\theta}{\partial t^2}$ and that the force is also function of the angle through $F_\perp(\theta) = -mg\sin\theta$, where g is the gravity acceleration, the equation rewrites as the following 2nd order ODE:

$$\frac{\partial^2\theta}{\partial t^2} = -\frac{g}{l}\sin\theta. \tag{6.5}$$

If we define $\rho := \frac{\partial\theta}{\partial t}$ and consider the symplectic structure $(S^1 \times \mathbb{R}, \omega = d\theta \wedge d\rho)$ of the phase space, Equation (6.5) is equivalent to the Hamiltonian first order system of ODE's:

$$\begin{cases} \frac{\partial\theta}{\partial t} = \rho \\ \frac{\partial\rho}{\partial t} = -\frac{g}{l}\sin\theta \end{cases}, \tag{6.6}$$

whose Hamiltonian is

$$\hat{H}(\theta,\rho) = \frac{\rho^2}{2} - \frac{g}{l}\cos\theta. \tag{6.7}$$

The first equilibrium point of (6.6) is found at $\theta = \rho = 0$ and it is an stable point. Dropping out the physical constants, the Hamiltonian there has the normal form $\bar{H} = \frac{1}{2}(\rho^2 + \theta^2)$, which corresponds to an elliptic singularity like in the harmonic oscillator. We are more interested in the second equilibrium point, found at $\theta = \pi, \rho = 0$.

The Hamiltonian there can be locally expanded as:

$$H(\theta,\rho) = \frac{1}{2}\left(\rho^2 - \frac{g}{l}\theta^2\right). \tag{6.8}$$

Dropping the physical constants g and l, this Hamiltonian corresponds to the normal form of a one-dimensional system with a hyperbolic singularity at the origin.

6.3 The spherical pendulum

The most basic physical example of a singularity of focus-focus type comes from the spherical pendulum. Consider a point of mass m attached to an end of a rigid massless rod of length l and assume that the other end of the rod is fixed at the origin and that the mass can move freely as long as it remains attached to the rod, as in Figure 6. The mass can move, then, on a sphere of radius l.

The natural phase space is the cotangent bundle T^*S^2 and, while spherical coordinates are the optimal setting to study the dynamics of the spherical pendulum, Cartesian coordinates are more appropriated to analyze the singularities of the system. In Cartesian, the position of the point of mass will be given by $\vec{r} = (x, y, z)$, with $\|\vec{r}\| = l$. The conjugate variable to $\vec{r}$ is the linear momentum of the point, $\vec{p} = (p_x, p_y, p_z) = m\dot{\vec{r}}$, which has to satisfy $\vec{r}\cdot\vec{p} = 0$ in order to be contained in the tangent space of the sphere.

The Hamiltonian of the system is the sum of kinetic and potential energies and in the symplectic setting $(\mathbb{R}^6, \omega = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z)$ writes as:

$$H(\vec{r},\vec{p}) = \frac{\|\vec{p}\|^2}{2m} + mgl\frac{\vec{r}\cdot\hat{z}}{\|\vec{r}\|}, \tag{6.9}$$

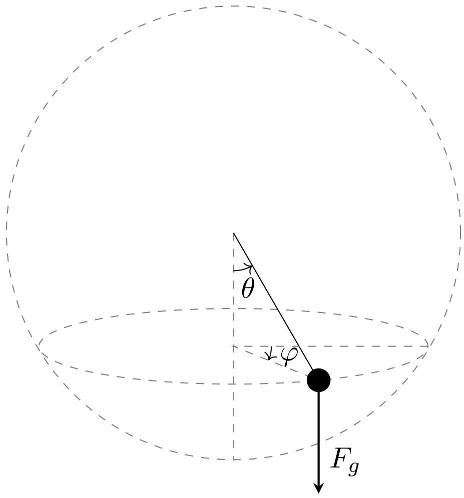

Figure 6: The spherical pendulum.

where g accounts for the gravity acceleration and $\hat{z}$ is the unit vector in the z direction. There is another conserved quantity, the angular momentum in the z direction: $L := L_z = xp_y - yp_x$. H and L satisfy $\{H, L\} = 0$ and are independent almost everywhere. Hence, they form the Liouville integrable system corresponding the spherical pendulum.

There are two singularities in the pendulum system, one corresponding to $z = -l$ (or to $\vec{r}_- = (0, 0, -l)$) and the other one to $z = l$ (or to $\vec{r}_+ = (0, 0, l)$). We are interested in $\vec{r}_1$, the unstable equilibrium, where we are going to identify the focus-focus singularity.

To study the system near $z = l$, we use that $z = \sqrt{l^2 - x^2 - y^2}$ and take local coordinates $(x, y, z) = (x, y, \sqrt{l^2 - x^2 - y^2})$. The conjugate momentum $\vec{p} = (p_x, p_y, p_z)$ satisfies locally that $p_z = 0$. In these symplectic coordinates the symplectic form is $\omega = dx \wedge dp_x + dy \wedge dp_y$ and the Hamiltonian of the system writes as:

$$H = \frac{1}{2ml^2}\left(p_x^2(l^2 - x^2) + p_y^2(l^2 - y^2) - 2xyp_xp_y\right) + mg(\sqrt{l^2 - x^2 - y^2} - l). \quad (6.10)$$

At this point, it is convenient to apply a symplectic scaling in order to adimensionalize the Hamiltonian. We apply the following symplectic transformation:

$$\begin{cases} x = \frac{\xi}{\sqrt{m\nu}} \\ p_x = p_\xi\sqrt{m\nu} \\ y = \frac{\eta}{\sqrt{m\nu}} \\ p_y = p_\eta\sqrt{m\nu} \end{cases}, \quad (6.11)$$

where $\nu = \sqrt{g/l}$. In these local symplectic coordinates near the unstable equilibrium of the spherical pendulum, the symplecit form is rewritten as $\omega = d\xi \wedge dp_\xi + d\eta \wedge dp_\eta$

and the Hamiltonian becomes:

$$H = \nu \left(\frac{1}{2}(p_\xi^2 + p_\eta^2) - \frac{\kappa}{2}(\xi p_\xi + \eta p_\eta)^2 + \frac{1}{\kappa}(\sqrt{1-\kappa\rho^2} - 1) \right), \tag{6.12}$$

where $\rho^2 = \xi^2 + \eta^2$, $\nu^2 = g/l$ and $1/\kappa = ml^2\nu = mgl/\nu$ and they are all constants.

Finally, a last symplectic transformation reveals that the Williamson normal form at the unstable equilibrium of the spherical pendulum corresponds to the focus-focus singularity. It is the following:

$$\sqrt{2}\xi = q_1 - p_1, \quad \sqrt{2}p_\xi = q_1 + p_1, \qquad \sqrt{2}\eta = q_2 - p_2, \quad \sqrt{2}p_\eta = q_2 + p_2\,. \tag{6.13}$$

In these coordinates, where $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, the Hamiltonian is:

$$H = \nu \left(p_1 q_1 + p_2 q_2 - \kappa \frac{1}{8}(q^2 - p^2)^2 + \frac{1}{\kappa}\sqrt{1-\kappa\rho^2} + \frac{\rho^2}{2} - \frac{1}{\kappa} \right), \tag{6.14}$$

where $q^2 = q_1^2 + q_2^2$, $p^2 = p_1^2 + p_2^2$ and $\rho^2 = p^2/2 + q^2/2 - (p_1q_1 + p_2q_2)$.

Observe that the quadratic part of the potential has been absorbed in the terms $H' = \nu(p_1q_2 + p_2q_2)$ and that the remaining terms of the potential are of order 4 and higher. The quadratic part of H is simply H' and the angular momentum in the p, q variables is $L = q_1p_2 - q_2p_1$. So, the system $F = (H', L)$ has a singularity of focus-focus type.

6.4 The hyperbolic singularity as a cotangent lift

Take coordinates (x, y) on $T^*\mathbb{R}$ such that the symplectic form is $\omega = dx \wedge dy$ and the moment map is $f = xy$.

If we compute the Hamiltonian vector field associated to f, we obtain

$$X = -\frac{\partial f}{\partial y}\left(\frac{\partial}{\partial x}\right) + \frac{\partial f}{\partial x}\left(\frac{\partial}{\partial y}\right) = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = (-x, y). \tag{6.15}$$

Consider the action of $(\mathbb{R}, +)$ on $\mathbb{R}$ given by:

$$\begin{array}{rccc} \rho : & \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (t, x) & \longmapsto & e^{-t}x \end{array},$$

and the induced an action $\rho_t : \mathbb{R} \longrightarrow \mathbb{R}$. The differential of ρ_t at a point $x \in \mathbb{R}$ is:

$$\begin{array}{rccc} (d\rho_t)_x : & T_x\mathbb{R} & \longrightarrow & T_x\mathbb{R} \\ & y & \longmapsto & e^{-t}y \end{array},$$

Then, $((d\rho_t)_x^*)^{-1}$ acts as $y \longmapsto e^t y$, and the cotangent lift $\hat{\rho}_t$ associated to the group action ρ_t, in coordinates (x, y) of $T^*\mathbb{R}$ is exactly:

$$\begin{array}{rccc} \hat{\rho} : & T^*\mathbb{R} & \longrightarrow & T^*\mathbb{R} \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} e^{-t}x \\ e^t y \end{pmatrix} \end{array}.$$

Deriving the last vector with respect to t and evaluating at $t = 0$, we obtain exactly $X = (-x, y)$, the vector field associated to the hyperbolic singularity.

6.5 The elliptic singularity as a cotangent lift

The cotangent lift in the elliptic case uses a complex moment map which is not holomorphic. It is a formal development and by no means holomorphicity is assumed.

Take complex coordinates $(z, \bar{z}) = (x + iy, x - iy)$ such that the symplectic form is $\omega = dz \wedge d\bar{z}$. The moment map corresponding to the elliptic singularity is $f = x^2 + y^2 = z\bar{z}$.

The Hamilton's equations in this complex setting are:

$$\iota_X \omega = -df \iff \iota_{a\frac{\partial}{\partial z}+b\frac{\partial}{\partial \bar{z}}} dz \wedge d\bar{z} = -\frac{\partial f}{\partial z}dz - \frac{\partial f}{\partial \bar{z}}d\bar{z} \iff \begin{cases} a = -\frac{\partial f}{\partial \bar{z}} \\ b = \frac{\partial f}{\partial z} \end{cases} . \tag{6.16}$$

If we compute the Hamiltonian vector field associated to f, we obtain

$$X = -\frac{\partial f}{\partial \bar{z}}\left(\frac{\partial}{\partial z}\right) + \frac{\partial f}{\partial z}\left(\frac{\partial}{\partial \bar{z}}\right) = -z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} = (-z, \bar{z}). \tag{6.17}$$

Now, consider the following action, which is the same that we used for the hyperbolic cotangent lift but in complex coordinates:

$$\begin{array}{rccc} \rho: & \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (t, z) & \longmapsto & e^{-t}z \end{array},$$

And consider the induced an action $\rho_t : \mathbb{R} \longrightarrow \mathbb{R}$. The differential of ρ_t at a point $z \in \mathbb{R}$ is:

$$\begin{array}{rccc} (d\rho_t)_z: & T_z\mathbb{R} & \longrightarrow & T_z\mathbb{R} \\ & \bar{z} & \longmapsto & e^{-t}\bar{z} \end{array}.$$

Then, $((d\rho_t)_z^*)^{-1}$ acts as $\bar{z} \longmapsto e^t\bar{z}$, and the cotangent lift $\hat{\rho}_t$ associated to the group action ρ_t, in coordinates $(z, \bar{z})$ of $T^*\mathbb{R}$ is:

$$\begin{array}{rccc} \hat{\rho}: & T^*\mathbb{R} & \longrightarrow & T^*\mathbb{R} \\ & \begin{pmatrix} z \\ \bar{z} \end{pmatrix} & \longmapsto & \begin{pmatrix} e^{-t}z \\ e^t\bar{z} \end{pmatrix} \end{array}.$$

Deriving the last vector with respect to t and evaluating at $t = 0$ we obtain $X = (-z, \bar{z})$, the vector field associated to the hyperbolic singularity.

6.6 The focus-focus singularity as a cotangent lift

To describe the basic singularity of focus-focus type in a manifold of dimension 4 we take coordinates (x_1, x_2, y_1, y_2). The symplectic form is $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and the moment map associated to this singularity is $F = (f_1, f_2) = (x_1y_2 - x_2y_1, x_1y_1 + x_2y_2)$.

If we compute the Hamiltonian vector field associated to f_1 and f_2, we obtain

$$X_1 = -\frac{\partial f_1}{\partial y_1}\left(\frac{\partial}{\partial x_1}\right) - \frac{\partial f_1}{\partial y_2}\left(\frac{\partial}{\partial x_2}\right) + \frac{\partial f_1}{\partial x_1}\left(\frac{\partial}{\partial y_1}\right) + \frac{\partial f_1}{\partial x_2}\left(\frac{\partial}{\partial y_2}\right) = \tag{6.18}$$

$$= x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2} = (x_2, -x_1, y_2, -y_1), \tag{6.19}$$

and

$$X_2 = -\frac{\partial f_2}{\partial y_1}\left(\frac{\partial}{\partial x_1}\right) - \frac{\partial f_2}{\partial y_2}\left(\frac{\partial}{\partial x_2}\right) + \frac{\partial f_2}{\partial x_1}\left(\frac{\partial}{\partial y_1}\right) + \frac{\partial f_2}{\partial x_2}\left(\frac{\partial}{\partial y_2}\right) = \tag{6.20}$$

$$= -x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} = (-x_1, -x_2, y_1, y_2). \tag{6.21}$$

Now consider the action of a rotation and a radial dilation on $\mathbb{R}^2$ given by:

$$\begin{array}{cccc} \rho: & (S^1 \times \mathbb{R}) \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & ((\theta, t), \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) & \longmapsto & \rho_{\theta,t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{array}.$$

The differential of the induced action $\rho_{\theta,t}$ at a point $x = (x_1, x_2)$ is the following linear map:

$$\begin{array}{cccc} d\rho_{\theta,t}: & T_x\mathbb{R}^2 & \longrightarrow & T_x\mathbb{R}^2 \\ & \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} & \longmapsto & e^{-t}\begin{pmatrix} y_1\cos\theta + y_2\sin\theta \\ -y_1\sin\theta + y_2\cos\theta \end{pmatrix} \end{array}.$$

Then, $((d\rho_{\theta,t})^*)^{-1}$ acts as:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto e^{t}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

And the cotangent lift $\hat{\rho}_{\theta,t}$ associated to the group action $\rho_{\theta,t}$ is:

$$\begin{array}{cccc} \hat{\rho}_{\theta,t}: & T^*\mathbb{R}^2 & \longrightarrow & T^*\mathbb{R}^2 \\ & \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} & \longmapsto & \begin{pmatrix} e^{-t}(x_1\cos\theta + x_2\sin\theta) \\ e^{-t}(-x_1\sin\theta + x_2\cos\theta) \\ e^{t}(y_1\cos\theta + y_2\sin\theta) \\ e^{t}(-y_1\sin\theta + y_2\cos\theta) \end{pmatrix} \end{array}.$$

Finally, deriving the last vector with respect to θ and evaluating at 0 and deriving the vector with respect to t and evaluating at 0 we obtain, respectively, $X_1 = (x_2, -x_1, y_2, -y_1)$ and $X_2 = (-x_1, -x_2, y_1, y_2)$, the vector fields associated with f_1 and f_2, the components of the moment map of the hyperbolic singularity.

7. The focus-focus singularity, an overview

Following the idea of stability and classification of Hamiltonian systems, an important result by Atiyah [Ati82] and Guillemin and Sternberg [GS82] states that, in Hamiltonian systems where the moment map F is an action of a k-dimensional torus and the manifold M^{2n} is compact, $F(M)$ is a *convex polytope*. Later, Delzant proved in [Del88] that, if M is a toric variety, this polytope determines M up to isomorphism. In the case where the dimension of the torus is n (half the dimension of M), $F(M)$ was called a *Delzant polytope*. When trying to classify integrable Hamiltonian systems at the non-regular points, namely, at the singularities, finding objects such as the Delzant polytope which are invariants of the manifold is one of the main goals.

A special class of integrable Hamiltonian systems which has been of interest of the author is the class of semitoric integrable systems. There exist results concerning invariant objects in this special class of systems which are explained in this section.

Definition 7.1. A *semitoric integrable system* consists of a symplectic connected manifold (M,ω) of dimension 4 and two independent and smooth functions J, H defined on M, with J generating a Hamiltonian S^1 action and $\{J,H\}$=0.

Remark 7.2. For simplicity, in the study of semitoric integrable systems it is assumed that F has only non-degenerate singularities and that none of them is hyperbolic. It is the same as to say that the three possible singularities in a semitoric integrable system, written in an appropriate basis, are:

$$\begin{array}{ll} F_1 = (q_1^2+p_1^2, p_2) & (\text{transversally} - \text{elliptic}), \\ F_2 = (q_1^2+p_1^2, q_2^2+p_2^2) & (\text{elliptic} - \text{elliptic}), \\ F_3 = (q_1p_1+q_2p_2, q_1p_2-q_2p_1) & (\text{focus} - \text{focus}). \end{array}$$

It is also general to assume that the focus-focus singularities are on different level sets of F.

For the sake of completeness, and before starting with the particular case of the focus-focus singularity, we mention that there are 5 invariants that San Vu Ngoc and Alvaro Pelayo declared to be the complete family of invariants that determines a semitoric integrable system in a manifold of dimension 4. The rest of the section is devoted to the description of one of these invariants. For more details on the other invariants, see [PN09].

A Hamiltonian system (M,ω,F) is said to be *singular* at a point $m \in M$ if m is a critical point for the momentum map F, i.e., if each function f_i has a critical point at m. Without loss of generality, we will assume from now on that $f_i(m)=0$ at the critical point m. The *simple focus-focus singularity* is defined to be the singularity that appears in a manifold M of dimension 4 when there is a critical point m which is non-degenerate and the momentum map can be written locally in a basis (f_1,f_2) such that, in the Williamson coordinates (q_1,q_2,p_1,p_2) the components f_1 and f_2 are expressed as [Eli90]:

$$\begin{cases} f_1 = q_1 p_1 + q_2 p_2 \\ f_2 = q_1 p_2 - q_2 p_1 \end{cases} \tag{7.1}$$

It is clear from the local chart of coordinates that the focus-focus singularity is isolated as a critical point of F. In particular, one can see that the punctured neighbourhood of m in M is foliated by leafs of the form $\Lambda_c = F^{-1}(c)$ which are smooth Lagrangian submanifolds of M. In the rest of the section, we describe the leaf Λ_0 of the singular foliation and its neighbourhood.

7.1 The focus-focus invariant leaf

With the notation of Equation 7.1, we denote the Hamiltonian vector fields of f_1 and f_2 by X_1 and X_2, respectively, and define the complex coordinates $z_1 := q_1 + iq_2$, $z_2 := p_1 + ip_2$. The flows of X_1 and X_2 if the system has a focus-focus singularity are, respectively:

$$\begin{cases} \varphi_t^1(z_1, z_2) = (e^t z_1, e^{-t} z_2) \\ \varphi_t^2(z_1, z_2) = (e^{it} z_1, e^{it} z_2) \end{cases} \tag{7.2}$$

In these complex coordinates, it is easy to see that the complex function associated to the momentum map, $\tilde{F} := f_1 + if_2$, is written as $\tilde{F}(z_1, z_2) = \bar{z}_1 z_2$. Hence, the singular leaf $\Lambda_0 = F^{-1}(0)$ is the union of the complex planes $\{z_1 = 0\}$ and $\{z_2 = 0\}$, which, besides, correspond to the stable and unstable manifolds of the flow of X_1.

Although the focus-focus singularities are isolated, it can happen that a Lagrangian leaf Λ_0 contains several of them. Assuming Λ_0 to be compact, there can be only finitely many of them. However, in this section, we will discuss the case where Λ_0 contains only one critical point, denoted by m. In this case, the leaf is called simple focus-focus leaf.

We notice that the punctured leaf $\Lambda_0 \setminus \{m\}$, which is smooth and invariant under Hamiltonian flows of the system, is not compact. This prevents the neighboring leaves to be diffeomorphic to $\Lambda_0 \setminus \{m\}$ because by the Liouville-Arnold-Mineur Theorem (see Theorem 3.3) they are compact. Nevertheless, we with are going to show that the local structure of Λ_0 gives enough information to describe the leaf globally. Namely, we are going to review the proof of the following theorem [San98].

Theorem 7.3. *Let Λ_0 be a simple focus-focus leaf of the momentum map and let m be the singularity. Then, the connected component of m in Λ_0 is the image of an immersion of a 2-sphere with a double point. Besides, the punctured connected component of m in Λ_0 is an orbit of the Hamiltonian action of the system, with the structure of an affine infinite cylinder.*

Proof. We start supposing, without loss of generality, that Λ_0 is connected, because any other connected components in its neighbourhood are necessarily regular tori and the Arnold-Liouville-Mineur Theorem applies there.

Now, notice that $\Lambda_0 \setminus \{m\}$ can have at most two connected components, as Λ_0 is the union of two complex planes through their respective origins. On $\Lambda_0 \setminus \{m\}$, the action of the momentum map is locally free, as it does not contain any singularity. It implies that each connected component of $\Lambda_0 \setminus \{m\}$ is an orbit of the system, on which all isotropy subgroups I are conjugated. This orbit, then, is diffeomorphic to $\mathbb{R}^2/I$. Therefore, each connected component of $\Lambda_0 \setminus \{m\}$ is either diffeomorphic to $\mathbb{R}^2$ if I only contains the identity, or it is diffeomorphic to $\mathbb{R}^2/\mathbb{Z} = \mathbb{R} \times S^1$ (an infinite cylinder) if $I = \mathbb{Z}$. Notice that the possibility that $I = \mathbb{Z}^2$ is not considered because $\mathbb{R}^2/\mathbb{Z}^2$ is a torus, hence compact, contradicting the non-compactness of $\Lambda_0 \setminus \{m\}$.

To solve if the connected components of $\Lambda_0 \setminus \{m\}$ are infinite planes or infinite cylinders, we study the neighbourhood of the critical point m. The action of the system near m is symplectically linearizable (as Eliasson explains in [Eli90]), i.e. there exists a symplectic chart in which the Hamiltonian vector fields of f_1 and f_2 are linear combinations of the standard focus-focus vector fields X_1 and X_2 associated to the quadratic forms f_1 and f_2. Moreover, the coefficients of these linear combinations form an invertible 2×2 matrix M_c which is locally constant along each fiber Λ_c. As f_2 has periodic orbits in any neighbourhood of m (see Equation 7.2), the isotropy group is necessarily the group formed by the integer multiples of the period, so it's isomorphic to Z. In conclusion, the connected components of $\Lambda_0 \setminus \{m\}$ are infinite cylinders.

Now, notice that in one of these infinite cylinders there are two infinitesimal generators of the action of the system which are globally defined, X_1 and X_2, constructed as a combination of the initial generators X_{f_1} and X_{f_2} of the system. In particular they are built exactly through the combination provided by M_c^{-1}. As we see in Equation 7.2, these vector fields are transversal to each other and, hence, as X_2 generates the periodic orbits, X_1 describes an axis of the cylinder.

Finally, we check that there is only one connected component in $\Lambda_0 \setminus \{m\}$, by showing that the entire $\Lambda_0 \setminus \{m\}$ is connected. As we already know, the punctured set $\Lambda_0 \setminus \{m\}$, by construction, has only one or two connected components. This is because Λ_0 is connected, but removing the point m could divide the leaf into two connected components. One of these two would be associated to the stable manifold of X_1 at m and the other one to the unstable manifold of X_1 at m (recall that we know that the local behaviour of the system near m is given by 7.2).

We fix now a point x in the unstable manifold of X_1, so close to m that it is contained in a neighbourhood U of m which is S^1 invariant (i.e.: invariant through X_2). The flow of X_1 with initial condition x, as the time increases, goes out of U. But, as the dynamics on the infinite cylinder force the flow take only finite time to leave any compact subset, and since the manifold with boundary $\Lambda_0 \setminus U$ is compact, the image $x(t)$ of x through the flow of X_1 must necessarily leave $\Lambda_0 \setminus U$ at some finite time t_0. In other words, it has to enter U again. And it is clear from the local structure of the flow near m that $x(t)$ can enter U (a neighbourhood of m only via the stable manifold.

Therefore, the stable and the unstable manifolds are connected to each other,

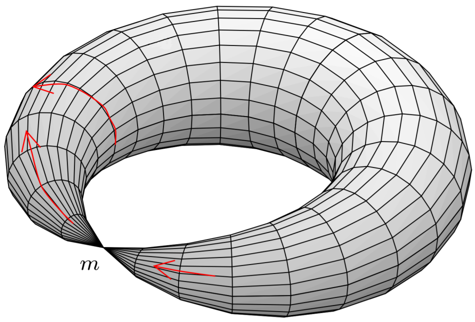

Figure 7: A simple singular leaf of focus-focus type.

and hence they are equal and there is only one connected component in $\Lambda_0 \setminus \{m\}$. We conclude that Λ_0 is homeomorphic to a cylinder whose ends are compactified in a unique point, the focus-focus singularity m (see Figure 7). It can also be thought as a pinched torus, or as a sphere with two points identified.

□

7.2 The neighbourhood of a simple focus-focus leaf

As we have seen, the leaf Λ_0 of a simple focus-focus singularity is topologically a pinched torus. In this section, we present a characterization of the neighbouring leaves through some invariants.

First of all, we recall that the Liouville-Arnold-Mineur Theorem states that as neighbouring leaves of a singular simple leaf Λ_0 are regular, they are tori if they are compact manifolds.

Now, we define the Hamiltonian vector field $t_1X_1 + t_2X_2$ in coordinates (c_1, c_2) on $\mathbb{R}^2$, where X_1 and X_2 are the vector fields associated to the components f_1 and f_2 of the momentum map F (which defines the foliation). If (c_1, c_2) is a regular value of F we know by the Liouville-Arnold-Mineur Theorem that the set of values (t_1, t_2) which make the flow of the Hamiltonian vector field $t_1X_1 + t_2X_2$ be periodic of period 1 are a sublattice of $\mathbb{R}^2$ called the period lattice [VuN03].

But this period lattice collapses when c tends to the critical value $(0, 0)$, as we already know that in the leaf Λ_0 only one period survives (the one associated to the periodic orbits of the transversal circles in the pinched torus) and the other period goes to infinity (the one associated to the axis of the torus or the homoclinic orbit).

Nevertheless, we see that we can "follow" the collapse of the period lattice as c tends to 0. Let Ω be a small neighbourhood of m where the symplectic linearization of the system ($F = (f_1, f_2)$) explained in the previous section is valid and fix a point A in the intersection of the regular leaf Λ_c and Ω. We denote by $S^1(A)$ the orbit of A through the action of f_2, which we know that is 2π-periodic. For $c \neq 0$, let $\tau_1(c) > 0$ be the time necessary for the flow of X_1 to send $S^1(A)$ to itself for first time (after an entire loop around the torus leaf Λ_c, see Figure 8). As X_1 and X_2 commute, τ_1

is independent of the point $A \in S^1(A)$ that is chosen. We denote A' the image of A through this map and denote by $\tau_2(c)$ the time necessary for the flow of X_2 to send A' to A. $\tau_2(c)$, again, does not depend on the choice of $p \in S^1(A)$.

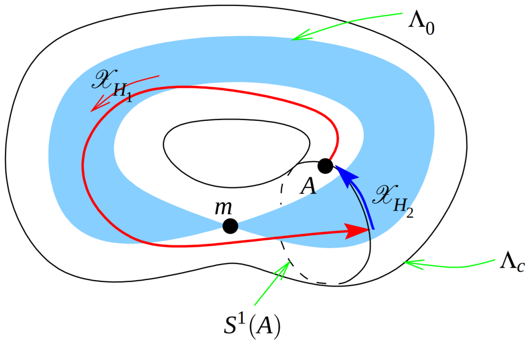

Figure 8: Neighbourhood of a simple focus-focus singularity.

Now, we have that the vector fields $\tau_1 X_1 + \tau_2 X_2$ and $2\pi X_2$ both define 1-periodic flows. For this reason, the basis $\{(\tau_1, \tau_2), (0, 2\pi)\}$ is a $\mathbb{Z}-$basis for the period lattice. This is important because the singularity classification depends on the nature of this basis when c tends to 0.

Finally, we present a result that quantifies the divergence of the period when the collapse happens. In this way, we will be able to define a 1-form that we can assure is closed.

Theorem 7.4. *The following functions extend to smooth and single-valued functions in a neighbourhood of $c = 0$.*

$$\begin{cases} \sigma_1(c) = \tau_1(c) + \mathcal{R}(\ln c) \\ \sigma_2(c) = \tau_2(c) - \mathcal{I}(\ln c) \end{cases} \tag{7.3}$$

Where $\ln c$ is any determination of the complex logarithm and $c = (c_1, c_2)$ is identified with $c_1 + ic_2$. Besides, the 1-form defined by $\sigma := \sigma_1 dc_1 + \sigma_2 dc_2$ is closed.

At this point, we can state the main result that allows to characterize the neighbourhood of a simple focus-focus leaf

Theorem 7.5. *Let S be the unique smooth function defined in some neighbourhood of $0 \in \mathbb{R}^2$ such that $dS = \sigma$ (its existence is guaranteed by the Poincaré lemma) and such that $S(0) = 0$. Then, the Taylor expansion of S at $c = 0$ is a symplectic invariant of the singular Liouville foliation of focus-focus type at a simple focus-focus leaf. It is denoted by (S).*

This result, which goes down to the level of 1-forms, can be more elegantly stated in the following way:

Theorem 7.6. *The set of equivalence classes of germs of singular Liouville foliations of focus-focus type at a simple focus-focus leaf is in natural bijection with $R[[X, Y]]_0$, which is the algebra of real formal power series in two variables with vanishing constant term.*

Our rigidity theorems of Section 4 apply for compact actions and, hence, we can not use them to state rigidity for the focus-focus singular leaf, which arises from a non-compact action. The invariants defined by San Vu Ngoc (see Theorem 7.5) characterize not only the focus-focus singularity but its whole invariant leaf. Then, it is a natural question to ask if rigidity of the singular focus-focus leaf could be proved using some closeness conditions on these invariants.

On the other hand, the cotangent lift (see the cotangent models for non-degenerate singularities presented in Section 6) provides a local modal valid in the neighbourhood of a point. To get the symplectic normal form for the system in the neighbourhood of a leaf one needs to cover this neighbourhood by action-angle neighbourhoods and glue them back. In [MVuN05] it is proved that there exist a moduli of symplectic structures. In a future work we plan to identify Vu Ngoc's invariants with an action on the cotangent models.

8. The saddle-focus singularity, a physical example

In the restricted three body problem (*R3BP*), it occurs one of the most studied bifurcations in the sense of dynamical systems, the Hopf Bifurcation. It is a bifurcation that happens at two fixed points denoted by $\mathcal{L}_4$ and $\mathcal{L}_5$ when a one-dimensional parameter μ goes through a constant value μ_1, the so-called *Routh Mass.* This section is devoted to describe this singularity, which is sometimes also called the *saddle-focus singularity.*

A brief summary of what it will be explained in this section os the following. At points $\mathcal{L}_4$ and $\mathcal{L}_5$, the matrix of the linearized system of the R3BP can have three types of eigenvalues:

- $\pm i\omega_1, \pm i\omega_2$, if $0 < \mu < \mu_1 = (1 - \sqrt{69}/9)/2$,
- $\pm \omega i$ with double multiplicity, if $\mu = \mu_1$, and
- $\pm\alpha \pm i\beta$, if $\mu_1 < \mu < 1/2$.

For the case $\mu < \mu_1$, the Lyapunov Center Theorem applies and it gives the existence of a family of periodic orbits, while for the case $\mu > \mu_1$, the classical Stable Manifold Theorem explains the behaviour of the system at the critical point. Finally, for the case $\mu = \mu_1$, it is necessary to analyze the system through some symplectic changes of coordinates[6] and scalings.

We will introduce the restricted three body problem and then we are going to focus on the discussion of the Hopf Bifurcation at $\mathcal{L}_4$ and $\mathcal{L}_5$. For that, we will previously state the Lyapunov Center Theorem and its proof. After a quite general study of the Hopf Bifurcation in a general quadratic Hamiltonian system, we are going to check that it is exactly the bifurcation that arises at $\mathcal{L}_4$ when μ goes through μ_1 in the R3BP. At the end, we will conclude with real examples of objects found near the points $\mathcal{L}_4$ and $\mathcal{L}_5$ of different subsystems of the Solar System.

8.1 The restricted planar three body problem

The restricted planar three body problem is a classical problem in Celestial Mechanics which, even being a quite simple model, suffices to explain accurately enough some phenomena that occur, for instance, in the Solar System.

The R3BP can be seen as the classical three body problem (*3BP*) in rotating coordinates and restricted to the plane, with the assumption that one of the three bodies has infinitesimally small mass while the other two, the *primaries*, have finite mass. The motion of the two primaries is assumed, for this model, to be in circular orbits around each other. For this reason, is natural to take rotating coordinates, with the center of mass of the two finite masses placed at the origin. The aim of the

[6]If $\varphi : (z,t) \mapsto \varphi(z,t) : U \subset \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ is a change of variables and satisfies that $D_z\varphi \in Sp(2n, \mathbb{R})$, i.e., is a symplectic matrix, then we say that it is a *symplectic change of coordinates.*

model is precisely to study the motion of the third body due to the attraction of the primaries.

The formulation of the R3BP is the following. Consider the rotating coordinates $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}^2$, the u's accounting for the positions and the v's accounting for the conjugate momenta of the particles $1, 2, 3$ in the plane. Let m_1, m_2 be the masses of the two primaries and $\varepsilon^2 << 1$ the mass of the small particle. The Hamiltonian of the R3BP is

$$H_3 = \frac{\|v_3\|^2}{2\varepsilon^2} - u_3^T K v_3 - \sum_{i=1}^{2} \frac{\varepsilon^2 m_i}{\|u_3 - u_i\|} + \sum_{i=1}^{2} u_i^T K v_i + \frac{m_1 m_2}{\|u_1 - u_2\|} \tag{8.1}$$

where $K = J_2$.

Through a symplectic change of variables, it is possible to pass from the total version of the R3BP to a partial version that focuses on the dynamics of the small mass [MO17]. The Hamiltonian of the small mass, assuming its coordinates to be (x_1, x_2) for the position and (y_1, y_2) for the conjugate momentum is

$$H = \frac{\|y\|^2}{2} - x^T K y - \frac{\mu}{d_1} - \frac{1-\mu}{d_2} \tag{8.2}$$

where $\mu \in (0, 1/2]$ is the mass of the first primary, $1-\mu$ is the mass of the second primary, $d_1^2 = (x_1 - 1 + \mu)^2 + x_2^2$ and $d_2^2 = (x_1 + \mu)^2 + x_2^2$. We will denote $\frac{\mu}{d_1} + \frac{1-\mu}{d_2}$ by $U(x)$. In this rotating coordinates setting, the first primary is placed at $(1-\mu, 0)$ and second primary is at $(-\mu, 0)$, in such a way that the origin is the center of mass of the two primaries. The equations of motion derived from this Hamiltonian are

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} = y + Kx \\ \dot{y} = -\frac{\partial H}{\partial x} = Ky + \frac{\partial U(x)}{\partial x} \end{cases} \tag{8.3}$$

In search of the equilibrium, one solves the equations in (8.3) for $\dot{x} = \dot{y} = 0$ and obtains the new equation $x + \partial U/\partial x = 0$. This equation has five solutions, the *libration points*, as it can be seen in Figure 9. Three of them lie on the line through the two primaries and are called $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$. The other two are symmetric to each other with respect to the horizontal axis $x_2 = 0$ and both of them form an equilateral triangle with the primaries. They are called $\mathcal{L}_4, \mathcal{L}_5$ and their coordinates are $(1/2 - \mu, \pm\sqrt{3}/2)$.

Among these five equilibrium points of the R3BP, $\mathcal{L}_4, \mathcal{L}_5$ are stable but $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are not. For this reason, one expects periodic orbits around $\mathcal{L}_4$ and $\mathcal{L}_5$. The next step consists on calculating the spectrum of these two equilibrium points, which is equivalent to calculate the eigenvalues of the linear part of the system (8.3) there.

To study the linearized system at $\mathcal{L}_4$ (for $\mathcal{L}_5$ it is equivalent), we denote its coordinates by $(\xi_1, \xi_2, -\xi_2, \xi_1)$ and apply a translation to the original coordinates to center the system there. Hence, the transformation we apply is the following

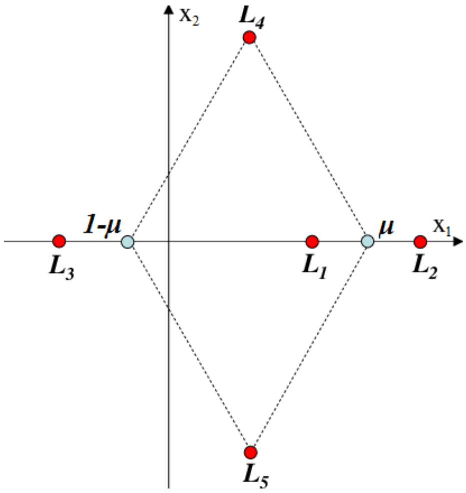

Figure 9: Position of the five equilibrium points in the R3BP.

$$u_1 = x_1 - \xi_1, u_2 = x_2 - \xi_2, v_1 = y_1 + \xi_2, v_2 = y_2 - \xi_1 \tag{8.4}$$

which is symplectic. Now, we expand the Hamiltonian H in (8.2) in the new variables u, v and obtain

$$\mathcal{H} = \frac{1}{2}(v_1 + v_2) + u_2 v_1 - u_1 v_2 - \frac{1}{2}(U_{11}u_1^2 + 2U_{12}u_1u_2 + U_{22}u_2^2) + \mathcal{O}_3(u) \tag{8.5}$$

Computing the linear part of the system, and recalling that for $\mathcal{L}_4$ we have that $(\xi_1, \xi_2) = (1/2 - \mu, \sqrt{3}/2)$, we obtain that the characteristic polynomial of the linear part of (8.5) at the libration point $\mathcal{L}_4$ is

$$p(\lambda) = \lambda^4 + \lambda^2 + \frac{27}{4}\mu(1/\mu) \tag{8.6}$$

The solutions of (8.6) are $\lambda^2 = 1/2(-1 \pm \sqrt{1 - 27\mu(1-\mu)})$. They depend importantly on μ, that we recall that is the mass-ratio parameter and takes values in $(0, 1/2]$. For $\mu = \mu_1 = (1 - \sqrt{69}/9)/2$, the Routh Mass, the discriminant vanishes and the solutions are $\lambda = \pm i/\sqrt{2}$ with multiplicity two. For $\mu < \mu_1$, the solutions are purely imaginary eigenvalues of the form $\lambda = \pm i\omega_1, \pm i\omega_2$. For $\mu > \mu_1$, the solutions are complex with non zero real part and can be written in the form $\mu = \pm\alpha \pm i\beta$.

We can conclude that, for $\mu > \mu_1$, the system is not stable at a neighbourhood of $\mathcal{L}_4$, because their linearization has eigenvalues with positive real part. On the contrary, for $\mu \leq \mu_1$, the eigenvalues of the linearization of the system are purely imaginary, so it is necessary to discuss if they give rise to some kind of central equilibrium or periodic orbits at $\mathcal{L}_4$. This has to be analyzed through the higher order terms of (8.5) and one discovers, as we show in the next sections, that a special bifurcation occurs in this system for $\mu = \mu_1$.

8.2 The Hopf Bifurcation

The change in the nature of the eigenvalues of the linear part of the Hamiltonian system corresponding to the R3BP when the mass ratio parameter μ goes through $\mu_1 = (1 - \sqrt{69}/9)/2$ leads to an essential change in the behaviour of the system (a bifurcation). In order to study it, we will need an important result, the Lyapunov Center Theorem, basic in dynamical systems.

8.3 Lyapunov Center Theorem

Theorem 8.1 (Lyapunov Center Theorem)**.** *Suppose that the system $\dot{x} = f(x)$ has a non-degenerate integral and has an equilibrium point with exponents (eigenvalues of the linear part) $+\omega i, -\omega i, \lambda_3, \ldots, \lambda_m$. If $\lambda_i/i\omega$ is not an integer for any $i = 3, \ldots, m$, then it exists a one-parameter family of periodic orbits that are born at the equilibrium point. Besides, the periods of the orbits in this family tend exactly to $2\pi/\omega$ when approaching the equilibrium point along the family.*

Proof. Take $x = 0$ as the equilibrium point. The equation $\dot{x} = f(x)$ can be rewritten as $\dot{x} = Ax + g(x)$, with A a constant matrix accounting for the linear part and $g(x)$ a function of x without linear part at the critical point, i.e. $\partial g/\partial x(0) = 0$.

Now, do a scaling of the system by the change $x \to \varepsilon x$ and get $\dot{x} = Ax + \mathcal{O}(\varepsilon)$. When $\varepsilon = 0$, the system is linear and by hypothesis it ha as solution of the form $\phi(t) = \exp{(At)}a$, with a a constant non zero vector and such that its period is $2\pi/\omega$. The multipliers of the periodic solution are the eigenvalues of the matrix $\exp{(A2\pi/\omega)}$, which, by hypothesis, are 1,1,$\exp{(2\pi\lambda_i/\omega)}$ for $i = 3, \ldots, m$. Again, by hypothesis, they are not 1 for any $i = 3, \ldots, m$, because $\lambda_i/i\omega$ is not an integer for any $i = 3, \ldots, m$. This implies that the periodic solution $\phi(t)$ is elementary.

An elementary orbit in a system with a non-degenerate integral can be continued, so there exists a periodic solution of the scaled system of the form $\varphi(t) = \exp{(At)}a + \mathcal{O}(\varepsilon)$ which, deescalating, becomes $\varphi(t) = \varepsilon \exp{(At)}a + \mathcal{O}(\varepsilon^2)$, which is exactly what we wanted to prove. □

8.4 Study of the Hopf Bifurcation

In the case of the R3BP, when $\mu < \mu_1$ and μ is close to μ_1, the Lyapunov Center Theorem states that, as the quotient of eigenvalues (frequencies) is not an integer, there exist two one-parameter families of periodic solutions emanating from the libration point $\mathcal{L}_4$ (also from $\mathcal{L}_5$). They are associated to the frequencies ω_1 and ω_2, respectively.

On the contrary, when $\mu_1 < \mu < 1/2$, the fact that the eigenvalues of the linear system have non-zero real part implies by the Stable Manifold Theorem that there are no periodic orbits around $\mathcal{L}_4$. The following part of the essay is dedicated to study what happens to the periodic solutions when μ goes through $\mu_1 = (1 - \sqrt{69}/9)/2$.

At $\mu = \mu_1 = (1 - \sqrt{69}/9)/2$, the linear part of the Hamiltonian system of the restricted three body problem has as eigenvalues $\pm\sqrt{2}/2i$ with multiplicity two. This is the value of μ for which we say that the bifurcation occurs. As the Lyapunov Center Theorem doesn't apply here, the analysis has to be carried studying the system for values of μ close to μ_1, in order to see what exactly happens with the two families of periodic solutions that emerge from the origin when $\mu < \mu_1$ and escape away when $\mu > \mu_1$.

To perform the analysis, we will use the canonical form of a Hamiltonian with eigenvalues $\pm i\omega$ with multiplicity two, which is exactly the case of the R3BP when $\mu = \mu_1$.

In canonical form, the Hamiltonian of the system in the variables $(\xi_1, \xi_2, \eta_1, \eta_2)$ is [MO17]

$$H_0 = \sqrt{2}/2(\xi_2\eta_1 - \xi_1\eta_2) + 1/2(\xi_1^2 + \xi_2^2) \tag{8.7}$$

and the corresponding Hamiltonian linear system of differential equations is $\dot{z} = Az$, where

$$A = \begin{pmatrix} 0 & \sqrt{2}/2 & 0 & 0 \\ -\sqrt{2}/2 & 0 & 0 & 0 \\ -1 & 0 & 0 & \sqrt{2}/2 \\ 0 & -1 & -\sqrt{2}/2 & 0 \end{pmatrix} \tag{8.8}$$

The plan now is to study smooth perturbations of the Hamiltonian H_0. First, we define three quantities that are function of the $(\xi_1, \xi_2, \eta_1, \eta_2)$ variables and will simplify the notation:

$$\Gamma_1 = \xi_2\eta_1 - \xi_1\eta_2, \quad \Gamma_2 = 1/2(\xi_1^2 + \xi_2^2), \quad \Gamma_3 = 1/2(\eta_1^2 + \eta_2^2)$$

In function of these quantities, the Hamiltonian H_0 is simply written as $H_0 = \sqrt{2}/2\Gamma_1 + \Gamma_2$. Now, assume initially, for simplicity, that the perturbation of H_0 is quadratic and can be expanded through the parameter ν as $H = H_0 + \nu H_1 + \cdots$. If H is in normal form, the terms of order 1 and higher (in ν) can always be expressed as functions of Γ_1 and Γ_3 only, i.e.:

$$H(\nu) = H_0 + \nu H_1 + \cdots = \sqrt{2}/2\Gamma_1 + \Gamma_2 + \nu(a\Gamma_1 + b\Gamma_3) + \cdots \tag{8.9}$$

Here, we apply a change to complex coordinates, which is symplectic:

$$y_1 = \xi_1 + i\xi_2, \quad y_2 = \xi_1 - i\xi_2, \quad y_3 = \eta_1 + i\eta_2, \quad y_4 = \eta_1 - i\eta_2,$$

Notice that $y_1 = \bar{y}_2$ and $y_3 = \bar{y}_4$. In these new coordinates, if $w = (y_1, y_2, y_3, y_4)$, the system of differential equations is $\dot{w} = B_0 w + \nu B_1 w + \cdots$, where

$$B_0 = \begin{pmatrix} -\sqrt{2}/2i & 0 & 0 & 0 \\ 0 & \sqrt{2}/2i & 0 & 0 \\ 0 & -1 & \sqrt{2}/2i & 0 \\ -1 & 0 & 0 & -\sqrt{2}/2i \end{pmatrix} \tag{8.10}$$

and

$$B_1 = \begin{pmatrix} -ai & 0 & 0 & b \\ 0 & ai & b & 0 \\ 0 & 0 & ai & 0 \\ 0 & 0 & 0 & -ai \end{pmatrix} \tag{8.11}$$

Considering the terms at order zero and one in ν, the characteristic polynomial of the system is

$$\left(\lambda^2 + (\sqrt{2}/2 + \nu a)^2\right)^2 + 2\nu b\left(\lambda^2 - (\sqrt{2}/2 + \nu a)^2\right) + \nu^2 b^2 \tag{8.12}$$

and has roots $\lambda = \pm(\sqrt{2}/2 + \nu a)i \pm \sqrt{-b\nu}$. These eigenvalues are pure imaginary if $b\nu > 0$, whereas they acquire a non-zero real part if $b\nu < 0$. This is coherent with the behaviour at $\mathcal{L}_4$ that we already know near $\mu = \mu_1$, as, for values of μ bigger than μ_1, the eigenvalues acquire a real part, and for values of μ lower than μ_1, the eigenvalues remain pure imaginary.

Now we consider a general perturbation to the Hamiltonian H_0, i.e. it has $H(\nu)$ as the quadratic part but has also the rest of the terms than appear in general when written in normal form:

$$\mathcal{H}(\nu) = \sqrt{2}/2\Gamma_1 + \Gamma_2 + \nu(a\Gamma_1 + b\Gamma_3) + 1/2(c\Gamma_1^2 + 2d\Gamma_1\Gamma_3 + e\Gamma_3^2) + \cdots \tag{8.13}$$

Now, we notice that the Γ's, in the y's variables are written as

$$\Gamma_1 = i(y_2y_4 - y_1y_3), \quad \Gamma_2 = y_1y_2, \quad \Gamma_3 = y_3y_4$$

and we apply the following scaling to the y's variables and also to the ν, which is symplectic (with multiplier ε^3):

$$y_1 \to \varepsilon^2 y_1, \quad y_2 \to \varepsilon^2 y_2, \quad y_3 \to \varepsilon y_3, \quad y_4 \to \varepsilon y_4, \quad \nu \to \varepsilon^2 \nu$$

Thanks to this scaling, we obtain a new Hamiltonian system (8.14) which will satisfy some convenient properties that will allow us to analyze the behaviour of the system near the value 0 of all the introduced parameters.

$$\mathcal{H}(\nu) = \sqrt{2}/2\Gamma_1 + \varepsilon(\Gamma_2 + \nu b\Gamma_3 + 1/2e\Gamma_3^2) + O(\varepsilon^2) \tag{8.14}$$

Dropping the terms in $O(\varepsilon^2)$, the differential equations of motion associated to $\mathcal{H}(\nu)$ become:

$$\begin{cases} \dot{y}_1 = -\sqrt{2}/2iy_1 + \varepsilon(\nu by_4 + ey_3y_4^2) \\ \dot{y}_2 = +\sqrt{2}/2iy_2 + \varepsilon(\nu by_3 + ey_3^2y_4) \\ \dot{y}_3 = +\sqrt{2}/2iy_3 - \varepsilon y_2 \\ \dot{y}_4 = -\sqrt{2}/2iy_4 - \varepsilon y_1 \end{cases} \tag{8.15}$$

In compact notation, if $w = (y_1, y_2, y_3, y_4)$, equation 8.15 can be written as

$$\dot{w} = Cw + \varepsilon f(w, \nu) \tag{8.16}$$

where $C = \text{diag}(-\sqrt{2}/2i, +\sqrt{2}/2i, +\sqrt{2}/2i, -\sqrt{2}/2i)$
and $f(w, \nu) = (\nu by_4 + ey_3y_4^2,\ \nu by_3 + ey_3^2y_4,\ y_2,\ y_1)$.

The matrix C satisfies, naturally, that $\exp(CT) = I$, for $T = 2\pi/(\sqrt{2}/2) = 2\sqrt{2}\pi$. And the function $f(w, \nu)$ has the property that $f(e^{Ct}w, \nu) = e^{Ct}f(w, \nu)$ for all t. For the equation (8.16), we try with an ansatz of the form $w(t) = e^{(1-\varepsilon\tau)Ct}v$, where τ is a parameter that will be linked with the correction of the period of the emanating orbits, and v is a constant vector (that, like in w, satisfies that $v_1 = \bar{v}_2$ and $v_3 = \bar{v}_4$. The function $w(t)$ is a solution of (8.16) if the following equality is satisfied:

$$\tau Cv + f(v, \nu) = 0 \tag{8.17}$$

Besides, if v satisfies (8.17), then the solution $w(t) = e^{(1-\varepsilon\tau)Ct}v$ is a periodic solution of the system (8.16), with period $2\sqrt{2}\pi/(1-\varepsilon\tau) = 2\sqrt{2}\pi \cdot (1 + \varepsilon\tau + \cdots)$. Now, we find the conditions that $v = (v_1, v_2, v_3, v_4)$ and τ have to fulfill to solve (8.17):

$$\begin{cases} -i\sqrt{2}/2\tau v_1 + \nu bv_4 + ev_3v_4^2 = 0 \\ +i\sqrt{2}/2\tau v_2 + \nu bv_3 + ev_3^2v_4 = 0 \\ +i\sqrt{2}/2\tau v_3 - v_2 = 0 \\ -i\sqrt{2}/2\tau v_4 - v_1 = 0 \end{cases} \tag{8.18}$$

The condition that emerges from combining the first and fourth equations (and also from the second and third ones) is:

$$\tau^2/2 - ev_3v_4 = b\nu \tag{8.19}$$

Now, we set $v_3 = \alpha_1 + i\alpha_2$ and, hence, $v_4 = \alpha_1 - i\alpha_2$. Then, we compute $v_1 = -\sqrt{2}/2\tau(\alpha_2 + i\alpha_1)$ and $v_2 = -\sqrt{2}/2\tau(\alpha_2 - i\alpha_1)$. We can state now that the solutions of (8.16) are of the form

$$w(t) = e^{(1-\varepsilon\tau)Ct}\cdot(-\sqrt{2}/2\tau(\alpha_2+i\alpha_1),\ -\sqrt{2}/2\tau(\alpha_2-i\alpha_1),\ \alpha_1+i\alpha_2,\ \alpha_1-i\alpha_2),\ (8.20)$$

as long as $\tau^2/2 - e(\alpha_1^2 + \alpha_2^2) = b\nu$.

We see that this is a family of periodic solutions that depends on 3 parameters, α_1, α_2, τ. If we set $r^2 := v_3v_4 = \alpha_1^2 + \alpha_2^2$, then we can fix r to determine a circle of periodic solutions that corresponds to one periodic orbit. Because of this, $w(t)$ can bee seen as a 2-parameter family of periodic orbits parametrized by the parameters r and τ.

The condition $\tau^2/2 - er^2 = b\nu$, which the family of solutions has to satisfy, has to be analyzed by separate in essentially two different cases, depending on the sign of e. The one that interests us is the first case, when $e > 0$ (see Figure 10), because it corresponds to the bifurcation at $\mathcal{L}_4$.

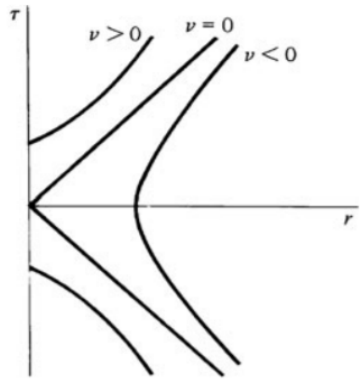

Figure 10: Solutions of 8.19 (where $v_3v_4 = r^2$) for $e > 0$.

In Figure 10 we sketch the curves that solve (8.19) in the case $e > 0$, assuming that $b > 0$ without loss of generality, as a change in the sign of b is equivalent to a change in the sign of ν. We only plot solutions for $r > 0$, as it is the only domain that is necessary to study.

We see that, for a fixed $\nu > 0$ the graph of the points (r, τ) that are solution of (8.19) form a hyperbola with one branch at the upper plane and the other one in the lower plane. For $\nu = 0$, the solutions form two straight lines through the origin, one at each half-plane. For a fixed $\nu < 0$, the solutions form a hyperbola with only one branch at $r > 0$.

We recall that τ is a parameter that accounts for the deviation of the period $(T = 2\sqrt{2}\pi)$ of the periodic orbits that emanate from the origin, in particular through the expression $T/(1 - \varepsilon\tau)$. Meanwhile, r is a parameter that determines the length of v_3, and hence the length of v_1, v_2 and v_4. More precisely, fixing r determines the circles where the coordinates v_1, v_2, v_3, v_4 of the periodic orbits remain. In summary, a point (r, τ) in the graph at Figure 10 corresponds to a periodic orbit of the system 8.16 with period $2\sqrt{2}\pi/(1 - \varepsilon\tau)$ and *radius* r. We remark that $r = 0$ corresponds to the origin of the system, so it is the value that corresponds to $\mathcal{L}_4$.

Knowing this, we can proceed to study the bifurcation, i.e. the change in the behaviour of solutions where ν goes through 0.

We observe that, when $\nu > 0$, there are two curves of solutions emanating from the axis $r = 0$. It means that there are two families of periodic solutions that grow from $\mathcal{L}_4$. Indeed, the characteristic polynomial (8.12) has two pairs of pure imaginary conjugated solutions and, by Lyapunov Center Theorem, there have to exist these two families of periodic solutions that emanate from the origin.

When $\nu = 0$, we observe that the two branches of the hyperbola that appeared for $\nu > 0$ converge to the origin and become two lines of solutions that meet exactly there, at the origin, and correspond to two families of periodic solutions. This is really interesting and a discovery achieved thanks to the change of coordinates and scaling, because for $\nu = 0$, the characteristic polynomial (8.12) has two conjugated solutions of multiplicity two, hence the Lyapunov Center Theorem didn't apply.

Finally, when $\nu < 0$, the two lines at the origin are degenerated and converted into a single curve, a branch of the hyperbola that doesn't go through $r = 0$. Hence, in the case $\nu < 0$, there is a unique family of periodic solutions that, besides, don't pass through the origin. This is consistent with the fact that, for $\nu < 0$, the solutions of (8.12) have non-zero real part, and so there is a neighbourhood of the origin where there can not be periodic solutions.

Condensing, there are two families of periodic orbits that emanate from the origin when $\nu > 0$, these two remain when $\nu = 0$ and transform to a single family of periodic solutions that moves away from the origin when $\nu < 0$. This bifurcation is called the Hopf Bifurcation.

One can show now that the conclusions that we achieved are still valid if we consider the terms in $\mathcal{O}(\varepsilon^2)$ in Equation (8.14). We don't do it here but it can be found on [MO17]. The tracing back of the solutions to get them in the original coordinates $\xi_1, \xi_2, \eta_1, \eta_2$ can be only performed implicitly, but we notice that there is no problem in its definition domain because the changes applied all the way where scalings and symplectic transformations, so invertible.

8.5 Application to the restricted three body problem

In the restricted problem, what occurs at $\mathcal{L}_4$ and $\mathcal{L}_5$ when the mass-ratio parameter μ passes through the value of $\mu_1 = (1 - \sqrt{69}/9)/2 \simeq 0.03852$ is exactly the Hopf Bifurcation. For $\mu < \mu_1$, there are two Lyapunov families of periodic solutions emanating from $\mathcal{L}_4$, that still exist when $\mu = \mu_1$, and that converge to a single family that moves away from $\mathcal{L}_4$ when $\mu > \mu_1$. Let us show it.

The Hamiltonian (8.2), up to quadratic order, can be expanded the following way at the libration point $\mathcal{L}_4$ [MS71]

$$H = \frac{y_1^2 + y_2^2}{2} - x_1 y_2 + x_2 y_1 + \frac{1}{8}x_1^2 - \frac{3\sqrt{3}}{4}(1 - 2\mu)x_1 x_2 - \frac{5}{8}x_2^2 \tag{8.21}$$

A small perturbation of the parameter μ of the form $\mu = \mu_1 + \nu$ leads to the Hamiltonian matrix $B_0 + \nu B_1 + \cdots$, where B_0 and B_1 are (8.10) and (8.11) respectively. This matrix has eigenvalues $\lambda = \pm\sqrt{2}/2i \pm \sqrt{-b\nu}$. As we know that the characteristic polynomial at $\mathcal{L}_4$ is (8.6) and its roots are $\lambda = \pm\sqrt{(1/2)(-1 \pm \sqrt{1 - 27\mu(1-\mu)})}$, by term comparison we obtain that $b = -3\sqrt{69}/2$, so $b < 0$.

One can also compute terms and obtain that the sign of what we called e in (8.19) is positive [Dep68]. Hence, we can apply the analysis from the Hopf Bifurcation here and conclude that the bifurcation that occurs at $\mathcal{L}_4$ is precisely this one.

The last detail that we have to take into account is that in the development of the Hopf Bifurcation and, by the way, also in Figure 10, we assumed b to be positive. But we have found that $b < 0$ in the R3BP, so the conclusions that we extracted for $\nu > 0$ there are valid for $\nu < 0$ here, and vice-versa. This doesn't suppose any contradiction, but on the contrary, as we know that in the R3BP, there are two families of periodic orbits that emanate from $\mathcal{L}_4$ when the perturbation $\mu = \mu_1 + \nu$ goes through negative ν's, and there are no orbits growing from $\mathcal{L}_4$ when ν is negative. And now we know more, because the characterization of the Hopf Bifurcation allows us to say that for the particular case $\mu = \mu_1$, the two families of periodic orbits that emanate from $\mathcal{L}_4$ still exist.

9. Conclusions and open problems

We demonstrated the new Theorem 4.2 on the equivalence of the cotangent lifts of close actions of a compact group on a compact manifold. It is a result on rigidity of group actions that contributes to the stability theory for actions started by Palais [Pal61b]. In particular, it is an analogue version to Theorem 3.11 for the case of actions obtained via the cotangent lift of close group actions. This makes it really interesting for applications in the field of mathematical physics, as the technique of the cotangent lift is a big tool in the study of Hamiltonian systems. Indeed, we applied it to obtain Theorems 4.4 and 4.13 which state rigidity of a class of singular compact leaves of integrable systems.

While proving Theorem 4.2, we wondered whether a slightly stronger result might also be true. In particular, we thought that relaxing the condition of compactness of the group and, instead, asking the actions to be proper, we could still obtain equivalence. In precision, what we conjectured is the following result.

Conjecture 9.1. *Let G be a locally compact Lie group and (M, ω) a compact symplectic. Let $\rho_1, \rho_2 : G \times M \longrightarrow M$ be two proper actions which are C^1-close. Let $\hat{\rho}_1, \hat{\rho}_2$ be the cotangent lifts of ρ_1, ρ_2, respectively. Then, there exists a symplectomorphism that conjugates $\hat{\rho}_1$ and $\hat{\rho}_2$.*

Some of the most interesting cases of integrable Hamiltonian systems, for instance the semitoric integrable system with a focus-focus singularity considered in Section 7 or the integrable system with a saddle-focus singularity considered in Section 8 are given by actions of the form $\mathbb{T}^k \times \mathbb{R}^l$. These are not compact actions because including any $\mathbb{R}^l$ component makes the Lie group non-compact, although local compactness is not lost. The conjecture, then, would also have an immediate application on integrable Hamiltonian systems.

When trying to prove this conjecture with the same procedure used in the proof of Theorem 5.5, we could not overcome the problem of averaging over a non-compact Lie group. That is, averaging over the elements of a Lie group is well defined for compact Lie groups but it may not converge for groups which are only locally compact. More precisely, Equation 5.10 may not converge if the Lie group G is non-compact. Nevertheless, we think that the averaging of the vector field over the elements of a non-compact group can be well defined if the vector field satisfies some conditions. For instance, in our case, X_t satisfies $\iota_{X_t}\omega_t = -\alpha$ for a 1-form α which, at its turn, satisfies $\omega_1 - \omega_0 = d\alpha$. It could mean, in some way, that the integrand in 5.10, $\hat{\rho}(g)_*(X_t)$, is so small that the integral does not diverge. The proof, however, is still lacking and we hope it is a future step in the way of developing new results on rigidity of Lie group actions on symplectic manifolds.

The Hamiltonian formulation of mechanical systems is based on considering the dynamical problems on cotangent bundles and then, the use of the cotangent lift becomes natural. The cotangent models considered in Section 6 for the three basic types of non-degenerate singularities can be the starting point of a new formulation of integrable systems with singularities, considering that there is still a huge work to do in the study of degenerate singularities. By the way, because of this lack of a complete theory for degenerate singularities, a little new result concerning degenerate singularities such as the one we obtained in Theorem 4.13 is already a nice improvement.

At this point, we hope that it has become clear for the reader that, in general, a result on rigidity of group actions usually has interesting applications in the theory of integrable Hamiltonian systems, and that the symplectic approach is really powerful. Proving stability for a class of singular Hamiltonian systems is just one little example of what symplectic geometry and, in general, Poisson geometry can provide to the theory of Hamiltonian systems.

From a dynamical point of view, the results included in this paper can be understood as a weak KAM theorem where Hamiltonian perturbations occur in the subclass of integrable systems. It would be interesting to explore the weak analogues for focus-focus singularities which can be seen as a cotangent lift as shown in Section 6.6. Those singularities are infinitesimally stable [MVuN05] and stable [Mir14] however it is not possible to follow the guidelines above due to the lack of compactness of the group $S^1 \times \mathbb{R}$.

The mathematical models for non-degenerate singularities based on the cotangent lift, namely the elliptic, hyperbolic and focus-focus models constructed in Section 6, are formally correct and well-defined as the cotangent lift of Lie group actions. Nevertheless, there is work to do in order to apply rigidity results on these models because the three arise from the cotangent lift of a non-compact action. This is not a problem for the elliptic singularity, because it is already a compact action by itself (without the cotangent lift). Indeed in this case we have been able to apply our rigidity result (Theorem 4.2). The idea that we have is that it is possible to work on the focus-focus model to obtain that it is actually possible to also apply to it the rigidity result for compact close actions.

The focus-focus model comes from the cotangent lift of $S^1 \times \mathbb{R}$ (see Section 6.6), and the complexification of $S^1 \cong SO(n, \mathbb{R})$ gives $SO(n, \mathbb{C})$, which is diffeomorphic to $S^1 \times \mathbb{R}$. The cotangent model of the focus-focus singularity, then, can be seen as the cotangent lift of the complexification of the compact Lie group S^1. Our idea is to use the fact that Lie group action of S^1 is compact in order to apply a complex version of our rigidity results, which require compactness. If we could prove that rigidity of the S^1 action is preserved along a complexification and a cotangent lift, then we could arrive to some rigidity also for the singular focus-focus model.

When we were studying the model of the simple focus-focus singularity in complex coordinates $z_1 = q_1 + iq_2, z_2 = p_1 + ip_2$ (see Section 7), we wondered if its structure motivated a slightly more complex model, the *quaternionic* model. We considered the *quaternionification* of the cotangent lift, which can be created through the following change of variables:

$$q_i = x_i + x_{i+1}i + y_ij + y_{i+1}k, \tag{9.1}$$

where $q_i \in \mathbb{H}$ is a *quaternion.* Nevertheless, until the moment we have observed that the quaternionic formulation is way too tight to adapt to a proper model for this singularity, so we leave this approach, as well as a hypothetical *octonionic* approach (inspired by the constructions of Cohl Furey in [Fur12]) for a future research.

Another application of the cotangent models of non-degenerate singularities which we plan to explore is that of geometric quantization. It is clear how to compute a cotangent bundle (see for instance [Woo92]) but the role of the cotangent lift in this classical approach to quantization seems not be explored yet. In [HM10, MP15, MPS20] the authors follow the recipe due to Kostant with the aid of a sheaf cohomology computation obtaining models with infinite dimensional contributions in the case of hyperbolic and focus-focus singularities. We plan to adopt a more primitive approach and take advantage of the cotangent models for non-degenerate singularities with the hope of reducing out the infinite dimensional contributions in the Kostant model. We are investigating these models with professor Eva Miranda.

Finally, the b-symplectic version of cotangent models for integrable systems, which has not been included in this master thesis but can be found on [KM17], is one of the tools being exploited by the author in a paper which is in preparation, in collaboration with Baptiste Coquinot from École Normale Supérieure and Eva Miranda. We are working on a physical interpretation of the twisted b-cotangent lift model in the context of magnetism. Besides, we believe that results obtained by Victor Ginzburg in [Gin96] can be adapted to the b-symplectic case, obtaining existence of periodic orbits in manifolds with boundary also in the context of a charged particle in a magnetic field.

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