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Title: Topics on periodic solutions to integro-differential equations

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Topics on periodic solutions to integro-differential equations

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Supervised by Xavier Cabré Vilagut and Albert Mas Blesa

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Abstract

This Master's Degree Thesis investigates periodic solutions to nonlinear equations involving integro-differential operators. We show the existence and we describe these solutions for generalized Benjamin-Ono type nonlinearities, using a variational formulation and a constrained minimization argument. We show that there exists a minimal period for which nontrivial solutions exist, and we also provide stability and qualitative properties of these solutions. Furthermore, in the case of the fractional Laplacian and with suitable exponents of the nonlinearity, we prove that the period where constrained minimizers change from constant to nonconstant is strictly smaller than the period for which the unique positive constant solution loses stability. Within the literature, the articles [5, 10], which concern two problems closely related to ours, claimed that these two values of the period coincide. Their arguments to prove such claim were incomplete but, if they could be completed, they would also work for our equation. In this work we show that this task cannot be carried out, since we find an explicit range of parameters (concerning the fraction of the fractional Laplacian and the pure power in the nonlinearity) for which the equality does not hold.

Keywords

Integro-differential operators, periodic solutions, nonlinear equations

Contents

1 Introduction

This Master's Degree Thesis concerns the study of periodic solutions to nonlinear equations for integro-differential operators. This kind of equations appear in many physical contexts where the modelling involves nonlocal interactions. Our motivation is to better comprehend the class of periodic solutions of such equations through a variational approach. We summarize our two main contributions in this direction.

- The existence of periodic solutions to integro-differential equations with Benjamin-Ono nonlinearities is shown in [6, 7]. In this work we generalise this results by considering integro-differential operators (such as, but not restricted to the Fractional Laplacian) and nonlinearities other than pure powers but with similar properties. Using a constrained minimization argument called the Nehari manifold method, we prove the existence of periodic solutions and of minimal periods for which nontrivial solutions exist. This procedure also gives us stability and qualitative properties of these periodic solutions. With these results we try to overcome the lack of a phase plane analysis in the nonlocal framework, in comparison to the local setting, where it is given by the theorem of existence and uniqueness of ODEs.

- In the articles [5, 10], which concern two problems closely related to ours, the value of the period where constrained minimizers change from constant to nonconstant was claimed to coincide with the period for which the trivial solution loses stability. Their arguments to prove such claim were incomplete but, if they could be completed, they would also work for our equation. The main achievement of this work is to show that such task cannot be carried out. We show this by finding an explicit range of parameters (concerning the fraction of the fractional Laplacian and the pure power in the nonlinearity) for which the equality does not hold.

We will primarily study the structure of periodic solutions to semilinear equations of the form

$$\mathcal{L}_K u = f(u) \quad \text{in } \mathbb{R}, \tag{1}$$

where $u : \mathbb{R} \to \mathbb{R}$ and $\mathcal{L}_K$ is an integro-differential operator defined by

$$\mathcal{L}_K u(x) := \lim_{\varepsilon \searrow 0} \int_{\{y \in \mathbb{R} : |x-y| > \varepsilon\}} \big(u(x) - u(y)\big) K(|x-y|)\, dy, \tag{2}$$

whenever the integral and the limit make sense. In the sequel, we will omit the limit of truncated integrals in the definition of $\mathcal{L}_K$, that is, the principal value sense. The kernel K is such that

$$\frac{\lambda}{t^{1+2s}} \leq K(t) \leq \frac{\Lambda}{t^{1+2s}} \quad \text{for all } t > 0, \tag{3}$$

for some constants $0 < s < 1$ and $0 < \lambda \leq \Lambda$. Such condition implies that $\lim_{t \to 0} tK(t) = +\infty$ and $\lim_{t \to +\infty} tK(t) = 0$. We will further assume that $tK(t)$ is strictly decreasing in t. That is to say,

$$t_2 K(t_2) < t_1 K(t_1) \quad \text{for all } 0 < t_1 < t_2. \tag{4}$$

The well-known s-fractional Laplacian $(-\Delta)^s$ corresponds to $K(t) = c_s t^{-1-2s}$, where

$$c_s := \frac{s 4^s \Gamma(1/2 + s)}{\sqrt{\pi}\, \Gamma(1-s)} \tag{5}$$

and clearly satisfies (4).

In the periodic setting, a crucial remark is that the standard Fourier basis of sinus and cosinus are eigenfunctions of $(-\Delta)^s$. Consequently, the integro-differential operators have a simple representation in the Fourier side through multipliers. This led Cabré, Mas and Solà-Morales to consider in [6] semilinear equations in the form $\mathcal{L}u = f(u)$ for general multiplier operators given by

$$\mathcal{L}u(x) = \sum_{k\in\mathbb{Z}} \ell\left(\tfrac{\pi k}{L}\right) u_k e^{\frac{i\pi k}{L}x}, \tag{6}$$

where $u(x) = \sum_{k\in\mathbb{Z}} u_k e^{\frac{i\pi k}{L}x}$ is a $2L$-periodic function regular enough, and $\ell : \mathbb{R} \to \mathbb{R}$ denotes the symbol of $\mathcal{L}$. As it was seen in [6], if we set

$$\ell_K(\xi) := \frac{1}{|\xi|}\int_{\mathbb{R}} \big(1-\cos(y)\big) K\left(\frac{|y|}{|\xi|}\right) dy \quad \text{for } \xi \in \mathbb{R}\setminus\{0\}\,, \quad \text{and } \ell_K(0) := 0, \tag{7}$$

then we recover $\mathcal{L}_K$ from (6) simply by taking $\ell = \ell_K$. In the literature, the operator $\mathcal{L}$ represents the dispersion in simplified models for wave propagation. For instance, Bona and Chen in [8] studied the general nonlinear wave equation

$$v_t - \mathcal{L}v_x + g(v)_x = 0 \quad \text{for } x \in \mathbb{R},\ t \geq 0, \tag{8}$$

and it is easily seen that travelling-wave solutions $v(x,t) \equiv u(x-ct)$ of (8) are given by solutions to $\mathcal{L}u = f(u)$, for suitable nonlinearities f. This is precisely the motivation behind this thesis and the works [6, 7].

There are many models in the literature that are represented by equations like (1). Such is the case of the Benjamin-Ono equation $(-\Delta)^{1/2}u = -u+u^2$ from theoretical hydrodynamics [4, 12]. This model can be extended to the generalized Benjamin-Ono equation $(-\Delta)^s u = -u + |u|^{p-1}u$ for $0 < s < 1$ and $1 < p < \frac{1+2s}{1-2s}$.

In this direction, we look for periodic, even and positive solutions to

$$\mathcal{L}_K u = -u + g(u) \quad \text{in } \mathbb{R}, \tag{9}$$

for some nonlinearity g satisfying certain properties to be mentioned afterwards. This equation is the integro-differential version of the generalized Benjamin-Ono equation, replacing the nonlinear term u^p for a more general nonlinearity $g(u)$. The case $g(u) = u^p$ where $1 < p < \frac{1+2s}{1-2s}$ is fully covered in the work of Cabré, Mas and Solà-Morales [7].

We will prove the existence of solutions to (9) as critical points of an associated functional applied to 2π-periodic functions. Consequently, we shall study how $\mathcal{L}_K$ behaves with respect to rescaling. Given $\mu > 0$ and $u : \mathbb{R} \to \mathbb{R}$, define $v : \mathbb{R} \to \mathbb{R}$ by $u(x) = v(\mu x)$. This way, for $x' = \mu x$, a change of variables shows that

$$\begin{aligned} \mathcal{L}_K u(x) &= \int_{\mathbb{R}} \big(v(\mu x) - v(\mu y)\big) K(|x-y|)\, dy \\ &= \int_{\mathbb{R}} \big(v(x') - v(y)\big) \frac{1}{\mu} K\left(\frac{|x'-y|}{\mu}\right) dy = \mu^{2s}\mathcal{L}_{K_\mu} v(\mu x), \end{aligned} \tag{10}$$

where we have set

$$K_\mu(t) := \frac{1}{\mu^{1+2s}} K\left(\frac{t}{\mu}\right) \quad \text{for all } t > 0.$$

Let us note that, by (3),

$$\frac{\lambda}{t^{1+2s}} \leq K_\mu(t) \leq \frac{\Lambda}{t^{1+2s}} \quad \text{for all } t > 0 \text{ and all } \mu > 0, \tag{11}$$

so both K and K_μ have the same growth estimates independently of $\mu > 0$. In the case of the fractional Laplacian, we have $K = K_\mu$. Moreover, by (4),

$$\mu_1^{2s} K_{\mu_1}(t) < \mu_2^{2s} K_{\mu_2}(t) \quad \text{for all } 0 < \mu_1 < \mu_2 \text{ and all } t > 0. \tag{12}$$

We develop a constrained minimization method where all the functions under consideration have the same period, namely 2π. In this direction, let $L > 0$ and $u : \mathbb{R} \to \mathbb{R}$ be a positive and $2L$-periodic solution to (9). If we write

$$u(x) = w(\tfrac{\pi}{L} x), \tag{13}$$

then $w : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and considering (10) for $\mu = \pi/L$,

$$\mathcal{L}_{K_{\pi/L}} w = -\left(\frac{L}{\pi}\right)^{2s} w + \left(\frac{L}{\pi}\right)^{2s} g(w). \tag{14}$$

With this we observe that finding nontrivial 2π-periodic solutions to (14) is equivalent to finding nontrivial $2L$-periodic solutions to (9). This allows us to consider (14) as the starting point of the constrained minimization method, with the benefit that all the functions under consideration have the same period, independently of $L > 0$.

Let E_L be the functional defined by

$$E_L(v) := \frac{1}{2}\left(\int_{-\pi}^{\pi} \mathcal{L}_{K_{\pi/L}} v \, v + \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} v^2\right) - \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} G(v) \quad \text{for } v \in H^s_{\mathrm{ep},\pi}, \tag{15}$$

with $G(v) = \int_0^v g$. The space $H^s_{\mathrm{ep},L}$, which will be described in more detail in Section 2, refers to the space of $2L$-periodic and even functions $v : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-L}^{L} |v|^2 + \int_{-L}^{L} \int_{\mathbb{R}} \frac{|v(x) - v(y)|^2}{|x-y|^{1+2s}} dy\, dx < +\infty.$$

Let $c(L)$ be the minimal value for E_L on $H^s_{\mathrm{ep},L} \setminus \{0\}$, that is to say,

$$c(L) := \inf\left\{E_L(v) : v \in H^s_{\mathrm{ep},\pi},\ v \not\equiv 0\right\}. \tag{16}$$

If $w \in H^s_{\mathrm{ep},\pi}$ is a positive minimizer of $c(L)$, we will show that w solves (14). As a result, every positive minimizer w of $c(L)$ yields a $2L$-periodic and even solution u to (9). From the minimization procedure we will also prove that there exists a unique period L_* for which the only minimizers of E_L are constant functions if $L < L_*$ and nonconstant functions if $L > L_*$.

In the sequel, we will further assume that the nonlinearity $g \in C^{1+\varepsilon}(\mathbb{R})$ for some $\varepsilon > 0$ and is such that

($g1$) $g(0) = g'(0) = 0$,

($g2$) $g(u)/|u|$ is strictly increasing in $(-\infty, 0)$ and $(0, +\infty)$,

($g3$) $G(u)/u^2 \to +\infty$ as $|u| \to +\infty$,

($g4$) There exist $\mu > 2$ and $R > 0$ such that $0 < \mu G(u) \leq g(u)u$ for all $|u| \geq R$,

$(g5)$ $|g(u)| \le C(1+|u|^p)$ for some $C>0$ and $1<p<\frac{1+2s}{1-2s}$,

$(g6)$ $\int_{-a}^{a} g \ge 0$ for all $a>0$.

Note that $g(u)=|u|^{p-1}u$ with $1<p<\frac{1+2s}{1-2s}$ satisfies all the above conditions. Under these assumptions, we will see that there exists a unique real number $u_0>0$ such that $g(u_0)=u_0$, therefore a unique constant solution to (9) and a critical point of (15). In particular, its energy is

$$E_L(u_0) = 2\pi \left(\frac{L}{\pi}\right)^{2s} \left(\frac{u_0^2}{2} - G(u_0)\right). \tag{17}$$

Our approach to find minimizers of the energy is based on the Nehari manifold method, since a constrained minimization argument like that of [7] is no longer available for nonlinearities more general than pure powers. We give a description of the Nehari manifold method in Section 3, while further details may be found in [15], a complete survey on this method. Denoting $\Phi(v)=E_L(v)$, this method consists of finding a minimizer for Φ amongst a subset $\mathcal{N}$ called *Nehari manifold* that contains all critical values of Φ. In particular, we can write our functional Φ as

$$\Phi(v) = \frac{1}{2}\|v\|^2 - I(v), \tag{18}$$

with

$$\|v\|^2 := \int_{-\pi}^{\pi} \mathcal{L}_{K_{\pi/L}} v\, v + \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} v^2 \quad \text{and} \quad I(v) := \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} G(v). \tag{19}$$

We will see that $\|v\|$ defines a norm in H^s_{ep} and that, if g satisfies $(g1)$-$(g6)$, we will be able to apply the Nehari manifold method on Φ.

Once the minimization procedure is done, we study the stability of the solutions u obtained through the constrained minimizers of E_L. This leads to the study of the spectrum of the linearized operator associated to (9) acting on $H^s_{\mathrm{ep},L}$, that is,

$$\mathcal{L}_{u,L}\psi := \mathcal{L}_K\psi + \psi - g'(u)\psi \quad \text{for } \psi \in H^s_{\mathrm{ep},L}. \tag{20}$$

One can easily see that the eigenvalues of $\mathcal{L}_{u,L}$ form a nondecreasing sequence $\sigma_1(\mathcal{L}_{u,L}) \le \sigma_2(\mathcal{L}_{u,L}) \le \sigma_3(\mathcal{L}_{u,L}) \le \ldots$ We will prove that, for minimizers, the first eigenvalue is always strictly negative while the second one is nonnegative. As a result, we can find an upper bound for the threshold L_* introduced above looking at the linearized operator at the constant solution $u\equiv u_0$. More precisely, under the assumption (4), there exists a unique $L_0>0$ such that

$$\ell_K(\tfrac{\pi}{L_0}) = g'(u_0) - 1, \tag{21}$$

where ℓ_K is given in (7). We will show that $L_* \le L_0$. This constant L_0 is the precise period for which the nontrivial constant solution to (9) loses stability, that is, $\sigma_2(\mathcal{L}_{u_0,L})>0$ for $L<L_0$ and $\sigma_2(\mathcal{L}_{u_0,L})<0$ for $L>L_0$. For the fractional Laplacian case, we get

$$L_0 = \pi \left(g'(u_0)-1\right)^{-\frac{1}{2s}}.$$

All these results are the generalization of Theorem 1.4 of [7] and are fully described in Theorem 1.1 below, whose proof is given in Section 4.

Theorem 1.1. *Let g satisfy $(g1)$-$(g6)$. Then, for every $L>0$, we have $0<c(L)<+\infty$ and the infimum in (16) is attained. Every minimizer is of class C^{1+2s} and does not vanish in $\mathbb{R}$. In particular, there always exists a positive minimizer.*

Let $w \in H^s_{ep,\pi}$ be such that $E_L(w) = c(L)$ and $w > 0$ in $\mathbb{R}$ and set $u(x) = w(\frac{\pi}{L}x)$. Then, u is a $2L$-periodic, of class C^{1+2s}, even and positive solution to $\mathcal{L}_K u = -u + g(u)$ in $\mathbb{R}$. Furthermore, $\sigma_1(\mathcal{L}_{u,L}) < 0 \le \sigma_2(\mathcal{L}_{u,L})$, where $\mathcal{L}_{u,L}$ is the linearized operator at u given by (20).

Finally, there exists a unique $L_* > 0$ for which the following holds:

(i) *If $0 < L < L_*$ then $c(L) = E_L(u_0)$ and it is only attained at constant functions.*

(ii) *If $L > L_*$ then $c(L) < E_L(u_0)$ and $c(L)$ is only attained at nonconstant functions.*

(iii) *It holds that $L_* \le L_0$, where L_0 is given in (21). Moreover, for $\mathcal{L}_K = (-\Delta)^s$, we have that $L_0 = \pi\,(g'(u_0) - 1)^{-\frac{1}{2s}}$.*

In all three cases $E_L(u_0)$ given by (17) is the energy of u_0, the constant positive nontrivial solution to (9).

However, from our arguments above we cannot tell whether the values L_* and L_0 coincide or not. From now on, we will focus on positive solutions to

$$(-\Delta)^s u = -u + u^p, \tag{22}$$

for some $1 < p < \frac{1+2s}{1-2s}$ if $0 < s < 1/2$ and $1 < p < +\infty$ if $1/2 \le s < 1$. In [7] Cabré, Mas, and Solà-Morales find $2L$-periodic and even positive solutions to (22) through a constrained minimization method. Their arguments are the following: for $L > 0$ and $u : \mathbb{R} \to \mathbb{R}$ a positive and $2L$-periodic solution to (22), we set

$$u(x) = \left(\tfrac{\pi}{L}\right)^{\frac{2s}{p-1}} v\left(\tfrac{\pi}{L}x\right).$$

Then, $v : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and solves

$$(-\Delta)^s v = -\left(\frac{L}{\pi}\right)^{2s} v + v^p. \tag{23}$$

Thus, as in our case, finding nontrivial $2L$-periodic solutions to (22) is equivalent to finding nontrivial 2π-periodic solutions to (23). This allows them to consider (23) as the starting point to develop a constrained minimization method, where all the functions under consideration in (23) are 2π-periodic, independently of $L > 0$.

Then they define the functional

$$E_L(w) := \frac{\int_{-\pi}^{\pi} (-\Delta)^s w\, w + \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} w^2}{\left(\int_{-\pi}^{\pi} |w|^{p+1}\right)^{\frac{2}{p+1}}} \quad \text{for } w \in H^s_{\mathrm{ep},\pi},\ w \not\equiv 0, \tag{24}$$

and also

$$c(L) := \inf\left\{E_L(w) : w \in H^s_{\mathrm{ep},\pi},\ w \not\equiv 0\right\}. \tag{25}$$

They see that any positive minimizer w of $c(L)$ in $H^s_{\mathrm{ep},\pi}$ satisfies the associated Euler-Lagrange equation

$$(-\Delta)^s w = -\left(\frac{L}{\pi}\right)^{2s} w + c(L)\left(\int_{-\pi}^{\pi} |w|^{p+1}\right)^{-\frac{p-1}{p+1}} w^p, \tag{26}$$

thus taking

$$v = c(L)^{\frac{1}{p-1}} \left(\int_{-\pi}^{\pi} |w|^{p+1}\right)^{-\frac{1}{p+1}} w \quad \text{and} \quad u(x) = \left(\tfrac{\pi}{L}\right)^{\frac{2s}{p-1}} v\left(\tfrac{\pi}{L}x\right)$$

they see that v and u solve (23) and (22), respectively.

From their minimization procedure, they are able to prove that there exists a unique period L_* for which the only minimizers of $c(L)$ are constant functions if $L < L_*$ and nonconstant functions if $L > L_*$. Moreover, they also see that

$$L_0 = \pi\,(p-1)^{-\frac{1}{2s}} \tag{27}$$

is the precise period for which $u \equiv 1$, the nontrivial positive constant solution to (22), loses stability, hence $L_* \leq L_0$.

In [5] Berestycki and Wei use a constrained minimization argument to find solutions to $-\Delta u = -u + u^p$ in an infinite strip with Neumann boundary conditions. In their work they also show the existence of a critical period for which the minimizers change from constant to nonconstant (that is, the analog of our L_*), and a unique period for which the trivial solution loses stability (that is, the anlalog of our L_0). Despite that their scenario is different than ours, the approach used in [7] strongly relies on the ideas from [5]. In [5] they claim that these two special periods coincide, which in our scenario would read as $L_* = L_0$, although their proof of the equality is not complete. Indeed, they prove the analog of $L_* \leq L_0$ and that, if $L_* > L_0$, then there is a contradiction.

In [10], for the energy functional associated to the fractional Yamabe problem, it is also claimed that $L_* = L_0$. Although our scenario differs from the one they consider, the arguments they use to prove such claim would also apply to our case. However, there is a gap in their arguments for proving the inequality analogous to $L_0 \leq L_*$, at some point they assume the positiveness of an eigenfunction that must be orthogonal to a nontrivial constant function. Consequently, their arguments only yield an estimate from above, which, in our setting, correspond to $L_* \leq L_0$.

These difficulties when proving $L_* = L_0$ suggested that perhaps one would have $L_* < L_0$ in some cases. Here, we indeed find a specific range of parameters s and p for which $L_* < L_0$. To do so, we shall find a period $L < L_0$ and a function $u \in H^s_{\mathrm{ep},\pi}$ such that $E_L(u) < E_L(1)$, thus asserting that $c(L)$ is not attained by the nontrivial positive constant solution, yielding $L_* \leq L < L_0$.

In Theorem 1.2 below we see that the nontrivial positive constant solution bifurcates to a local family of nonconstant periodic solutions to (22) exactly when it loses its stability. Thus, suitable candidates $u \in H^s_{\mathrm{ep},\pi}$ to have less energy would be those belonging to the bifurcated branch.

Theorem 1.2. *Let $f \in C^6(\mathbb{R})$ satisfy $f(0) = 0$ and $f'(0) > 0$. Assume $s \in (0,1)$ and let $\alpha \in (0,1)$ be such that $1 < 2s + \alpha < 2$. Then, for some small $\nu > 0$ there exist maps $a \in (-\nu,\nu) \mapsto u_a \in C^{2s+\alpha}(\mathbb{R})$ and $a \in (-\nu,\nu) \mapsto L(a) \in (0,+\infty)$ of class C^4 for which u_a is a C^4 periodic solution of*

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R} \tag{28}$$

having, for $a \neq 0$, minimal period $2L(a)$ and such that $u_0 \equiv 0$ and $L(0) = L_0$, which is given by (27).

Moreover, for each a, u_a is an even function of x of the form

$$u_a(x) = a\cos\left(\tfrac{\pi}{L(a)}x\right) + a v_a\left(\tfrac{\pi}{L(a)}x\right)$$

where v_a is an even and 2π-periodic function such that $\int_{-\pi}^{\pi} v_a(y)\cos(y)\,dy = 0$ and approaching $v_0 \equiv 0$ in the norm of $C^{2s+\alpha}(\mathbb{R})$ as $a \to 0$. Also, $L(a) = L(-a)$, that is, it is an even function with respect to a.

We require f to be at least of class $C^6(\mathbb{R})$ in Theorem 1.2 because, in order to show that these bifurcated solutions have strictly less energy that the constant solution, we will need to take up to four derivatives with respect to a of $u_a(x)$. The most convenient way to allow that is to prove that $a \mapsto u_a$ is of class C^4 into the space $C^{2s+\alpha}(\mathbb{R})$. This justifies the need of at least five continuous derivatives of f. However, there will be one more derivative needed to apply the theorem on bifurcation from a simple eigenvalue.

Next, we will apply the method and ideas behind this result to $f(u) = -u + u^p$ whenever $f \in C^6(\mathbb{R})$. We note that $u_0 \equiv 1$ satisfies $f(u_0) = 0$ and $f'(u_0) = p - 1 > 0$, which are precisely the hypothesis of Theorem 1.2.

In Section 5 we will prove that, for a small enough, $E_{L(a)}(u_a) < E_{L(a)}(1)$ is equivalent to $Q(p) > 0$, where $Q(p) := 1 + \big(1 - \frac{3}{4(2^{2s}-1)}\big)p$. The coefficient of p in $Q(p)$ changes sign, from negative to positive, at $s = \frac{1}{2}\frac{\ln(7/4)}{\ln 2} \approx 0.4036...$ With this, a simple calculation yields that $Q(p) > 0$ whenever

$$\begin{cases} 0 \le p < \frac{4(2^{2s}-1)}{3-4(2^{2s}-1)} & \text{if } s < \frac{1}{2}\frac{\ln(7/4)}{\ln 2}, \\ 0 \le p < +\infty & \text{if } s \ge \frac{1}{2}\frac{\ln(7/4)}{\ln 2}. \end{cases}$$

Therefore, for these range of parameters s and p (and recalling that we also need $f \in C^6$), the bifurcated solutions will have strictly less energy that the constant solution. Hence, it remains to be seen when do these bifurcated solutions have smaller periods than L_0. For $\lambda(a) = (L(a)/\pi)^{2s}$ for the family of solutions obtained in Theorem 1.2 we have $\lambda(0) = (L(0)/\pi)^{2s}$ and $\lambda'(0) = 0$. Moreover, we have that $\lambda''(0) > 0$ if $s \ge 1/2$, so the period grows initially independently of the value of $p > 1$. But when $s < 1/2$ we see that $\lambda''(0) < 0$ in the region $p > \frac{2(2^{2s}-1)}{2-2^{2s}}$ and so the period decreases initially, that is, $L(a) < L(0) = L_0$ for a small enough.

The first statement of the following theorem gives conditions on (s, p) for which, for a small enough, the energy of the bifurcated solution u_a is strictly smaller than the energy of the constant solution $u_0 \equiv 1$, that is, $E_{L(a)}(u_a) < E_{L(a)}(1)$. The second statement gives conditions on (s, p) for which, for a small enough, $E_{L(a)}(u_a) < E_{L(a)}(1)$ and the periodicity of this bifurcated solution is strictly smaller than L_0, that is $L(a) < L_0$, and therefore $L_* < L_0$.

Theorem 1.3. *Let $0 < s < 1$, $p > 1$ be such that for $f(u) = -u + u^p$, $f \in C^6(\mathbb{R})$ and let $\widetilde{s} \in (0, 1/2)$ be the unique solution to $\frac{1+2\widetilde{s}}{1-2\widetilde{s}} = \frac{4(2^{2\widetilde{s}}-1)}{3-4(2^{2\widetilde{s}}-1)}$. Then,*

(i) For a small enough, we have $E_{L(a)}(u_a) < E_{L(a)}(1)$ if one of the following holds:

(i-a) $0 < s < \frac{1}{2}\frac{\ln(7/4)}{\ln(2)}$ and $1 < p < \frac{4(2^{2s}-1)}{3-4(2^{2s}-1)}$,

or

(i-b) $\frac{1}{2}\frac{\ln(7/4)}{\ln(2)} < s < 1$ and $1 < p < +\infty$,

which are nonempty regions in (s, p).

(ii) We have $L_* < L_0$ if one of the following holds:

(ii-a) $\frac{1}{2}\frac{\ln(3/2)}{\ln(2)} < s \le \widetilde{s}$ and $\frac{2(2^{2s}-1)}{2-2^{2s}} < p < \frac{4(2^{2s}-1)}{3-4(2^{2s}-1)}$,

or

(ii-b) $\widetilde{s} < s < 1/2$ and $\frac{2(2^{2s}-1)}{2-2^{2s}} < p < \frac{1+2s}{1-2s}$,

which are nonempty regions in (s,p), see Figure 1.

Note that in the first statement we do not distinguish whether $1 < p < \frac{1+2s}{1-2s}$ or not, even though we consider $\frac{1}{2}\frac{\ln(7/4)}{\ln(2)} < s < 1/2$. This is so because E_L is well defined independently of p and the theorem on the existence of the bifurcating branch, namely Theorem 1.2, does not require the nonlinearity to be subcritical from the point of view of the fractional Sobolev embedding. Nevertheless, in order to have L_* well defined and to compare it with L_0, we must require $1 < p < \frac{1+2s}{1-2s}$ if $0 < s < 1/2$.

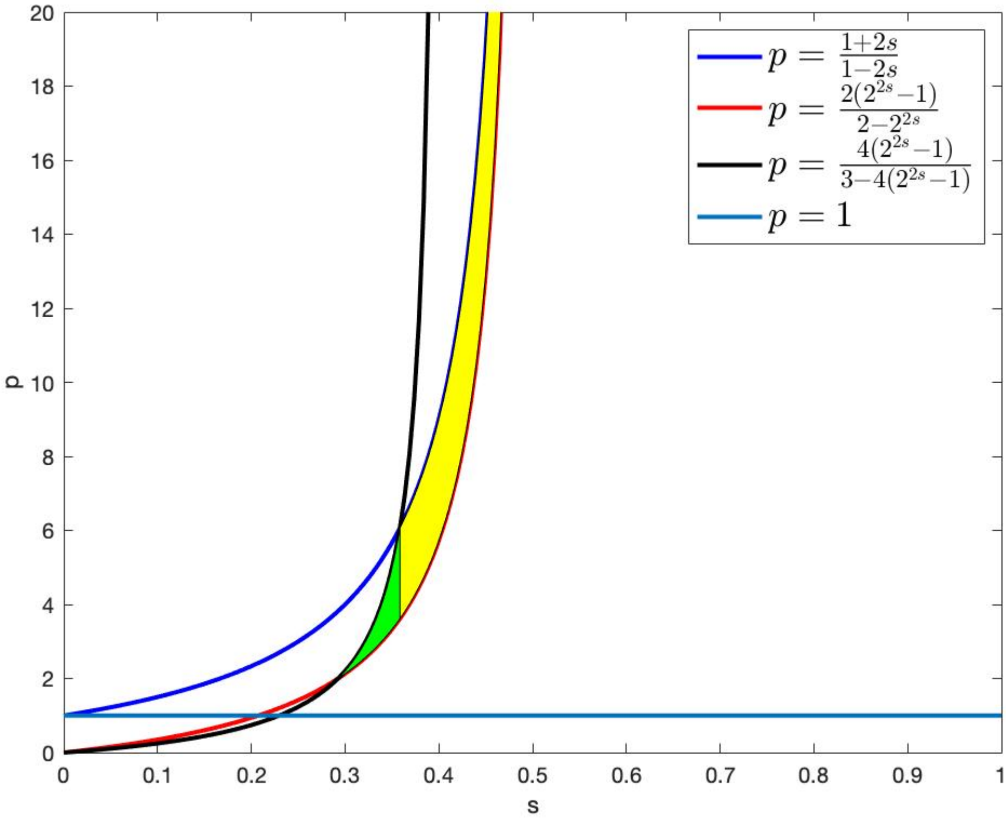

Figure 1: Whenever $f \in C^6(\mathbb{R})$, for a small enough, the region between the red and blue curves is where $L(a) < L_0$ and the region below the black curve is where $E_{L(a)}(u_a) < E_{L(a)}(1)$. The intersection, shaded green and yellow, is where $L_* < L_0$ and corresponds to the regions described in (*ii-a*) and (*ii-b*) in Theorem 1.3, respectively.

Regarding the structure of the Thesis, in Section 2 we state the functional preliminaries that will be used in the sequel. There we present the well-known function spaces where we will work, in addition to enunciating some properties of integro-differential operators on the Fourier side and regularity results for periodic solutions.

In Section 3 we give the basic definitions and results of critical point theory, as well as a brief introduction to the Nehari manifold method. There we state the main results on this method, which will be used in some parts of this work. Finally, Sections 4 and 5 are entirely devoted to the proofs of the three original results of this work, namely Theorem 1.1, Theorem 1.2 and Theorem 1.3.

2 Functional preliminaries

In this section we give some preliminaries regarding the spaces of periodic functions and the regularity results that will be used throughout Sections 4 and 5. All these results, and their corresponding proofs can be found in more detail in [7].

2.1 Function spaces

Given $1 \le p \le \infty$ and $L > 0$ we define

$$\begin{aligned} L^p_{\mathrm{p},L} &:= \left\{ u : \mathbb{R} \to \mathbb{R} : u(x) = u(x+2L) \text{ for a.e. } x \in \mathbb{R},\ \|u\|_{L^p(-L,L)} < +\infty \right\}, \\ L^p_{\mathrm{ep},L} &:= \left\{ u \in L^p_{\mathrm{p},L} : u(x) = u(-x) \text{ for a.e. } x \in \mathbb{R} \right\}, \end{aligned}$$

where

$$\|u\|_{L^p(-L,L)} := \int_{-L}^{L} |u|^p \quad (1 \le p < \infty), \qquad \|u\|_{L^\infty(-L,L)} := \operatorname*{ess\,sup}_{x \in (-L,L)} |u(x)|.$$

Note that $L^p_{\mathrm{p},L}$ refers to $2L$-periodic functions and $L^p_{\mathrm{ep},L}$ to $2L$-periodic functions which are even with respect to $x = 0$, this is the reason behind the subscripts p,L and ep,L. Given $0 < s < 1$ we define

$$\begin{aligned} H^s_{\mathrm{p},L} &:= \left\{ u \in L^2_{\mathrm{p},L} : [u]_{s,L} < +\infty \right\}, \\ H^s_{\mathrm{ep},L} &:= H^s_{\mathrm{p},L} \cap L^2_{\mathrm{ep},L}, \end{aligned}$$

where

$$[u]^2_{s,L} := \int_{-L}^{L} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}} dy\, dx.$$

In a similar manner, for $\alpha > 0$ and $0 < \beta \le 1$ we define

$$\begin{aligned} C^\alpha_{\mathrm{p},L} &:= \left\{ u \in L^\infty_{\mathrm{p},L} : u^{(j)} \text{ is continuous in } \mathbb{R}\ \forall j \le \lfloor \alpha \rfloor,\ \|u\|_{C^\alpha} < +\infty \right\}, \\ C^\alpha_{\mathrm{ep},L} &:= C^\alpha_{\mathrm{p},L} \cap L^\infty_{\mathrm{ep},L}, \end{aligned}$$

where

$$\begin{aligned} \|u\|_{C^\alpha} &:= \max_{j \in \mathbb{N} \cup \{0\},\ j \le \lfloor \alpha \rfloor} \sup_{x \in \mathbb{R}} |u^{(j)}(x)| + [u^{(\lfloor \alpha \rfloor)}]_{C^{\alpha - \lfloor \alpha \rfloor}(\mathbb{R})}, \\ [u]_{C^\beta(\mathbb{R})} &:= \sup_{x,y \in \mathbb{R}\ x \ne y} \frac{|u(x) - u(y)|}{|x-y|^\beta}. \end{aligned} \tag{29}$$

Here $\lfloor \cdot \rfloor$ denotes the integer part and $u^{(j)}$ stands for the jth derivative of u. One has the chain of inclusions $C^{s+\varepsilon}_{\mathrm{p},L} \subset H^s_{\mathrm{p},L} \subset L^2_{\mathrm{p},L}$, for all $\varepsilon > 0$. For the sake of simplicity, we will omit the subscript L when $L = \pi$, that is, we will simply write

$$L^p_{\mathrm{p}},\ L^p_{\mathrm{ep}},\ \|\cdot\|_{L^p},\ H^s_{\mathrm{p}},\ H^s_{\mathrm{ep}},\ [\cdot]_s,\ C^\alpha_{\mathrm{p}},\ C^\alpha_{\mathrm{ep}}$$

to refer to the different spaces and norms of 2π-periodic functions. We recall that the standard norm in $W^{s,2}(-\pi,\pi)$ is given by

$$\|u\|^2_{W^{s,2}(-\pi,\pi)} := \|u\|^2_{L^2} + [u]^2_{W^{s,2}(-\pi,\pi)},$$

where $[\cdot]^2_{W^{s,2}(-\pi,\pi)}$ is the classical Gagliardo $W^{s,2}$-seminorm defined by

$$[u]^2_{W^{s,2}(-\pi,\pi)} := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}dy\,dx.$$

It is clear that $[u]_{W^{s,2}(-\pi,\pi)} \le [u]_s$ for all u. Moreover, the following lemma shows that, in fact, $[\cdot]_s$ and $[\cdot]_{W^{s,2}(-\pi,\pi)}$ are comparable when applied to even 2π-periodic functions.

Lemma 2.1. *There exists a constant $C>0$, depending only on s, such that*

$$[u]_{W^{s,2}(-\pi,\pi)} \le [u]_s \le C[u]_{W^{s,2}(-\pi,\pi)} \tag{30}$$

for all even 2π-periodic function $u:\mathbb{R}\to\mathbb{R}$.

From (30) we deduce that $H^s_{\mathrm{ep}} = L^2_{\mathrm{ep}} \cap W^{s,2}(-\pi,\pi)$. The following lemma and (3) show the intuitive fact that $\int_{-L}^{L} u\mathcal{L}_K u$ and $[u]^2_{s,L}$ are comparable, and therefore, that $\int_{-L}^{L} u\mathcal{L}_K u$ and $[u]^2_{W^{s,2}(-\pi,\pi)}$ are comparable when applied to 2π-periodic even functions.

Lemma 2.2 ([6])**.** *Let $0<s<1$ and $\mathcal{L}_K$ be as in (2) and (3). Let $u,v:\mathbb{R}\to\mathbb{R}$ be $2L$-periodic functions in $L^2(-L,L)$ for which $\mathcal{L}_K u$ and $\mathcal{L}_K v$ belong to $L^2(-L,L)$. Then,*

$$\begin{aligned}\int_{-L}^{L} v\mathcal{L}_K u &= \int_{-L}^{L} u\mathcal{L}_K v\\ &= \frac{1}{2}\int_{-L}^{L}\int_{\mathbb{R}}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)K(|x-y|)dy\,dx.\end{aligned} \tag{31}$$

Now we shall define the norm in H^s_{ep} with which we will work in order to apply the Nehari method. Let $L>0$ and $0<s<1$, we define

$$[u]^2 := \int_{-\pi}^{\pi}\int_{\mathbb{R}}\big(u(x)-u(y)\big)^2 K_{\pi/L}(x-y)\,dy\,dx$$

and

$$\|u\|^2 := \left(\frac{L}{\pi}\right)^{2s}\|u\|^2_{L^2} + \frac{1}{2}[u]^2. \tag{32}$$

Note that for K as in (3), we have $K_{\pi/L}$ as in (11). Then, (11) yields that $[u]$ is comparable to $[u]_s$ and thus to $[u]_{W^{s,2}(-\pi,\pi)}$ by (30) if u is even. As a result, $\|u\|$ is comparable to $\|u\|_{W^{s,2}(-\pi,\pi)}$ in H^s_{ep}, with the constants of comparability depending on L, λ and Λ. In the sequel we will use several properties of the fractional Sobolev space $W^{s,2}$, such as the fractional Sobolev embeddings and the fractional Poincaré Inequality. These results can be found in [11].

In the periodic setting, the standard Fourier basis of sinus and cosinus are eigenfunctions of $\mathcal{L}_K$. This property will be particularly used in several computations of Section 5. More precisely, we have the following result.

Lemma 2.3 ([6])**.** *Let K satisfy (3) and let $L>0$. Then,*

$$\begin{aligned}\mathcal{L}_K\left(\cos\left(\tfrac{\pi k}{L}\cdot\right)\right)(x) &= \ell\left(\tfrac{\pi k}{L}\right)^{2s}\cos\left(\tfrac{\pi k}{L}x\right)\\ \mathcal{L}_K\left(\sin\left(\tfrac{\pi k}{L}\cdot\right)\right)(x) &= \ell\left(\tfrac{\pi k}{L}\right)^{2s}\sin\left(\tfrac{\pi k}{L}x\right)\end{aligned} \tag{33}$$

for all $k\in\mathbb{Z}$ and all $x\in\mathbb{R}$. Moreover,

$$\frac{\lambda}{c_s}|\xi|^{2s} \le \ell_K(\xi) \le \frac{\Lambda}{c_s}|\xi|^{2s} \tag{34}$$

for all $\xi\in\mathbb{R}$, where c_s is given in (5). If $\mathcal{L}_K = (-\Delta)^s$ then $\ell_K(\xi) = |\xi|^{2s}$.

2.2 Regularity Results

We conclude this section by giving some regularity results for weak solutions involving integro-differential operators on periodic solutions. We start by recalling the suitable notion of weak solution introduced in [6] for the periodic framework.

Definition 2.4. Let $L > 0$ and set $I = (-L, L)$, given $2L$-periodic functions u and f belonging to $L^1(I)$, we say that u *is a weak* $2L$*-periodic solution to* $\mathcal{L}_K u = f$ *in* $\mathbb{R}$ if

$$\int_I u\mathcal{L}_K\varphi = \int_I f\varphi$$

for all $2L$-periodic $\varphi \in C^\infty(I)$.

Theorem 2.5 ([6]). *Let $0 < s < 1$ and $f \in C^\beta(\mathbb{R})$ for some $\beta > 0$. Let $L > 0$ and $u \in L^\infty(\mathbb{R})$ be a $2L$-periodic function. If u is a weak periodic solution to $\mathcal{L}_K u = f(u)$ in $\mathbb{R}$ then $u \in C^{\beta+2s-\varepsilon}(\mathbb{R})$, for all $\varepsilon > 0$.*

Lemma 2.6 ([7]). *Let $0 < s < 1/2$, $1 \le p < \frac{1+2s}{1-2s}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be such that*

$$|f(x,t)| \le C_0\,(1 + |t|^p) \tag{35}$$

for some $C_0 > 0$ and all $(x,t) \in \mathbb{R} \times [0, +\infty)$. Let $u : \mathbb{R} \to \mathbb{R}$ be a positive 2π-periodic function such that $\|u\|_{L^2} + [u]_s < +\infty$. Assume that u is a weak periodic solution of

$$\mathcal{L}_K u = f(x,u) \quad \text{in } \mathbb{R}.$$

Then, $\|u\|_{L^\infty} \le C$ for some constant $C > 0$ depending only on s, p, C_0 and $\|u\|_{W^{s,2}(-\pi,\pi)}$.

Corollary 2.7 ([7]). *Let $0 < s < 1$ and let $f \in C^\beta(\mathbb{R})$ for some $\beta > 0$. Let $u \in H^s_{ep}$ be a positive weak periodic solution of $\mathcal{L}_K u = f(u)$ in $\mathbb{R}$. Assume that one of the following holds:*

(i) $1/2 < s < 1$,

(ii) $0 < s < 1/2$ *and*

$$|f(t)| \le C\,(1 + |t|^p) \tag{36}$$

for some $C > 0$ and $1 \le p < \frac{1+2s}{1-2s}$, and all $t \ge 0$,

(iii) $s = 1/2$ *and (36) holds for some $C > 0$ and $1 \le p < +\infty$, and all $t \ge 0$.*

Then $u \in C^{\beta+2s-\varepsilon}(\mathbb{R})$ for all $\varepsilon > 0$.

With these results, we can assert that any weak periodic solution to (9), obtained minimizing a suitable functional and thus solving the associated Euler-Lagrange equation, is in fact a classical solution to (9).

3 The Nehari manifold method

As we have mentioned before, we want to find solutions to (14) as minimizers, and hence critical points, of the energy functional (15). The purpose of this section is to explain the Nehari manifold method, which will be used in Section 4 to find minimizers of the above-mentioned energy functional. In the following, we present the definitions and results of critical point theory that will be further used to develop the Nehari manifold method.

Let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$ a functional such that $\Phi(0) = 0$. The Fréchet derivative of Φ at u, $\Phi'(u)$ is an element of the dual space E^* and we will denote $\Phi'(u)$ evaluated at $v \in E$ by $\Phi'(u)v$.

In Section 4 we will take $E = H^s_{\text{ep}}$, a real Hilbert space, in particular a real Banach space and $\Phi(v)$ defined by (18) and (19), belonging to $C^1(H^s_{\text{ep}}, \mathbb{R})$.

Definition 3.1. A point $u \in E$ is called *critical* if $\Phi'(u) = 0$. The corresponding value $c = \Phi(u)$ is a *critical value* or a *critical level.*

Definition 3.2. We say that $(u_n) \subset E$ is a *Palais-Smale sequence* if $(\Phi(u_n))$ is bounded and $\Phi'(u_n) \to 0$ in the operator norm sense in E^*, the continuous dual of E. Moreover, if $\Phi(u_n) \to c \in \mathbb{R}$ and $\Phi'(u_n) \to 0$, then (u_n) is a *$(PS)_c$-sequence.*

Definition 3.3. We say that the functional Φ satisfies the Palais-Smale condition (or *$(PS)_c$-condition*) if each Palais-Smale sequence (or *$(PS)_c$-sequence*) has a convergent subsequence.

It is therefore clear that if a Palais-Smale sequence converges to u (maybe through a subsequence), then u is a critical point. Let us now assume that the unit sphere S in E is a submanifold of class (at least) C^1 and let $\Phi \in C^1(S, \mathbb{R})$.

Theorem 3.4. *If Φ is bounded below and satisfies the Palais-Smale condition, then $c := \inf_S \Phi$ is attained and is a critical value of Φ.*

The Nehari manifold method, like the Mountain Pass Theorem and the Linking Theorem (see, [14, 16]) is a variational technique used to find critical points of an energy functional through a minimization procedure. We present the main definitions and results of this method following the approach of Szulkin and Weth in [15].

Assume we have a critical point $u \neq 0$ of Φ. Then, u must belong to

$$\mathcal{N} := \left\{u \in E \setminus \{0\} : \Phi'(u)u = 0\right\}.$$

Therefore, $\mathcal{N}$ becomes a natural constraint when looking for nontrivial (that is, $u \neq 0$) critical points of Φ. We shall call $\mathcal{N}$ the Nehari manifold (even though in general it may not be a manifold). Setting

$$c := \inf_{u \in \mathcal{N}} \Phi(u) \tag{37}$$

one hopes that c is attained at some $u_0 \in \mathcal{N}$ and that u_0 is a critical point, all this under appropriate conditions on Φ.

Definition 3.5. A function $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ is said to be a *normalization function* if $\varphi(0) = 0$, φ is strictly increasing and $\varphi(t) \to +\infty$ as $t \to +\infty$.

Let us denote $S := S_1(0) = \{u \in E : \|u\|_E = 1\}$, we assume that

(A_1) There exits a normalization function φ such that

$$u \mapsto \psi(u) := \int_0^{\|u\|} \varphi(t)dt \in C^1(E \setminus \{0\}, \mathbb{R}),$$

$J := \psi'(u)$ is bounded on bounded sets and $J(w)w = 1$ for all $w \in S$.

(A_2) For each $w \in E \setminus \{0\}$ there exists $s_w \in (0, +\infty)$ such that if $a_w(s) := \Phi(s_w w)$, then $\alpha'_w(s) > 0$ for $0 < s < s_w$ and $a'_w(s) < 0$ for $s > s_w$.

(A_3) There exists $\delta > 0$ such that $s_w \geq \delta$ for all $w \in S$ and for each compact subset $\mathcal{W} \subset S$ there exists a constant $C_{\mathcal{W}}$ such that $s_w \leq C_{\mathcal{W}}$ for all $w \in \mathcal{W}$.

From (A_1) one can see that S is a C^1-submanifold of E and the tangent space of S at w is given by

$$T_w(S) = \{z \in E : J(w)z = 0\}.$$

The intuition behind these assumptions is that there exists $\delta > 0$ such that for all $w \in S$, $\alpha_w(s)$ attains a unique maximum $s_w \geq \delta > 0$ in $(0, +\infty)$ and thus $0 = \alpha'_w(s_w) = \Phi'(s_w w)w$. Consequently, $s_w w$ is the unique point on the ray $s \mapsto sw$, $s > 0$ that intersects $\mathcal{N}$, with $\Phi(s_w w) > \Phi(0) = 0$. From this, one deduces that c in (37), if attained, is positive and that $u_0 \in \mathcal{N}$ is a critical point whenever $\Phi(u_0) = c$, which motivates the following definition.

Definition 3.6. We say $u_0 \in \mathcal{N}$ is a *ground state* solution if $\Phi(u_0) = c$.

We will now state the main results of the abstract Nehari manifold theory.

Definition 3.7. We define the mappings $\widehat{m} : E \setminus \{0\} \to \mathcal{N}$ and $m : S \to \mathcal{N}$ by setting

$$\widehat{m}(w) := s_w w \quad \text{and} \quad m := \widehat{m}|_S$$

Under these assumptions, for any $w \in E \setminus \{0\}$ the map $\widehat{m}$ sends w to the unique point $s_w w$ where $\mathcal{N}$ intersects the ray $s \mapsto sw$. It also does so with certain regularity, as stated in the following result.

Proposition 3.8. *Let Φ satisfy (A_2) and (A_3). Then,*

(i) The mapping $\widehat{m}$ is continuous.

(ii) The mapping m is a homeomorphism between S and $\mathcal{N}$, and the inverse of m is given by $m^{-1}(u) = u/\|u\|$ for all $u \in \mathcal{N}$.

Definition 3.9. We define the functionals $\widehat{\Psi} : E \setminus \{0\} \to \mathbb{R}$ and $\Psi : S \to \mathbb{R}$ given by

$$\widehat{\Psi}(w) := \Phi(\widehat{m}(w)) \quad \text{and} \quad \Psi := \widehat{\Psi}|_S.$$

This way, for any $w \in E \setminus \{0\}$ the functional $\widehat{\Psi}$ gives us the value of Φ at the unique point $s_w w$ where $\mathcal{N}$ intersects the ray $s \mapsto sw$. This gives us a relationship between critical points of Ψ and that of Φ, as will be stated in the next two results.

Proposition 3.10. *Let E be a Banach space that satisfies (A_1) and let Φ satisfy (A_2) and (A_3). Then, $\widehat{\Psi} \in C^1(E \setminus \{0\}, \mathbb{R})$ and*

$$\widehat{\Psi}'(w)z = \frac{\|\widehat{m}(w)\|}{\|w\|} \Phi'(\widehat{m}(w))z \quad \textit{for all } w, z \in E, \ w \neq 0.$$

Corollary 3.11. *Let E be a Banach space that satisfies (A_1) and let Φ satisfy (A_2) and (A_3). Then,*

(i) $\Psi \in C^1(S, \mathbb{R})$ *and*

$$\Psi'(w)z = \|m(w)\|\Phi(m(w))z \quad \text{for all } z \in T_w(S).$$

(ii) If (w_n) is a Palais-Smale sequence for Ψ, then $(m(w_n))$ is a Palais-Smale sequence for Φ. Conversely, if $(u_n) \subset \mathcal{N}$ is a bounded Palais-Smale sequence for Φ, then $(m^{-1}(u_n))$ is a Palais-Smale sequence for Ψ.

(iii) w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of Φ. Moreover, the corresponding values of Ψ and Φ coincide and $\inf_S \Psi = \inf_{\mathcal{N}} \Phi$.

Note that with this result we have a minimax characterization for the infimum of Φ over $\mathcal{N}$. Indeed,

$$c = \inf_{u \in \mathcal{N}} \Phi(u) = \inf_{w \in E \setminus \{0\}} \max_{s>0} \Phi(sw) = \inf_{w \in S} \max_{s>0} \Phi(sw).$$

Finally, we present one of the main results of the Nehari manifold method and the one that we will use, which gives sufficient conditions for the existence of a ground state.

Theorem 3.12. *Let E be a Hilbert space and suppose that $\Phi(u) = \frac{1}{2}\|u\|^2 - I(u)$, for $u \in E$, where*

(i) $I'(u) = o(\|u\|)$ *as* $u \to 0$,

(ii) $s \mapsto I'(su)u/s$ *is strictly increasing for all* $u \neq 0$ *and* $s > 0$,

(iii) $I(su)/s^2 \to +\infty$ *uniformly for u on weakly compact subsets of* $E \setminus \{0\}$ *as* $s \to +\infty$,

(iv) The operator I' is completely continuous (or weak-to-strong continuous), that is, if $u_n \rightharpoonup u$, then $I'(u_n) \to I'(u)$.

Then, equation $\Phi'(u) = 0$ has a ground state solution.

4 Periodic solutions to nonlinear integro-differential equations

This section is wholly devoted to the proof of Theorem 1.1. We will use the function spaces and notation introduced in Section 2 and the critical point theory and the Nehari manifold method developed in Section 3. Throughout all this section, we will assume that the nonlinearity g satisfies $(g1)$-$(g6)$ and that E_L and $c(L)$ are defined by (15) and (16), respectively. The following course of action to prove Theorem 1.1 is inspired by the proof of Theorem 1.4 in [7].

The first step is to show that, using the Nehari manifold method, the infimum in (16) is always attained, which is the purpose of Lemma 4.1. For this we need g to satisfy $(g1)$, $(g2)$, $(g3)$ and $(g5)$. Next, if $(g5)$ and $(g6)$ hold, in Lemma 4.2 we prove that there is a strictly positive minimizer of $E_L(w)$ belonging to C^{1+2s}_{ep} and solving (14). Then, defining u as in (13) we finally get a positive solution to (9), which was our first objective in this work.

Assuming $(g1)$, $(g2)$, $(g3)$ and $(g4)$ we are able in, Lemma 4.3, to find the unique nontrivial positive constant solution u_0 to (9) and its associated energy $E_L(u_0)$. Clearly, u_0 is also the unique nontrivial solution to (14). Afterwards, in Lemma 4.4 we see that if for some period L the infimum $c(L)$ is not attained by u_0 then it will no longer be attained by u_0 for any period greater than L.

In a similar direction, if g satisfies $(g5)$ we see in Lemma 4.5 that the minimizers of $E_L(w)$ are constant if the period is small enough. In Lemma 4.7 we find the sign of the first and second eigenvalues of $\mathcal{L}_{u,L}$ when u is a minimizer, as mentioned in the introduction. To this purpose, a technical result is needed, namely Lemma 4.6.

Lemma 4.1. *Given $L > 0$, we have $0 < c(L) < +\infty$ and there exists $w \in H^s_{ep}$ such that $w \not\equiv 0$ and $E_L(w) = c(L)$.*

Proof. We will apply the Nehari manifold theory we have developed so far. In order to use Theorem 3.12, let us take $E = H^s_{\text{ep}}$ a Hilbert space with the norm $\|v\|^2 = \left(\frac{L}{\pi}\right)^{2s}\|v\|^2_{L^2} + \frac{1}{2}[v]^2$ previously defined in (32), we shall see that $\Phi(v)$ defined in (18) satisfies the hypothesis of the theorem. To check them, we will follow the ideas for a similar problem presented in [15].

By the sake of comfort, let us denote $\widehat{I}(v) = \int_{-\pi}^{\pi} G(v)$. Then, $I(v)$ defined in (19) satisfies the hypothesis of the theorem if and only if $\widehat{I}(v)$ does so, the constant $\left(\frac{L}{\pi}\right)^{2s}$ does not play any role at all here. Abusing of notation, we rename $I(v)$ to $\widehat{I}(v)$.

We endow the dual continuous space of H^s_{ep} with the usual norm $\|F\|_{H^s_{\text{ep}},\mathbb{R}}$ defined by

$$\|F\|_{H^s_{\text{ep}},\mathbb{R}} = \sup_{v \in H^s_{\text{ep}} \setminus \{0\}} \frac{|F(v)|}{\|v\|},$$

for all $F : H^s_{\text{ep}} \to \mathbb{R}$. To prove (i), we shall see that for all $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\|u\| \le \delta$ then

$$\frac{\|I'(u)\|_{H^s_{\text{ep}},\mathbb{R}}}{\|u\|} \le \varepsilon.$$

Since g satisfies $(g1)$ we have that $g(u) = o(u)$ as $u \to 0$, which together with $(g5)$ implies that for each $\varepsilon > 0$ there is C_ε such that

$$|g(u)| \le \varepsilon|u| + C_\varepsilon|u|^p, \quad \text{for some } 1 < p < \tfrac{1+2s}{1-2s}.$$

It is clear that $I'(u)v = \int_{-\pi}^{\pi} g(u)v$. Combining this, $(g5)$, Hölder inequality and the fractional Sobolev embeddings (Theorem 6.7 of [11], note that $p+1 < \frac{2}{1-2s}$, where the right hand side the fractional critical Sobolev exponent) we have

$$
\begin{aligned}
|I'(u)v| \le \int_{-\pi}^{\pi} |g(u)||v| \le \int_{-\pi}^{\pi} (\varepsilon|u||v| + C_\varepsilon |u|^p|v|) &\le \varepsilon \|u\|_{L^2}\|v\|_{L^2} + C_\varepsilon \|u\|_{L^{p+1}}^p \|v\|_{L^{p+1}} \\
&\le C\|v\|_{W^{s,2}} \left(\varepsilon \|u\|_{W^{s,2}} + C_\varepsilon \|u\|_{W^{s,2}}^p\right) \\
&\le C\|v\| \left(\varepsilon\|u\| + C_\varepsilon \|u\|^p\right),
\end{aligned}
$$

which gives us

$$
\frac{\|I'(u)\|_{H^s_{\mathrm{ep}},\mathbb{R}}}{\|u\|} \le C\left(\varepsilon + C_\varepsilon \|u\|^{p-1}\right) \to C\varepsilon \quad \text{for } \|u\| \to 0,
$$

because $p > 1$, from which we obtain (i).

Let $0 < s < r$ and $u \neq 0$, let us recall that g satisfies $(g2)$, that is, $\frac{g(u)}{|u|}$ is strictly increasing in $(-\infty, 0)$ and $(0, +\infty)$. Then,

$$
\begin{aligned}
I'(su)\frac{u}{s} = \int_{-\pi}^{\pi} g(su)\frac{u}{s} &= \int_{[-\pi,\pi]\cap\{u\neq 0\}} \frac{g(su)}{su} u^2 \\
&< \int_{[-\pi,\pi]\cap\{u\neq 0\}} \frac{g(ru)}{ru} u^2 = \int_{-\pi}^{\pi} g(ru)\frac{u}{r} = I'(ru)\frac{u}{r},
\end{aligned}
$$

which gives us (ii).

To check (iii), we proceed by contradiction; let $\mathcal{W} \subset H^s_{\mathrm{ep}} \setminus \{0\}$ be weakly compact and assume there exist $M > 0$, $s_n \to \infty$ and $(u_n)_{n\ge 1} \subset \mathcal{W}$ such that $I(s_n u_n)/s_n^2 < M$, for all $n \ge 1$. Note that $(u_n)_{n\ge1} \subset \mathcal{W}$ has a weakly convergent subsequence, which we rename u_n and it is bounded, because it is included in the weakly compact subset $\mathcal{W}$. If u denotes its limit, then $\|u\| \le \liminf \|u_n\|$. In particular, we have a bounded subsequence, in H^s_{ep}, which is compactly embedded in $L^2(-\pi,\pi)$. Therefore, $u_n \to u$ in $L^2(-\pi,\pi)$ and, maybe through another subsequence, we have $u_n \to u$ a.e in $(-\pi,\pi)$. In particular, $u(x) \neq 0$ in a set of positive measure.

We have $|s_n u_n(x)| \to +\infty$ for almost every x such that $u(x) \neq 0$ because $\mathcal{W}$ is a weakly compact subset of $H^s_{\mathrm{ep}} \setminus \{0\}$. Then, Fatou's Lemma and $(g3)$ yield

$$
\begin{aligned}
M \ge \liminf_n \frac{I(s_n u_n)}{s_n^2} = \liminf_n \int_{-\pi}^{\pi} \frac{G(s_n u_n)}{s_n^2} &= \liminf_n \int_{-\pi}^{\pi} \frac{G(s_n u_n)}{(s_n u_n)^2} u_n^2 \\
&\ge \int_{-\pi}^{\pi} \liminf_n \frac{G(s_n u_n)}{(s_n u_n)^2} u_n^2 \to +\infty,
\end{aligned}
$$

which is a contradiction, and so we have that I satisfies (iii).

Finally, to see (iv), that I is completely continuous, we must see that if $(u_n) \subset H^s_{\mathrm{ep}}$ converges weakly to $u \in H^s_{\mathrm{ep}}$ then the linear functional $I'(u_n)$ converges to $I'(u)$ in the operator norm. Let $u_n \rightharpoonup u$ in H^s_{ep}, then (u_n) is a bounded sequence in H^s_{ep}, which is compactly embedded in $L^{p+1}(-\pi,\pi)$. Therefore, we can find a subsequence $(u_{n_k}) \subset H^s_{\mathrm{ep}}$ converging almost everywhere to a 2π-periodic even function u such that $\|u_{n_k} - u\|_{L^{p+1}} \to 0$.

Let $v \in H^s_{\mathrm{ep}}$, an application of Hölder inequality, together with the Fractional Sobolev Embedding yields

$$
\begin{aligned}
|I'(u_n)v - I'(u)v| \le \int_{-\pi}^{\pi} |g(u_n) - g(u)||v| &\le \|g(u_n) - g(u)\|_{L^{\frac{p+1}{p}}} \|v\|_{L^{p+1}} \\
&\le C\|g(u_n) - g(u)\|_{L^{\frac{p+1}{p}}} \|v\|_{W^{s,2}} \\
&\le C\|g(u_n) - g(u)\|_{L^{\frac{p+1}{p}}} \|v\|,
\end{aligned}
$$

from which we deduce that

$$\|I'(u_n) - I'(u)\|_{H^s_{\text{ep}},\mathbb{R}} \le C\|g(u_n) - g(u)\|_{L^{\frac{p+1}{p}}}.$$

Furthermore, since $(g5)$ holds and g is continuous, by Krasnoselskii's Theorem (see Theorem C.2 in [14]) we have that

$$\|g(u_n) - g(u)\|_{L^{\frac{p+1}{p}}} \le C\|u_n - u\|_{L^{p+1}} \to 0, \quad \text{as} \quad n \to \infty.$$

Hence, $I'(u_n)$ converges to $I'(u_n)$, we have checked that I satisfies (iv).

Finally we apply Theorem 3.12 to our functional Φ; there exists $w \in H^s_{\text{ep}} \setminus \{0\}$ a ground state solution, that is, $c(L) = E_L(w)$. Note that $c(L)$ being positive is a direct consequence of the Nehari manifold method. □

The following result states that the minimizers of E_L are strictly positive and regular enough to be classical solutions to (14).

Lemma 4.2. *Let $L > 0$ and $w \in H^s_{ep}$ be such that $E_L(w) = c(L)$. Then $w \in C^{1+2s}_{ep}$ and w does not vanish. Moreover, there always exists minimizer w such that $w > 0$ in $\mathbb{R}$ and solves (14).*

Proof. We first prove that if $w \in H^s_{ep}$ is such that $E_L(w) = c(L)$ then either $w \ge 0$ or $w \le 0$ in $\mathbb{R}$. Suppose that there exist measurable sets U, $V \subset [-\pi, \pi]$, both with positive Lebesgue measure, such that $w(x) > 0 > w(y)$ for all $x \in U$, $y \in V$. Then $\big||w(x)| - |w(y)|\big| < |w(x) - w(y)|$ for all $(x, y) \in U \times V$, hence

$$\int_{-\pi}^{\pi} \mathcal{L}_{K_{\pi/L}}|w|\,|w| < \int_{-\pi}^{\pi} \mathcal{L}_{K_{\pi/L}} w\, w,$$

where we also used that the kernel $K(t)$ is positive and Lemma 2.2. Note also that condition $(g6)$ implies

$$-I(w) + I(|w|) = \int_{-\pi}^{\pi} G(|w|) - G(w) = \int_{-\pi}^{\pi}\int_{w}^{|w|} g(y)dy \ge 0.$$

But then $c(L) \le E_L(|w|) < E_L(w) = c(L)$, which is a contradiction with the fact that w is assumed to be a minimizer. This means that either $w \ge 0$ or $w \le 0$ almost everywhere. In fact, we have $-I(-|w|) + I(|w|) \ge 0$. Moreover, since w does not change sign, $E_L(|w|) \le E_L(-|w|)$. Hence, we can always take $|w|$ as the minimizer of E_L.

If we take $\varphi \in H^s_{\text{ep}}$ then $w + \varepsilon\varphi \in H^s_{\text{ep}}$ for all $\varepsilon \in \mathbb{R}$. Since $E_L(w) = c(L)$, a computation shows that

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E_L(w + \varepsilon\varphi) \\ &= \int_{-\pi}^{\pi} \mathcal{L}_{K_{\pi/L}} w\, \varphi + \left(\frac{L}{\pi}\right)^{2s}\int_{-\pi}^{\pi} w\varphi - \left(\frac{L}{\pi}\right)^{2s}\int_{-\pi}^{\pi} g(w)\varphi \\ &= \int_{-\pi}^{\pi}\left(\mathcal{L}_{K_{\pi/L}} w + \left(\frac{L}{\pi}\right)^{2s} w - \left(\frac{L}{\pi}\right)^{2s} g(w)\right)\varphi. \end{aligned}$$

Since this holds for all $\varphi \in H^s_{\text{ep}}$, w weakly solves in $(0, \pi)$, and thus in $\mathbb{R}$ by parity and periodicity, the semilinear equation

$$\mathcal{L}_{K_{\pi/L}} w = f_L(w), \quad \text{where } f_L(t) := -\left(\frac{L}{\pi}\right)^{2s} t + \left(\frac{L}{\pi}\right)^{2s} g(t). \tag{38}$$

Note that $f_L \in C^{1+\varepsilon}(\mathbb{R})$ because $g \in C^{1+\varepsilon}(\mathbb{R})$. Additionally, f_L satisfies the growth estimate (36). Since $w \geq 0$, using Corollary 2.7 we deduce that $\|w\|_{C^{1+2s}} < +\infty$. In particular, $\mathcal{L}_{K_{\pi/L}} w$ makes sense pointwise and (38) holds in the classical sense.

Let us now prove that actually w does not vanish anywhere. If there exists $x \in \mathbb{R}$ such that $w(x) = 0$ then, using (38), (11) and $w(y) \geq 0$ for all $y \in \mathbb{R}$, we deduce that

$$0 = f_L(w(x)) = \mathcal{L}_{K_{\pi/L}} w(x) = -\int_{\mathbb{R}} w(y) K_{\pi/L}(|x-y|) dy \leq 0,$$

which shows that w vanishes identically. But we already know that $w \not\equiv 0$ because it is a critical point of E_L on the Nehari manifold, thus we conclude that $w > 0$ on $\mathbb{R}$, as desired. □

The following lemma assures the existence of a special nontrivial solution to (9), the positive constant solution.

Lemma 4.3. *There exists a unique $u_0 > 0$ such that $g(u_0) = u_0$. Moreover, its energy is that given by (17) and u_0 is the unique positive constant solution to (9).*

Proof. Thanks to $(g1)$ and $(g2)$ we have that $h(u) := \frac{g(u)}{|u|}$ is a strictly increasing well defined continuous function for all $u \geq 0$, with $h(0) = 0$. Moreover, $(g3)$ and $(g4)$ imply that $h(u) \to +\infty$ as $u \to +\infty$. Therefore, there exists a unique $u_0 > 0$ such that $h(u_0) = 1$ because $h(u)$ is strictly increasing. □

Thanks to the following lemma we can prove that if for some $L_w > 0$ the minimizer w is nonconstant, then the minimizer for $L > L_w$ is also nonconstant.

Lemma 4.4. *Suppose that there exists a nonconstant function $w \in H^s_{ep}$ such that $c(L_w) = E_{L_w}(w)$ for some $L_w > 0$. Then, $c(L) < E_L(u_0)$ for all $L > L_w$.*

Proof. Since K is positive and w is nonconstant, by (31) and (12) we see that

$$0 < \int_{-\pi}^{\pi} \mathcal{L}_{K_{\pi/L}} w\, w < \left(\frac{L}{L_w}\right)^{2s} \int_{-\pi}^{\pi} \mathcal{L}_{K_{\pi/L_w}} w\, w.$$

Therefore, using $E_L(u_0)$ given in (17),

$$\begin{aligned} c(L) \leq E_L(w) &= \frac{1}{2}\left(\int_{-\pi}^{\pi} \mathcal{L}_{K_{\pi/L}} w\, w + \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} w^2\right) - \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} G(w) \leq \left(\frac{L}{L_w}\right)^{2s} E_{L_w}(w) \\ &= \left(\frac{L}{L_w}\right)^{2s} c(L_w) \leq \left(\frac{L}{L_w}\right)^{2s} E_{L_w}(u_0) = E_L(u_0). \end{aligned}$$

□

The following result proves that the only minimizers of $c(L)$ are constant functions if L is small enough.

Lemma 4.5. *If $w \in H^s_{ep}$, $w > 0$, $E_L(w) = c(L)$ and $L > 0$ is sufficiently small, then w is constant.*

Proof. We begin recalling that our functional $\Phi(v)$ is given by $\Phi(v) = \frac{1}{2}\|v\|^2 - I(v)$, with $\|v\|^2 = \left(\frac{L}{\pi}\right)^{2s}\|v\|^2_{L^2} + \frac{1}{2}[v]^2$ and $I(v) = \left(\frac{L}{\pi}\right)^{2s}\int_{-\pi}^{\pi} G(v)$, and that we write $E_L(v) = \Phi(v)$. A priori, $[v]^2$ depends on L, that is

$$[v]^2 := \int_{-\pi}^{\pi}\int_{\mathbb{R}} \big(u(x) - u(y)\big)^2 K_{\pi/L}(x-y)\,dy\,dx,$$

but (11) ensures us that $[v]^2$ is comparable to $[v]^2_{W^{s,2}(-\pi,\pi)}$ and these constants of comparability do not depend on L. Let $w > 0$ be a minimizer of E_L and denote $c := E_L(w)$. By Lemma 4.2 we know that w solves

$$\mathcal{L}_{K_{\pi/L}} w + \left(\tfrac{L}{\pi}\right)^{2s} w = \left(\tfrac{L}{\pi}\right)^{2s} g(w). \tag{39}$$

Then, since $g(w) \geq 0$ for $w \geq 0$, g satisfies $(g4)$ and $w > 0$ solves (39) we have that

$$\begin{aligned}
\frac{1}{2}\|w\|^2 &= c + I(w) = c + \left(\tfrac{L}{\pi}\right)^{2s}\int_{-\pi}^{\pi} G(w)\\
&= c + \left(\tfrac{L}{\pi}\right)^{2s}\int_{\{|w|\leq R\}\cap[-\pi,\pi]} G(w) + \left(\tfrac{L}{\pi}\right)^{2s}\int_{\{|w|>R\}\cap[-\pi,\pi]} G(w)\\
&\leq c + 2\pi\left(\tfrac{L}{\pi}\right)^{2s}\|G\|_{L^\infty(-R,R)} + \frac{1}{\mu}\left(\tfrac{L}{\pi}\right)^{2s}\int_{\{|w|>R\}\cap[-\pi,\pi]} g(w)w\\
&\leq c + 2\pi\left(\tfrac{L}{\pi}\right)^{2s}\|G\|_{L^\infty(-R,R)} + \frac{1}{\mu}\left(\tfrac{L}{\pi}\right)^{2s}\int_{-\pi}^{\pi} g(w)w\\
&= c + 2\pi\left(\tfrac{L}{\pi}\right)^{2s}\|G\|_{L^\infty(-R,R)} + \frac{1}{\mu}\int_{-\pi}^{\pi}\left(\mathcal{L}_{K_{\pi/L}} w\, w + \left(\tfrac{L}{\pi}\right)^{2s} w^2\right)\\
&= c + 2\pi\left(\tfrac{L}{\pi}\right)^{2s}\|G\|_{L^\infty(-R,R)} + \frac{1}{\mu}\|w\|^2,
\end{aligned}$$

and thus, we estimate

$$\|w\|^2 \leq \frac{2\mu}{\mu-2}\left(c + 2\pi\left(\tfrac{L}{\pi}\right)^{2s}\|G\|_{L^\infty(-R,R)}\right).$$

To prove the result, we argue by contradiction. Assume that there exist $L_j \searrow 0$ and nonconstant functions $w_j \in H^s_{\text{ep}}$ such that $E_{L_j}(w_j) = c(L_j)$, for all $j \in \mathbb{N}$. Define $b_j := \left(\frac{L_j}{\pi}\right)^{2s}$ and $M := \max(1, \lambda^{-1})$. Now, for $j \geq j_0$ big enough we have $b_j < 1$ and

$$\begin{aligned}
\|w_j\|^2_{W^{s,2}(-\pi,\pi)} &\leq \|w_j\|^2_{L^2} + [w_j]^2_s \leq \frac{2}{b_j}\left(b_j\|w_j\|^2_{L^2} + \frac{1}{2}[w_j]^2_s\right) \leq \frac{2}{b_j}\max\left(1,\lambda^{-1}\right)\|w_j\|^2\\
&\leq \frac{2M}{b_j}\frac{2\mu}{\mu-2}\left(c_j + 2\pi b_j\|G\|_{L^\infty(-R,R)}\right) = 4M\frac{\mu}{\mu-2}\left(\frac{c_j}{b_j} + 2\pi\|G\|_{L^\infty(-R,R)}\right)
\end{aligned}$$

Notice that $c_j \leq E_{L_j}(u_0) = 2\pi b_j\left(\frac{u_0^2}{2} - G(u_0)\right)$. Then, we have that

$$\begin{aligned}
\|w_j\|^2_{W^{s,2}(-\pi,\pi)} &\leq 4M\frac{\mu}{\mu-2}\left(\frac{c_j}{b_j} + 2\pi\|G\|_{L^\infty(-R,R)}\right)\\
&\leq 2\pi 4M\frac{\mu}{\mu-2}\left(\frac{u_0^2}{2} + \|G\|_{L^\infty(-R,R)} + |G(u_0)|\right).
\end{aligned}$$

Therefore, $\|w_j\|^2_{W^{s,2}(-\pi,\pi)} \leq C$ for all j big enough. In case that $1/2 < s < 1$ the Sobolev inequality directly yields that $\|w_j\|_{L^\infty} \leq K$ for all j big enough, and if $0 < s \leq 1/2$ we can appeal to Lemma 2.6 to get a uniform bound of the L^∞ norm.

Thanks to Lemma 4.2 we can set $h_j := w_j' \in C^{2s}(\mathbb{R})$. By differentiating in (39) we see that h_j solves

$$\mathcal{L}_{K_{\pi/L}} h_j = -b_j h_j + b_j g'(w_j) h_j. \tag{40}$$

Observe that $\int_{-\pi}^{\pi} h_j = 0$ because w_j is 2π-periodic. Combining the fractional Poincaré inequality [2] with (40) and the uniform bound A of $\|w_j\|_{L^\infty}$, we see that

$$\begin{aligned} \|h_j\|_{L^2}^2 &\le C \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|h_j(x)-h_j(y)|^2}{|x-y|^{1+2s}} dydx \le C \int_{-\pi}^{\pi} \mathcal{L}_{K_{\pi/L}} h_j \, h_j \\ &\le C \left\{ b_j \|h_j\|_{L^2}^2 + b_j \|g'\|_{L^\infty(-A,A)} \|h_j\|_{L^2}^2 \right\} \\ &\le C b_j \|h_j\|_{L^2}^2, \end{aligned} \tag{41}$$

where $C > 0$ does not depend on j. Recall that $\lim_{j\to\infty} b_j = \lim_{j\to\infty} \left(\frac{L_j}{\pi}\right)^{2s} = 0$. Then, (41) shows that there exists $j_0 \in \mathbb{N}$ such that $\|h_j\|_{L^2}^2 \le \frac{1}{2}\|h_j\|_{L^2}^2$ for all $j \ge j_0$, which means that $w_j' = h_j = 0$ for all $j \ge j_0$. This contradicts the fact that w_j is a nonconstant function for all $j \in \mathbb{N}$. ☐

Now we would like to obtain some spectral properties of the linearized operator associated to (9) and the minimizers of $c(L)$. Before we do that, we first need the following technical result.

Lemma 4.6. *Let $L > 0$, $w \in H^s_{ep}$, $w > 0$ be such that $E_L(w) = c(L)$ and $\varphi \in H^s_{ep}$. Then,*

$$(s_w)' := \frac{d}{dh}\bigg|_{h=0} s_{w+h\varphi} = -\frac{1}{\int_{-\pi}^{\pi}(g'(w)w - g(w))w} \int_{-\pi}^{\pi} (g'(w)w - g(w))\varphi.$$

Proof. Let $v \in H^s_{\text{ep}} \setminus \{0\}$ and define $\alpha_v(s) : [0,+\infty) \to \mathbb{R}$ by $\alpha_v(s) = \Phi(sv)$. By the Nehari manifold method, condition $(A2)$ assures that there exists $s_v > 0$ such that $\Phi(s_v v) = \max_{s\in(0,\infty)} \Phi(sv)$, so it must be $\alpha_v'(s_v) = 0$ and $\alpha_v''(s_v) \le 0$. Then,

$$0 = \frac{d}{ds}\bigg|_{s=s_v} \left(\frac{1}{2}\|s_v v\|^2 - \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} G(sv) \right) = s_v\|v\|^2 - \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} g(s_v v)v,$$

which gives us an implicit expression for s_v. For $v = w + h\varphi$, we have that $s_{w+h\varphi}$ is given by

$$\|w+h\varphi\|^2 s_{w+h\varphi} = \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} g(s_{w+h\varphi}(w+h\varphi))(w+h\varphi).$$

Let us differentiate with respect to h and evaluate at $h = 0$, taking into account that $s_w = 1$ (because w is in the Nehari manifold). Then, we have

$$2\langle w, \varphi\rangle + \|w\|^2 (s_w)' = \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} \left(g'(w)(\varphi + (s_w)'w)w + g(w)\varphi\right),$$

where $\langle w, \varphi\rangle$ denotes the scalar product in H^s_{ep} that induces $\|\cdot\|$ and given by

$$\langle w, \varphi\rangle = \int_{-\pi}^{\pi} \mathcal{L}_{K_{\pi/L}} w \, \varphi + \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} w\varphi.$$

Since w is a minimizer, by Lemma 4.2 we have w solves (14) and hence $\langle w, \varphi\rangle = \left(\frac{L}{\pi}\right)^{2s} \int_{-\pi}^{\pi} g(w)\varphi$, for all $\varphi \in H^s_{\text{ep}}$. Then,

$$2\int_{-\pi}^{\pi} g(w)\varphi + (s_w)' \int_{-\pi}^{\pi} g(w)w = \int_{-\pi}^{\pi} \left(g'(w)(\varphi + (s_w)'w)w + g(w)\varphi\right),$$

and finally, solving for $(s_w)'$, we obtain the desired result. Now, $(g2)$ implies $g'(u)u > g(u)$ for all $u > 0$ and since $w > 0$, note that

$$0 > \alpha_w''(1) = -\int_{-\pi}^{\pi} (g'(w)w - g(w))w,$$

which assures us that $(s_w)'$ is indeed well defined. $\square$

Lemma 4.7. *Let $w \in H^s_{ep}$ be such that $E_L(w) = c(L)$ and $w > 0$. Let u be as in (13). Then, for every $\psi \in H^s_{ep,L}$,*

$$0 \leq \int_{-L}^{L} \mathcal{L}_{u,L}\psi\ \psi + \frac{1}{\int_{-L}^{L} (g'(u)u - g(u))\, u} \left(\int_{-L}^{L} (g'(u)u - g(u))\, \psi \right)^2. \tag{42}$$

In particular, $\sigma_1(\mathcal{L}_{u,L}) < 0 \leq \sigma_2(\mathcal{L}_{u,L})$, where $(\mathcal{L}_{u,L})$ denotes the linearized operator acting on $H^s_{ep,L}$ functions, given by (20), and $\sigma_1(\mathcal{L}_{u,L})$ and $\sigma_2(\mathcal{L}_{u,L})$ denote its first and second eigenvalues, respectively.

Proof. To obtain this kind of result, one would study the variation of the functional under a small perturbation of its minimizer. However, our constrained minimization method consists of finding minimizers of the functional Φ restricted to the Nehari subset $\mathcal{N}$, so a general perturbation on a minimizer w would fall outside $\mathcal{N}$ and therefore its energy would not be comparable to that of w, because w may not be a minimizer in the whole space.

However, let $\varphi \in H^s_{\text{ep}} \setminus \{0\}$ and define $\rho(h) := \widehat{\Psi}(w + h\varphi)$. It has been seen in Proposition 3.10 that

$$\rho'(h) = s_{w+h\varphi}\Phi'(s_{w+h\varphi}(w + h\varphi))\varphi.$$

Since w is a minimizer on $\mathcal{N}$ we now have that $\rho(h)$ has a minimum at $h = 0$, that is, $\rho'(0) = 0$ and $\rho''(0) \geq 0$. Hence, we compute

$$\begin{aligned}\rho''(0) &= (s_w)'\Phi'(s_w w)[\varphi] + s_w\Phi''(s_w w)[s_w\varphi + (s_w)'w, \varphi]\\ &= \Phi''(w)[\varphi,\varphi] + (s_w)'\Phi''(w)[w,\varphi],\end{aligned}$$

because $s_w = 1$ and $\Phi'(w) = 0$. Here, $\Phi''(w)[\xi,\eta]$ denotes the second Fréchet derivative and it is given by

$$\Phi''(w)[\xi,\eta] = \langle \xi,\eta\rangle - \left(\frac{L}{\pi}\right)^{2s}\int_{-\pi}^{\pi} g'(w)\xi\eta, \quad \text{for all } \xi,\eta \in H^s_{\text{ep}}.$$

Furthermore, note that the linearized equation of (14) is given by

$$\mathfrak{L}_{w,L}\psi := \mathcal{L}_{K_{\pi/L}}\psi + \left(\tfrac{L}{\pi}\right)^{2s}\psi - \left(\tfrac{L}{\pi}\right)^{2s} g'(w)\psi \quad \text{for } \psi \in H^s_{\text{ep}},$$

and, therefore,

$$\int_{-\pi}^{\pi} \mathfrak{L}_{w,L}\varphi\ \varphi = \Phi''(w)[\varphi,\varphi].$$

If we use the value of $(s_w)'$ found in Lemma 4.6 and the fact that $\langle w,\varphi\rangle = \left(\frac{L}{\pi}\right)^{2s}\int_{-\pi}^{\pi} g(w)\varphi$ given by Lemma 4.2, we obtain

$$0 \leq \int_{-\pi}^{\pi} \mathfrak{L}_{w,L}\varphi\ \varphi + \frac{\left(\frac{L}{\pi}\right)^{2s}}{\int_{-\pi}^{\pi} (g'(w)w - g(w))\, w} \left(\int_{-\pi}^{\pi} (g'(w)w - g(w))\, \varphi \right)^2. \tag{43}$$

Using the change $\psi(x) = \varphi(\frac{\pi}{L}x)$ and that $u(x) = w(\frac{\pi}{L}x)$, one can rewrite (43) as (42), we omit the details.

Using the expression of $\sigma_2(\mathcal{L}_{u,L})$ in terms of a Rayleigh quotient, we have

$$\sigma_2(\mathcal{L}_{u,L}) = \max_{\substack{V \subset H^s_{\mathrm{ep},L} \text{ subspace,} \\ \dim(V)=1}} \inf_{\substack{\psi \in H^s_{\mathrm{ep},L} \\ \int_{-L}^{L} v\psi = 0,\, \forall v \in V}} \frac{\int_{-L}^{L} \mathcal{L}_{u,L}\psi\, \psi}{\int_{-L}^{L} \psi^2}.$$

Take $V = \operatorname{span}\{g'(u)u - g(u)\} \subset H^s_{\mathrm{ep}}$. Given $\psi \in V^{\perp}$, (42) yields

$$0 \le \int_{-L}^{L} \mathcal{L}_{u,L}\psi\, \psi.$$

Therefore, $\sigma_2(\mathcal{L}_{u,L}) \ge 0$ as desired.

The estimate for $\sigma_1(\mathcal{L}_{u,L})$ is easier because

$$\sigma_1(\mathcal{L}_{u,L}) = \inf_{\psi \in H^s_{\mathrm{ep},L}} \frac{\int_{-L}^{L} \mathcal{L}_{u,L}\psi\, \psi}{\int_{-L}^{L} \psi^2} \le \frac{\int_{-L}^{L} \mathcal{L}_{u,L}u\, u}{\int_{-L}^{L} u^2} = \frac{\int_{-\pi}^{\pi} u(g(u) - g'(u)u)}{\int_{-L}^{L} u^2} < 0.$$

where we have used that $u > 0$ solves the equation and that g satisfies $(g2)$. $\square$

With all these Lemmas we are now able to give the proof of Theorem 1.1 by combining them properly.

Proof of Theorem 1.1. The first statements in the theorem follow from Lemmas 4.1, 4.2 and 4.7. Hence, it remains to show (i), (ii) and (iii).

Let u_0 be as in Lemma 4.3. Observe that

$$\mathcal{L}_{u_0,L}\psi = \mathcal{L}_K\psi + (1 - g'(u_0))\psi \quad \text{for } \psi \in H^s_{\mathrm{ep},L}.$$

Since $\left\{\cos\left(\frac{\pi k}{L}x\right)\right\}_{k\ge 0}$ is an orthogonal basis of H^s_{ep}, thanks to (33) we can easily find all the eigenvalues of $\mathcal{L}_{u_0,L}$ when acting on $H^s_{\mathrm{ep},L}$. More precisely, (7) gives us

$$\ell_K(\tfrac{\pi k}{L}) = \int_{\mathbb{R}} \frac{1 - \cos(y)}{|y|} \frac{L|y|}{\pi|k|} K\left(\frac{L|y|}{\pi|k|}\right) dy$$

for all $k \in \mathbb{Z} \setminus \{0\}$ and $\ell_K(0) = 0$. By (4) we can see that $0 \le \ell_K(\frac{\pi k_1}{L}) \le \ell_K(\frac{\pi k_2}{L})$ for all $|k_1| \le |k_2|$. From this and (33) we obtain that

$$\sigma_1(\mathcal{L}_{u_0,L}) = 1 - g'(u_0) < 0,$$
$$\sigma_2(\mathcal{L}_{u_0,L}) = \ell_K(\tfrac{\pi}{L}) + 1 - g'(u_0),$$

where the constant function and $\cos\left(\frac{\pi}{L}x\right)$ are the corresponding eigenfunctions in $H^s_{\mathrm{ep},L}$ for $\sigma_1(\mathcal{L}_{u_0,L})$ and $\sigma_2(\mathcal{L}_{u_0,L})$, respectively. The strictly negative sign of $\sigma_1(\mathcal{L}_{u_0,L})$ is due to the fact that, since $g(u_0) = u_0 > 0$ and g satisfies $(g2)$, namely $g'(u)u > g(u)$ for all $u > 0$, then $g'(u_0) > 1$. Observe that

$$\sigma_2(\mathcal{L}_{u_0,L}) = \ell_K(\tfrac{\pi}{L}) + 1 - g'(u_0) \tag{44}$$

is a continuous and strictly decreasing function with respect to $L > 0$ because of (4). Moreover, using (34) we get

$$\left(\frac{\pi}{L}\right)^{2s} \frac{\lambda}{c_s} + 1 - g'(u_0) \le \sigma_2(\mathcal{L}_{u_0,L}) \le \left(\frac{\pi}{L}\right)^{2s} \frac{\Lambda}{c_s} + 1 - g'(u_0),$$

from which we deduce that $\lim_{L\to 0}\sigma_2(\mathcal{L}_{u_0,L}) > 0 > \lim_{L\to+\infty}\sigma_2(\mathcal{L}_{u_0,L})$ since $g'(u_0) > 1$. As a result, there exists a unique $L_0 > 0$, for which $\sigma_2(\mathcal{L}_{u_0,L_0}) = 0$, that is, (21) holds for a unique $L_0 > 0$. Moreover, $\sigma_2(\mathcal{L}_{u_0,L}) > 0$ if $L < L_0$ and $\sigma_2(\mathcal{L}_{u_0,L}) < 0$ if $L > L_0$. Furthermore, if $E_L(u_0) = c(L)$, then $\sigma_2(\mathcal{L}_{u_0,L}) \geq 0$ by Lemma 4.7, which leads to $L \leq L_0$.

Define

$$L_* := \sup\{L > 0 : c(l) = E_l(u_0) \quad \text{for all } 0 < l \leq L\}. \tag{45}$$

Assume v_0 is a constant positive minimizer of E_l, by Lemma 4.2 v_0 is a constant positive solution to (14), which, by Lemma 4.3 has a unique constant positive solution, namely u_0. Therefore, it must be $v_0 = u_0$ and $c(l) = E_l(u_0)$.

We claim $0 < L_* \leq L_0 < +\infty$. Indeed, Lemma 4.5 yields $L_* > 0$ and if it were $L_* > L_0$, then for some $L_0 < L < L_*$ we would have $c(L) = E_L(u_0)$, so that u_0 would be a minimizer, with $\sigma_2(\mathcal{L}_{u_0,L}) < 0$, a contradiction with the characterization of minimizers of Lemma 4.7.

By definition, there exists $\{L_k\}_{k\in\mathbb{N}}$ such that $L_k > L_*$ for all $k \in \mathbb{N}$, $L_k \to L_*$ when $k \to \infty$ and $c(L_k) < E_{L_k}(u_0)$. This means that $c(L_k)$ is attained by a nonconstant positive function, thus Lemma 4.4 yields $c(L) < E_L(u_0)$ for all $L > L_k$ and all $k \in \mathbb{N}$. Hence,

$$L_* = \inf\{L > 0 : c(l) < E_l(u_0) \quad \text{for all } l \geq L\}. \tag{46}$$

It is clear that any L_* satisfying both (45) and (46) must be unique. Moreover, (45) and Lemma 4.4 ensure that if $L < L_*$ then $c(L)$ is only attained by constant functions, and (46) shows that if $L_* < L$ then $c(L)$ is only attained by nonconstant functions. We have proven (i) and (ii).

For (iii), we have already seen that $L_* \leq L_0$, the inequality $L_{**} \leq L_*$ is a direct consequence of (ii). For $\mathcal{L}_K = (-\Delta)^s$ we have $\ell_K(\frac{\pi}{L}) = (\frac{\pi}{L})^{2s}$ and thus $0 = \sigma_2(\mathcal{L}_{u_0,L_0})$ yields $L_0 = \pi\,(g'(u_0) - 1)^{-\frac{1}{2s}}$. $\square$

5 Bifurcated solutions of small amplitude

This whole section is entirely devoted to the proof of Theorems 1.2 and 1.3. We start directly with the proof of Theorem 1.2, a result inspired by Cabré, Mas and Solà-Moarles in [7].

Proof of Theorem 1.2. This result is a slight modification of Theorem 1.5 in [7] and hence the proof will be basically that of [7], except for some functional setting changes. Here we will just comment on these changes we have done to adapt the aforementioned proof to our case.

We begin with the same strategy: in order to find $2L$-periodic solutions to (28) we define $u(x) = \overline{u}(\frac{\pi}{L}x)$ so that (28) becomes $(-\Delta)^s\overline{u} = \left(\frac{L}{\pi}\right)^{2s} f(\overline{u})$. As a matter of fact, let us call $\lambda = \left(\frac{L}{\pi}\right)^{2s}$ and u to this new unknown $\overline{u}$. We remark that from now on u will be 2π-periodic and our unknowns will be pairs (λ, u) such that

$$(-\Delta)^s u - \lambda f(u) = 0 \tag{47}$$

The statement of the theorem is just a result of bifurcation from a simple eigenvalue. The proof of Theorem 1.5 in [7] is based on Theorem 4.1 of [3, Chapert 5] and uses the same notation. Recall that we write L^2_{ep} to denote the space of $L^2(-\pi,\pi)$ functions that are (a.e.) even and 2π-periodically extended to all of $\mathbb{R}$. We define the functional spaces $X := C^{2s+\alpha}(\mathbb{R}) \cap L^2_{ep}$ and $Y := C^\alpha(\mathbb{R}) \cap L^2_{ep}$. We also define

$$\mathcal{F}(\lambda, u) = (-\Delta)^s u - \lambda f(u),$$

for which we want to solve $\mathcal{F}(\lambda, u) = 0$.

Let us see that $\mathcal{F} \in C^5(\mathbb{R}\times X, Y)$. Thanks to our hypothesis in s and α, namely $1 < 2s+\alpha < 2$ we have that $(-\Delta)^s + \mathrm{Id}$ is an isomorphism between X and Y that is of class C^5 with respect to λ as a bounded linear map from X to Y. The isomorphism property is deduced from the linear inhomogeneous equation $(-\Delta)^s u + u = f_1$, when $f_1 \in Y$. A bounded weak solution can be first obtained in Fourier series and then the regularity results of Theorem 2.5 can be applied. The fact that the operator is of class C^5 with respect to λ is clear because $\mathcal{F}$ is linear in λ.

Now we shall see that the map $u \mapsto f(u)$ is of class C^5 from Y to itself. This will yield that the map $u \mapsto f(u)$ is of class C^5 from X to Y thanks to the continuous linear embedding $X \subset Y$. We will use the Banach algebra property of $C^\alpha(\mathbb{R})$, namely that if $u_1, u_2 \in C^\alpha(\mathbb{R})$ then also $u_1u_2 \in C^\alpha(\mathbb{R})$ and $\|u_1u_2\|_{C^\alpha} \le C\|u_1\|_{C^\alpha}\|u_2\|_{C^\alpha}$, for some $C > 0$.

The fifth Fréchet derivative of $u \overset{F}{\mapsto} f(u)$ is the multilinear mapping defined by

$$\begin{array}{rcc} F^{(5)}(u) : C^\alpha(\mathbb{R}) \times C^\alpha(\mathbb{R}) \times C^\alpha(\mathbb{R}) \times C^\alpha(\mathbb{R}) \times C^\alpha(\mathbb{R}) & \longrightarrow & C^\alpha(\mathbb{R}) \\ (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) & \mapsto & f^{(5)}(u)\varphi_1\varphi_2\varphi_3\varphi_4\varphi_5 \end{array}$$

so we shall see that indeed $f^{(5)}(u)\varphi_1\varphi_2\varphi_3\varphi_4\varphi_5$ belongs to $C^\alpha(\mathbb{R})$ for all $u \in C^\alpha(\mathbb{R})$ and that $F^{(5)}$ is continuous as an operator from $C^\alpha(\mathbb{R})$ to the space of continuous multilinear maps from $\bigotimes_{i=1}^5 C^\alpha(\mathbb{R})$ to $C^\alpha(\mathbb{R})$, that we will denote by $\mathcal{L}(\bigotimes_{i=1}^5 C^\alpha(\mathbb{R}), C^\alpha(\mathbb{R}))$ endowed with its usual operator norm $\|\cdot\|_{\mathcal{L}(\bigotimes_{i=1}^5 C^\alpha(\mathbb{R}), C^\alpha(\mathbb{R}))}$. The other four first Fréchet derivatives are treated equally, we omit the details.

By simplicity, let us denote $g := f^{(5)}$ and $h := f^{(6)}$, hence $g \in C^1(\mathbb{R})$ and $h \in C(\mathbb{R})$. Note that if $g(u)$ belongs to $C^\alpha(\mathbb{R})$ then the Banach algebra property of $C^\alpha(\mathbb{R})$ will give us $g(u)\varphi_1\varphi_2\varphi_3\varphi_4\varphi_5 \in C^\alpha(\mathbb{R})$. Let's see this, namely, that $g(u)$ is continuous and $\|g(u)\|_{C^\alpha} < \infty$. The continuity of $g(u)$ is clear by being a composition of two continuous functions.

Regarding the C^α-norm, every $u \in Y = C^\alpha(\mathbb{R}) \cap L^2_{ep}$ must be bounded because it is continuous and periodic, so let $M > 0$ and $K := [-M, M]$ such that $u(x) \in K$, for all $x \in \mathbb{R}$. Then, we have $\|g(u)\|_{L^\infty(\mathbb{R})} \leq \|g\|_{L^\infty(K)} < \infty$ and $\|h(u)\|_{L^\infty(\mathbb{R})} \leq \|h\|_{L^\infty(K)} < \infty$. Now, for all $x, y \in \mathbb{R}$ such that $x \neq y$ the mean value theorem gives us

$$\frac{|g(u(x)) - g(u(y))|}{|x-y|^\alpha} \leq |h(\xi)| \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq \|h\|_{C(K)}[u]_{C^\alpha} < +\infty,$$

for some ξ in the segment joining $u(x)$ and $u(y)$, hence belonging to K. As a result, we obtain that $\|g(u)\|_{C^\alpha} < +\infty$ and thus $f^{(5)}(u) \in C^\alpha(\mathbb{R})$.

Now let us see that $F^{(5)}$ is continuous. We will see that, for every $u \in C^\alpha(\mathbb{R})$,

$$\|F^{(5)}(u) - F^{(5)}(v_n)\|_{\mathcal{L}(\bigotimes_{i=1}^5 C^\alpha(\mathbb{R}), C^\alpha(\mathbb{R}))} \to 0,$$

for all $v_n \to u$ in $C^\alpha(\mathbb{R})$. Given $u \in C^\alpha(\mathbb{R})$, let $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \in C^\alpha(\mathbb{R})$ and $v_n \in C^\alpha(\mathbb{R})$ with $v_n \to u$ in $C^\alpha(\mathbb{R})$. In particular, we have that $\|v_n\|_{C^\alpha} \leq \|u\|_{C^\alpha} + 1 < M + 1$, for all n big enough. Now,

$$\begin{aligned}\|\big(F^{(5)}(u) - F^{(5)}(v_n)\big)(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)\|_{C^\alpha} &= \|\big(f^{(5)}(u) - f^{(5)}(v_n)\big)\varphi_1\varphi_2\varphi_3\varphi_4\varphi_5\|_{C^\alpha} \\ &\leq C\|f^{(5)}(u) - f^{(5)}(v_n)\|_{C^\alpha} \prod_{i=1}^5 \|\varphi_i\|_{C^\alpha},\end{aligned}$$

from which we deduce that

$$\|F^{(5)}(u) - F^{(5)}(v_n)\|_{\mathcal{L}(\bigotimes_{i=1}^5 C^\alpha(\mathbb{R}), C^\alpha(\mathbb{R}))} \leq C\|g(u) - g(v_n)\|_{C^\alpha},$$

Let us study $\|g(u) - g(v_n)\|_{C^\alpha}$. Given $x, y \in \mathbb{R}$ we have

$$\begin{aligned}|\big(g(u) - g(v_n)\big)(x) - \big(g(u) - g(v_n)\big)(y)| &= |\big(g(u(x)) - g(u(y))\big) - \big(g(v_n(x)) - g(v_n(y))\big)| \\ &= \Bigg|(u(x) - u(y)) \int_0^1 h\big(tu(x) + (1-t)u(y)\big)\, dt \\ &\qquad -(v_n(x) - v_n(y)) \int_0^1 h\big(tv_n(x) + (1-t)v_n(y)\big)\, dt\Bigg|.\end{aligned}$$

Adding and subtracting $(u(x) - u(y)) \int_0^1 h\big(tv_n(x) + (1-t)v_n(y)\big)dt$ and using the triangular inequality, we can estimate the expression above by two terms that we bound independently. The first one is

$$\left|(u(x) - u(y)) \int_0^1 h\big(tu(x) + (1-t)u(y)\big) - h\big(tv_n(x) + (1-t)v_n(y)\big)\, dt\right| \leq [u]_{C^\alpha}|x-y|^\alpha \varepsilon.$$

This is so because h is uniformly continuous, it is continuous and defined in $K_1 := [-M-1, M+1]$ a compact set, note that both $tu(x) + (1-t)u(y)$ and $tv_n(x) + (1-t)v_n(y)$ are bounded by $M+1$ for all n big enough, and $|u(z) - v_n(z)| < \delta(\varepsilon)$ uniformly for all $z \in \mathbb{R}$, for all n big enough. The second term is

$$\begin{aligned}&\left|\big((u(x) - u(y)) - (v_n(x) - v_n(y))\big) \int_0^1 h\big(tv_n(x) + (1-t)v_n(y)\big)dt\right| \\ &\qquad \leq \|h\|_{C(K_1)} \left|\frac{u(x) - u(y)}{|x-y|^\alpha} - \frac{v_n(x) - v_n(y)}{|x-y|^\alpha}\right| |x-y|^\alpha \leq \|h\|_{C(K_1)}[u - v_n]_{C^\alpha}|x-y|^\alpha.\end{aligned}$$

Finally, we see that

$$\frac{|\big(g(u)-g(v_n)\big)(x)-\big(g(u)-g(v_n)\big)(y)|}{|x-y|^\alpha}\le \varepsilon[u]_{C^\alpha}+\|h\|_{C(K_1)}[u-v_n]_{C^\alpha}\to 0,$$

for $n\to+\infty$. Thus $[g(u)-g(v_n)]_{C^\alpha}\to 0$ as $n\to+\infty$. Moreover,

$$|g(u(x))-g(v_n(x))|\le |h(\xi_x)|\,|u(x)-v_n(x)|\le \|h\|_{C(K)}\|u-v_n\|_{C(\mathbb{R})},$$

for some ξ_x between $u(x)$ and $v_n(x)$, in particular $\xi_x\in K$ for all n big enough. As a result, $\|g(u)-g(v_n)\|_{L^\infty(\mathbb{R})}\to 0$ as $n\to+\infty$. From these two estimates we finally obtain

$$\|F^{(5)}(u)-F^{(5)}(v_n)\|_{\mathcal{L}(\bigotimes_{i=1}^5 C^\alpha(\mathbb{R}),C^\alpha(\mathbb{R}))}\to 0 \text{ as } n\to+\infty,$$

for all $v_n\to u$ in $C^\alpha(\mathbb{R})$, that is, $F^{(5)}$ is continuous and therefore $F:u\mapsto f(u)$ is of class C^5 from Y to itself.

From this point onwards, one can check the other hypothesis exactly as in [7] and obtain the desired result. Note that the bifurcation point is $(\lambda^*,0)$, with $\lambda^*=f'(0)^{-1}$, and thus, the set $\mathcal{F}^{-1}\{0\}$ consists, in a neighbourhood of $(\lambda,u)=(\lambda^*,0)$, solely of $\{u=0\}$ and a curve $(\lambda(a),u_a)$ with $(\lambda(0),u_0)=(\lambda^*,0)$ and of class C^4 in the parameter a. There is a loss of one degree of differentiability due to a factorization that is needed in the method of Lyapunov-Shmidt, see the details in [3]. □

In the sequel, we will assume that E_L and $c(L)$ are defined by (24) and (25) respectively. To determine whether $L_*<L_0$ or not we just need to see that that there exists a function $u\in H^s_{\text{ep}}$ and a period $L<L_0$ such that u has less energy than the constant nontrivial solution $u_0\equiv 1$. That is, we need

$$E_L(u)=\frac{\int_{-\pi}^{\pi}u(-\Delta)^s u+\left(\frac{L}{\pi}\right)^{2s}\int_{-\pi}^{\pi}u^2}{\|u\|^2_{L^{p+1}}}<\left(\tfrac{L}{\pi}\right)^{2s}(2\pi)^{\frac{p-1}{p+1}}=E_L(u_0),$$

which is the same as asking

$$\left(\int_{-\pi}^{\pi}u(-\Delta)^s u+\left(\tfrac{L}{\pi}\right)^{2s}\int_{-\pi}^{\pi}u^2\right)^{p+1}<\left(\|u\|^{p+1}_{L^{p+1}}\right)^2(E_L(u_0))^{p+1}. \tag{48}$$

If (48) holds, since E_L is homogeneous, then the minimizer of E_L will automatically be nonconstant and thus $L_*\le L<L_0$. To this purpose, we consider the scalar $\lambda(a)=(L(a)/\pi)^{2s}$ and the 2π-periodic and even function u_a given by the proof of Theorem 1.2, that is, $(\lambda(a),u_a)$ is the unique curve of solutions to (47) of class C^4 with respect to the parameter a. The following results focus on obtaining suitable expressions for the terms appearing in (48) when we take $u=u_a$ and $L=L(a)$.

As usual, we say that $g_a=o(a^4)$ provided $\frac{|g_a|}{a^4}\to 0$ as $a\to 0$.

Lemma 5.1. *Under the setting of Theorem 1.2, u_a is a 2π-periodic even function of x of the form*

$$\begin{aligned}u_a(x)=1&+a\cos(x)+\frac{1}{2!}\left(c_0^2+c_2^2\cos(2x)\right)a^2\\&+\frac{1}{3!}c_3^3\cos(3x)a^3+\frac{1}{4!}\left(c_0^4+c_2^4\cos(2x)+c_4^4\cos(4x)\right)a^4+w_a(x),\end{aligned}\tag{49}$$

with constants c_0^2, c_2^2, c_3^3, c_0^4, c_2^4, c_4^4 depending on s,p and $w_a\in C^{2s+\alpha}(\mathbb{R})$ such that $\frac{\|w_a\|_{C^{2s+\alpha}}}{a^4}\to 0$ as $a\to 0$, that is, $\|w_a\|_{C^{2s+\alpha}}=o(a^4)$.

We denote c_k^n the constant accompanying the term $\cos(kx)a^n$ in the above expression.

Proof. We know that $a \mapsto u_a \in C^{2s+\alpha}(\mathbb{R})$ is of class C^4, with $u_0 = 1$. Hence, Taylor's theorem for Banach spaces (see [9]) gives us

$$u_a = 1 + (\partial_a u_0)a + \frac{1}{2}(\partial_a^2 u_0)a^2 + \frac{1}{3!}(\partial_a^3 u_0)a^3 + \frac{1}{4!}(\partial_a^4 u_0)a^4 + v_a,$$

with $v_a \in C^{2s+\alpha}(\mathbb{R})$ and $\|v_a\|_{C^{2s+\alpha}} = o(a^4)$. In order to find the Taylor coefficients for u_a, we will take derivatives with respect to a in the identity $(-\Delta)^s u_a = \lambda(a) f(u_a)$. That is,

$$(-\Delta)^s \partial_a u_a = \lambda'(a) f(u_a) + \lambda(a) f'(u_a)\partial_a u_a \tag{50}$$

Since $L(a)$ is even, so is $\lambda(a)$, hence $\lambda'(0) = 0$. Moreover, $\lambda(0) = f'(1)^{-1}$ and $f(1) = 0$. Thus, writing it for $a = 0$ we have

$$(-\Delta)^s \partial_a u_0 - \partial_a u_0 = 0$$

Now, $(-\Delta)^s - \mathrm{Id}$ diagonalizes in the basis of cosinus and, furthermore, $\cos(x)$ is its kernel. Hence, we have that

$$\partial_a u_0 = \cos(x).$$

Differentiating (50) with respect to a, we obtain

$$\begin{aligned}(-\Delta)^s \partial_a^2 u_a = {}& \lambda''(a) f(u_a) + 2\lambda'(a) f'(u_a)\partial_a u_a \\ & + \lambda(a) f''(u_a)\left(\partial_a u_a\right)^2 + \lambda(a) f'(u_a)\partial_a^2 u_a.\end{aligned} \tag{51}$$

Writing it for $a = 0$ we have

$$\begin{aligned}(-\Delta)^s \partial_a^2 u_0 - \partial_a^2 u_0 &= \frac{f''(1)}{f'(1)}\cos^2(x) \\ &= \frac{f''(1)}{2f'(1)} + \frac{f''(1)}{2f'(1)}\cos(2x)\end{aligned}$$

and therefore, knowing that $\cos(x)$ is in the kernel of the right hand side and it must be orthogonal to $\partial_a^2 u_0$ by Theorem 1.2, we have that

$$\partial_a^2 u_0 = -\frac{f''(1)}{2f'(1)} + \frac{f''(1)}{2\left(2^{2s}-1\right)f'(1)}\cos(2x),$$

from which we deduce that

$$c_0^2 = -\frac{f''(1)}{2f'(1)}, \quad c_2^2 = \frac{f''(1)}{2(2^{2s}-1)f'(1)}. \tag{52}$$

Differentiating (51) with respect to a we find

$$\begin{aligned}(-\Delta)^s \partial_a^3 u_a = {}& \lambda'''(a) f(u_a) + 3\lambda''(a) f'(u_a)\partial_a u_a + 3\lambda'(a) f''(u_a)\left(\partial_a u_a\right)^2 \\ & + 3\lambda'(a) f'(u_a)\partial_a^2 u_a + \lambda(a) f'''(u_a)\left(\partial_a u_a\right)^3 \\ & + 3\lambda(a) f''(u_a)\partial_a u_a \partial_a^2 u_a + \lambda(a) f'(u_a)\partial_a^3 u_a.\end{aligned} \tag{53}$$

Writing it for $a = 0$ and using the expressions we already know for $\partial_a u_0$ and $\partial_a^2 u_0$ and the fact that $\lambda'(a) = 0$ we obtain

$$\begin{aligned}(-\Delta)^s \partial_a^3 u_0 - \partial_a^3 u_0 = {}& 3\lambda''(0) f'(1)\cos(x) + f'''(1) f'(1)^{-1}\cos^3(x) \\ & + 3\frac{f''(1)}{f'(1)}\cos(x)\left(-\frac{f''(1)}{2f'(1)} + \frac{f''(1)}{2\left(2^{2s}-1\right)f'(1)}\cos(2x)\right).\end{aligned}$$

Using that $\cos^3(x) = \frac{3}{4}\cos(x) + \frac{1}{4}\cos(3x)$ and $\cos(x)\cos(2x) = \frac{1}{2}\cos(x) + \frac{1}{2}\cos(3x)$ we get

$$(-\Delta)^s \partial_a^3 u_0 - \partial_a^3 u_0 = 3\left(\lambda''(0)f'(1) + \frac{1}{4}\frac{f'''(1)}{f'(1)} - \left(\frac{f''(1)}{f'(1)}\right)^2\left(\frac{1}{2} - \frac{1}{4}\frac{1}{2^{2s}-1}\right)\right)\cos(x)$$
$$+\left(\frac{1}{4}\frac{f'''(1)}{f'(1)} + \frac{3}{4}\frac{1}{2^{2s}-1}\left(\frac{f''(1)}{f'(1)}\right)^2\right)\cos(3x).$$

Now, by the same reason as before, the right hand side has to be orthogonal to $\cos(x)$, the kernel of the left hand side. Also, since $\cos(3x)$ is an eigenfunction to $(-\Delta)^s - \mathrm{Id}$ with eigenvalue $3^{2s}-1$, we obtain

$$\lambda''(0) = \frac{1}{f'(1)}\left[-\frac{1}{4}\frac{f'''(1)}{f'(1)} + \left(\frac{1}{2} - \frac{1}{4}\frac{1}{2^{2s}-1}\right)\left(\frac{f''(1)}{f'(1)}\right)^2\right] \tag{54}$$

and

$$\partial_a^3 u_0 = \frac{1}{3^{2s}-1}\left(\frac{1}{4}\frac{f'''(1)}{f'(1)} + \frac{3}{4}\frac{1}{2^{2s}-1}\left(\frac{f''(1)}{f'(1)}\right)^2\right)\cos(3x),$$

from which we deduce that

$$c_3^3 := \frac{1}{3^{2s}-1}\left(\frac{1}{4}\frac{f'''(1)}{f'(1)} + \frac{3}{4}\frac{1}{2^{2s}-1}\left(\frac{f''(1)}{f'(1)}\right)^2\right).$$

Differentiating (53) with respect to a, we find

$$\begin{aligned}(-\Delta)^s \partial_a^4 u_a = {} & \lambda^{(4)}(a)f(u_a) + 4\lambda'''(a)f'(u_a)\partial_a u_a + 6\lambda''(a)f''(u_a)\left(\partial_a u_a\right)^2 \\ & + 6\lambda''(a)f'(u_a)\partial_a^2 u_a + 4\lambda'(a)f'''(u_a)\left(\partial_a u_a\right)^3 \\ & + 12\lambda'(a)f''(u_a)\partial_a u_a \partial_a^2 u_a + 4\lambda'(a)f'(u_a)\partial_a^3 u_a \\ & + \lambda(a)f^{(4)}(u_a)\left(\partial_a u_a\right)^4 + 6\lambda(a)f'''(u_a)\left(\partial_a u_a\right)^2\partial_a^2 u_a \\ & + 3\lambda(a)f''(u_a)\left(\partial_a^2 u_a\right)^2 + 4\lambda(a)f''(u_a)\partial_a u_a \partial_a^3 u_a + \lambda(a)f'(u_a)\partial_a^4 u_a.\end{aligned}$$

Writing it for $a = 0$, since $\lambda'''(0) = 0$ because $\lambda(a)$ is even, we obtain

$$\begin{aligned}(-\Delta)^s \partial_a^4 u_0 - \partial_a^4 u_0 = {} & 6\lambda''(0)f''(1)\cos^2(x) + 6\lambda''(0)f'(1)\partial_a^2 u_0 \\ & + \frac{f^{(4)}(1)}{f'(1)}\cos^4(x) + 6\frac{f'''(1)}{f'(1)}\cos^2(x)\partial_a^2 u_0 \\ & + 3\frac{f''(1)}{f'(1)}\left(\partial_a^2 u_0\right)^2 + 4\frac{f''(1)}{f'(1)}\cos(x)\partial_a^3 u_0.\end{aligned}$$

Using the expressions we have for $\partial_a^2 u_0$ and $\partial_a^3 u_0$, and that

$$\begin{aligned}\cos^2(2x) &= \frac{1}{2} + \frac{1}{2}\cos(4x) \\ \cos(x)\cos(3x) &= \frac{1}{2}\cos(2x) + \frac{1}{2}\cos(4x) \\ \cos^4(x) &= \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\end{aligned}$$

we end up having

$$(-\Delta)^s \partial_a^4 u_0 - \partial_a^4 u_0 = C_0 + C_2\cos(2x) + C_4\cos(4x)$$

for some constants C_0, C_2 and C_4. Since 1, $\cos(2x)$ and $\cos(4x)$ are eigenfunctions to $(-\Delta)^s - \mathrm{Id}$, whose kernel $\cos(x)$ is orthogonal to $\partial_a^4 u_0$, we finally obtain

$$\partial_a^4 u_0 = c_0^4 + c_2^4 \cos(2x) + c_4^4 \cos(4x).$$

which concludes the proof. $\square$

In the sequel there will appear expressions like $g_a := \int_{-\pi}^{\pi} v_a(x)dx$, where $v_a \in C^\beta(\mathbb{R})$, for some $\beta > 0$ and $\|v_a\|_{C^\beta} = o(a^4)$. Then, one has $g_a = o(a^4)$. Indeed, we can bound $|g_a| \leq 2\pi \|v_a\|_{L^\infty(-\pi,\pi)} \leq 2\pi \|v_a\|_{C^\beta}$ and the property follows.

Lemma 5.2. *Under the setting of Theorem 1.2, let u_a be as in (49). Then,*

$$\int_{-\pi}^{\pi} u_a(-\Delta)^s u_a = \pi \left(a^2 + 2^{2s} \left(\frac{1}{2} c_2^2 \right)^2 a^4 + o(a^4) \right).$$

Proof. Let us write u_a in terms of $\cos(x)$, $\cos(2x)$, ... We have

$$\begin{aligned} u_a = 1 + \frac{1}{2}c_0^2 a^2 + \frac{1}{4!}c_0^4 a^4 + a\cos(x) + \left(\frac{1}{2}c_2^2 a^2 + \frac{1}{4!}c_2^4 a^4 \right) \cos(2x) \\ + \frac{1}{3!}c_3^3 a^3 \cos(3x) + \frac{1}{4!}c_4^4 a^4 \cos(4x) + v_a(x), \end{aligned}$$

with $v_a = o(a^4)$. Applying $(-\Delta)^s$, one gets

$$\begin{aligned} (-\Delta)^s u_a = a\cos(x) + 2^{2s} \left(\frac{1}{2}c_2^2 a^2 + \frac{1}{4!}c_2^4 a^4 \right) \cos(2x) \\ + 3^{2s} \frac{1}{3!} c_3^3 a^3 \cos(3x) + 4^{2s} \frac{1}{4!} c_4^4 a^4 \cos(4x) + w_a(x), \end{aligned}$$

where $w_a(x) = (-\Delta)^s v_a(x) \in C^\alpha(\mathbb{R})$ and is such that $\|w_a\|_{C^\alpha} = o(a^4)$. This is so thanks to $\|v_a\|_{C^{2s+\alpha}} = o(a^4)$ and $\|w_a\|_{C^\alpha} = \|(-\Delta)^s v_a\|_{C^\alpha} \leq C\|v_a\|_{C^{2s+\alpha}}$ (see [13, Chapter 2]).

Now we multiply it by u_a and we integrate it. Using the orthogonality of the terms we get

$$\begin{aligned} \int_{-\pi}^{\pi} u_a(-\Delta)^s u_a &= \pi \left\{ a^2 + 2^{2s} \left(\frac{1}{2}c_2^2 a^2 + \frac{1}{4!}c_2^4 a^4 \right)^2 \right. \\ &\qquad \left. + 3^{2s} \left(\frac{1}{3!}c_3^3 a^3 \right)^2 + 4^{2s} \left(\frac{1}{4!}c_4^4 a^4 \right)^2 + o(a^4) \right\}. \\ &= \pi \left(a^2 + 2^{2s} \left(\frac{1}{2}c_2^2 \right)^2 a^4 + o(a^4) \right), \end{aligned}$$

which conclude the proof. $\square$

Lemma 5.3. *Under the setting of Theorem 1.2, let u_a be as in (49). Then,*

$$\begin{aligned} \lambda(a) \int_{-\pi}^{\pi} u_a^2 &= 2\pi\lambda(0) + \pi \Big(\left(1 + 2c_0^2\right) \lambda(0) + \lambda''(0) \Big) a^2 \\ &\quad + \pi \left(\lambda(0) \left(\frac{1}{3!}c_0^4 + \frac{1}{2}\left(c_0^2\right)^2 + \left(\frac{1}{2}c_2^2 \right)^2 \right) + \frac{1}{2}\lambda''(0)\left(1 + 2c_0^2\right) + \frac{2}{4!}\lambda^{(4)}(0) \right) a^4 + o(a^4). \end{aligned}$$

Proof. If we square u_a and then integrate it in $(-\pi,\pi)$, using again the orthogonality properties of $\{\cos(kx)\}_{k\geq 0}$ and their L^2-norms we find that

$$\begin{aligned}\int_{-\pi}^{\pi} u_a^2 &= \pi\left\{2\left(1+\frac{1}{2}c_0^2a^2+\frac{1}{4!}c_0^4a^4\right)^2 + a^2 + \left(\frac{1}{2}c_2^2a^2+\frac{1}{4!}c_2^4a^4\right)^2 + o(a^4)\right\}\\ &= \pi\left\{2\left(1+c_0^2a^2+\left(\frac{2}{4!}c_0^4+\left(\frac{1}{2}c_0^2\right)^2\right)a^4\right)+a^2+\left(\frac{1}{2}c_2^2\right)^2a^4+o(a^4)\right\}\\ &= \pi\left\{2+\left(1+2c_0^2\right)a^2+\left(\frac{1}{3!}c_0^4+\frac{1}{2}\left(c_0^2\right)^2+\left(\frac{1}{2}c_2^2\right)^2\right)a^4+o(a^4)\right\}.\end{aligned}$$

Moreover, thanks to Theorem 1.2 we have $a\mapsto\lambda(a)$ is an even function of class C^4 and thus Taylor's theorem yields

$$\lambda(a)=\lambda(0)+\frac{1}{2}\lambda''(0)a^2+\frac{1}{4!}\lambda^{(4)}(0)a^4+o(a^4).$$

Finally we multiply both terms. Arranging them in powers of a we obtain the result. □

Lemma 5.4. *Under the setting of Theorem 1.2, let u_a be as in (49). Then*

$$\left(\int_{-\pi}^{\pi}u_a(-\Delta)^su_a+\lambda(a)\int_{-\pi}^{\pi}u_a^2\right)^{p+1}=O_e^0+O_e^2a^2+O_e^4a^4+o(a^4),$$

with the coefficients

$$\begin{aligned}O_e^0 &= [2\pi\lambda(0)]^{p+1},\\ O_e^2 &= [2\pi\lambda(0)]^{p+1}\frac{p+1}{2\lambda(0)}\left[1+\left(1+2c_0^2\right)\lambda(0)+\lambda''(0)\right],\\ O_e^4 &= [2\pi\lambda(0)]^{p+1}\left\{\frac{p+1}{2\lambda(0)}\left[\frac{2}{4!}\lambda^{(4)}(0)+\frac{1}{2}\left(1+2c_0^2\right)\lambda''(0)\right.\right.\\ &\qquad\left.+\lambda(0)\left(\frac{1}{3!}c_0^4+\frac{1}{2}\left(c_0^2\right)^2+\left(\frac{1}{2}c_2^2\right)^2\right)+2^{2s}\left(\frac{1}{2}c_2^2\right)^2\right]\\ &\qquad\left.+\frac{1}{[2\lambda(0)]^2}\binom{p+1}{2}\left[1+\left(1+2c_0^2\right)\lambda(0)+\lambda''(0)\right]^2\right\}.\end{aligned}$$

Proof. Firstly, we use Lemma 5.2 and Lemma 5.3 to obtain

$$\int_{-\pi}^{\pi}u_a(-\Delta)^su_a+\lambda(a)\int_{-\pi}^{\pi}u_a^2=f_0+f_2a^2+f_4a^4+o\left(a^4\right),$$

with the coefficients

$$\begin{aligned}f_0 &= 2\pi\lambda(0),\\ f_2 &= \pi\left[1+\left(1+2c_0^2\right)\lambda(0)+\lambda''(0)\right],\\ f_4 &= \pi\left[\frac{2}{4!}\lambda^{(4)}(0)+\frac{1}{2}\left(1+2c_0^2\right)\lambda''(0)+\lambda(0)\left(\frac{1}{3!}c_0^4+\frac{1}{2}\left(c_0^2\right)^2+\left(\frac{1}{2}c_2^2\right)^2\right)+2^{2s}\left(\frac{1}{2}c_2^2\right)^2\right].\end{aligned}$$

Now, if we define

$$\begin{aligned}F(a) &:= \int_{-\pi}^{\pi}u_a(-\Delta)^su_a+\lambda(a)\int_{-\pi}^{\pi}u_a^2\\ &= f_0+f_2a^2+f_4a^4+o\left(a^4\right)\end{aligned}$$

then, using the binomial theorem for the real exponent $p+1$ (see [1]) and considering $\binom{p+1}{2} := \frac{p(p+1)}{2}$, we have that

$$
\begin{aligned}
F(a)^{p+1} &= f_0^{p+1} + (p+1) f_0^p f_2 a^2 + \left((p+1) f_0^p f_4 + \binom{p+1}{2} f_0^{p-1} f_2^2\right) a^4 + o\left(a^4\right) \\
&= [2\pi\lambda(0)]^{p+1} + \pi^{p+1}(p+1)\,[2\lambda(0)]^p \left[1 + \left(1+2c_0^2\right)\lambda(0) + \lambda''(0)\right] a^2 \\
&\quad + \pi^{p+1} \Bigg\{ (p+1)\,[2\lambda(0)]^p \Bigg[\frac{2}{4!}\lambda^{(4)}(0) + \frac{1}{2}\left(1+2c_0^2\right)\lambda''(0) \\
&\qquad\qquad + \lambda(0)\left(\frac{1}{3!}c_0^4 + \frac{1}{2}\left(c_0^2\right)^2 + \left(\frac{1}{2}c_2^2\right)^2\right) + 2^{2s}\left(\frac{1}{2}c_2^2\right)^2 \Bigg] \\
&\qquad \binom{p+1}{2} [2\lambda(0)]^{p-1} \left[1 + \left(1+2c_0^2\right)\lambda(0) + \lambda''(0)\right]^2 \Bigg\} a^4 + o\left(a^4\right) \\
&= O_e^0 + O_e^2 a^2 + O_e^4 a^4 + o\left(a^4\right),
\end{aligned}
$$

which concludes the proof. $\square$

Lemma 5.5. *Under the setting of Theorem 1.2, let u_a be as in (49). Then,*

$$
\begin{aligned}
\left(\|u_a\|_{L^{p+1}}^{p+1}\right)^2 &= 4\pi^2 + 4\pi^2\left[\binom{p+1}{2} + (p+1)c_0^2\right] a^2 \\
&\quad + \pi^2 \Bigg[4\left((p+1)\frac{2}{4!}c_0^4 + \frac{1}{4}\binom{p+1}{2}\left(2\left(c_0^2\right)^2 + \left(c_2^2\right)^2\right)\right) \\
&\qquad + \left(\binom{p+1}{2} + (p+1)c_0^2\right)^2 \Bigg] a^4 + o\left(a^4\right).
\end{aligned}
$$

Proof. Let us remember that u_a is of the form

$$
\begin{aligned}
u_a(x) &= 1 + a\cos(x) + \frac{1}{2!}\left(c_0^2 + c_2^2\cos(2x)\right) a^2 \\
&\quad + \frac{1}{3!}c_3^3\cos(3x) a^3 + \frac{1}{4!}\left(c_0^4 + c_2^4\cos(2x) + c_4^4\cos(4x)\right) a^4 + v_a(x),
\end{aligned}
$$

with $v_a \in C^{2s+\alpha}(\mathbb{R})$ and $\|v_a\|_{C^{2s+\alpha}} = o(a^4)$. Using the binomial theorem, we have that

$$
\begin{aligned}
(u_a(x))^{p+1} &= 1 + (p+1)a\cos(x) + \left[\binom{p+1}{2}\cos^2(x) + (p+1)\frac{1}{2!}\left(c_0^2 + c_2^2\cos(2x)\right)\right] a^2 \\
&\quad + \left[(p+1)\frac{1}{3!}c_3^3\cos(3x) + \binom{p+1}{2}\cos(x)\frac{1}{2!}\left(c_0^2 + c_2^2\cos(2x)\right)\right] a^3 \\
&\quad + \Bigg[(p+1)\frac{1}{4!}\left(c_0^4 + c_2^4\cos(2x) + c_4^4\cos(4x)\right) \\
&\qquad + \binom{p+1}{2}\left(\frac{1}{2!}\left(c_0^2 + c_2^2\cos(2x)\right)\right)^2 + \binom{p+1}{2}\frac{1}{3!}c_3^3\cos(x)\cos(3x)\Bigg] a^4 + \widetilde{v}_a,
\end{aligned}
$$

with $\widetilde{v}_a \in C^{2s+\alpha}(\mathbb{R})$ and $\|\widetilde{v}_a\|_{C^{2s+\alpha}} = o(a^4)$. Now we integrate it using the orthogonal properties of $\{\cos(kx)\}_{k\geq 0}$ in order to obtain

$$
\begin{aligned}
\int_{-\pi}^{\pi} u_a^{p+1} &= 2\pi + \pi\left[\binom{p+1}{2} + (p+1)c_0^2\right] a^2 \\
&\quad + \pi\left[(p+1)\frac{2}{4!}c_0^4 + \frac{1}{4}\binom{p+1}{2}\left(2\left(c_0^2\right)^2 + \left(c_2^2\right)^2\right)\right] a^4 + o\left(a^4\right).
\end{aligned}
$$

Finally, we square it and we arrange the terms in orders of a to obtain the result. $\square$

Lemma 5.6. *Under the setting of Theorem 1.2, let $u_0 \equiv 1$. Then,*

$$
\begin{aligned}
\left(E_{L(a)}(u_0)\right)^{p+1} &= (2\pi)^{p-1}\,\lambda(0)^{p+1} + (2\pi)^{p-1}\,\frac{p+1}{2}\lambda(0)^p\lambda''(0)a^2 \\
&\quad + (2\pi)^{p-1}\left[\frac{p+1}{4!}\lambda(0)^p\lambda^{(4)}(0) + \binom{p+1}{2}\lambda(0)^{p-1}\left(\frac{1}{2}\lambda''(0)\right)^2\right]a^4 + o\left(a^4\right).
\end{aligned}
$$

Proof. We know that $E_{L(a)}(u_0) = \lambda(a)\,(2\pi)^{\frac{p-1}{p+1}}$ and we recall the Taylor expansion for $\lambda(a)$, namely

$$
\lambda(a) = \lambda(0) + \frac{1}{2}\lambda''(0)a^2 + \frac{1}{4!}\lambda^{(4)}(0)a^4 + o(a^4).
$$

Then, we just raise $\lambda(a)\,(2\pi)^{\frac{p-1}{p+1}}$ to the power $p+1$ using the binomial theorem and we arrange the terms in orders of a as we have done in the proof of Lemma 5.4. $\square$

Lemma 5.7. *Under the setting of Theorem 1.2, let u_a be as in (49). Then,*

$$
\left(E_{L(a)}(u_0)\right)^{p+1}\left(\|u_a\|_{L^{p+1}}^{p+1}\right)^2 = O_d^0 + O_d^2a^2 + O_d^4a^4 + o\left(a^4\right)
$$

with the coefficients

$$
\begin{aligned}
O_d^0 &= [2\pi\lambda(0)]^{p+1}\,, \\
O_d^2 &= [2\pi\lambda(0)]^{p+1}\left\{(p+1)\frac{\lambda''(0)}{2\lambda(0)} + \binom{p+1}{2} + (p+1)c_0^2\right\}, \\
O_d^4 &= [2\pi\lambda(0)]^{p+1}\left\{(p+1)\frac{\lambda''(0)}{2\lambda(0)}\left[\binom{p+1}{2} + (p+1)c_0^2\right] + (p+1)\frac{2}{4!}c_0^4\right. \\
&\qquad + \frac{1}{4}\binom{p+1}{2}\left[2\left(c_0^2\right)^2 + \left(c_2^2\right)^2\right] + \frac{1}{4}\left[\binom{p+1}{2} + (p+1)c_0^2\right]^2 \\
&\qquad \left. +(p+1)\frac{\lambda^{(4)}(0)}{4!\lambda(0)} + \binom{p+1}{2}\left(\frac{\lambda''(0)}{2\lambda(0)}\right)^2\right\}.
\end{aligned}
$$

Proof. Notice that $\left(\|u_a\|_{L^{p+1}}^{p+1}\right)^2$ and $\left(E_{L(a)}(u_0)\right)^{p+1}$ are given by Lemma 5.5 and Lemma 5.6 respectively, we multiply them and we arrange the terms in orders of a. $\square$

Now, let us define the quantities

$$
\begin{aligned}
\mathbf{D}_a &= \left(E_{L(a)}(u_0)\right)^{p+1}\left(\|u_a\|_{L^{p+1}}^{p+1}\right)^2, \\
\mathbf{E}_a &= \left(\int_{-\pi}^{\pi} u_a(-\Delta)^s u_a + \lambda(a)\int_{-\pi}^{\pi} u_a^2\right)^{p+1}.
\end{aligned}
$$

In order to see if (48) holds, we shall see $0 < \mathbf{D}_a - \mathbf{E}_a$. The following lemma gives us an expression for this difference.

Lemma 5.8. *Let $C_p := [2\pi\lambda(0)]^{p+1} > 0$ and $f(u) = -u + u^p$, for some $p > 1$ such that $f \in C^6(\mathbb{R})$. Then,*

$$
\mathbf{D}_a - \mathbf{E}_a = \frac{C_p}{8}p(p+1)(p-1)\left\{1 + \left(1 - \frac{3}{4(2^{2s}-1)}\right)p\right\}a^4 + o\left(a^4\right).
$$

Proof. Note that we can use Theorem 1.2 and all the previous Lemmas for $f(u) = -u + u^p$ with $p > 1$ such that $f \in C^6(\mathbb{R})$. We will study the difference $\mathbf{D}_a - \mathbf{E}_a$ using the expressions we have thanks to Lemma 5.4 and Lemma 5.7. Moreover, for $f(u) = -u + u^p$ and the expressions for $\lambda''(0)$ in (54) and c_0^2, c_2^2 in (52) we now have

$$\begin{aligned}
\lambda(0) &= \frac{1}{p-1}, \\
\lambda''(0) &= \frac{1}{p-1}\left[-\frac{p(p-2)}{4} + \left(\frac{1}{2} - \frac{1}{4}\frac{1}{2^{2s}-1}\right)p^2\right], \\
c_0^2 &= -\frac{p}{2} \Longrightarrow \binom{p+1}{2} + (p+1)c_0^2 = 0 \text{ and } 1 + \left(1 + 2c_0^2\right)\lambda(0) = 0, \\
c_2^2 &= \frac{p}{2}\frac{1}{2^{2s}-1} \Longrightarrow \left(c_2^2\right)^2 = \frac{p^2}{4}\frac{1}{\left(2^{2s}-1\right)^2}.
\end{aligned} \tag{55}$$

Let us begin by writing

$$\mathbf{D}_a - \mathbf{E}_a = \left(O_d^0 - O_e^0\right) + \left(O_d^2 - O_e^2\right)a^2 + \left(O_d^4 - O_e^4\right)a^4 + o\left(a^4\right),$$

we easily see that the zero order term vanishes, so let us now study the second order term. We have

$$\begin{aligned}
O_d^2 - O_e^2 &= C_p\left((p+1)\frac{\lambda''(0)}{2\lambda(0)} + \binom{p+1}{2} + (p+1)c_0^2 - \frac{p+1}{2\lambda(0)}\left[1 + \left(1+2c_0^2\right)\lambda(0) + \lambda''(0)\right]\right) \\
&= 0,
\end{aligned}$$

and so the second order term also vanishes. Hence, we investigate the fourth order term.

$$\begin{aligned}
O_d^4 - O_e^4 = C_p \Bigg\{ & (p+1)\frac{\lambda''(0)}{2\lambda(0)}\left[\binom{p+1}{2} + (p+1)c_0^2\right] + (p+1)\frac{2}{4!}c_0^4 + \frac{1}{4}\binom{p+1}{2}\left[2\left(c_0^2\right)^2 + \left(c_2^2\right)^2\right] \\
& + \frac{1}{4}\left[\binom{p+1}{2} + (p+1)c_0^2\right]^2 + \frac{p+1}{4!\lambda(0)}\lambda^{(4)}(0) + \binom{p+1}{2}\left(\frac{\lambda''(0)}{2\lambda(0)}\right)^2 \\
& - \frac{1}{\left[2\lambda(0)\right]^2}\binom{p+1}{2}\left[1 + \left(1+2c_0^2\right)\lambda(0) + \lambda''(0)\right]^2 - \frac{p+1}{2\lambda(0)}\Bigg[\frac{2}{4!}\lambda^{(4)}(0) \\
& \quad + \frac{1}{2}\left(1+2c_0^2\right)\lambda''(0) + \lambda(0)\left(\frac{1}{3!}c_0^4 + \frac{1}{2}\left(c_0^2\right)^2 + \left(\frac{1}{2}c_2^2\right)^2\right) + 2^{2s}\left(\frac{1}{2}c_2^2\right)^2\Bigg]\Bigg\}.
\end{aligned}$$

Note that we are able to cancel out both $\lambda^{(4)}(0)$ and c_0^4. Grouping the other terms,

$$
\begin{aligned}
O_d^4 - O_e^4 &= C_p \Bigg\{ \frac{p}{4}(p+1)\left(c_0^2\right)^2 + \frac{p}{8}(p+1)\left(c_2^2\right)^2 + p\frac{p+1}{2}\left(\frac{\lambda''(0)}{2\lambda(0)}\right)^2 - \frac{p+1}{2}\left(1+2c_0^2\right)\frac{\lambda''(0)}{2\lambda(0)} \\
&\qquad -\frac{p+1}{4}\left(c_0^2\right)^2 - \frac{p+1}{2}\left(\frac{1}{2}c_2^2\right)^2 - \frac{p+1}{2\lambda(0)}2^{2s}\left(\frac{1}{2}c_2^2\right)^2 - p\frac{p+1}{2}\left(\frac{\lambda''(0)}{2\lambda(0)}\right)^2 \Bigg\} \\
&= C_p \Bigg\{ \frac{1}{4}(p+1)(p-1)\left(c_0^2\right)^2 + \frac{1}{8}(p+1)(p-1)\left(c_2^2\right)^2 \\
&\qquad -\frac{p+1}{4}\left(1+2c_0^2\right)\frac{\lambda''(0)}{\lambda(0)} - \frac{p+1}{2\lambda(0)}2^{2s}\left(\frac{1}{2}c_2^2\right)^2 \Bigg\} \\
&= C_p \Bigg\{ \frac{1}{16}p^2(p+1)(p-1) + \frac{1}{8}(p+1)(p-1)\left(c_2^2\right)^2 \\
&\qquad +\frac{1}{4}(p+1)(p-1)\frac{\lambda''(0)}{\lambda(0)} - \frac{1}{8}(p+1)(p-1)2^{2s}\left(c_2^2\right)^2 \Bigg\} \\
&= \frac{C_p}{4}(p+1)(p-1)\left\{\frac{p^2}{4} - \frac{1}{2}(2^{2s}-1)\left(c_2^2\right)^2 + \frac{\lambda''(0)}{\lambda(0)}\right\}.
\end{aligned}
$$

Finally we write the values for c_2^2 and $\lambda''(0)$ given in (52) and (54) respectively, to get

$$
\begin{aligned}
O_d^4 - O_e^4 &= \frac{C_p}{4}(p+1)(p-1)\left\{\frac{p^2}{4} - \frac{p^2}{8}\frac{1}{2^{2s}-1} - \frac{p(p-2)}{4} + \left(\frac{1}{2} - \frac{1}{4\left(2^{2s}-1\right)}\right)p^2\right\} \\
&= \frac{C_p}{4}(p+1)(p-1)\left\{\frac{p^2}{4} - \frac{p^2}{8}\frac{1}{2^{2s}-1} - \frac{p^2}{4} + \frac{p}{2} + \frac{p^2}{2} - \frac{p^2}{4}\frac{1}{2^{2s}-1}\right\} \\
&= \frac{C_p}{4}(p+1)(p-1)\left\{\frac{p^2}{2} - \frac{3}{8}\frac{p^2}{2^{2s}-1} + \frac{p}{2}\right\} \\
&= \frac{C_p}{8}p(p+1)(p-1)\left\{1 + (1 - \frac{3}{4(2^{2s}-1)})p\right\},
\end{aligned}
$$

and thus we conclude the proof. $\square$

We have just seen that

$$
\begin{aligned}
\mathbf{D}_a - \mathbf{E}_a &= \frac{C_p}{8}p(p+1)(p-1)\left\{1 + (1 - \frac{3}{4(2^{2s}-1)})p\right\}a^4 + o(a^4) \\
&= \frac{C_p}{8}p(p+1)(p-1)Q(p)a^4 + o(a^4),
\end{aligned}
$$

for $Q(p) := 1 + \big(1 - \frac{3}{4(2^{2s}-1)}\big)p$. Since $\frac{C_p}{8}p(p+1)(p-1) > 0$ for $p > 1$, the sign of $\mathbf{D}_a - \mathbf{E}_a$ will be that of $Q(p)$, for a small enough. With this we are finally able to give the proof of Theorem 1.3.

Proof of Theorem 1.3. As we have previously said, if (48) holds for some $u \in H^s_{\mathrm{ep}}$ and some $L < L_0$ then we will have that $L_* < L_0$. Using the bifurcated solution $(\lambda(a), u_a)$, we have seen that, for $u = u_a$ and $L = L(a)$, (48) holds whenever $\mathbf{D}_a - \mathbf{E}_a > 0$. Using Lemma 5.8, this will be the case whenever $Q(p) > 0$, for a small enough (and $f(u) = -u + u^p$ is of class $C^6(\mathbb{R})$).

Note that $Q(p) = 1 + \big(1 - \frac{3}{4(2^{2s}-1)}\big)p$ is a polynomial of degree one. Now, it can easily be seen that for $0 < s < \frac{1}{2}\frac{\ln(7/4)}{\ln 2}$ and $1 < p < \frac{4(2^{2s}-1)}{3-4(2^{2s}-1)}$ we have $Q(p) > 0$. Similarly, we also have $Q(p) > 0$ for $s \geq \frac{1}{2}\frac{\ln(7/4)}{\ln 2}$ and $1 < p < +\infty$. This shows (i).

Moreover, for $0 < s < 1/2$ we have that $\lambda''(0)$ given by (55) is such that $\lambda''(0) < 0$, whenever $\frac{2(2^{2s}-1)}{2-2^{2s}} < p$. This means that the period decreases initially with the amplitude, that is, $L(a) < L(0) = L_0$ for a small enough.

Then, we have $L_* < L_0$ if $p < \frac{1+2s}{1-2s}$ (to ensure L_* is well defined) and both $\mathbf{D}_a - \mathbf{E}_a > 0$ and $\lambda''(0) < 0$ hold, which take place exactly for the regions that are precisely described in (ii). $\square$

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