MASTER'S THESIS

Master of Science in Advanced Mathematics and Mathematical Engineering

Title: Numerical Simulation of worm-like motion

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In this thesis there have been a lot of people helping emotionally that without their support I would not have been able to complete it, so I would like to thank:

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Abstract

C. Elegans is a nematode worm which is a very popular organism used in all kinds of research in biology. It is widely used because its complete genome has been completely determined, as well as each of the cells of its body and the worms are transparent, permitting a variety of structural investigations.

In this project we will focus in the mechanics of the organisms that have a similar shape in their bodies, which are called worm-like organisms. This organisms are characterized by its sinusoidal shape and movement, and they can be analyzed in a simple one-dimensional model.

We will study with more detail from a theoretical and a practical point of view the different forces that act in the system and which characteristics allows the movement of the organism. We will develop a code in Matlab to have a better understanding of the forces and check our theoretical analysis.

Keywords

C.Elegans, Biological Model, Mechanics, Numerical Methods, Worm-like organism

Contents

1. Introduction

Motion is a short but powerful word in this world we live in. We can find motion from big objects like planets and galaxies to small ones like atoms. We even can find them in the amazing life around us, like birds flying in the sky, cats walking in a street or fish swimming in the sea. We also can find it in a microscopic level, like protozoos or bacteria trying to find new nutrients exploring the environment.

It is in this microorganism world where we will focus this study. In this microscopic level, the bacteria and rest of living creatures swim in a very low Reynolds number, which produces some interesting facts such as if we have a reciprocal motion, the organism stays in the same place, due to the negligible inertia as it can be read in more detail in [1]. Hence, this microorganism have adopted different types of locomotion that allow them to swim. In [2] we can see some different types of locomotion, but with these we can see that it is not trivial to study the motion of all organisms, because each species have developed different strategies to move.

Because of the complexity of each of this motions, we have to study each case thoughtfully using the model which is more accurate in each case. We have to choose carefully which aspects of the motion are more important to model in a simpler manner without forgetting important aspects. In [3] we can see different models of important aspects of the biology and the importance of choosing one model over another one when we are studying the same object.

In this thesis, we will study the movement of a certain case of organisms: the worm like organisms. For that we will assume that we have a one-dimensional rod representing the organism discretized in nodes in a two-dimensional environment (we will not allow deformations in the third dimension), so the motion will be restricted to a plane.

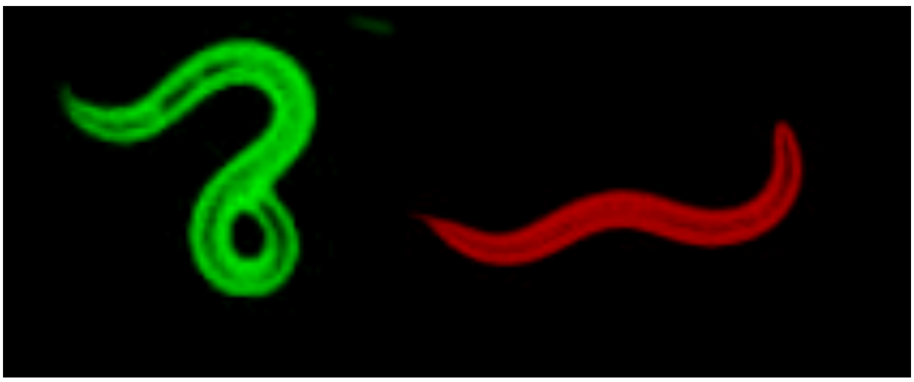


Figure 1: Two deformed Caenorhabditis elegans (C. elegans) worm stained and under different mutations. Courtesy of Michael Krieg's laboratory at Institut fo Fotonic Sciences (ICFO) [6]

1.1 Goal of the thesis

The main purpose of this thesis is to study the different forces that have an important outcome in the shape and deformation of the rod and to study the properties of the rod and the environment that allow the motion of its center of mass.

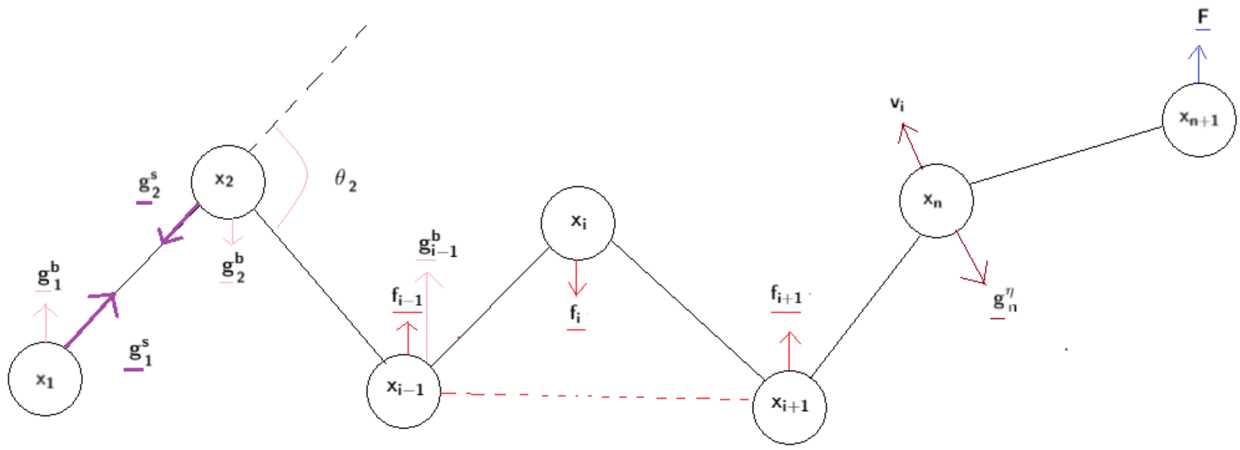


Figure 2: Rod discretized with all the forces we will take into account

In Figure 2 we show a representation of all the forces that we will study in this thesis. The forces $\mathbf{g}^b$ are the bending forces, the forces $\mathbf{g}^s$ are the stretching forces, the blue $\mathbf{F}$ is the external force of the system, the $\mathbf{g}^\eta$ are the viscous forces and the red $\mathbf{f}_{i-1}$, $\mathbf{f}_i$ and $\mathbf{f}_{i+1}$ are the forces in which each active moment M_i of a node $\mathbf{x_i}$ is decomposed. We also show in Figure 2 the velocity $\mathbf{v_i}$ of a node and the angle θ_i that forms a segment with the adjacent one (which we will use for the bending forces).

1.2 Outline of the thesis

At the beginning we will study the elastic forces, which will be the bending and stretching of the segments in which we will divide the rod. We will apply external forces and we will put some boundary conditions to see different deformations.

Later, we will assume that the organism is in a viscous environment, so we will have to add to the rest of forces a viscous force, which will be proportional to the velocity in which the rod is deforming and against it. In this case we will see that we would be able to erase the boundary conditions and the rod will be more free to move.

Finally, we will also apply some internal moments which represent the movement that the organism produce to move. We will study its deformations and shape and we will use different types of moment distribution along the nodes of the rod to see in which conditions the rod changes its center of mass and in which conditions it does not.

2. Methodology

We will divide the rod into n segments (hence we will have $n+1$ nodes). We will also assume that the discretized rod has initial total length $L = 1$.

First we will study the bending and the stretching forces. Both are elastic forces which come from elastic potentials energies which we will have to minimize to find the equilibrium of forces. The total elastic energy is:

$$W^{el} = W^b + W^s \tag{1}$$

and the balance equations that we are going to solve are:

$$\frac{dW^{el}}{d\mathbf{x}_i} = \mathbf{g}_i^{\eta} + \mathbf{g}_i^{m} + \mathbf{f}_i^{ext} \text{ , for } i = 1, ..., n+1 \tag{2}$$

where $\mathbf{g}_i^{\eta}$ is the viscous force, $\mathbf{g}_i^{m}$ is the force produced at node i by active moments and $\mathbf{f}_i^{ext}$ is the external force, all of them applied at node $\mathbf{x}_i$. In figure 3 we represented the elastic forces.

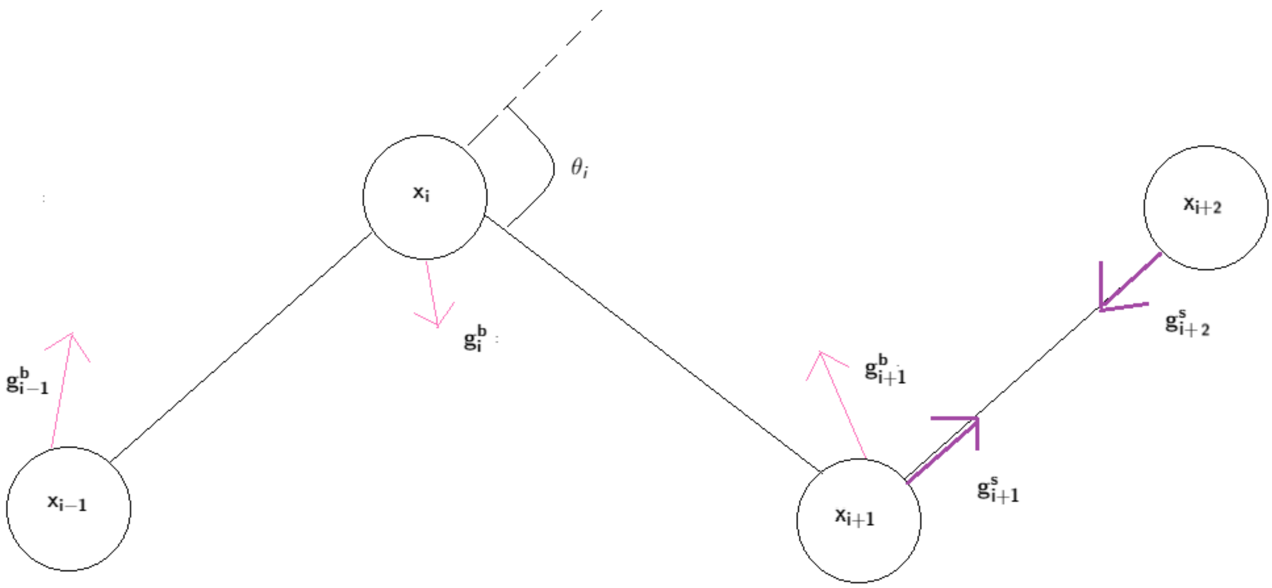


Figure 3: Elastic forces in the rod

Later, we will also take into account viscous forces, which are proportional to the velocity in which the rod is deforming and against it. In that section we will separate the forces in the case that we have an isotropic friction environment or the rod have different frictions with the environment.

Finally we will study the internal moments that the rod produce to change its shape. We will find the explicit equations and we will study the conditions under which the rod changes its center of mass or not.

2.1 Bending Forces

In this subsection we will assume the simplest case that we can have: we subject the rod to an external constant force $\mathbf{F}$ and we only take into account as interior forces the elastic force due to the bending of the row.

We will assume that the elastic bending force comes from the following potential:

$$W^b = \frac{1}{2} k_b \sum_{i=2}^{n} \theta_i^2 \tag{3}$$

where k_b is the elastic bending constant and θ_i is the angle that form the line that goes through $\mathbf{x_{i-1}}$ and $\mathbf{x_i}$ with the line that goes through $\mathbf{x_i}$ and $\mathbf{x_{i+1}}$.

We have chosen this potential because if the three nodes are aligned we have that the angle is 0 and therefore the elastic energy is also 0, while if we increase the bending of the system, the angle increases and with it the elastic potential. However, we could have chosen a different elastic bending potential and have a different bending force.

As an additional note, it is important to notice that we exclude the segments of the rod ($i = 1$ and $i = n + 1$) because the end nodes cannot form a segment with an adjacent segment.

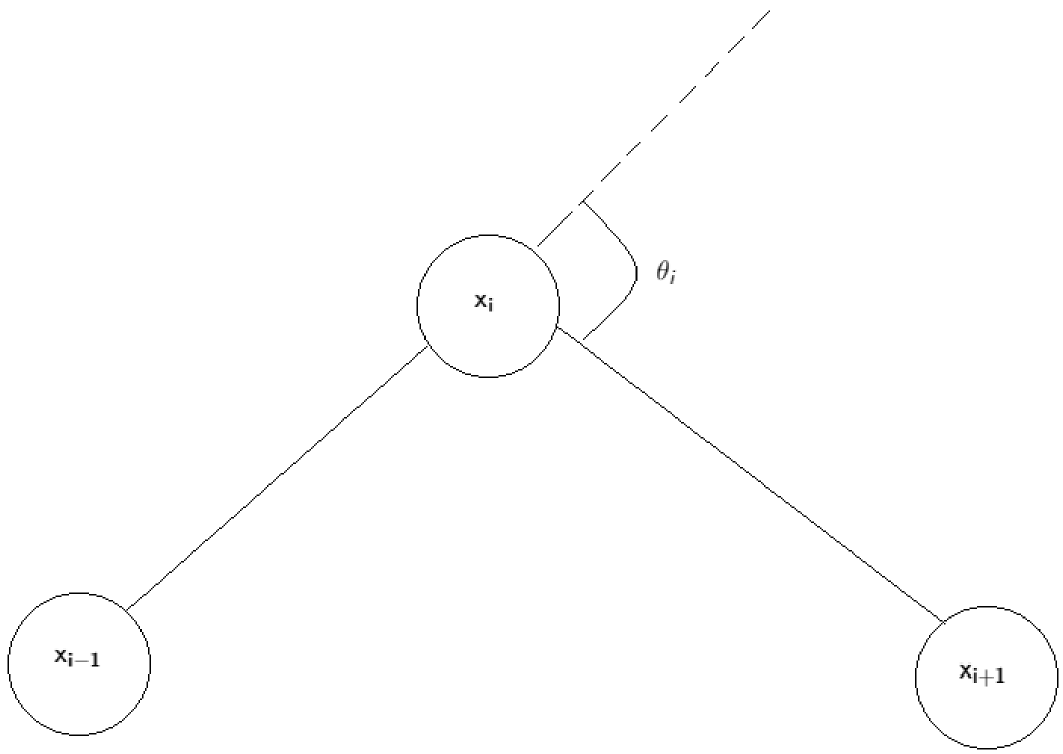


Figure 4: System of three nodes with the angle θ_i

Returning to the elastic potential, we will use an approximation of θ_i to have easier calculations, which is the following one:

$$\theta_i \approx sin(\theta_i) = \frac{(\mathbf{x_{i+1}} - \mathbf{x_i}) \wedge (\mathbf{x_i} - \mathbf{x_{i-1}})}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\| \; \|\mathbf{x_i} - \mathbf{x_{i-1}}\|} \tag{4}$$

This approximation is good enough because if we do not have small angles, we can use a larger number of nodes in the discretization to obtain it.

We even can go further and consider another approximation, that is that the nodes are equidistant and that the segments have norm:

$$\|\mathbf{x_i} - \mathbf{x_{i-1}}\| \approx \frac{L}{n} \tag{5}$$

Therefore, we obtain the approximation:

$$\theta_i \approx \frac{n^2}{L^2}(\mathbf{x_{i+1}} - \mathbf{x_i}) \wedge (\mathbf{x_i} - \mathbf{x_{i-1}}) = \frac{n^2}{L^2}(\mathbf{x_{i+1}} - \mathbf{x_i})^t \; \mathbf{J_2} \; (\mathbf{x_i} - \mathbf{x_{i-1}}) \tag{6}$$

where $\mathbf{J_2}$ is the matrix:

$$\mathbf{J_2} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}$$

Furthermore, we can rewrite it as:

$$\theta_i \approx \frac{1}{2} \; \frac{n^2}{L^2} \; (\mathbf{X}^i)^t \; \mathbf{J_6} \; \mathbf{X}^i \tag{7}$$

where $\mathbf{J_6}$ is the matrix:

$$\mathbf{J_6} = \begin{bmatrix} \mathbf{0} & -\mathbf{J_2} & \mathbf{J_2} \\ \mathbf{J_2} & \mathbf{0} & -\mathbf{J_2} \\ -\mathbf{J_2} & \mathbf{J_2} & \mathbf{0} \end{bmatrix}$$

and $\mathbf{X}^i$ is the vector:

$$\mathbf{X}^i = \begin{Bmatrix} \mathbf{x_{i-1}} \\ \mathbf{x_i} \\ \mathbf{x_{i+1}} \end{Bmatrix}$$

Therefore, we have an expression of the elastic potential that we will use:

$$W^b = \frac{1}{2} k_b \sum_{i=2}^{n} \left(\frac{1}{2} \; \frac{n^2}{L^2} \; (\mathbf{X}^i)^t \; \mathbf{J_6} \; \mathbf{X}^i \right)^2 \tag{8}$$

Now that we have the potential, we will compute the total elastic force $\mathbf{g^b}$, which is the assembly of the joint contribution:

$$\mathbf{g_i^b} = k_b \ \theta_i \ \frac{\partial \theta_i}{\partial \mathbf{X^i}} \tag{9}$$

where

$$\frac{\partial \theta_i}{\partial \mathbf{X_i}} = \begin{Bmatrix} \dfrac{\partial \theta_i}{\partial \mathbf{x_{i-1}}} \\ \\ \dfrac{\partial \theta_i}{\partial \mathbf{x_i}} \\ \\ \dfrac{\partial \theta_i}{\partial \mathbf{x_{i+1}}} \end{Bmatrix} = \frac{n^2}{L^2} \mathbf{J_6} \mathbf{X}^i \tag{10}$$

Therefore, with the assembly of the values of $\mathbf{g_i}$ for all the interior nodes, we arrive to the total elastic force $\mathbf{g^b}$. The linearization of this force necessary to solve the non-linear system in (2) can be found in Appendix A.1.

2.2 Stretching Forces

In this subsection we describe the forces of stretching that come from a potential:

$$W^s = \frac{1}{2} k_s \sum_{e=1}^{n} \frac{(l_e - l_0)^2}{l_0^2} = \frac{1}{2} k_s \sum_{i=1}^{n} \frac{(\|\mathbf{x_{i+1}} - \mathbf{x_i}\| - l_0)^2}{l_0^2} \tag{11}$$

with l_0 being the original size of the segment (we assume initial equidistant nodes, so it will be the same value for all the nodes) and k_s being a elastic constant due to stretching.

Hence, the forces are:

$$\mathbf{g_i^s} = \frac{\partial W^s}{\partial \mathbf{X}^i} = k_s \frac{\|\mathbf{x_{i+1}} - \mathbf{x_i}\| - l_0}{l_0^2} \begin{Bmatrix} -\mathbf{e_i} \\ \mathbf{e_i} \end{Bmatrix} \tag{12}$$

with $\mathbf{X}^i$ being

$$\mathbf{X}^i = \begin{Bmatrix} \mathbf{x_i} \\ \mathbf{x_{i+1}} \end{Bmatrix}$$

and $\mathbf{e_i} = \dfrac{\mathbf{x_{i+1}} - \mathbf{x_i}}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}$.

The linearization of this force contribution, necessary for the numerical solution of the non-linear equations in (2) is given in Appendix A.2.

2.3 Viscous forces

We will now assume that the rod is placed in a viscous environment. Therefore, we also have to take into account forces due to the interaction of the rod with the fluid around it or the substrate below it. It is important to remark that when using viscous forces, the problem becomes well-posed even if no Dirichlet conditions are employed. Mathematically, the viscous terms can be considered as a regularization term in the resulting Jacobian matrix.

We will divide this subsection in two cases: i) The interaction with the environment has an isotropic viscosity, and ii) The rod have a different viscosity depending on the tangential and normal movement of the rod.

2.3.1. Isotropic viscosity

This force will be in each node of the form:

$$\mathbf{g_i^\eta} = -\eta\ \mathbf{v_i} \tag{13}$$

with η being the viscosity parameter and $\mathbf{v_i}$ being the velocity of the node $\mathbf{x_i}$.

Hence, with the rest of forces, the balance equation in (2) becomes:

$$\mathbf{g^b} + \mathbf{g^s} = \mathbf{f} + \mathbf{g}^\eta \tag{14}$$

Replacing the expression of $\mathbf{g_i}^\eta$ from (13), we obtain:

$$-\eta\ \frac{d\mathbf{x}}{dt} = \mathbf{g^b} + \mathbf{g^s} - \mathbf{f} \tag{15}$$

which is an ODE of 1st order. To solve it, we will use Backward Euler (to avoid problems of stability):

$$\mathbf{x^{k+1}} = \mathbf{x^k} - \frac{h}{\eta}\left(\mathbf{g^b}(\mathbf{x^{k+1}}) + \mathbf{g^s}(\mathbf{x^{k+1}}) - \mathbf{f}\right) \tag{16}$$

with $\mathbf{x^0} = \mathbf{X}(\mathbf{0})$, the initial position of the nodes and h is the time-step.

We can also rewrite it as:

$$\mathbf{g^b} + \mathbf{g^s} - \mathbf{g}^\eta = \mathbf{f} \tag{17}$$

with $\mathbf{g}^\eta = \sum_{i=1}^{n} -\eta\ \frac{\mathbf{x_i^{k+1}} - \mathbf{x_i^k}}{h}$.

Now, for computing Newton-Raphson as before, we have to compute the Jacobian of the viscous force, which is:

$$\mathbf{K}^\eta = -\frac{\eta}{h}\ \mathbf{Id} \tag{18}$$

Hence, the Jacobian matrix of $\mathbf{g}(\mathbf{x^k}) = \mathbf{g^b} + \mathbf{g^s} - \mathbf{g}^\eta - \mathbf{f}$ is:

$$\mathbf{K} = \mathbf{K^b} + \mathbf{K^s} - \mathbf{K}^\eta \tag{19}$$

2.3.2. Non-Isotropic viscosity

Similar as before, because we have a different force in each node which is proportional to the velocity, but in this case we separate the velocity of the node in a tangential and a normal part:

$$\mathbf{g}_\mathbf{i}^\eta = -(\eta_t\ \mathbf{v_t} + \eta_n\ \mathbf{v_n}) \tag{20}$$

with $\mathbf{v_t}$ and $\mathbf{v_n}$ being the tangential and normal velocity of the node, each one multiplied by a different viscosity constant.

In this case, however, we do not have such a simple expression of the $\mathbf{K}^\eta$ as before. To compute it, we have to study separately the tangential and normal velocity of the nodes according if they are interior nodes or external nodes:

1. *Interior Nodes*

   For interior nodes, we divide the node velocity $\mathbf{v_i}$ into a tangential part and a normal part projecting the velocity into a normal vector $\mathbf{n}$:

$$\begin{aligned} \mathbf{v_n} &= (\mathbf{v_i} \cdot \mathbf{n})\ \mathbf{n} \\ \mathbf{v_t} &= \mathbf{v_i} - \mathbf{v_n} \end{aligned} \tag{21}$$

   where $\mathbf{n} = \mathbf{J}\ \mathbf{e_3}$ with $\mathbf{e_3} = \dfrac{\mathbf{x_{i+1}} - \mathbf{x_{i-1}}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}$.

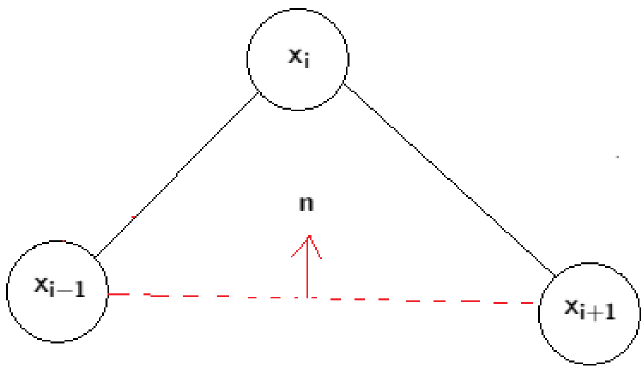


Figure 5: System of three nodes with the normal vector defined

2. *Exterior Nodes*

For the first and last node, we will project the velocity in the tangential part defined by the segment of the adjacent node and define the normal part as the substraction of that from the velocity:

$$\begin{aligned} \mathbf{v_t} &= (\mathbf{v_i} \cdot \mathbf{t})\, \mathbf{t} \\ \mathbf{v_n} &= \mathbf{v_i} - \mathbf{v_n} \end{aligned} \tag{22}$$

where $\mathbf{t} = \dfrac{\mathbf{x_{i+1}} - \mathbf{x_i}}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}$

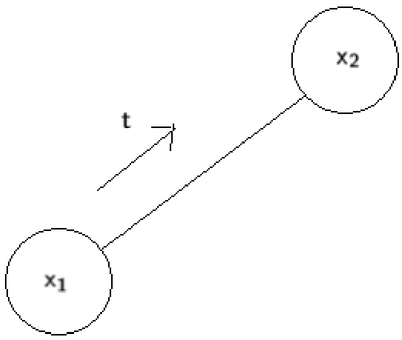


Figure 6: First node with tangential vector **t**

With this tangential and normal forces, we would be able to find the $\mathbf{K}^{\eta}$ assembling in each of the nodes. And with that, we would use Newton-Raphson to solve the non-linear system. The expression of the Jacobian is explicitely given in Appendix A.3.

2.4 Active moment forces

Now we will also impose active moments in the nodes of the rod. These represent the muscle activity that the worm-like organism exerts.

Given a scalar measure of the moment M_i at a node i, here we are going to derive the explicit equations of the three equivalent forces that we will apply in 3 consecutively nodes. We can see this forces in figure 7.

In each configuration of 3 nodes, $\mathbf{x_{i-1}}$, $\mathbf{x_i}$ and $\mathbf{x_{i+1}}$, we will apply forces, denoted respectively $\mathbf{f_{i-1}}$, $\mathbf{f_i}$ and $\mathbf{f_{i+1}}$, due to the active moment M_i. The sum of the three forces has to be equal to zero, so that no net force is actually applied. Therefore in each system of 3 consecutively nodes, we have that:

$$\mathbf{f_{i+1}} + \mathbf{f_i} + \mathbf{f_{i-1}} = \mathbf{0} \tag{23}$$

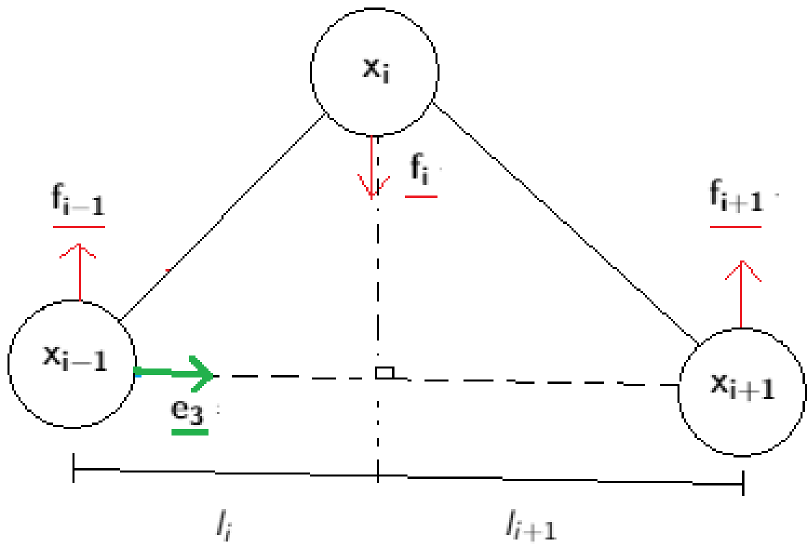


Figure 7: Active forces applied on 3 consecutive nodes, $\mathbf{x_{i-1}}$, $\mathbf{x_i}$ and $\mathbf{x_{i+1}}$, and equivalent to the active moment M_i at node $\mathbf{x_i}$.

We also impose that the sum of all the static moments at node **i** in the system of 3 consecutively nodes have to be 0 (otherwise we would induce an external net rotation). Therefore:

$$(\mathbf{x_{i-1}} - \mathbf{x_i}) \times \mathbf{f_{i-1}} + (\mathbf{x_{i+1}} - \mathbf{x_i}) \times \mathbf{f_{i+1}} = 0 \tag{24}$$

Hence:

$$(\mathbf{x_{i-1}} - \mathbf{x_i}) \times \mathbf{f_{i-1}} = (\mathbf{x_i} - \mathbf{x_{i+1}}) \times \mathbf{f_{i+1}} \tag{25}$$

We can rewrite $(\mathbf{x_{i-1}} - \mathbf{x_i}) \times \mathbf{f_{i-1}}$ as:

$$||\mathbf{f_{i-1}}|| \cdot ||\mathbf{x_{i-1}} - \mathbf{x_i}|| \cdot sin(\alpha_i) = f_{i-1}\ l_i \tag{26}$$

where f_i is the modulus of $\mathbf{f_i}$ and l_i is:

$$l_i = ||(\mathbf{x_{i-1}} - \mathbf{x_i}) - (\mathbf{x_{i-1}} - \mathbf{x_i})\ \mathbf{n}\ \mathbf{n}^t|| = |\mathbf{e_3}^t\ (\mathbf{x_i} - \mathbf{x_{i-1}})| \tag{27}$$

$$l_{i+1} = |\mathbf{e_3^t}\ (\mathbf{x_i} - \mathbf{x_{i+1}})| \tag{28}$$

where $\mathbf{e_3} = \dfrac{\mathbf{x_{i+1}} - \mathbf{x_{i-1}}}{||\mathbf{x_{i+1}} - \mathbf{x_{i-1}}||}$.

Hence, we have that the static moments satisfy the equation: $f_{i-1}\ l_i = f_{i+1}\ l_{i+1}$.

Finally, we have that in each system there is an input of moments, hence, that resultant moment is equal to the difference between the two moments involved in the system (which have different orientation of rotation). Therefore, we have:

$$f_{i-1}\ l_i + f_{i+1}\ l_{i+1} = M_i \tag{29}$$

In conclusion, for the active moments, we have to solve the system:

$$\begin{aligned} &\mathbf{f_{i+1}} + \mathbf{f_i} + \mathbf{f_{i-1}} = \mathbf{0} \\ &f_{i-1}\ l_i = f_{i+1}\ l_{i+1} \\ &f_{i-1}\ l_i + f_{i+1}\ l_{i+1} = M_i \end{aligned} \tag{30}$$

where M_i is the moment applied onto node $\mathbf{x_i}$.

If we add the second and third formula, we get:

$$f_{i-1}\ l_i = \frac{M_i}{2} \Rightarrow \mathbf{f_{i-1}} = \frac{M_i}{2}\ \frac{1}{l_i}\ \mathbf{n} \tag{31}$$

with $\mathbf{n}$ being a vector, that we will define later, that points into the direction of the force.

Now, if we substitute it in the second formula we have:

$$f_{i+1} = \frac{l_i}{l_{i+1}}\ f_{i-1} = \frac{M_i}{2}\ \frac{1}{l_{i+1}} \Rightarrow \mathbf{f_{i+1}} = \frac{M_i}{2}\frac{1}{l_{i+1}}\ \mathbf{n} \tag{32}$$

Finally, if we plug-in both explicit formulas of the forces in the first one, we obtain:

$$\mathbf{f_i} = -\frac{M_i}{2}\left(\frac{1}{l_i} + \frac{1}{l_{i+1}}\right)\mathbf{n} \tag{33}$$

We decide that the forces will be in the direction perpendicular to the vector from $\mathbf{x_{i-1}}$ to $\mathbf{x_{i+1}}$, so we would have that $\mathbf{n}$ will be:

$$\mathbf{n} = \mathbf{J}\ \frac{\mathbf{x_{i+1}} - \mathbf{x_{i-1}}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|} \tag{34}$$

As before, we compute the Newton-Raphson method to solve the non-linear system of equations. The explicit expression of the Jacobian $\mathbf{K^m}$ can be found in Appendix A.4.

2.5 Center of Mass of the system

In this subsection we will show under which conditions, with all the forces that we have explain before, the center of mass of the system does not change.

We remember that the *center of mass* of a body is defined as:

$$\mathbf{x}_{CM} = \frac{\sum_{i=1}^{n+1} m_i\ \mathbf{x}_i}{\sum_{i=1}^{n+1} m_i} \tag{35}$$

We will also say that a system of forces is *self-equilibrated* if the sum of nodal contributions is equal to 0.

Before the proposition, we have to make the following remarks:

- Remark 1: The elastic potentials are invariant by a rigid body motion.

  Proof. Under rigid body motions, angles and lengths remain unchanged. Since our elastic potentials depend solely on lengths and angles, the potential energy will be invariant if the body is subjected to such rigid body motions. That is, for an arbitrary constant displacement $\bar{\mathbf{u}}$, (not necessary equal along x and y,), we have that

$$W^b(\mathbf{x}+\bar{\mathbf{u}}) = W^b(\mathbf{x}) \tag{36}$$

  and

$$W^s(\mathbf{x}+\bar{\mathbf{u}}) = W^s(\mathbf{x}) \tag{37}$$

  □

- Remark 2: The active moments and the elastic forces are self-equilibrated.

  Proof. By construction the active moments satisfy:

$$\sum_{i=1}^{n+1} \mathbf{g}_i^m = \mathbf{0} \tag{38}$$

  Therefore, they are self-equilibrated.

  The elastic forces come from a potential which is invariant for a rigid motion (*Remark 1*), and thus are not affected by an arbitrary constant displacement. Indeed, if we use a Taylor approximation for the bending potential:

$$W^b(\mathbf{x}+\bar{\mathbf{u}}) = W^b(\mathbf{x}) + \bar{\mathbf{u}}^t\ \frac{\partial W^b(\mathbf{x})}{\partial \mathbf{x}} + \bar{\mathbf{u}}^t\ \frac{\partial^2 W^b(\mathbf{x})}{\partial \mathbf{x}^2}\ \bar{\mathbf{u}} + ... \tag{39}$$

  and we use that $W^b(\mathbf{x}+\bar{\mathbf{u}}) = W^b(\mathbf{x})$ (*Remark 1*), we have that all the terms except the first one must be 0. Therefore, we have, for arbitrary constant translation

$$\bar{\mathbf{u}} = \bar{u}_x\ \{1,0,1,0,...,1,0\}^t + \bar{u}_y\ \{0,1,0,1,...,0,1\}^t \tag{40}$$

  we have:

$$\bar{\mathbf{u}}^t\ \frac{\partial W^b(\mathbf{x})}{\partial \mathbf{x}} = \bar{\mathbf{u}}^t\ \mathbf{g}^b(\mathbf{x}) = (\bar{u}_x)^t\ \sum_{i=1}^{n+1} g^b_{i,x} + (\bar{u}_y)^t\ \sum_{i+1}^{n+1} g^b_{i,y} = 0 \tag{41}$$

  We can do the same for the stretching forces to see that they are also self-equilibrated.

  □

Proposition The center of masses of an elastic body in an isotropic and homogeneous frictional or viscous environment and contracting by a system of self-equilibrated active forces remains still, i.e. $\mathbf{u}_{CM} = \mathbf{0}$

Proof. For each instant of time t, we have the balance equation:

$$\sum_{i=1}^{n+1} \left(\mathbf{g}_i^b + \mathbf{g}_i^s - \mathbf{g}_i^\eta - \mathbf{g}_i^m \right) = \mathbf{0} \tag{42}$$

where $\mathbf{g}_i^b$ is the bending force, $\mathbf{g}_i^s$ is the stretching force, $\mathbf{g}_i^\eta$ is the viscous force and $\mathbf{g}_i^m$ is the moment force at the node $\mathbf{x}_i$.

From *Remark 2* we know that the elastic forces and the active moments are self-equilibrated. Therefore, we can reduce equation (42) to:

$$-\sum_{i=1}^{n+1} \mathbf{g}_i^\eta = \mathbf{0} \tag{43}$$

As we recall from (13) we have that $\mathbf{g}_i^\eta = -\eta\ \mathbf{v}_i$ with $\mathbf{v}_i$ being the velocity of the node $\mathbf{x}_i$. Hence. we have that:

$$\mathbf{0} = -\sum_{i=1}^{n+1} \mathbf{g}_i^\eta = \eta \sum_{i=1}^{n+1} \mathbf{v}_i = \eta\ (n+1)\ \mathbf{v}_{CM} \tag{44}$$

because all the masses of the nodes m_i are constant. Therefore, we have $\mathbf{v}_{CM} = \mathbf{0}$, which implies $\mathbf{x}_{CM} = \mathit{constant}$, so $\mathbf{u}_{CM} = \mathbf{0}$.

□

Finally, it is important to notice that in the case that viscosity or friction with the environment is not isotropic or homogeneous (different values in different directions or positions), the proposition fails. In the result section we will show that indeed the center of mass can move for non-isotropic viscosity depending on the distribution of moments that we apply to the system.

3. Numerical Tests

In this section we will explain the results we have arrived using a code made in Matlab to compute the deformation of the worm-like organism and its center of mass.

We will assume that the total initial length of the rod is $L = 1$, it is discretized in $n + 1$ nodes and the initial position of the rod is horizontal (unless we say something else). We will also assume that the deformations and displacements are only two-dimensional.

The method that we will use to solve the non-linear systems is the Newton-Raphson method (mentioned before), and in the case of viscosity, we will use the Backward Euler method to solve the ODE that we obtain.

3.1 Bending Forces

In this subsection we will assume that we only have bending forces acting in the rod and that we have an external vertical force F in the last node of the rod.

As the bending forces are translation and rotation-invariant, we have to impose some Dirichlet conditions on the bar to have only one solution of the system. On the one hand, we will assume that the first node and the vertical component of the second node are clamped ($\mathbf{u}_1^x = \mathbf{u}_1^y = \mathbf{u}_2^y = \mathbf{0}$). On the other hand, as we only have bending forces and a vertical external force, we have that the Jacobian matrix $\mathbf{K}^b$ have zero values in the rows corresponding to the horizontal components, so we have to assume that the horizontal components of the nodes are fixed, i.e. $\mathbf{u}_i^x = 0$, to have a unique solution.

In Figure 8 we can see different deformations according to different values of the parameter k_b with the same force value $F = 0.1$. We can see that if he have a larger value, the rod is more rigid and it bends less, while if the value is smaller, the rod is easier to bend.

If we keep the same value of k_b but change the number of nodes of the discretization, we see that the number of nodes also affects the bending of the rod. In Figure 9 we see the if we increase the number of nodes the rod bends more. That's because when we add new nodes, we are adding more joints and thus increasing the compliance of the rod.

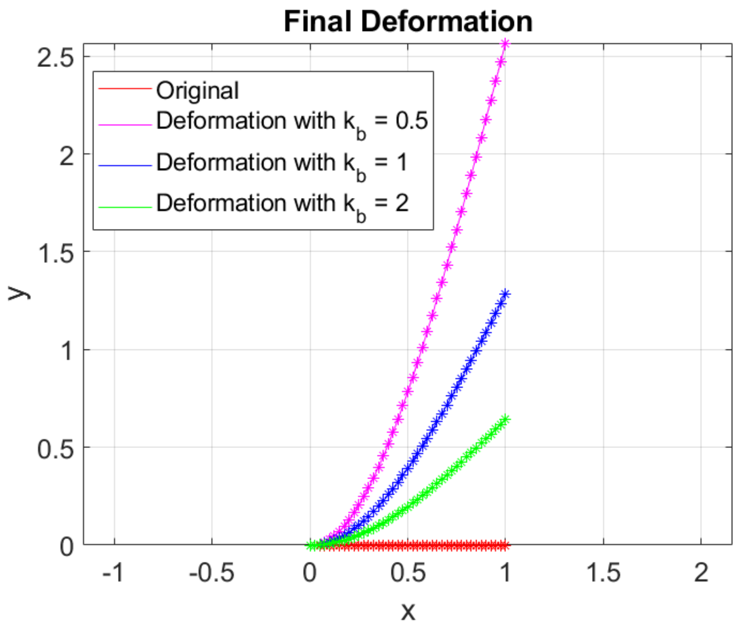


Figure 8: Equilibrium of the rod discretized in 41 nodes and with an external force $F = 0.1$ changing the parameter k_b

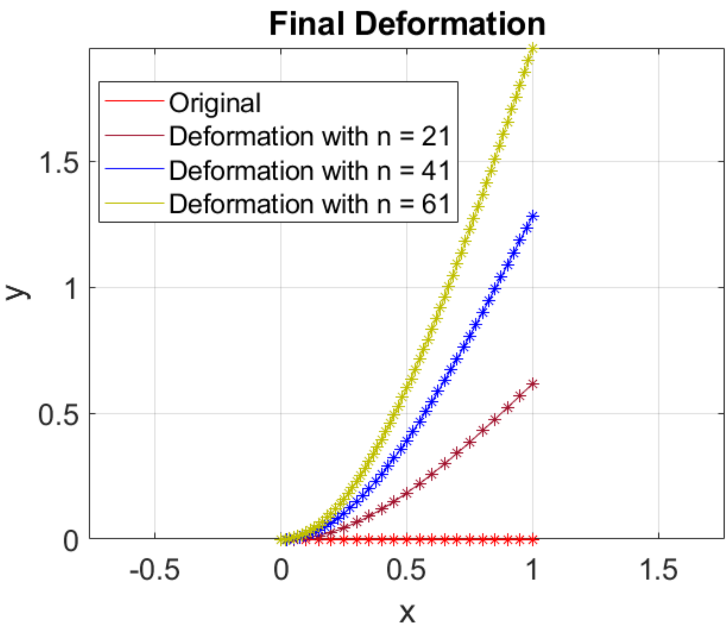


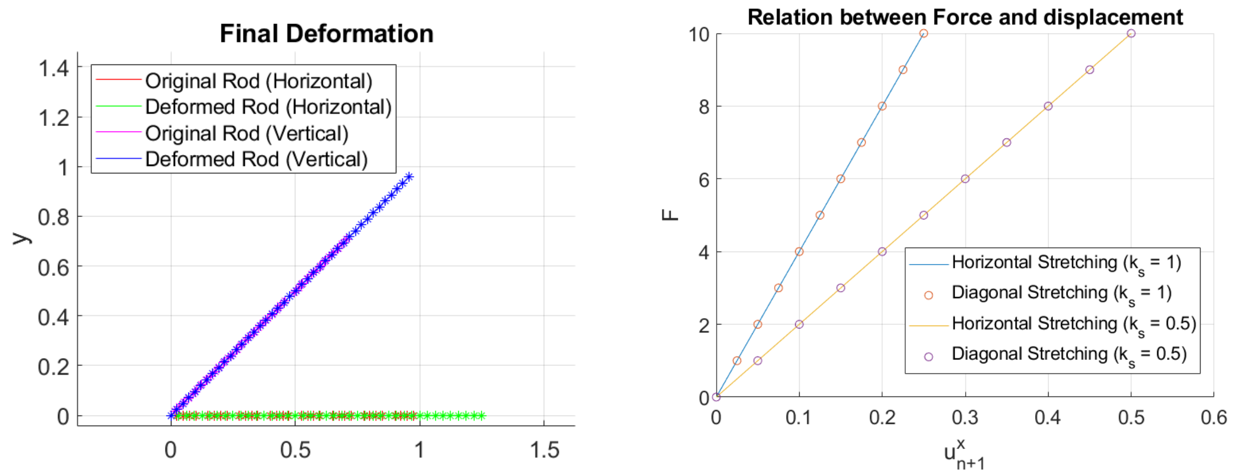
Figure 9: Equilibrium of the rod with fixed $k_b = 1$ and $F = 0.1$ changing the number of nodes.

3.2 Stretching Forces

In this subsection we will assume that we have all the elastic forces (bending and stretching) and we will apply different external forces in the last node. In all the cases we have to impose some Dirichlet conditions to have a unique solution, so we will assume (as before) that the first node is fixed.

First, we will study the stretching forces under a horizontal external force in the last node. It will help us to study the parameter k_s, because we will have bending energy equals to zero, so it will not interfere with the results. We will assume a bigger force, $F = 10$, to have a more visualized picture. We can see in Figure 10 the rod at equilibrium when $k_s = 1$ and the relation between the parameter k_s, the force F and how much the end node deforms.

We can see that the deformation is proportional to the force applied in subfigure 10*b* and that if we decrease the parameter k_s, the rod can deform more.



(a) Equilibrium when $k_s = 1$

(b) Relation between the displacement of the horizontal component of the end node and the force F.

Figure 10: At right: Equilibrium when $F = 10$ in the horizontal and diagonal stretching with $n + 1 = 41$ nodes, $k_b = 1$. At left: Relation of F and u^x_{n+1} for different values of k_s.

Second, we will study the stretching forces when the force is diagonal, to check that the orientation of the rod does not affect the equations, i.e. it can handle arbitrary rotations. In this case, we will start with the rod diagonal as initial condition because it is a good initial condition for the Newton-Raphson method. If we had the rod horizontal, for any force F (as small as we could) we would have a big displacement of the nodes, so the method may fail its convergence.

In Figure 10*a* we can see that the the constant k_s behaves as in the horizontal case (in terms of stretching) and the relation between force and deformation (subfigure 10*b*) is the same that in the horizontal case, so it does not depend on the orientation of the rod.

3.3 Viscous forces

Now we will also assume that the rod is placed in a viscous environment. This produces viscous forces that go against the displacement of the nodes. This forces depend on the velocity of the nodes, so we will have to use a method to discretize the time and solve the ODE. We have decided to use Backward Euler to avoid any problem of stability, but we could have used others as the mid-point rule or Forward Euler.

It is important to notice that the jacobian matrix of the viscous forces relaxes the matrix $\mathbf{K}$ of the Newton-Raphson method. Therefore, the requirement of Dirichlet conditions becomes unnecessary in this case.

We are going to consider a horizontal rod with a vertical external force $F = 10$ which will be applied in $nt = 16$ steps. As in the previous subsections, we are going to show the role of η in the deformation of the rod by looking a the different deformations that we obtain for different values of η.

In Figure 11 we see the variations of the parameter η. We can see that if we increase η the viscous forces are stronger, so it allows less to the rod to deform (while if we decrease it, the nodes have more freedom to move). Of course, if we had a very large F we would have that after some steps the rod would only displace in the vertical direction (while the x-component would remain into the center of the rod).

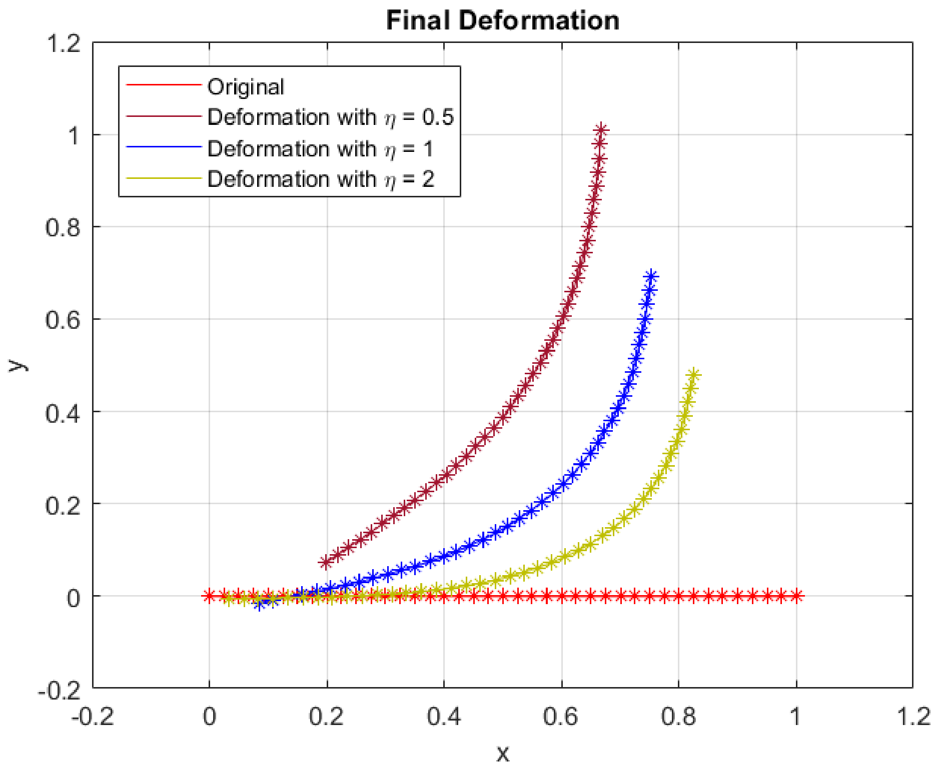
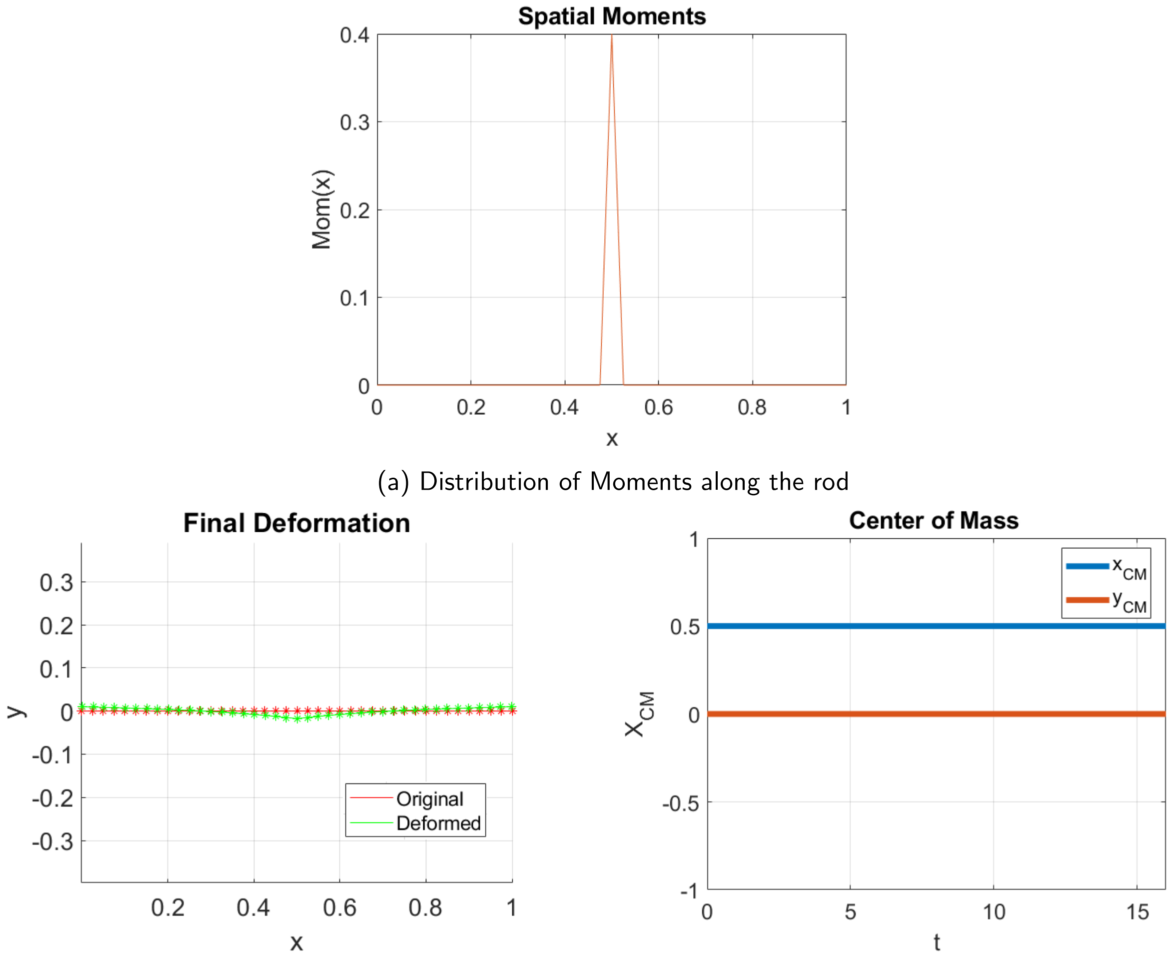


Figure 11: Equilibrium with $n + 1 = 41$ nodes, $k_b = 1$, $k_s = 1$ and a vertical force $F = 10$

3.4 Active Moments

In this subsection we will assume that we do not have the external force F, but the rod itself produces active moments to change its form. From the Methodology section we know that the active moments are self-equilibrated, so if we have an isotropic environment we are in the conditions of the proposition and therefore the center of mass will remain fixed in all the examples.

3.4.1. Single Active Moment



(a) Distribution of Moments along the rod

(b) Equilibrium of the rod

(c) Center of Mass of the system

Figure 12: Simulation of a single active moment with $n+1 = 41$ nodes, $k_b = 1$, $k_s = 1$ and $\eta = 0.5$ and constant active moment in the middle node $M = 0.4$, applied in 16 steps.

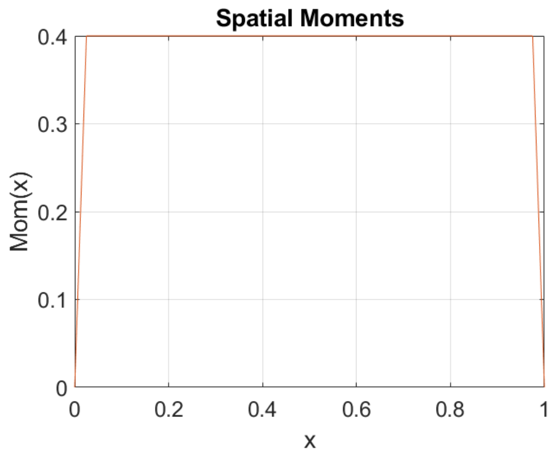
Here we will consider the case where we put only an active moment in the middle node (we can see it in Figure 12*a*, where we have the distribution of moments along the nodes of the rod). We will continue to assume that we have viscous forces and with the nodes free to move. Then, we can see that it makes as spike, as we can see in Figure 12*b*.

Notice that the spike is in the upward direction in 12*a* and in the down direction in 12*b* because we impose the active moments in the down direction, as we can see in the Methodology section.

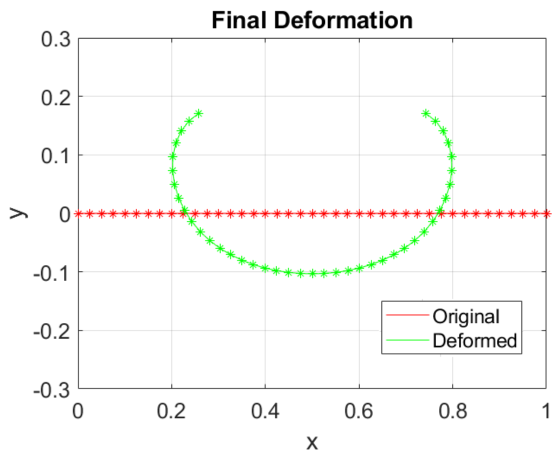
If we increase the moment applied M we would have a larger spike. We also can see numerically in 12*c* that the center of mass remains fixed always, as we expected.

3.4.2. Constant Active Moment

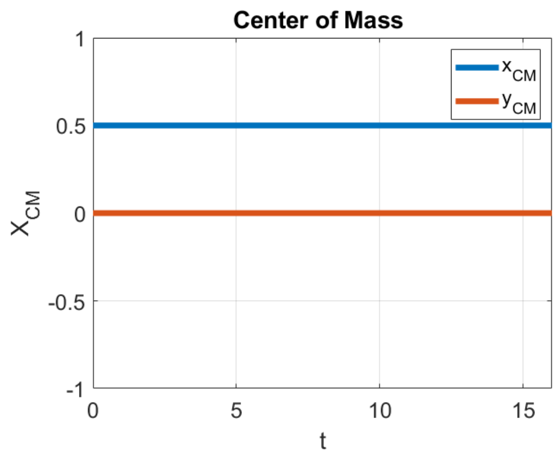
Now we will consider the case where all the nodes (except the first and last node) have the same active moment value (see subfigure 13*a*). We can see in subfigure 13*b* that the equilibrium is satisfied when we have an elliptic arc. Finally, we can see in subfigure 13*c* that the center of mass remains fixed, as a numeric check of the proposition before.



(a) Distribution of moments along the rod



(b) Equilibrium of the rod



(c) Center of Mass of the system

Figure 13: Moment distribution, equilibrium and center of mass of the system with $n+1 = 41$ nodes, $k_b = 1$, $k_s = 1$ and $\eta = 0.5$ and constant active moment in all node $M = 0.4$, applied in 16 steps.

From now on in this section we will study three different distributions of moments and we will study the differences that they have in its center of mass according to if the organism have different viscosities to move in the environment or not. In all the cases we will use $n+1 = 41$ nodes, $k_b = 1$, $k_s = 1$ and $\eta = 0.5$.

We will start with an isotropic viscosity environment. In this case, as the active moment forces are self-equilibrated, we are in the conditions of the proposition before and therefore we must have that the center of mass remains fixed in the three types of moment distribution.

3.4.3. Oscillatory Spatial Active Moment in Isotropic Environment

First we are going to consider that we have a spatial distribution of moments of the form:

$$M_i = A \ \sin(k\mathbf{x}_i) \ , \text{ with } i = 2, ..., n \tag{45}$$

where M_i is the active moment applied at node i, A is the amplitude of the active moment and k is the wave number.

In Figure 14 we can see the moment distribution, deformation and the variation of the center of mass when $A = 0.4$ and with 2 different values of k.

3.4.4. Oscillatory Temporal Active Moment in Isotropic Environment

Second, we will consider that we have that all the active moments in the nodes are the same constant value, but this time we will change this constant value along the time. For example, we will consider a sinusoidal function such as:

$$M_i = A \ \sin(\omega t) \ , \text{ for } i = 2, ..., n \text{ and for } t = 1, ..., nt \tag{46}$$

with ω being the angular frequency. In Figure 15 we can see some plots (moment distribution, deformation and center of mass) changing the parameter ω when $A = 0.4$.

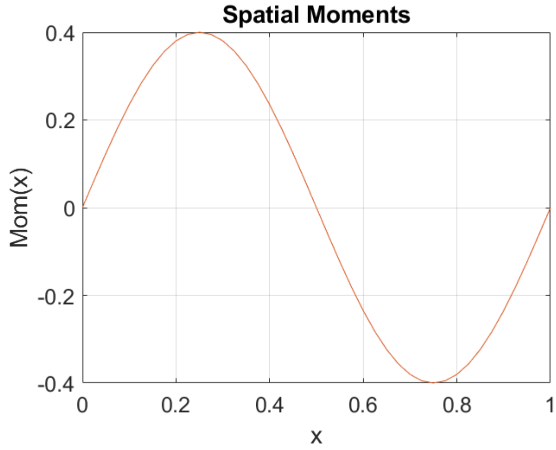
3.4.5. Oscillatory Time-Space Active Moment (Wave) in Isotropic Environment

Finally, we are going to consider a wave across time and space of the form:

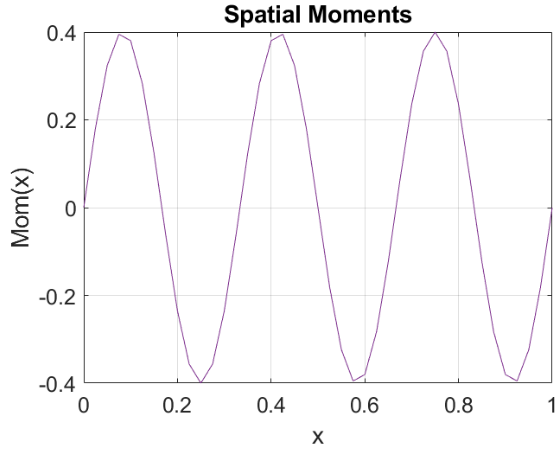
$$M_i = A \ \sin(\omega t - k\mathbf{x_i}) \ , \text{ for } i = 2, ..., n \text{ and for } t = 1, ..., nt \tag{47}$$

with ω being the angular frequency and k the wave number.

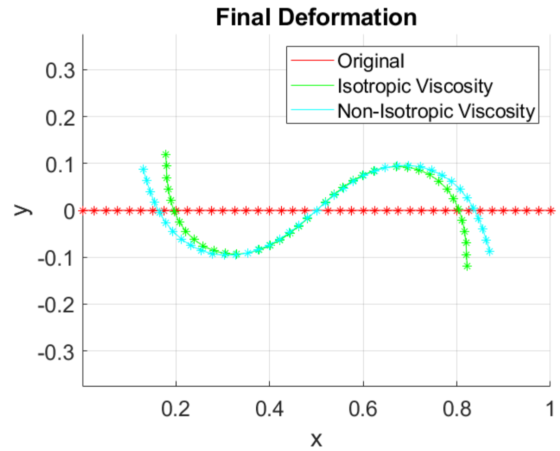
In Figure 16 and 17 we see some plots with different values of k and ω. On the one hand the subfigures 16*a* and 17*a* we have represented the evolution of the moment at space (each color represents a fixed time t). On the other hand, the subfigures 16*b* and 17*b* represents the evolution of the moments in each node (at the left part we can see from which node position starts and how it evolves along time).



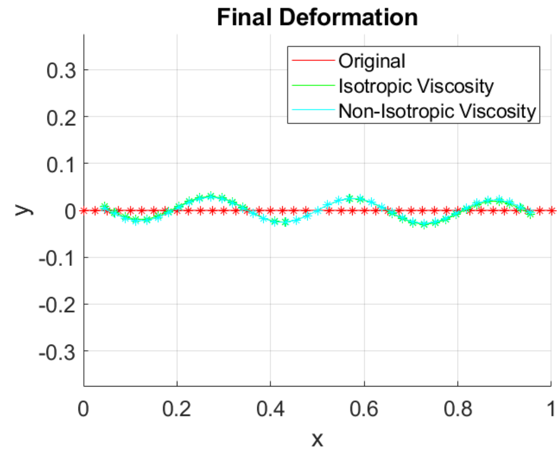
(a) Distribution of moments along the rod



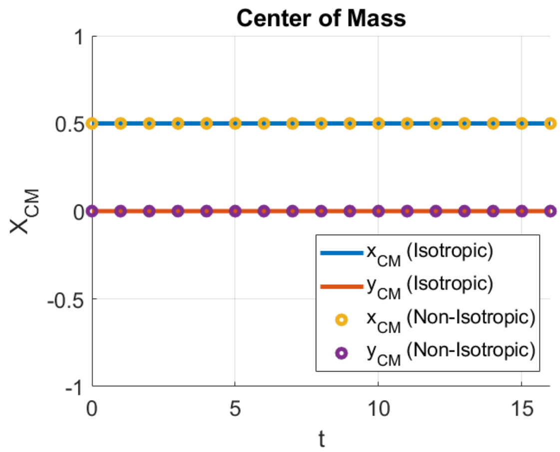
(b) Distribution of moments along the rod



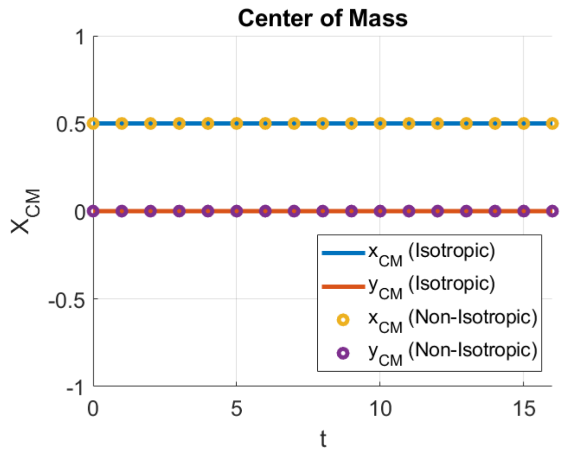
(c) Equilibrium of the rod



(d) Equilibrium of the rod

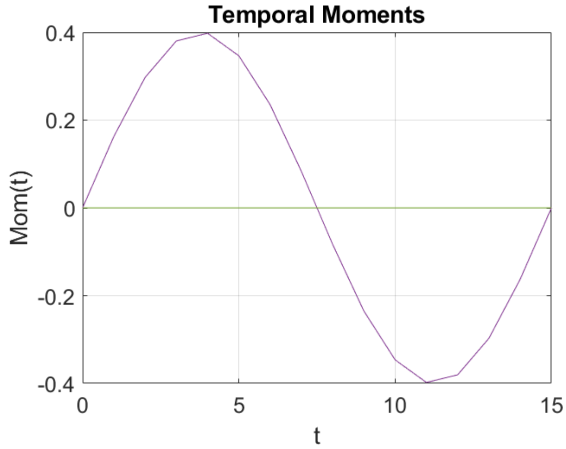


(e) Center of Mass of the system

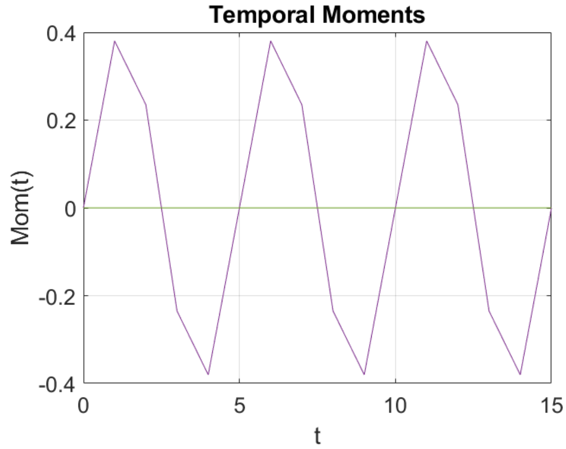


(f) Center of Mass of the system

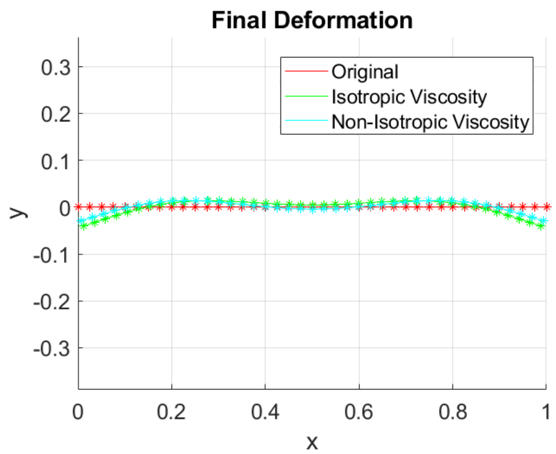
Figure 14: Simulations with an oscillatory spatial distribution of moments with isotropic and non-isotropic viscosity. Left: $k = 2\pi$. Right: $k = 6\pi$.



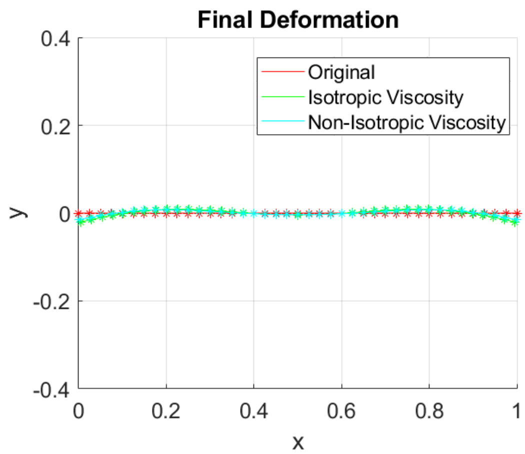
(a) Distribution of moments along time



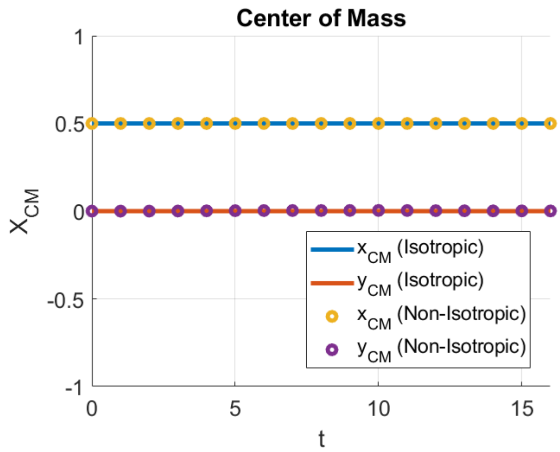
(b) Distribution of moments along time



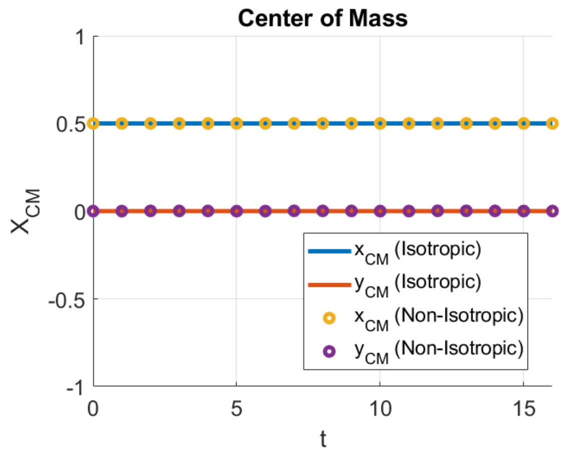
(c) Equilibrium of the rod



(d) Equilibrium of the rod

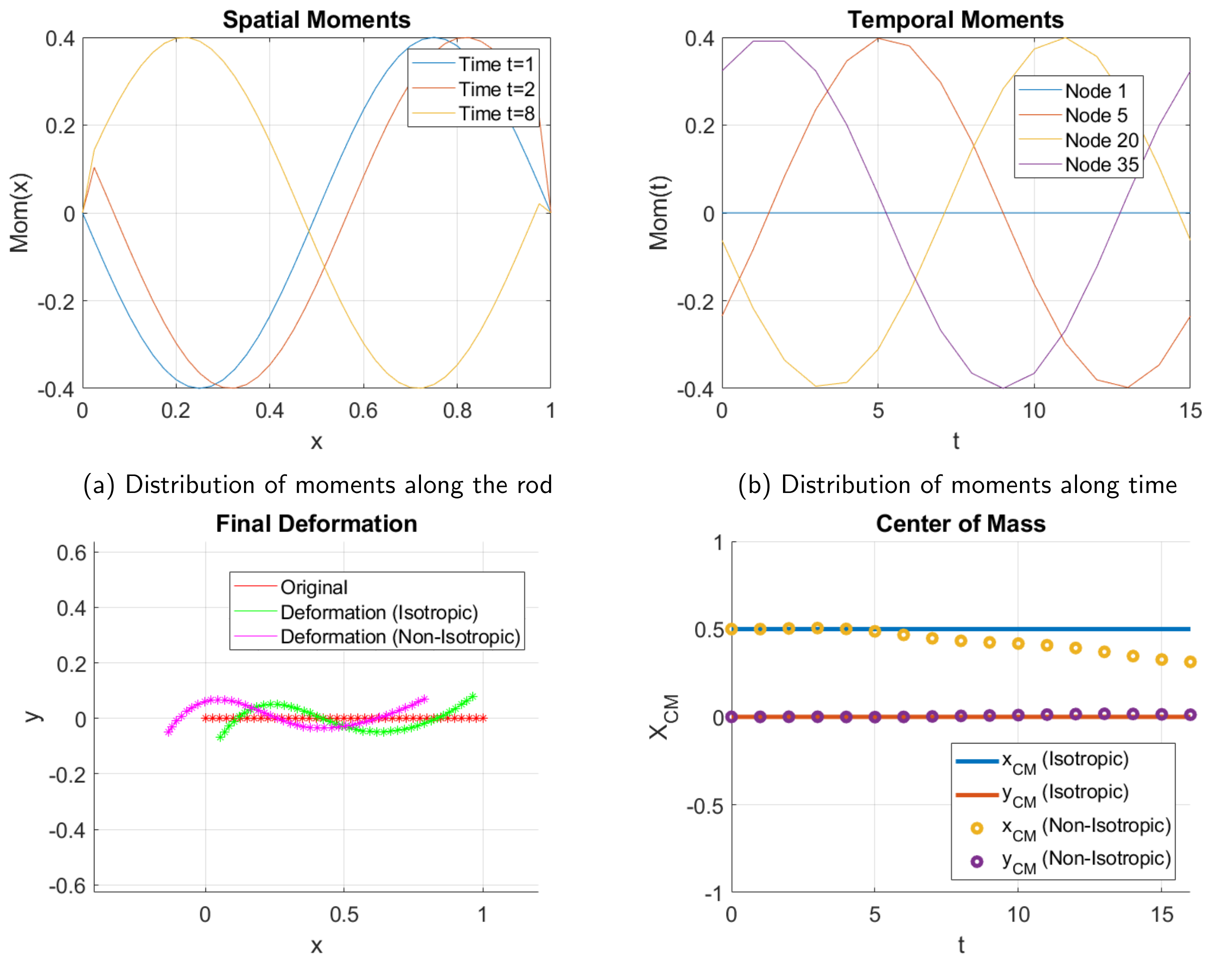


(e) Center of Mass of the system



(f) Center of Mass of the system

Figure 15: Simulations with an oscillatory temporal distribution of moments with isotropic and non-isotropic viscosity. Left: $\omega = 2\pi$. Right: $\omega = 6\pi$.



(a) Distribution of moments along the rod

(b) Distribution of moments along time

(c) Equilibrium of the system

(d) Center of Mass of the system

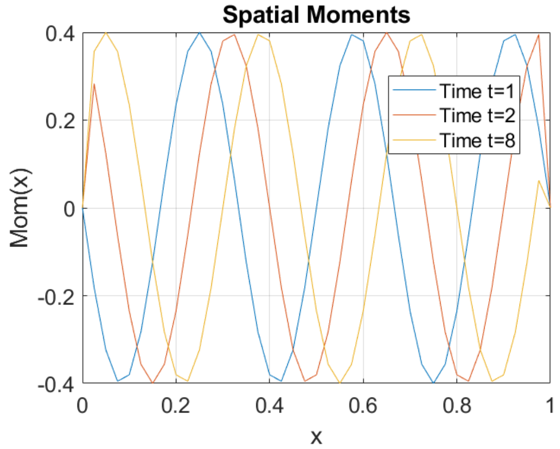
Figure 16: Simulations of the wave with $\omega = 2\pi$ and $k = 2\pi$ with isotropic and non-isotropic viscosity.

As we can see in 16*d* and 17*d* we can see that the center of mass remains fixed always (the red and blue lines) because we are in an isotropic viscous environment (see Proposition of the Methodology Section).

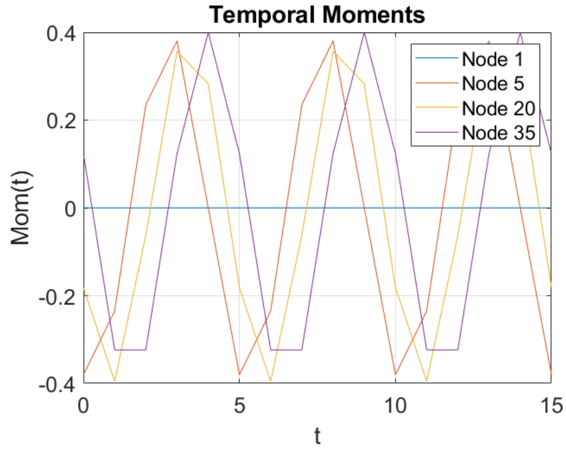
3.4.6. Oscillatory Active Moments in Non-Isotropic Environment

Finally, we are going to see what happens in the case that we have different viscosity in the tangential and normal direction. The fact that we do not have isotropic viscosity translates into the fact that we cannot apply the proposition that we gave before, so we have to study each case in particular to see if the center of mass changes or not. In all the simulations we will use $\eta_t = 0.1$ and $\eta_n = 1$.

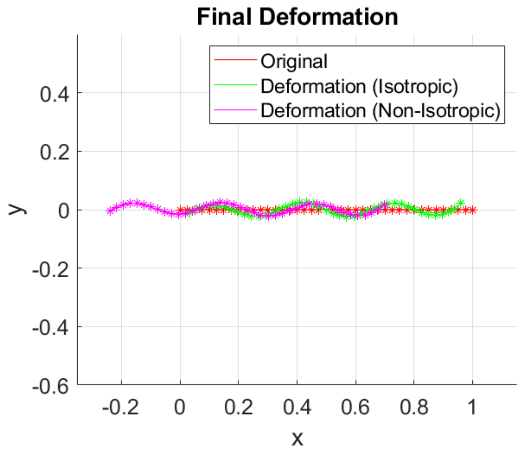
We can check again Figures 14, 15 and 16 to see how the non-isotropic viscosity affects in an oscillatory spatial distribution of moments, an oscillatory temporal and a wave-like distribution. We can see that the difference of viscosities change the final deformation of the rod at the spatial and temporal oscillations (see subfigures 14*e*, 14*f*, 15*e* and 15*f*), but the center of mass remains fixed (we have that the yellow and purple



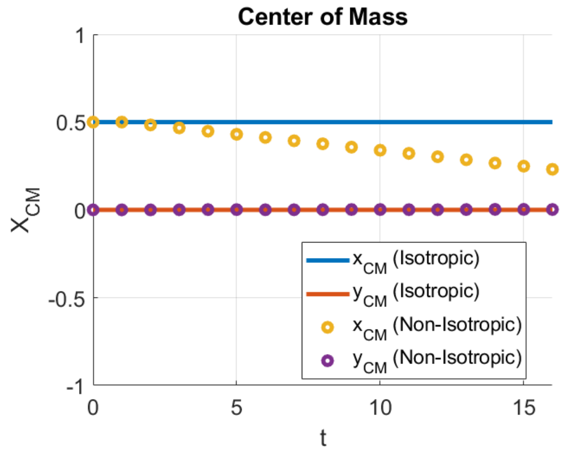
(a) Distribution of moments along the rod



(b) Distribution of moments along time



(c) Equilibrium of the system



(d) Center of Mass of the system

Figure 17: Simulations of the wave with $\omega = 6\pi$ and $k = 6\pi$ with isotropic and non-isotropic viscosity.

cercles that correspond to the non-isotropic environment overlap the blue and red ones corresponding to the isotropic one).

However, the same does not happen in the wave-like case. There we can see in 16*c* and 17*c* that the rod has moved to the left direction, hence the center of mass have changed in both cases. We can see it more clearly in 16*d* and 17*d*, where we see that the lines do not overlap anymore and the horizontal component have decreased in both.

4. Conclusions and future work

In this thesis we have modelled the main forces that act in a worm-like organism and check from a computational point of view the meaning of the parameters involved.

We have also shown analytically and numerically that the main difference between an organism that can change its viscosity along the tangent direction and another one that cannot is that the first one is able to swim through the viscous environment while the second one will always remain in the same place. This is crucial for the survival of the organism, because if it cannot swim it will not be able to search and find new nutrients to live.

In future analysis, we can study the affectation of the different parameters between them and study which of them are better for the organism to be able to swim from one place to another one. In this work we have not searched for optimal parameters for displacement, and considered strategies other than sinusoidal or periodic active moments. The study of optimal strategies is relevant not only to understand how worm-like organism move, but also for the design of similar artificial robots or snake-like shapes may be able to crawl at a minimum energy cost in microscopic (drug delivery) or macroscopic systems (risk emergencies or hazardous situations) systems. We could use Machine Learning techniques to find the optimal parameters for the organism to move along the optimal path.

In this thesis we have considered the main forces that act in an organism when we have a small Reynolds number, so we have neglected all the inertial forces. In a future work we could add this forces and study how changing the inertia and the viscosity affect the deformation and the swimming of the organism.

Finally, can also try to find new ways to move along the viscous environment. We have only considered when the deformations are all in a 2-dimensional plane, but it can also move in a third dimension. For example, the worm-like organism could bend in a way that a part of its body is in the third dimension and therefore that part is not affected by the viscosity of the environment, so it would be able to move its center of mass. We could also improve the model considering instead of a 1-dimensional rod, a 3-dimensional object that can rotate in the tangent direction, doing a movement similar to a screwdriver when we apply it into a nail.

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A. Appendix

A.1 Bending Jacobian

In the bending section of the methodology of the thesis we have seen that the expression of the bending force is:

$$\mathbf{g_i^b} = k_b\ \theta_i\ \frac{\partial \theta_i}{\partial \mathbf{X}^i} = k_b\ \left(\frac{1}{2}\ \frac{n^2}{L^2}\ \left(\mathbf{X}^i\right)^t\ \mathbf{J_6}\ \mathbf{X}^i\right)\ \left(\frac{n^2}{L^2}\mathbf{J_6}\mathbf{X}^i\right)$$

where $\mathbf{J_6}$ is the matrix:

$$\mathbf{J_6} = \begin{bmatrix} \mathbf{0} & -\mathbf{J_2} & \mathbf{J_2} \\ \mathbf{J_2} & \mathbf{0} & -\mathbf{J_2} \\ -\mathbf{J_2} & \mathbf{J_2} & \mathbf{0} \end{bmatrix}$$

and $\mathbf{X}^i$ is the vector:

$$\mathbf{X}^i = \begin{Bmatrix} \mathbf{x_{i-1}} \\ \mathbf{x_i} \\ \mathbf{x_{i+1}} \end{Bmatrix}$$

Therefore, if we derivate this force, we will have the matrices $\mathbf{K_b^i}$, and the total matrix $\mathbf{K_b}$ will be the assembly of all this matrices:

$$\mathbf{K_b^i} = \frac{\partial \mathbf{g_i}}{\partial \mathbf{X}^i} = \frac{n^2}{L^2} k_b \left(\mathbf{J_6}\theta_i + \frac{n^2}{L^2}\mathbf{J_6}\mathbf{X^i}\left(\mathbf{J_6}\mathbf{X}^i\right)^t\right)$$

A.2 Stretching Jacobian

Here we will do the calculations of the Jacobian matrix that comes from the stretching forces ($\mathbf{K^s}$). This matrix will be the assembly of different matrices $\mathbf{K_i^s}$:

$$\mathbf{K_i^s} = \begin{bmatrix} (1) & (2) \\ (3) & (4) \end{bmatrix}$$

where:

$$\begin{aligned}
(1) &= \frac{\partial}{\partial \mathbf{x_i}}\left(\frac{\partial}{\partial \mathbf{x_i}}(W_s)\right) = k_s\ \frac{\partial}{\partial \mathbf{x_i}}\left(\frac{\|\mathbf{x_{i+1}} - \mathbf{x_i}\| - l_0}{(l_0)^2}(-\mathbf{e_i})\right) \\
&= k_s \frac{\partial}{\partial \mathbf{x_i}}\left(\frac{\|\mathbf{x_{i+1}} - \mathbf{x_i}\| - l_0}{(l_0)^2}\right)(-\mathbf{e_i})^t + k_s \frac{\|\mathbf{x_{i+1}} - \mathbf{x_i}\| - l_0}{(l_0)^2}\frac{\partial}{\partial \mathbf{x_i}}(-\mathbf{e_i}) \\
&= \frac{k_s}{(l_0)^2}\left(\mathbf{e_i}\ \mathbf{e_i}^t + \frac{(\|\mathbf{x_{i+1}} - \mathbf{x_i}\| - l_0)(\mathbf{Id} - \mathbf{e_i}\ \mathbf{e_i}^t)}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}\right) \\
&= \frac{k_s}{(l_0)^2}\left(\frac{(\|\mathbf{x_{i+1}} - \mathbf{x_i}\| - l_0)\mathbf{Id} + l_0\ \mathbf{e_i}\ \mathbf{e_i}^t}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}\right)
\end{aligned}$$

$$
\begin{aligned}
(2) &= \frac{\partial}{\partial \mathbf{x_{i+1}}}\left(\frac{\partial}{\partial \mathbf{x_i}}(W_s)\right) = k_s \ \frac{\partial}{\partial \mathbf{x_{i+1}}}\left(\frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}(-\mathbf{e_i})\right)\\
&= k_s \frac{\partial}{\partial \mathbf{x_{i+1}}}\left(\frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}\right)(-\mathbf{e_i})^t + k_s \frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}\frac{\partial}{\partial \mathbf{x_{i+1}}}(-\mathbf{e_i})\\
&= k_s\left(\frac{\mathbf{e_i}}{(l_0)^2}(-\mathbf{e_i})^t + \frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}\ \frac{(-\mathbf{Id}+\mathbf{e_i}\ \mathbf{e_i}^t)}{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|}\right)\\
&= -\frac{k_s}{(l_0)^2}\left(\frac{(\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0)\mathbf{Id}+l_0\mathbf{e_i}\ \mathbf{e_i}^t}{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|}\right)
\end{aligned}
$$

$$
\begin{aligned}
(3) &= \frac{\partial}{\partial \mathbf{x_i}}\left(\frac{\partial}{\partial \mathbf{x_{i+1}}}(W_s)\right) = k_s \ \frac{\partial}{\partial \mathbf{x_i}}\left(\frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}(\mathbf{e_i})\right)\\
&= k_s \frac{\partial}{\partial \mathbf{x_i}}\left(\frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}\right)(\mathbf{e_i})^t + k_s \frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{e_i})\\
&= k_s\left(\frac{(-\mathbf{e_i})}{(l_0)^2}\mathbf{e_i}^t + \frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}\ \frac{(-\mathbf{Id}+\mathbf{e_i}\ \mathbf{e_i}^t)}{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|}\right)\\
&= -\frac{k_s}{(l_0)^2}\left(\frac{(\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0)\mathbf{Id}+l_0\mathbf{e_i}\ \mathbf{e_i}^t}{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|}\right)
\end{aligned}
$$

$$
\begin{aligned}
(4) &= \frac{\partial}{\partial \mathbf{x_{i+1}}}\left(\frac{\partial}{\partial \mathbf{x_{i+1}}}(W_s)\right) = k_s \ \frac{\partial}{\partial \mathbf{x_{i+1}}}\left(\frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}(\mathbf{e_i})\right)\\
&= k_s \frac{\partial}{\partial \mathbf{x_{i+1}}}\left(\frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}\right)(\mathbf{e_i})^t + k_s \frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{e_i})\\
&= k_s\left(\frac{\mathbf{e_i}}{(l_0)^2}\mathbf{e_i}^t + \frac{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0}{(l_0)^2}\ \frac{(\mathbf{Id}-\mathbf{e_i}\ \mathbf{e_i}^t)}{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|}\right)\\
&= \frac{k_s}{(l_0)^2}\left(\frac{(\|\mathbf{x_{i+1}}-\mathbf{x_i}\|-l_0)\mathbf{Id}+l_0\ \mathbf{e_i}\ \mathbf{e_i}^t}{\|\mathbf{x_{i+1}}-\mathbf{x_i}\|}\right)
\end{aligned}
$$

Therefore, if we rewrite this matrix expression a bit, we have:

$$
\mathbf{K_i^s} = \frac{k_s}{l_0^2\ l_e}\begin{bmatrix} ((l_e-l_0)\ \mathbf{Id}+l_0\mathbf{e_i}\mathbf{e_i}^\mathbf{t}) & -((l_e-l_0))\ \mathbf{Id}+l_0\mathbf{e_i}\mathbf{e_i}^\mathbf{t}) \\ -((l_e-l_0)\ \mathbf{Id}+l_0\mathbf{e_i}\mathbf{e_i}^\mathbf{t}) & ((l_e-l_0)\ \mathbf{Id}+l_0\mathbf{e_i}\mathbf{e_i}^\mathbf{t}) \end{bmatrix}
$$

with $l_e = \|\mathbf{x_{i+1}}-\mathbf{x_i}\|$.

A.3 Viscosity Jacobian

Here we will do the calculations of the viscosity jacobian matrix $\mathbf{K}^{\eta}$ when we have a different viscosity in the tangential and normal velocity of the nodes.

1. *Interior Nodes*

   We remember that for the interior nodes we had that the normal and tangential velocity are:

$$\mathbf{v_n} = (\mathbf{v_i} \cdot \mathbf{n})\ \mathbf{n}$$

$$\mathbf{v_t} = \mathbf{v_i} - \mathbf{v_n}$$

   where $\mathbf{n} = \mathbf{J}\ \mathbf{e_3}$ with $\mathbf{e_3} = \dfrac{\mathbf{x_{i+1}} - \mathbf{x_{i-1}}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}$.

   On the one hand, with some calculatios, we obtain:

$$\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{e_3}) = \frac{-\mathbf{Id} + \mathbf{e_3}\ \mathbf{e_3}^t}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|} = \frac{-\mathbf{N}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}$$

$$\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{e_3}) = \mathbf{0}$$

$$\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{e_3}) = \frac{\mathbf{Id} - \mathbf{e_3}\ \mathbf{e_3}^t}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|} = \frac{\mathbf{N}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}$$

   where we have defined $\mathbf{N} := \mathbf{Id} - \mathbf{e_3}\ \mathbf{e_3}^t$ to simplify calculations.

   On the other hand, we have that:

$$\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{v_i}) = \mathbf{0}$$

$$\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{v_i}) = \frac{\mathbf{Id}}{h}$$

$$\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{v_i}) = \mathbf{0}$$

   Therefore, as:

$$\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{v_i} \cdot \mathbf{n}) = \mathbf{J}\left(\frac{-\mathbf{N}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right)\mathbf{v_i}$$

$$\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{v_i} \cdot \mathbf{n}) = \frac{\mathbf{Id}}{h}\ \mathbf{n} = \frac{1}{h}\ \mathbf{n}$$

$$\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{v_i} \cdot \mathbf{n}) = \mathbf{J}\ \frac{\mathbf{N}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\mathbf{v_i}$$

we have that:

$$\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{v_n}) = \mathbf{n}\left(\mathbf{J}\ \left(\frac{-\mathbf{N}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right)\mathbf{v_i}\right)^t + (\mathbf{v_i} \cdot \mathbf{n})\left(\frac{-\mathbf{N}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right)$$

$$\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{v_n}) = \mathbf{n}\left(\frac{\mathbf{Id}}{h}\ \mathbf{n}\right)^t$$

$$\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{v_n}) = \mathbf{n}\left(\mathbf{J}\frac{\mathbf{N}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\mathbf{v_i}\right)^t + (\mathbf{v_i} \cdot \mathbf{n})\ \mathbf{J}\ \frac{\mathbf{N}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}$$

For the tangential forces, we have:

$$\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{v_t}) = \frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{v_i}) - \frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{v_n}) = -\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{v_n})$$

$$\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{v_t}) = \frac{\partial}{\partial \mathbf{x_i}}(\mathbf{v_i}) - \frac{\partial}{\partial \mathbf{x_i}}(\mathbf{v_n}) = \frac{\mathbf{Id}}{h} - \frac{\partial}{\partial \mathbf{x_i}}(\mathbf{v_n})$$

$$\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{v_t}) = \frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{v_i}) - \frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{v_n}) = -\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{v_n})$$

2. *Exterior Nodes*

The expressions for the exterior nodes (first and last) are:

$$\mathbf{v_t} = (\mathbf{v_i} \cdot \mathbf{t})\ \mathbf{t}$$

$$\mathbf{v_n} = \mathbf{v_i} - \mathbf{v_n}$$

where $\mathbf{t} = \dfrac{\mathbf{x_{i+1}} - \mathbf{x_i}}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}$

In this case we arrive to a similar result as before:

$$\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{t}) = \frac{-\mathbf{Id} + \mathbf{t}\ \mathbf{t}^t}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|} = \frac{-\mathbf{M}}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}$$

$$\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{t}) = \frac{\mathbf{Id} - \mathbf{t}\ \mathbf{t}^t}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|} = \frac{\mathbf{M}}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}$$

with $\mathbf{M} = \mathbf{Id} - \mathbf{t}\ \mathbf{t}^t$. We also have:

$$\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{v_i} \cdot \mathbf{t}) = \frac{\mathbf{Id}}{h}\ \mathbf{t} + \frac{-\mathbf{M}}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}\mathbf{v_i}$$

$$\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{v_i} \cdot \mathbf{t}) = \frac{\mathbf{M}}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}\mathbf{v_i}$$

Therefore:

$$\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{v_t}) = \mathbf{t}\left(\frac{\mathbf{Id}}{h}\ \mathbf{t} + \frac{-\mathbf{M}}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}\mathbf{v_i}\right)^t + (\mathbf{v_i} \cdot \mathbf{t})\left(\frac{-\mathbf{M}}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}\right)$$

$$\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{v_t}) = \mathbf{t}\left(\frac{\mathbf{M}}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}\mathbf{v_i}\right)^t + (\mathbf{v_i} \cdot \mathbf{t})\left(\frac{\mathbf{M}}{\|\mathbf{x_{i+1}} - \mathbf{x_i}\|}\right)$$

$$\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{v_n}) = \frac{\mathbf{Id}}{h} - \frac{\partial}{\partial \mathbf{x_i}}(\mathbf{v_t})$$

$$\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{v_n}) = -\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{v_t})$$

A.4 Torque Jacobian

In this subsection of the appendix we will compute the derivatives that we need in order to compute the total jacobian $\mathbf{K^m}$.

Previous to that we are going to compute some of the derivatives that we will need later:

$$\begin{aligned}
&\frac{\partial}{\partial \mathbf{x_{i-1}}}\left(\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|\right) = -\frac{\mathbf{x_{i+1}} - \mathbf{x_{i-1}}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|} = -\mathbf{e_3}\\
&\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{e_3}) = \frac{\partial}{\partial \mathbf{x_{i-1}}}\left(\frac{\mathbf{x_{i+1}} - \mathbf{x_{i-1}}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right) = \frac{-Id + \mathbf{e_3}\ \mathbf{e_3}^t}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\\
&\frac{\partial}{\partial \mathbf{x_i}}\left(\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|\right) = \mathbf{0}\\
&\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{e_3}) = \mathbf{0}\\
&\frac{\partial}{\partial \mathbf{x_{i+1}}}\left(\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|\right) = \frac{\mathbf{x_{i+1}} - \mathbf{x_{i-1}}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|} = \mathbf{e_3}\\
&\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{e_3}) = \frac{\partial}{\partial \mathbf{x_{i+1}}}\left(\frac{\mathbf{x_{i+1}} - \mathbf{x_{i-1}}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right) = \frac{Id - \mathbf{e_3}\ \mathbf{e_3}^t}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}
\end{aligned}$$

To simplify calculations (and formulas), we will define:

$$\mathbf{N} = \mathbf{Id} - \mathbf{e_3}\ \mathbf{e_3}^t$$

$$\mathbf{U_i} = \mathbf{x_{i-1}} - \mathbf{x_i}$$

$$\mathbf{U_{i+1}} = \mathbf{x_i} - \mathbf{x_{i+1}}$$

Then:

$$\frac{\partial}{\partial \mathbf{x_{i-1}}}(l_i^{-1}) = -\frac{\mathbf{U_i}\cdot\mathbf{e_3}}{l_i^3}\left(\mathbf{e_3} - \frac{\mathbf{N\ U_i}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right)$$

$$\frac{\partial}{\partial \mathbf{x_i}}(l_i^{-1}) = \frac{\mathbf{U_i}\cdot\mathbf{e_3}}{l_i^3}\ \mathbf{e_3}$$

$$\frac{\partial}{\partial \mathbf{x_{i+1}}}(l_i^{-1}) = -\frac{\mathbf{U_i}\cdot\mathbf{e_3}}{l_i^3}\ \frac{\mathbf{N\ U_i}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}$$

$$\frac{\partial}{\partial \mathbf{x_{i-1}}}(l_{i+1}^{-1}) = \frac{\mathbf{U_{i+1}}\cdot\mathbf{e_3}}{(l_{i+1})^3}\left(\frac{\mathbf{N\ U_{i+1}}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right)$$

$$\frac{\partial}{\partial \mathbf{x_i}}(l_{i+1}^{-1}) = -\frac{\mathbf{U_{i+1}}\cdot\mathbf{e_3}}{(l_{i+1})^3}\ \mathbf{e_3}$$

$$\frac{\partial}{\partial \mathbf{x_{i+1}}}(l_{i+1}^{-1}) = \frac{\mathbf{U_{i+1}}\cdot\mathbf{e_3}}{(l_{i+1})^3}\ \left(\mathbf{e_3} - \frac{\mathbf{N\ U_{i+1}}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right)$$

Therefore, we obtain that the derivatives of the forces are:

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{f_{i-1}}) &= \frac{M_i}{2}\left(\frac{\partial}{\partial \mathbf{x_{i-1}}}(l_i^{-1})\ \mathbf{n} + \frac{1}{l_i}\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{n})\right)\\
&= \frac{M_i}{2}\left[\left(-\frac{\mathbf{U_i}\cdot\mathbf{e_3}}{(l_i)^3}\left[\mathbf{e_3} - \frac{\mathbf{N\ U_i}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right]\right)\ \mathbf{n} - \frac{1}{l_i}\ \mathbf{J}\ \frac{\mathbf{N}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right]\\
\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{f_{i-1}}) &= \frac{M_i}{2}\left(\frac{\partial}{\partial \mathbf{x_i}}(l_i^{-1})\ \mathbf{n} + \frac{1}{l_i}\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{n})\right)\\
&= \frac{M_i}{2}\left[\left(\frac{\mathbf{U_i}\cdot\mathbf{e_3}}{l_i^3}\ \mathbf{e_3}\right)\ \mathbf{n}\right]\\
\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{f_{i-1}}) &= \frac{M_i}{2}\left(\frac{\partial}{\partial \mathbf{x_{i+1}}}(l_i^{-1})\ \mathbf{n} + \frac{1}{l_i}\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{n})\right)\\
&= \frac{M_i}{2}\left[\left(-\frac{\mathbf{U_i}\cdot\ \mathbf{e_3}}{l_i^3}\frac{\mathbf{N\ U_i}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right)\mathbf{n} + \frac{1}{l_i}\ \mathbf{J}\ \frac{\mathbf{N}}{\|\mathbf{x_{i+1}} - \mathbf{x_{i-1}}\|}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{f_i}) &= \frac{M_i}{2}\left[\left(\frac{\partial}{\partial \mathbf{x_{i-1}}}\left(l_i^{-1}\right) + \frac{\partial}{\partial \mathbf{x_{i-1}}}\left((l_{i+1})^{-1}\right)\right)\ \mathbf{n} + \left(\frac{1}{l_i} + \frac{1}{l_{i+1}}\right)\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{n})\right]\\
\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{f_i}) &= \frac{M_i}{2}\left[\left(\frac{\partial}{\partial \mathbf{x_i}}(l_i^{-1}) + \frac{\partial}{\partial \mathbf{x_i}}((l_{i+1})^{-1})\right)\ \mathbf{n} + \left(\frac{1}{l_i} + \frac{1}{l_{i+1}}\right)\frac{\partial}{\partial \mathbf{x_i}}(\mathbf{n})\right]\\
\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{f_i}) &= \frac{M_i}{2}\left[\left(\frac{\partial}{\partial \mathbf{x_{i+1}}}(l_i^{-1}) + \frac{\partial}{\partial \mathbf{x_{i+1}}}((l_{i+1})^{-1})\right)\ \mathbf{n} + \left(\frac{1}{l_i} + \frac{1}{l_{i+1}}\right)\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{n})\right]
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{f_{i+1}}) &= \frac{M_i}{2}\left(\frac{\partial}{\partial \mathbf{x_{i-1}}}((l_{i+1})^{-1})\ \mathbf{n} + \frac{1}{l_{i+1}}\frac{\partial}{\partial \mathbf{x_{i-1}}}(\mathbf{n})\right) \\
&= \frac{M_i}{2}\left[\left(\frac{\mathbf{U_{i+1}}\cdot\mathbf{e_3}}{(l_{i+1})^3}\ \frac{\mathbf{N}\ \mathbf{U_{i+1}}}{\|\mathbf{x_{i+1}}-\mathbf{x_{i-1}}\|}\right)\ \mathbf{n} - \frac{1}{l_{i+1}}\ J\ \frac{\mathbf{N}}{\|\mathbf{x_{i+1}}-\mathbf{x_{i-1}}\|}\right] \\
\frac{\partial}{\partial \mathbf{x_{i}}}(\mathbf{f_{i+1}}) &= \frac{M_i}{2}\left(\frac{\partial}{\partial \mathbf{x_{i}}}((l_{i+1})^{-1})\ \mathbf{n} + \frac{1}{l_{i+1}}\frac{\partial}{\partial \mathbf{x_{i}}}(\mathbf{n})\right) \\
&= -\frac{M_i}{2}\left[\left(\frac{\mathbf{U_{i+1}}\cdot\mathbf{e_3}}{l_{i+1}^3}\ \mathbf{e_3}\right)\ \mathbf{n}\right] \\
\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{f_{i+1}}) &= \frac{M_i}{2}\left(\frac{\partial}{\partial \mathbf{x_{i+1}}}((l_{i+1})^{-1})\ \mathbf{n} + \frac{1}{l_{i+1}}\frac{\partial}{\partial \mathbf{x_{i+1}}}(\mathbf{n})\right) \\
&= \frac{M_i}{2}\left[\left(\frac{\mathbf{U_{i+1}}\cdot\ \mathbf{e_3}}{(l_{i+1})^3}\left[\mathbf{e_3} - \frac{\mathbf{N}\ \mathbf{U_{i+1}}}{\|\mathbf{x_{i+1}}-\mathbf{x_{i-1}}\|}\right]\right)\mathbf{n} + \frac{1}{l_{i+1}}J\frac{\mathbf{N}}{\|\mathbf{x_{i+1}}-\mathbf{x_{i-1}}\|}\right]
\end{aligned}
$$

A.5 Center of Masses (Alternate proof)

Here we can see a proof of the proposition of the center of mass using potentials.

Proposition The center of masses of an elastic body in an isotropic and homogeneous frictional or viscous environment, upon a suitable time discretization of the viscous forces, and contracting by a system of self-equilibrated active forces remains still, i.e. $\mathbf{u}_{CM} = 0$.

Proof. We will prove it when the self-equilibrated active forces are constant.

We first show that our mechanical problem is equivalent to finding the minimum of a potential $V(\mathbf{u})$. We have discretized the time due to the viscous forces, so we have to show that our problem is equivalent to find:

$$
\min_{\mathbf{u}^{k+1}} V(\mathbf{u}^{k+1}) = \min_{\mathbf{u}^{k+1}}\left\{W^b(\mathbf{u}^{k+1}) + W^s(\mathbf{u}^{k+1}) + W^\eta(\mathbf{u}^k,\mathbf{u}^{k+1}) - (\mathbf{u}^{k+1})^t\ \mathbf{f^m}\right\} \tag{48}
$$

with W^b and W^s being the bending and stretching potentials respectively that we have seen in the Methodology section, with $\mathbf{f^m}$ being the vector of moment forces and with W^η being defined as:

$$
W^\eta(\mathbf{u}^k,\mathbf{u}^{k+1}) = \frac{\eta\ h}{2}\left(\frac{\mathbf{u}^{k+1}-\mathbf{u}^k}{h}\right)^t\left(\frac{\mathbf{u}^{k+1}-\mathbf{u}^k}{h}\right) \tag{49}
$$

We are going to show that our problem is indeed equivalent to minimize the term in brackets in equation (48).

On the one hand, we know from the bending section that the derivative of $W^b(\mathbf{x})$ is $\mathbf{g}^b(\mathbf{x})$ and from the stretching section that the derivative of $W^s(\mathbf{x})$ is $\mathbf{g}^s(\mathbf{x})$. On the other hand, we have that

$$
\frac{\partial W^\eta(\mathbf{u}^{k+1})}{\partial \mathbf{u}^{k+1}} = \eta\ \frac{\mathbf{u}^{k+1}-\mathbf{u}^k}{h} = -\mathbf{g}^\eta(\mathbf{u}^{k+1}) \tag{50}
$$

Finally, if we derive $(\mathbf{u}^{k+1})^t\ \mathbf{f}^\mathbf{m}$ with respect to $\mathbf{u}^{k+1}$ we obtain $\mathbf{f}^\mathbf{m}$ (remember that as hypothesis we have assume that the self-equilibrated active forces are constant). Therefore we would arrive to the equation:

$$\mathbf{g}^\mathbf{b}(\mathbf{u}^{k+1}) + \mathbf{g}^\mathbf{s}(\mathbf{u}^{k+1}) - \mathbf{g}^\eta(\mathbf{u}^{k+1}) - \mathbf{g}^\mathbf{m} = \mathbf{0} \tag{51}$$

which is the balance equation that we have to solve in our problem if active forces are constant (independent of displacements).

Now, we will decompose the optimal solution $\mathbf{u}^* = \bar{\mathbf{u}} + \delta\mathbf{u}$ with $\bar{\mathbf{u}}$ being a constant vector field equal to the mean value of $\mathbf{u}^*$, i.e.

$$\bar{\mathbf{u}} = \frac{1}{|\Gamma|}\int_\Gamma \rho\ \mathbf{u}^* d\Gamma = \frac{\displaystyle\sum_{i=1}^{n+1} m_i\ \mathbf{u}_i}{\displaystyle\sum_{i=1}^{n+1} m_i} \tag{52}$$

where Γ is the domain of the rod, and due to the discretization, we equalise to the last term. Therefore the mean value of $\delta\mathbf{u}$ is 0, that is $\overline{\delta\mathbf{u}} = \mathbf{0}$.

We will show that $V(\delta\mathbf{u}) < V(\bar{\mathbf{u}} + \delta\mathbf{u})$ for $\|\mathbf{u}\| > 0$ and therefore, the optimization problem must satisfy $\bar{\mathbf{u}} = \mathbf{0}$.

Notice that:

$$(\bar{\mathbf{u}}^{k+1})^t\ \mathbf{f}^\mathbf{m} = (\bar{\mathbf{u}}^x\mathbf{e_x} + \bar{\mathbf{u}}^y\mathbf{e_y})^t \begin{Bmatrix} \dots \\ \mathbf{f}^\mathbf{m}_\mathbf{i-1} \\ \mathbf{f}^\mathbf{m}_\mathbf{i} \\ \mathbf{f}^\mathbf{m}_\mathbf{i+1} \\ \dots \end{Bmatrix} = 0 \tag{53}$$

where $\mathbf{e_x} = \{\mathbf{1}\ \mathbf{0}\ \dots\ \mathbf{1}\ \mathbf{0}\}^t$ and $\mathbf{e_y} = \{\mathbf{0}\ \mathbf{1}\ \dots\ \mathbf{0}\ \mathbf{1}\}^t$ and both are constant values of all the components of $\bar{\mathbf{u}}$.

We also have that the potentials of the bending and stretching energy are invariant by a translation, as we have seen in *Remark 1* before the proposition in the Methodology section.

Finally, we will see the viscosity pseudo-potential. Since we are considering the incremental displacement within a time-step, and without loss of generality, we can assume that $\mathbf{u}^k = \mathbf{0}$. Therefore:

$$W^\eta(\mathbf{u}^k, \mathbf{u}^{k+1}) = \frac{\eta h}{2}\left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{h}\right)^t\left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{h}\right) = \frac{\eta}{2}\left(\frac{\left(\mathbf{u}^{k+1}\right)^t\ \mathbf{u}^{k+1}}{h}\right) =: W^\eta(\mathbf{u}^{k+1}) \tag{54}$$

Hence:

$$
\begin{aligned}
W^{\eta}(\bar{\mathbf{u}}^{k+1} + \delta\mathbf{u}^{k+1}) &= \frac{\eta}{2\ h}\left(\bar{\mathbf{u}}^{k+1} + \delta\mathbf{u}^{k+1}\right)^t \left(\bar{\mathbf{u}}^{k+1} + \delta\mathbf{u}^{k+1}\right) \\
&= \frac{\eta}{2\ h}\left((\bar{\mathbf{u}}^{k+1})^t\ \bar{\mathbf{u}}^{k+1} + 2(\bar{\mathbf{u}}^{k+1})^t\ \delta\mathbf{u}^{k+1}\right) + (\delta\mathbf{u}^{k+1})^t\ \delta\mathbf{u}^{k+1} \\
&= \frac{\eta}{2\ h}\|\bar{\mathbf{u}}^{k+1}\|^2 + W^{\eta}(\delta\mathbf{u}^{k+1})
\end{aligned}
\tag{55}
$$

where we have applied that $\bar{\mathbf{u}}^t\ \delta\mathbf{u} = 0$ because the mean value of $\delta\mathbf{u}$ is zero and $\bar{\mathbf{u}}$ is a constant vector.

Therefore, if we add all together, we have that at each increment k of the Backward Euler method we have:

$$
\begin{aligned}
V(\bar{\mathbf{u}} + \delta\mathbf{u}) &= W^b(\delta\mathbf{u}) + W^s(\delta\mathbf{u}) + \frac{\eta}{2\ h}\|\bar{\mathbf{u}}^{k+1}\|^2 + W^{\eta}(\delta\mathbf{u}) - (\delta\mathbf{u})^t\ \mathbf{f^m} \\
&= V(\delta\mathbf{u}) + \frac{\eta}{2\ h}\|\bar{\mathbf{u}}^{k+1}\|^2 > V(\delta\mathbf{u})
\end{aligned}
\tag{56}
$$

for all $\|\bar{\mathbf{u}}\| > 0$.

Hence, as we have said before, we have from the optimality condition of the solution that $\bar{\mathbf{u}} = \mathbf{0}$, which implies that the center of mass of the rod remains fixed for each increment, and therefore also throughout the motion..

We can extend this proof for the case when the self-equilibrated active forces are displacement dependent $\mathbf{f}^m(\mathbf{u})$ linearizing the non-linear system and applying a similar proof of the independent-displacement case for each iteration.

□