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An a priori error analysis of a Lord-Shulman poro-thermoelastic problem with microtemperatures

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Abstract In this paper we deal with the numerical analysis of the Lord-Shulman thermoelastic problem with porosity and microtemperatures. The thermomechanical problem leads to a coupled system composed of linear hyperbolic partial differential equations written in terms of transformations of the displacement field and the volume fraction, the temperature and the microtemperatures. An existence and uniqueness result is stated. Then, a fully discrete approximation is introduced using the finite element method and the implicit Euler scheme. A discrete stability property is shown, and an a priori error

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analysis is provided, from which the linear convergence is derived under suitable regularity conditions. Finally, some numerical simulations are presented to demonstrate the accuracy of the approximation, the comparison with the classical Fourier theory and the behavior of the solution in two-dimensional examples.

Keywords Thermoelasticity · microtemperatures · Lord-Shulman · finite elements · discrete stability · a priori estimates

1 Introduction

Causality principle fails when we consider the heat equation based on the Fourier law and the usual heat conduction:

$$c\dot{\theta} = q_{i,i},$$

where θ is the temperature, c is the thermal capacity and q_i is the heat flux vector. Indeed, this fact is well known because this theory allows the existence of thermal waves propagating instantaneously. To overcome this drawback, several scientists have proposed alternative constitutive laws to the classical Fourier law. The most known is the one introduced by Cattaneo and Maxwell [1] which propose a relaxation parameter to consider the constitutive law

$$\tau\dot{q}_i + q_i = \kappa\theta_{,i},$$

where $\tau > 0$ is the relaxation parameter usually assumed small and κ is the thermal conductivity. It is also well known that this theory has been extended to consider thermoelastic effects and it is known as Lord-Shulman thermoelasticity [2].

Firsts theories of materials with microstructure date back to the begin of the past century. They received a big impulse from people like Eringen, Maugin or Ieşan [3,4] (among others) in the second part of that century and the beginning of the current one. The basic idea is that we can see the microstructure as a level where several deformations and temperature can be produced. The applicability of these materials makes that they are currently in fashion and many researchers consider these effects in their studies. One possible effect for the microstructure is the so-called *microtemperatures* [5–8]. In fact, many studies have been developed to understand the qualitative behavior of this kind of materials [9–23]. However, almost whenever microtemperatures effects we find again a parabolic system and therefore these effects also propagate with unbounded speed and we have another model where the causality principle is violated. In this contribution, we propose to save this drawback in the same way that Maxwell and Cattaneo did in the case of the heat conduction. That is, we will consider another relaxation parameter.

Another effect related to the microstructure is the porosity. Elastic materials with voids were proposed by Cowin and Nunziato [24–26]. This theory try to model solids with small distributed porous. Rocks, soils, woods, ceramics or biological materials as bones are examples where this theory can be applied. We can recall several contributions concerning this theory [27–40, 20, 41–43].

In this paper, we want to study thermo-porous-elastic materials with microtemperatures where the heat conduction is determined by the Cattaneo-Maxwell proposition, and where the microtemperatures satisfy a similar proposition. Therefore, it should be suitable to recall the evolution equations. They are:

$$\begin{aligned}
&\rho \ddot{u}_i = t_{ij,j} + \rho f_i,\\
&J\ddot{\phi} = h_{i,i} + g + \rho l,\\
&\rho T_0 \dot{\eta} = q_{i,i} + \rho s,\\
&\rho \dot{\epsilon}_i = q_{ij,j} + q_i - Q_i - \rho G_i.
\end{aligned}$$

In this system, ρ means the mass density, u_i is the displacement vector, t_{ij} is the stress tensor, J is the equilibrated inertia, η is the entropy. ϵ_i is the first heat flux moment, T_0 is the reference temperature, h_i is the equilibrated stress vector, g is the equilibrated body force, Q_i is the microheat flux average, q_{ij} is the first heat flux moment tensor, and f_i, l, s, G_i are supply terms.

For the constitutive equations, we consider

$$\begin{aligned}
&t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + \mu_0 \phi \delta_{ij} - \beta_0 \theta \delta_{ij},\\
&h_i = a_0 \phi_{,i} - \mu_2 T_i,\\
&g = -\mu_0 e_{ii} - \xi \phi + \beta_1 \theta,\\
&\rho \eta = \beta_0 e_{ii} + \beta_1 \phi + a\theta,\\
&\rho \epsilon_i = -\mu_2 \phi_{,i} - bT_i,
\end{aligned}$$

where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the strain tensor and δ_{ij} is the Kronecker symbol.

For the thermal and microthermal effects, we propose the following equations:

$$\begin{aligned}
&\tau_1 \dot{q}_i + q_i = \kappa \theta_{,i} + \kappa_1 T_i,\\
&\tau_2 \dot{q}_{ij} + q_{ij} = -\kappa_4 T_{r,r}\delta_{ij} - \kappa_5 T_{j,i} - \kappa_6 T_{i,j},\\
&\tau_2 \dot{Q}_i + Q_i = (\kappa - \kappa_3)\theta_{,i} + (\kappa_1 - \kappa_2) T_i.
\end{aligned}$$

Here, τ_1 and τ_2 are the relaxation parameters which are assumed small, but positive, and $\kappa_i,\ i = 1, \dots, 6$ are constitutive parameters. We point out that, in general, we can assume that τ_1 and τ_2 are different. However, as this contribution is pioneering in this sense, we want to study the easier case which corresponds to assume that $\tau_1 = \tau_2 = \tau$. After introduction of the constitutive equations into the evolutions equations, and assuming that $T_0 = 1$ to simplify the calculations,[1] we obtain the system proposed in the second section.

It is worth noting that this system was studied in a qualitative way in a recent paper [44]. It was obtained existence and stability results. Therefore, we also recall the kind of assumption they are imposed in this case. They are:

$$\begin{aligned}
&\rho > 0, \quad J > 0, \quad a > 0, \quad b > 0, \quad \mu > 0, \quad 3\lambda + 2\mu > 0, \quad a_0 > 0,\\
&(3\lambda + 2\mu)\xi > 3\mu_0^2, \quad \kappa > 0, \quad 3\kappa_4 + \kappa_5 + \kappa_6 > 0, \quad \kappa_5 + \kappa_6 > 0,\\
&\kappa_6 - \kappa_5 > 0, \quad (\kappa_1 + \kappa_3)^2 < 4\kappa\kappa_3.
\end{aligned} \tag{1}$$

[1] This assumption is not relevant in the analysis proposed here. It only allows to simplify the calculations

Hence, in this work we continue the research started in [44]. We introduce a fully discrete approximation by using the finite element method and the implicit Euler scheme, we prove a discrete stability property, we obtain a priori error estimates, from which the linear convergence of the approximation is derived under suitable regularity conditions, and we perform some numerical simulations in one and two dimensions.

2 The model and its variational formulation

In this section, we present the mathematical model, we derive its variational formulation and we state an existence and uniqueness result (see [44]).

Let us consider an elastic solid determined by a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, with boundary $\Gamma = \partial\Omega$ so smooth to apply the divergence theorem in the two- or three-dimensional case. Moreover, let $[0, T_f]$, $T_f > 0$, be the time interval of interest.

Let $\boldsymbol{u} = (u_i)_{i=1}^d$, ϕ, θ and $\boldsymbol{T} = (T_i)_{i=1}^d$ be the displacement, the volume fraction, the temperature and the thermal displacement, respectively.

According to [44], for isotropic and homogeneous materials the system of equations becomes, in $\Omega \times (0, T_f)$ and for $i = 1, \dots, d$,

$$
\begin{aligned}
&\rho\ddot{u}_i = \mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \mu_0\phi_{,i} - \beta_0\theta_{,i} + \rho f_i,\\
&J\ddot{\phi} = a_0\phi_{,jj} - \mu_2 T_{i,i} - \mu_0 u_{i,i} - \xi\phi + \beta_1\theta + \rho l,\\
&\tau a\ddot{\theta} + a\dot{\theta} = -\tau\beta_0\ddot{u}_{i,i} - \beta_0\dot{u}_{i,i} - \tau\beta_1\ddot{\phi} - \beta_1\dot{\phi} + \kappa\theta_{,jj} + \kappa_1 T_{i,i} + \rho\hat{s},\\
&\tau b\ddot{T}_i + b\dot{T}_i = -\tau\mu_2\ddot{\phi}_{,i} - \mu_2\dot{\phi}_{,i} + \kappa_6 T_{i,jj} + (\kappa_4+\kappa_5)T_{j,ji} - \kappa_2 T_i - \kappa_3\theta_{,i} + \rho\hat{G}_i,
\end{aligned}
$$

where $\boldsymbol{f} = (f_i)_{i=1}^d$, s, l, and $\boldsymbol{G} = (G_i)_{i=1}^d$ are supply forces, ρ is the density of the material, λ and μ denote the classical Lame's coefficients, J is the product of the mass density by the equilibrated inertia, a_0 is the porosity diffusion parameter, κ represents the thermal diffusion parameter, a is the heat capacity and μ_0, β_0, μ_2, ξ, β_1, κ_1, b, κ_6, κ_4, κ_5, κ_2 and κ_3 are constitutive parameters. $\tau > 0$ is the relaxation parameter introduced by the Lord-Shulman theory. Moreover, we used the notation $\hat{f} = f + \tau\dot{f}$.

Therefore, the previous system takes the form:

$$
\begin{aligned}
&\rho\ddot{\hat{u}}_i = \mu\hat{u}_{i,jj} + (\lambda+\mu)\hat{u}_{j,ji} + \mu_0\hat{\phi}_{,i} - \beta_0(\tau\dot{\theta}_{,i} + \theta_{,i}) + \rho\hat{f}_i,\\
&J\ddot{\hat{\phi}} = a_0\hat{\phi}_{,jj} - \mu_2(\tau\dot{T}_{i,i} + T_{i,i}) - \mu_0\hat{u}_{i,i} - \xi\hat{\phi} + \beta_1(\tau\dot{\theta} + \theta) + \rho\hat{l},\\
&\tau a\ddot{\theta} + a\dot{\theta} = -\beta_0\dot{\hat{u}}_{i,i} - \beta_1\dot{\hat{\phi}} + \kappa\theta_{,jj} + \kappa_1 T_{i,i} + \rho\hat{s},\\
&\tau b\ddot{T}_i + b\dot{T}_i = -\mu_2\dot{\hat{\phi}}_{,i} + \kappa_6 T_{i,jj} + (\kappa_4+\kappa_5)T_{j,ji} - \kappa_2 T_i - \kappa_3\theta_{,i} + \rho\hat{G}_i.
\end{aligned}
$$

It is clear that from the solutions to this system we obtain the solutions to the primitive system. Therefore, in order to simplify the notation, we drop the hat.

To study a well posed problem we should impose initial and boundary conditions. That is, we assume that, for $i=1,\dots,d$,

$$
\begin{aligned}
&u_i(\boldsymbol{x},0)=u_{0i}(\boldsymbol{x}),\quad \dot u_i(\boldsymbol{x},0)=v_{0i}(\boldsymbol{x}),\quad \phi(\boldsymbol{x},0)=\phi_0(\boldsymbol{x})\quad \text{for a.e. } \boldsymbol{x}\in\Omega,\\
&\dot\phi(\boldsymbol{x},0)=e_0(\boldsymbol{x}),\quad \theta(\boldsymbol{x},0)=\theta_0(\boldsymbol{x}),\quad \dot\theta(\boldsymbol{x},0)=\xi_0(\boldsymbol{x})\quad \text{for a.e. } \boldsymbol{x}\in\Omega,\\
&T_i(\boldsymbol{x},0)=T_{0i}(\boldsymbol{x}),\quad \dot T_i(\boldsymbol{x},0)=M_{0i}(\boldsymbol{x})\quad \text{for a.e. } \boldsymbol{x}\in\Omega,\\
&u_i(\boldsymbol{x},t)=\phi(\boldsymbol{x},t)=\theta(\boldsymbol{x},t)=T_i(\boldsymbol{x},t)=0\quad \text{for a.e. } \boldsymbol{x}\in\partial\Omega,\ t\in[0,T_f],
\end{aligned}
$$

where $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, ϕ_0, e_0, θ_0, ξ_0, $\boldsymbol{T}_0$ and $\boldsymbol{M}_0$ are given initial conditions.

We note that we have assumed homogeneous Dirichlet boundary conditions for the sake of simplicity. The analysis provided in the next section could be extended to a more general case in a straightforward way.

In order to obtain the variational formulation of the above thermomechanical problem, let $Y=L^2(\Omega)$, $H=[L^2(\Omega)]^d$ and $Q=[L^2(\Omega)]^{d\times d}$, and denote by $(\cdot,\cdot)_Y$, $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Q$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$, $\|\cdot\|_H$ and $\|\cdot\|_Q$. Moreover, let us define the variational spaces E and V as follows,

$$
\begin{aligned}
E&=\{z\in H^1(\Omega)\,;\ z=0\quad \text{on}\quad \Gamma\},\\
V&=\{\boldsymbol{w}\in[H^1(\Omega)]^d\,;\ \boldsymbol{w}=\boldsymbol{0}\quad \text{on}\quad \Gamma\},
\end{aligned}
$$

with respective scalar products $(\cdot,\cdot)_E$ and $(\cdot,\cdot)_V$, and norms $\|\cdot\|_E$ and $\|\cdot\|_V$.

By using Green's formula and taking into account the above boundary conditions, we write the variational formulation of the thermomechanical problem in terms of variables $\boldsymbol{v}=\dot{\boldsymbol{u}}$ and $e=\dot\phi$, the temperature speed $\xi=\dot\theta$ and the microtemperatures $\boldsymbol{M}=\dot{\boldsymbol{T}}$.

Problem VP. *Find the function $\boldsymbol{v}:[0,T_f]\to V$, the function $e:[0,T_f]\to E$, the temperature speed $\xi:[0,T_f]\to E$ and the microtemperatures $\boldsymbol{M}:[0,T_f]\to V$ such that $\boldsymbol{v}(0)=\boldsymbol{v}_0$, $e(0)=e_0$, $\xi(0)=\xi_0$ and $\boldsymbol{M}(0)=\boldsymbol{M}_0$, and, for a.e. $t\in(0,T_f)$ and $\boldsymbol{w}$, $\boldsymbol{\psi}\in V$, r, $z\in E$,*

$$
\begin{aligned}
&\rho(\dot{\boldsymbol{v}}(t),\boldsymbol{w})_H+\mu(\nabla\boldsymbol{u}(t),\nabla\boldsymbol{w})_Q+(\lambda+\mu)(\operatorname{div}\boldsymbol{u}(t),\operatorname{div}\boldsymbol{w})_Y\\
&\qquad-\mu_0(\nabla\phi(t),\boldsymbol{w})_H+\beta_0(\nabla(\tau\xi(t)+\theta(t)),\boldsymbol{w})_H=\rho(\boldsymbol{f}(t),\boldsymbol{w})_H, \qquad (2)\\
&(J\dot e(t),r)_Y+a_0(\nabla\phi(t),\nabla r)_H+\xi(\phi(t),r)_Y=-\mu_2(\tau\operatorname{div}\boldsymbol{M}(t)+\operatorname{div}\boldsymbol{T}(t),r)_Y\\
&\qquad-\mu_0(\operatorname{div}\boldsymbol{u}(t),r)_Y+\beta_1(\tau\xi(t)+\theta(t),r)_Y+\rho(l(t),r)_Y, \qquad (3)\\
&(\tau a\dot\xi(t)+a\xi(t),z)_Y+\kappa(\nabla\theta(t),\nabla z)_H=-\beta_0(\operatorname{div}\boldsymbol{v}(t),z)_Y-\beta_1(e(t),z)_Y\\
&\qquad+\kappa_1(\operatorname{div}\boldsymbol{T}(t),z)_Y+\rho(s(t),z)_Y, \qquad (4)\\
&(\tau b\dot{\boldsymbol{M}}(t)+b\boldsymbol{M}(t),\boldsymbol{\psi})_H+\kappa_6(\nabla\boldsymbol{T}(t),\nabla\boldsymbol{\psi})_Q+(\kappa_4+\kappa_5)(\operatorname{div}\boldsymbol{T}(t),\operatorname{div}\boldsymbol{\psi})_Y\\
&\qquad+\kappa_2(\boldsymbol{T}(t),\boldsymbol{\psi})_H=-\kappa_3(\nabla\theta(t),\boldsymbol{\psi})_H-\mu_2(\nabla e(t),\boldsymbol{\psi})_H+\rho(\boldsymbol{G}(t),\boldsymbol{\psi})_H, \qquad (5)
\end{aligned}
$$

where functions $\boldsymbol{u}$, ϕ, θ and $\boldsymbol{T}$ are then recovered from the relations:

$$
\begin{aligned}
\boldsymbol{u}(t)&=\int_0^t\boldsymbol{v}(s)\,ds+\boldsymbol{u}_0,\quad \phi(t)=\int_0^t e(s)\,ds+\phi_0,\\
\boldsymbol{T}(t)&=\int_0^t\boldsymbol{M}(s)\,ds+\boldsymbol{T}_0,\quad \theta(t)=\int_0^t\xi(s)\,ds+\theta_0.
\end{aligned}
\qquad (6)
$$

In [44] we proved the following existence and uniqueness result.

Theorem 1 *Assume that the coefficients satisfy the properties (1), the regularity on the supply forces, for $i = 1, \ldots, d$,*

$$f_i,\ s,\ l,\ G_i \in C^1([0,T_f];Y) \cap C([0,T_f];H^1(\Omega)), \tag{7}$$

and the following regularity on the initial conditions:

$$\boldsymbol{u}_0, \boldsymbol{v}_0, \boldsymbol{T}_0, \boldsymbol{M}_0 \in [H^2(\Omega)]^d, \quad e_0, \phi_0, \theta_0, \xi_0 \in H^2(\Omega). \tag{8}$$

Then, there exists a unique solution to Problem VP with the regularity:

$$\boldsymbol{v}, \boldsymbol{M} \in C([0,T_f];V) \cap C^1([0,T_f];H), \quad \xi, e \in C([0,T_f];E) \cap C^1([0,T_f];Y).$$

In fact, we shown that in the general case the solutions to Problem VP are generated by a quasi-contractive semigroup even in the non-isotropic and non-homogeneous case. In the particular case that we assume the Onsager Postulate (which implies in our case that $k_1 = k_3$), the energy of the system dissipates and the semigroup is contractive. In this situation, we have guaranteed the stability of solutions when the supply terms vanish and the decay of the solutions can be expected in the generic case. Therefore, it is natural to ask ourselves by the *rate of decay* of the solutions. In the reference [44] we analyzed this aspect for the one-dimensional case and we proved (by means of the semigroup arguments) that the solutions decay in an exponential way. That is, the energy of the system can be controlled by an exponential function. Certainly, the analysis of the energy decay should be different in the case that the dimension is greater than one, but in view of the well-known results for the classical thermoelasticity one cannot expect the exponential decay for dimensions two or three.

In fact, for the multidimensional case the picture is much more complicated. As far as we know, in this case only the classical thermoelasticity and type III thermoelasticity have been studied [45,46]. In general, we cannot expect that the thermal dissipation would be a strong enough mechanism to stabilize in an exponential way the thermoelastic solutions. What it is known until this moment in this case is:

1. For suitable domains there exist undamped isothermal solutions.
2. Generically we can expect asymptotic stability.
3. For most domains the exponential decay cannot expected.
4. Exponential stability can be obtained whenever the domain satisfies a condition that may described in terms of Geometric Optics (see [45,46]).
5. For most two-dimensional domains the energy of smooth solutions decays polynomially.

One thinks that, in our case, some of these facts also hold. For instance, it is clear that points 1,2 and 3 would hold; however, this study has not been developed yet even in the case of the Lord Shulman thermoelasticity (without porous neither microtemperatures effects). The difficulty to study this problem increases if we take into account that the heat equation is of hyperbolic type. We really believe that the multi-dimensional problem is a task which requires a big and large work that we cannot develop here.

3 Fully discrete approximations: an a priori error analysis

In this section, we will analyze a finite element algorithm for the approximation of Problem VP. Therefore, we consider the finite element spaces $V^h \subset V$and $E^h \subset E$ given by

$$V^h = \{\boldsymbol{w}^h \in [C(\overline{\Omega})]^d \; ; \; \boldsymbol{w}^h_{|_{Tr}} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h, \quad \boldsymbol{w}^h = \boldsymbol{0} \text{ on } \Gamma\}, \quad (9)$$

$$E^h = \{r^h \in C(\overline{\Omega}) \; ; \; r^h_{|_{Tr}} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad r^h = 0 \text{ on } \Gamma\}, \quad (10)$$

where Ω is assumed to be a polyhedral domain, $\mathcal{T}^h$ denotes a triangulation of $\overline{\Omega}$, and $P_1(Tr)$ represents the space of polynomials of global degree less or equal to 1 in element Tr. Here, $h > 0$ denotes the spatial discretization parameter.

In order to discretize the time derivatives, we use a uniform partition of the time interval $[0, T_f]$ denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, with time step $k = T/N$. Moreover, for a continuous function $f(t)$ we denote $f_n = f(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Using the classical backward Euler scheme, the fully discrete approximation of Problem VP is the following.

Problem VP^{hk}. *Find the discrete function $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$, the discrete function $e^{hk} = \{e_n^{hk}\}_{n=0}^N \subset E^h$, the discrete temperature speed $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset E^h$ and the discrete microtemperatures $\boldsymbol{M}^{hk} = \{\boldsymbol{M}_n^{hk}\}_{n=0}^N \subset V^h$ such that $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h$, $e_0^{hk} = e_0^h$, $\xi_0^{hk} = \xi_0^h$ and $\boldsymbol{M}_0^{hk} = \boldsymbol{M}_0^h$, and, for all $n = 1, \ldots, N$ and $\boldsymbol{w}^h$, $\boldsymbol{\psi}^h \in V^h$, r^h, $z^h \in E^h$,*

$$\begin{aligned}
&\rho(\delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + \mu(\nabla \boldsymbol{u}_n^{hk}, \nabla \boldsymbol{w}^h)_Q + (\lambda + \mu)(\operatorname{div} \boldsymbol{u}_n^{hk}, \operatorname{div} \boldsymbol{w}^h)_Y \\
&\quad -\mu_0(\nabla \phi_n^{hk}, \boldsymbol{w}^h)_H + \beta_0(\nabla(\tau \xi_n^{hk} + \theta_n^{hk}), \boldsymbol{w}^h)_H = \rho(\boldsymbol{f}_n, \boldsymbol{w}^h)_H, \quad (11)\\
&(J\delta e_n^{hk}, r^h)_Y + a_0(\nabla \phi_n^{hk}, \nabla r^h)_H + \xi(\phi_n^{hk}, r^h)_Y = -\mu_2(\tau \operatorname{div} \boldsymbol{M}_n^{hk} + \operatorname{div} \boldsymbol{T}_n^{hk}, r^h)_Y \\
&\quad -\mu_0(\operatorname{div} \boldsymbol{u}_n^{hk}, r^h)_Y + \beta_1(\tau \xi_n^{hk} + \theta_n^{hk}, r^h)_Y + \rho(l_n, r^h)_Y, \quad (12)\\
&(\tau a \delta \xi_n^{hk} + a \xi_n^{hk}, z^h)_Y + \kappa(\nabla \theta_n^{hk}, \nabla z^h)_H = -\beta_0(\operatorname{div} \boldsymbol{v}_n^{hk}, z^h)_Y - \beta_1(e_n^{hk}, z^h)_Y \\
&\quad +\kappa_1(\operatorname{div} \boldsymbol{T}_n^{hk}, z^h)_Y + \rho(s_n, z^h)_Y, \quad (13)\\
&(\tau b \delta \boldsymbol{M}_n^{hk} + b \boldsymbol{M}_n^{hk}, \boldsymbol{\psi}^h)_H + \kappa_6(\nabla \boldsymbol{T}_n^{hk}, \nabla \boldsymbol{\psi}^h)_Q + (\kappa_4 + \kappa_5)(\operatorname{div} \boldsymbol{T}_n^{hk}, \operatorname{div} \boldsymbol{\psi}^h)_Y \\
&\quad +\kappa_2(\boldsymbol{T}_n^{hk}, \boldsymbol{\psi}^h)_H = -\kappa_3(\nabla \theta_n^{hk}, \boldsymbol{\psi}^h)_H - \mu_2(\nabla e_n^{hk}, \boldsymbol{\psi}^h)_H + \rho(\boldsymbol{G}_n, \boldsymbol{\psi}^h)_H, \quad (14)
\end{aligned}$$

where discrete functions $\boldsymbol{u}_n^{hk}$, ϕ_n^{hk}, θ_n^{hk} and $\boldsymbol{T}_n^{hk}$ are then recovered from the relations:

$$\begin{aligned} \boldsymbol{u}_n^{hk} &= k\sum_{j=1}^{n}\boldsymbol{v}_j^{hk} + \boldsymbol{u}_0^h, \quad \phi_n^{hk} = k\sum_{j=1}^{n} e_j^{hk} + \phi_0^h, \\ \boldsymbol{T}_n^{hk} &= k\sum_{j=1}^{n}\boldsymbol{M}_j^{hk} + \boldsymbol{T}_0^h, \quad \theta_n^{hk} = k\sum_{j=1}^{n}\xi_j^{hk} + \theta_0^h, \end{aligned} \tag{15}$$

and the discrete initial conditions $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$, ϕ_0^h, e_0^h, θ_0^h, ξ_0^h, $\boldsymbol{T}_0^h$ and $\boldsymbol{M}_0^h$ are defined as follows,

$$\begin{aligned} &\boldsymbol{u}_0^h = \mathcal{P}^{1h}\boldsymbol{u}_0, \quad \boldsymbol{v}_0^h = \mathcal{P}^{1h}\boldsymbol{v}_0, \quad \phi_0^h = \mathcal{P}^{2h}\phi_0, \quad e_0^h = \mathcal{P}^{2h}e_0, \\ &\theta_0^h = \mathcal{P}^{2h}\theta_0, \quad \xi_0^h = \mathcal{P}^{2h}\xi_0, \quad \boldsymbol{T}_0^h = \mathcal{P}^{1h}\boldsymbol{T}_0, \quad \boldsymbol{M}_0^h = \mathcal{P}^{1h}\boldsymbol{M}_0. \end{aligned} \tag{16}$$

In the previous definitions, operators $\mathcal{P}^{1h}$ and $\mathcal{P}^{2h}$ are the projection operators over the finite element spaces V^h and E^h, respectively (see, for instance, [49]).

We point out that it is straightforward to obtain the existence and uniqueness of solution to Problem VP^{hk} using classical results on linear variational equations and conditions (1).

In order to provide the numerical analysis of the above discrete problem, we will prove the following stability result.

Lemma 1 *Under the assumptions of Theorem 1 and the following additional conditions on the constitutive coefficients:*

$$\lambda + \mu > 0, \quad (\lambda+\mu)\xi > \mu_0^2, \quad \kappa_4 + \kappa_5 > 0, \quad 4\kappa\kappa_2 > (\kappa_1+\kappa_3)^2, \tag{17}$$

it follows that the sequences $\{\boldsymbol{u}^{hk}, \boldsymbol{v}^{hk}, \phi^{hk}, e^{hk}, \theta^{hk}, \xi^{hk}, \boldsymbol{T}^{hk}, \boldsymbol{M}^{hk}\}$, generated by Problem VP^{hk}, satisfy the stability estimate:

$$\begin{aligned} &\|\nabla\boldsymbol{u}_n^{hk}\|_Q + \|\operatorname{div}\boldsymbol{u}_n^{hk}\|_Y + \|\boldsymbol{v}_n^{hk}\|_H + \|\nabla\phi_n^{hk}\|_H + \|\phi_n^{hk}\|_Y + \|e_n^{hk}\|_Y + \|\theta_n^{hk}\|_E \\ &\quad + \|\xi_n^{hk}\|_Y + \|\boldsymbol{T}_n^{hk}\|_H + \|\nabla\boldsymbol{T}_n^{hk}\|_Q + \|\operatorname{div}\boldsymbol{T}_n^{hk}\|_Y + \|\boldsymbol{M}_n^{hk}\|_H \le C, \end{aligned}$$

where C is a positive constant assumed to be independent of the discretization parameters h and k.

Remark 1 We note that additional conditions (17) are introduced to simplify the calculations in the following proof. It could be adapted to use conditions (1).

Proof Here, we assume that $\tau = 1$ for the sake of simplicity. We note that we can extend this result to the general case doing some simple modifications.

First, we estimate the terms on the discrete function $\boldsymbol{v}_n^{hk}$. Taking $\boldsymbol{w}^h = \boldsymbol{v}_n^{hk}$ as a test function in discrete variational equation (11) it follows that

$$\begin{aligned} &\rho(\delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n^{hk})_H + \mu(\nabla\boldsymbol{u}_n^{hk}, \nabla\boldsymbol{v}_n^{hk})_Q + (\lambda+\mu)(\operatorname{div}\boldsymbol{u}_n^{hk}, \operatorname{div}\boldsymbol{v}_n^{hk})_Y \\ &\quad -\mu_0(\nabla\phi_n^{hk}, \boldsymbol{v}_n^{hk})_H + \beta_0(\nabla(\xi_n^{hk}+\theta_n^{hk}), \boldsymbol{v}_n^{hk})_H = \rho(\boldsymbol{f}_n, \boldsymbol{v}_n^{hk})_H. \end{aligned}$$

So, using the estimates

$$
\begin{aligned}
&(\delta \boldsymbol{v}_n^{hk},\boldsymbol{v}_n^{hk})_H \geq \frac{1}{2k}\left\{\|\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\},\\
&(\nabla \boldsymbol{u}_n^{hk},\nabla \boldsymbol{v}_n^{hk})_Q \geq \frac{1}{2k}\left\{\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2-\|\nabla \boldsymbol{u}_{n-1}^{hk}\|_Q^2\right\},\\
&(\operatorname{div}\boldsymbol{u}_n^{hk},\operatorname{div}\boldsymbol{v}_n^{hk})_Y = \frac{1}{2k}\left\{\|\operatorname{div}\boldsymbol{u}_n^{hk}\|_Y^2-\|\operatorname{div}\boldsymbol{u}_{n-1}^{hk}\|_Y^2+\|\operatorname{div}(\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\},\\
&-\mu_0(\nabla\phi_n^{hk},\boldsymbol{v}_n^{hk})_H=\mu_0(\phi_n^{hk},\operatorname{div}\boldsymbol{v}_n^{hk})_Y,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{\rho}{2k}\left\{\|\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\}+\frac{\mu}{2k}\left\{\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2-\|\nabla \boldsymbol{u}_{n-1}^{hk}\|_Q^2\right\}\\
&\qquad+\frac{\lambda+\mu}{2k}\left\{\|\operatorname{div}\boldsymbol{u}_n^{hk}\|_Y^2-\|\operatorname{div}\boldsymbol{u}_{n-1}^{hk}\|_Y^2+\|\operatorname{div}(\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\}\\
&\qquad+\mu_0(\phi_n^{hk},\operatorname{div}\boldsymbol{v}_n^{hk})_Y+\beta_0(\nabla(\xi_n^{hk}+\theta_n^{hk}),\boldsymbol{v}_n^{hk})_H\leq C\|\boldsymbol{v}_n^{hk}\|_H.
\end{aligned}
$$

Secondly, we obtain the estimates on the discrete function e_n^{hk}. Thus, using $r^h=e_n^{hk}$ as a test function in discrete variational equation (12) it follows that

$$
\begin{aligned}
&(J\delta e_n^{hk},e_n^{hk})_Y+a_0(\nabla\phi_n^{hk},\nabla e_n^{hk})_H+\xi(\phi_n^{hk},e_n^{hk})_Y=-\mu_2(\operatorname{div}\boldsymbol{M}_n^{hk}+\operatorname{div}\boldsymbol{T}_n^{hk},e_n^{hk})_Y\\
&\qquad-\mu_0(\operatorname{div}\boldsymbol{u}_n^{hk},e_n^{hk})_Y+\beta_1(\xi_n^{hk}+\theta_n^{hk},e_n^{hk})_Y+\rho(l_n,e_n^{hk})_Y,
\end{aligned}
$$

and therefore, keeping in mind that

$$
\begin{aligned}
&(\delta e_n^{hk},e_n^{hk})_Y\geq\frac{1}{2k}\left\{\|e_n^{hk}\|_Y^2-\|e_{n-1}^{hk}\|_Y^2\right\},\\
&(\nabla\phi_n^{hk},\nabla e_n^{hk})_H\geq\frac{1}{2k}\left\{\|\nabla\phi_n^{hk}\|_H^2-\|\nabla\phi_{n-1}^{hk}\|_H^2\right\},\\
&(\phi_n^{hk},e_n^{hk})_Y\geq\frac{1}{2k}\left\{\|\phi_n^{hk}\|_Y^2-\|\phi_{n-1}^{hk}\|_Y^2+\|\phi_n^{hk}-\phi_{n-1}^{hk}\|_Y^2\right\},
\end{aligned}
$$

we find that

$$
\begin{aligned}
&\frac{J}{2k}\left\{\|e_n^{hk}\|_Y^2-\|e_{n-1}^{hk}\|_Y^2\right\}+\frac{a_0}{2k}\left\{\|\nabla\phi_n^{hk}\|_H^2-\|\nabla\phi_{n-1}^{hk}\|_H^2\right\}\\
&\qquad+\frac{\xi}{2k}\left\{\|\phi_n^{hk}\|_Y^2-\|\phi_{n-1}^{hk}\|_Y^2+\|\phi_n^{hk}-\phi_{n-1}^{hk}\|_Y^2\right\}+\mu_0(\operatorname{div}\boldsymbol{u}_n^{hk},e_n^{hk})_Y\\
&\quad\leq C\|e_n^{hk}\|_Y-\mu_2(\operatorname{div}\boldsymbol{M}_n^{hk}+\operatorname{div}\boldsymbol{T}_n^{hk},e_n^{hk})_Y+\beta_1(\xi_n^{hk}+\theta_n^{hk},e_n^{hk})_Y.
\end{aligned}
$$

Now, we proceed with the estimates on the discrete temperature speed ξ_n^{hk}. Thus, using $z^h=\xi_n^{hk}$ as a test function in discrete variational equation (13) it follows that

$$
\begin{aligned}
&(a\delta\xi_n^{hk}+a\xi_n^{hk},\xi_n^{hk})_Y+\kappa(\nabla\theta_n^{hk},\nabla\xi_n^{hk})_H=-\beta_0(\operatorname{div}\boldsymbol{v}_n^{hk},\xi_n^{hk})_Y-\beta_1(e_n^{hk},\xi_n^{hk})_Y\\
&\qquad+\kappa_1(\operatorname{div}\boldsymbol{T}_n^{hk},\xi_n^{hk})_Y+\rho(s_n,\xi_n^{hk})_Y,
\end{aligned}
$$

and taking into account that

$$
\begin{aligned}
&(\delta\xi_n^{hk},\xi_n^{hk})_Y\geq\frac{1}{2k}\left\{\|\xi_n^{hk}\|_Y^2-\|\xi_{n-1}^{hk}\|_Y^2\right\},\\
&(\nabla\theta_n^{hk},\nabla\xi_n^{hk})_H\geq\frac{1}{2k}\left\{\|\nabla\theta_n^{hk}\|_H^2-\|\nabla\theta_{n-1}^{hk}\|_H^2+\|\nabla(\theta_n^{hk}-\theta_{n-1}^{hk})\|_H^2\right\},\\
&-\beta_0(\operatorname{div}\boldsymbol{v}_n^{hk},\xi_n^{hk})_Y=\beta_0(\boldsymbol{v}_n^{hk},\nabla\xi_n^{hk})_H,
\end{aligned}
$$

we get

$$\frac{a}{2k}\left\{\|\xi_n^{hk}\|_Y^2-\|\xi_{n-1}^{hk}\|_Y^2\right\}+\frac{\kappa}{2k}\left\{\|\nabla\theta_n^{hk}\|_H^2-\|\nabla\theta_{n-1}^{hk}\|_H^2+\|\nabla(\theta_n^{hk}-\theta_{n-1}^{hk})\|_H^2\right\}$$
$$\le C\|\xi_n^{hk}\|_Y+\beta_0(\boldsymbol{v}_n^{hk},\nabla\xi_n^{hk})_H-\beta_1(e_n^{hk},\xi_n^{hk})_Y+\kappa_1(\operatorname{div}\boldsymbol{T}_n^{hk},\xi_n^{hk})_Y.$$

Finally, we obtain the estimates on the discrete microtemperatures $\boldsymbol{T}_n^{hk}$. Thus, using $\boldsymbol{\psi}^h=\boldsymbol{M}_n^{hk}$ as a test function in discrete variational equation (14) it leads

$$(b\delta\boldsymbol{M}_n^{hk}+b\boldsymbol{M}_n^{hk},\boldsymbol{M}_n^{hk})_H+\kappa_6(\nabla\boldsymbol{T}_n^{hk},\nabla\boldsymbol{M}_n^{hk})_Q+(\kappa_4+\kappa_5)(\operatorname{div}\boldsymbol{T}_n^{hk},\operatorname{div}\boldsymbol{M}_n^{hk})_Y$$
$$+\kappa_2(\boldsymbol{T}_n^{hk},\boldsymbol{M}_n^{hk})_H=-\kappa_3(\nabla\theta_n^{hk},\boldsymbol{M}_n^{hk})_H-\mu_2(\nabla e_n^{hk},\boldsymbol{M}_n^{hk})_H+\rho(\boldsymbol{G}_n,\boldsymbol{M}_n^{hk})_H.$$

So, using the estimates

$$(\delta\boldsymbol{M}_n^{hk},\boldsymbol{M}_n^{hk})_H\ge\frac{1}{2k}\left\{\|\boldsymbol{M}_n^{hk}\|_H^2-\|\boldsymbol{M}_{n-1}^{hk}\|_H^2\right\},$$
$$(\nabla\boldsymbol{T}_n^{hk},\nabla\boldsymbol{M}_n^{hk})_Q\ge\frac{1}{2k}\left\{\|\nabla\boldsymbol{T}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{T}_{n-1}^{hk}\|_Q^2\right\},$$
$$(\operatorname{div}\boldsymbol{T}_n^{hk},\operatorname{div}\boldsymbol{M}_n^{hk})_Y=\frac{1}{2k}\left\{\|\operatorname{div}\boldsymbol{T}_n^{hk}\|_Y^2-\|\operatorname{div}\boldsymbol{T}_{n-1}^{hk}\|_Y^2+\|\operatorname{div}(\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk})\|_Y^2\right\},$$
$$(\boldsymbol{T}_n^{hk},\boldsymbol{M}_n^{hk})_H\ge\frac{1}{2k}\left\{\|\boldsymbol{T}_n^{hk}\|_H^2-\|\boldsymbol{T}_{n-1}^{hk}\|_H^2\right\},$$
$$-\mu_2(\nabla e_n^{hk},\boldsymbol{M}_n^{hk})_H=\mu_2(e_n^{hk},\operatorname{div}\boldsymbol{M}_n^{hk})_Y,$$

we have

$$\frac{b}{2k}\left\{\|\boldsymbol{M}_n^{hk}\|_H^2-\|\boldsymbol{M}_{n-1}^{hk}\|_H^2\right\}+\frac{\kappa_6}{2k}\left\{\|\nabla\boldsymbol{T}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{T}_{n-1}^{hk}\|_Q^2\right\}+\frac{\kappa_2}{2k}\left\{\|\boldsymbol{T}_n^{hk}\|_H^2-\|\boldsymbol{T}_{n-1}^{hk}\|_H^2\right\}$$
$$+\frac{\kappa_4+\kappa_5}{2k}\left\{\|\operatorname{div}\boldsymbol{T}_n^{hk}\|_Y^2-\|\operatorname{div}\boldsymbol{T}_{n-1}^{hk}\|_Y^2+\|\operatorname{div}(\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk})\|_Y^2\right\}$$
$$\le C\|\boldsymbol{M}_n^{hk}\|_H-\kappa_3(\nabla\theta_n^{hk},\boldsymbol{M}_n^{hk})_H+\mu_2(e_n^{hk},\operatorname{div}\boldsymbol{M}_n^{hk})_Y.$$

Combining all these estimates we obtain

$$\frac{\rho}{2k}\left\{\|\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\}+\frac{\mu}{2k}\left\{\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{u}_{n-1}^{hk}\|_Q^2\right\}$$
$$+\frac{\lambda+\mu}{2k}\left\{\|\operatorname{div}\boldsymbol{u}_n^{hk}\|_Y^2-\|\operatorname{div}\boldsymbol{u}_{n-1}^{hk}\|_Y^2+\|\operatorname{div}(\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\}$$
$$+\mu_0(\phi_n^{hk},\operatorname{div}\boldsymbol{v}_n^{hk})_Y+\mu_0(\operatorname{div}\boldsymbol{u}_n^{hk},e_n^{hk})_Y+\frac{a}{2k}\left\{\|\xi_n^{hk}\|_Y^2-\|\xi_{n-1}^{hk}\|_Y^2\right\}$$
$$+\frac{J}{2k}\left\{\|e_n^{hk}\|_Y^2-\|e_{n-1}^{hk}\|_Y^2\right\}+\frac{a_0}{2k}\left\{\|\nabla\phi_n^{hk}\|_H^2-\|\nabla\phi_{n-1}^{hk}\|_H^2\right\}$$
$$+\frac{\xi}{2k}\left\{\|\phi_n^{hk}\|_Y^2-\|\phi_{n-1}^{hk}\|_Y^2+\|\phi_n^{hk}-\phi_{n-1}^{hk}\|_Y^2\right\}+\frac{\kappa_2}{2k}\left\{\|\boldsymbol{T}_n^{hk}\|_H^2-\|\boldsymbol{T}_{n-1}^{hk}\|_H^2\right\}$$
$$+\frac{\kappa}{2k}\left\{\|\nabla\theta_n^{hk}\|_H^2-\|\nabla\theta_{n-1}^{hk}\|_H^2+\|\nabla(\theta_n^{hk}-\theta_{n-1}^{hk})\|_H^2\right\}$$
$$+\frac{b}{2k}\left\{\|\boldsymbol{M}_n^{hk}\|_H^2-\|\boldsymbol{M}_{n-1}^{hk}\|_H^2\right\}+\frac{\kappa_6}{2k}\left\{\|\nabla\boldsymbol{T}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{T}_{n-1}^{hk}\|_Q^2\right\}$$
$$+\frac{\kappa_4+\kappa_5}{2k}\left\{\|\operatorname{div}\boldsymbol{T}_n^{hk}\|_Y^2-\|\operatorname{div}\boldsymbol{T}_{n-1}^{hk}\|_Y^2+\|\operatorname{div}(\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk})\|_Y^2\right\}$$
$$\le C\Big(1+\|\boldsymbol{v}_n^{hk}\|_H^2+\|\xi_n^{hk}\|_Y^2+\|e_n^{hk}\|_Y^2+\|\operatorname{div}\boldsymbol{T}_n^{hk}\|_Y^2+\|\nabla\theta_n^{hk}\|_H^2$$
$$+\|\theta_n^{hk}\|_Y^2+\|\boldsymbol{M}_n^{hk}\|_H^2\Big).$$

Using conditions (17) we easily find that

$$\mu_0(\phi_n^{hk}, \operatorname{div} \boldsymbol{v}_n^{hk})_Y + \mu_0(\operatorname{div} \boldsymbol{u}_n^{hk}, e_n^{hk})_Y = \frac{\mu_0}{k}\big\{(\phi_n^{hk}, \operatorname{div} \boldsymbol{u}_n^{hk})_Y - (\phi_{n-1}^{hk}, \operatorname{div} \boldsymbol{u}_{n-1}^{hk})_Y$$
$$+(\phi_n^{hk} - \phi_{n-1}^{hk}, \operatorname{div}(\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}))_Y\big\},$$
$$\frac{\lambda+\mu}{2k}\|\operatorname{div}(\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2 + \frac{\xi}{2k}\|\phi_n^{hk} - \phi_{n-1}^{hk}\|_Y^2 + \frac{\mu_0}{k}(\phi_n^{hk} - \phi_{n-1}^{hk}, \operatorname{div}(\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}))_Y \geq 0,$$

so multiplying the above estimates by k, summing up to n we obtain

$$\begin{aligned}
&\rho\|\boldsymbol{v}_n^{hk}\|_H^2 + \mu\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 + (\lambda+\mu)\|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2 + 2\mu_0(\phi_n^{hk}, \operatorname{div} \boldsymbol{u}_n^{hk})_Y + J\|e_n^{hk}\|_Y^2 \\
&\quad + a_0\|\nabla\phi_n^{hk}\|_H^2 + \xi\|\phi_n^{hk}\|_Y^2 + \kappa_2\|\boldsymbol{T}_n^{hk}\|_H^2 + a\|\xi_n^{hk}\|_Y^2 + \kappa\|\nabla\theta_n^{hk}\|_H^2 \\
&\quad + b\|\boldsymbol{M}_n^{hk}\|_H^2 + \kappa_6\|\nabla \boldsymbol{T}_n^{hk}\|_Q^2 + (\kappa_4+\kappa_5)\|\operatorname{div} \boldsymbol{T}_n^{hk}\|_Y^2 \\
&\leq Ck\sum_{j=1}^{n}\Big(\|\boldsymbol{v}_j^{hk}\|_H^2 + \|\xi_j^{hk}\|_Y^2 + \|e_j^{hk}\|_Y^2 + \|\operatorname{div} \boldsymbol{T}_j^{hk}\|_Y^2 + \|\theta_j^{hk}\|_E^2 + \|\boldsymbol{M}_j^{hk}\|_H^2\Big) \\
&\quad + C + C\Big(\|\boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0^h\|_V^2 + \|e_0^h\|_Y^2 + \|\phi_0^h\|_E^2 + \|\boldsymbol{T}_0^h\|_H^2 + \|\xi_0^h\|_Y^2 \\
&\quad + \|\theta_0^h\|_E^2 + \|\boldsymbol{M}_0^h\|_H^2 + \|\boldsymbol{T}_0^h\|_V^2\Big).
\end{aligned}$$

Now, using conditions (17), we can choose $\zeta > 0$ such that

$$\mu_0/(\lambda+\mu) < \zeta < \xi/\mu_0,$$

which implies that

$$(\lambda+\mu)\|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2 + \xi\|\phi_n^{hk}\|_Y^2 + 2\mu_0(\operatorname{div} \boldsymbol{u}_n^{hk}, \phi_n^{hk})_Y$$
$$\geq \left(\lambda+\mu - \frac{\mu_0}{\zeta}\right)\|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2 + (\xi - \mu_0\zeta)\,\|\phi_n^{hk}\|_Y^2.$$

Finally, using the Poincaré inequality in the term involving the gradient of the temperature and a discrete version of Gronwall's inequality (see, for instance, [47,48]), we conclude the desired stability property.

Now, we obtain some a priori error estimates on $\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}$, $e_n - e_n^{hk}$, $\xi_n - \xi_n^{hk}$ and $\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}$ that we state in the following.

Theorem 2 *Under the assumptions of Lemma 1, if we denote by* $(\boldsymbol{v}, e, \xi, \boldsymbol{M})$ *the solution to Problem* VP *and by* $(\boldsymbol{v}^{hk}, e^{hk}, \xi^{hk}, \boldsymbol{M}^{hk})$ *the solution to Problem* VP^{hk}*, then we have the following a priori error estimates, for all* $\{\boldsymbol{w}_j^h\}_{j=0}^N$,

$\{\boldsymbol{\psi}_j^h\}_{j=0}^N \subset V^h$ *and* $\{r_j^h\}_{j=0}^N, \{z_j^h\}_{j=0}^N \subset E^h$,

$$\max_{0\le n\le N} \Big\{ \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|e_n - e_n^{hk}\|_Y^2$$
$$+\|\phi_n - \phi_n^{hk}\|_Y^2 + \|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Y^2 + \|\xi_n - \xi_n^{hk}\|_Y^2$$
$$+\|\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}\|_H^2 + \|\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Q^2 + \|\theta_n - \theta_n^{hk}\|_E^2 + \|\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}\|_H^2 \Big\}$$
$$\le Ck\sum_{j=1}^n \Big(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_H^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 + \|\dot{\boldsymbol{u}}_j - \delta\boldsymbol{u}_j\|_V^2 + \|\dot{e}_j - \delta e_j\|_Y^2$$
$$+\|e_j - r_j^h\|_E^2 + \|\dot{\phi}_j - \delta\phi_j\|_E^2 + \|e_n - r_j^h\|_E^2 + \|\dot{\boldsymbol{M}}_j - \delta\boldsymbol{M}_j\|_H^2$$
$$+\|\dot{\xi}_j - \delta\xi_j\|_Y^2 + \|\dot{\boldsymbol{T}}_j - \delta\boldsymbol{T}_j\|_V^2 + \|\xi_j - z_j^h\|_E^2 + \|\dot{\theta}_j - \delta\theta_j\|_E^2 + \|\boldsymbol{M}_j - \boldsymbol{\psi}_j^h\|_V^2 \Big)$$
$$+\frac{C}{k}\sum_{j=1}^{N-1} \Big(\|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H^2 + \|e_j - r_j^h - (e_{j+1} - r_{j+1}^h)\|_Y^2$$
$$+\|\xi_j - z_j^h - (\xi_{j+1} - z_{j+1}^h)\|_Y^2 + \|\boldsymbol{M}_j - \boldsymbol{\psi}_j^h - (\boldsymbol{M}_{j+1} - \boldsymbol{\psi}_{j+1}^h)\|_H^2 \Big)$$
$$+C\max_{0\le n\le N} \Big(\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 + \|e_n - r_n^h\|_Y^2 + \|\xi_n - z_n^h\|_Y^2 + \|\boldsymbol{M}_n - \boldsymbol{\psi}_n^h\|_H^2 \Big)$$
$$+C\Big(\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|e_0 - e_0^h\|_Y^2 + \|\theta_0 - \theta_0^h\|_E^2$$
$$+\|\phi_0 - \phi_0^h\|_E^2 + \|\xi_0 - \xi_0^h\|_Y^2 + \|\boldsymbol{M}_0 - \boldsymbol{M}_0^h\|_H^2 + \|\boldsymbol{T}_0 - \boldsymbol{T}_0^h\|_V^2 \Big), \tag{18}$$

where $C > 0$ *is a positive constant which is be independent of the discretization parameters* h *and* k*, but depending on the continuous solution,* $\delta\xi_j = (\xi_j - \xi_{j-1})/k$*,* $\delta\phi_j = (\phi_j - \phi_{j-1})/k$*,* $\delta e_j = (e_j - e_{j-1})/k$*,* $\delta\boldsymbol{v}_j = (\boldsymbol{v}_j - \boldsymbol{v}_{j-1})/k$*,* $\delta\boldsymbol{u}_j = (\boldsymbol{u}_j - \boldsymbol{u}_{j-1})/k$*,* $\delta\boldsymbol{M}_j = (\boldsymbol{M}_j - \boldsymbol{M}_{j-1})/k$*,* $\delta\boldsymbol{T}_j = (\boldsymbol{T}_j - \boldsymbol{T}_{j-1})/k$ *and* $\delta\theta_j = (\theta_j - \theta_{j-1})/k$*.*

Proof As we did in the proof of Lemma 1, we also assume that the relaxation parameter $\tau = 1$.

First, we obtain some estimates for function $\boldsymbol{v}$. Then, we subtract variational equation (2) at time $t = t_n$ for a test function $\boldsymbol{w} = \boldsymbol{w}^h \in V^h \subset V$ and discrete variational equation (11) to obtain, for all $w^h \in V^h$,

$$\rho(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + \mu(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla\boldsymbol{w}^h)_Q - \mu_0(\nabla(\phi_n - \phi_n^{hk}), \boldsymbol{w}^h)_H$$
$$+(\lambda+\mu)(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,\boldsymbol{w}^h)_Y + \beta_0(\nabla(\xi_n - \xi_n^{hk} + \theta_n - \theta_n^{hk}), \boldsymbol{w}^h)_H = 0,$$

and therefore,

$$\rho(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + \mu(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q$$
$$+(\lambda+\mu)(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y$$
$$-\mu_0(\nabla(\phi_n - \phi_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + \beta_0(\nabla(\xi_n - \xi_n^{hk} + \theta_n - \theta_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H$$
$$= \rho(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H + \mu(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{w}^h))_Q$$
$$+(\lambda+\mu)(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{w}^h))_Y$$
$$-\mu_0(\nabla(\phi_n - \phi_n^{hk}), \boldsymbol{v}_n - \boldsymbol{w}^h)_H + \beta_0(\nabla(\xi_n - \xi_n^{hk} + \theta_n - \theta_n^{hk}), \boldsymbol{v}_n - \boldsymbol{w}^h)_H.$$

Keeping in mind that

$$
\begin{aligned}
&(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H \geq (\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + \frac{1}{2k}\left[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\right],\\
&(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \geq (\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n))_Q\\
&\qquad + \frac{1}{2k}\left[\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\right],\\
&(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y \geq (\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n))_Y\\
&\qquad + \frac{1}{2k}\left[\|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\right],\\
&\beta_0(\nabla(\xi_n - \xi_n^{hk}), \boldsymbol{v}_n - \boldsymbol{w}^h)_H = -\beta_0(\xi_n - \xi_n^{hk}, \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{w}^h))_Y,
\end{aligned}
$$

where we recall that $\boldsymbol{v}_n^{hk} = \delta \boldsymbol{u}_n^{hk} = (\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk})/k$, applying several times the following inequality:

$$
ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad a, b, \epsilon \in \mathbb{R},\ \epsilon > 0, \tag{19}
$$

we have, for all $\boldsymbol{w}^h \in V^h$,

$$
\begin{aligned}
&\frac{\rho}{2k}\left[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\right] + \beta_0(\nabla(\xi_n - \xi_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H\\
&\quad + \frac{\mu}{2k}\left[\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\right]\\
&\quad + \frac{\lambda + \mu}{2k}\left[\|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\right]\\
&\leq C\Big(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n\|_H^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2\\
&\quad + \|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + \|\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n\|_V^2 + (\delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H\\
&\quad + \|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\xi_n - \xi_n^{hk}\|_Y^2\Big).
\end{aligned} \tag{20}
$$

Now, we derive the error estimates on function e. Subtracting variational equation (3) at time $t = t_n$ for a test function $r = r^h \in E^h \subset E$ and discrete variational equation (12), we obtain, for all $r^h \in E^h$,

$$
\begin{aligned}
&(J(\dot{e}_n - \delta e_n^{hk}), r^h)_Y + a_0(\nabla(\phi_n - \phi_n^{hk}), \nabla r^h)_H + \xi(\phi_n - \phi_n^{hk}, r^h)_Y\\
&\qquad + \mu_2(\mathrm{div}\,(\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}) + \mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), r^h)_Y\\
&\qquad + \mu_0(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), r^h)_Y - \beta_1(\xi_n - \xi_n^{hk} + \theta_n - \theta_n^{hk}, r^h)_Y = 0,
\end{aligned}
$$

and so,

$$
\begin{aligned}
&(J(\dot{e}_n - \delta e_n^{hk}), e_n - e_n^{hk})_Y + a_0(\nabla(\phi_n - \phi_n^{hk}), \nabla(e_n - e_n^{hk}))_H\\
&\quad + \xi(\phi_n - \phi_n^{hk}, e_n - e_n^{hk})_Y + \mu_2(\mathrm{div}\,(\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}) + \mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), e_n - e_n^{hk})_Y\\
&\quad + \mu_0(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), e_n - e_n^{hk})_Y - \beta_1(\xi_n - \xi_n^{hk} + \theta_n - \theta_n^{hk}, e_n - e_n^{hk})_Y\\
&= (J(\dot{e}_n - \delta e_n^{hk}), e_n - r^h)_Y + a_0(\nabla(\phi_n - \phi_n^{hk}), \nabla(e_n - r^h))_H\\
&\quad + \xi(\phi_n - \phi_n^{hk}, e_n - r^h)_Y + \mu_2(\mathrm{div}\,(\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}) + \mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), e_n - r^h)_Y\\
&\quad + \mu_0(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), e_n - r^h)_Y - \beta_1(\xi_n - \xi_n^{hk} + \theta_n - \theta_n^{hk}, e_n - r^h)_Y.
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
&(\dot{e}_n - \delta e_n^{hk}, e_n - e_n^{hk})_Y \geq (\dot{e}_n - \delta e_n, e_n - e_n^{hk})_Y + \frac{1}{2k}\left[\|e_n - e_n^{hk}\|_Y^2 - \|e_{n-1} - e_{n-1}^{hk}\|_Y^2\right],\\
&(\nabla(\phi_n - \phi_n^{hk}), \nabla(\dot{\phi}_n - e_n^{hk}))_H = (\nabla(\phi_n - \phi_n^{hk}), \nabla(\dot{\phi}_n - \delta\phi_n))_H\\
&\quad + \frac{1}{2k}\Big[\|\nabla(\phi_n - \phi_n^{hk})\|_H^2 - \|\nabla(\phi_{n-1} - \phi_{n-1}^{hk})\|_H^2 + \|\nabla(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}))\|_H^2\Big],\\
&(\phi_n - \phi_n^{hk}, \dot{\phi}_n - e_n^{hk})_Y = (\phi_n - \phi_n^{hk}, \dot{\phi}_n - \delta\phi_n)_Y\\
&\quad + \frac{1}{2k}\Big[\|\phi_n - \phi_n^{hk}\|_Y^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|_Y^2 + \|\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})\|_Y^2\Big],\\
&(\operatorname{div}(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), e_n - r^h)_Y = -(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}, \nabla(e_n - r^h))_H,
\end{aligned}
$$

where the definition $e_n^{hk} = \delta\phi_n^{hk} = (\phi_n^{hk} - \phi_{n-1}^{hk})/k$ was used, applying again Young's inequality (19) we find that, for all $r^h \in E^h$,

$$
\begin{aligned}
&\frac{J}{2k}\left\{\|e_n - e_n^{hk}\|_Y^2 - \|e_{n-1} - e_{n-1}^{hk}\|_Y^2\right\} + \frac{\xi}{2k}\Big\{\|\phi_n - \phi_n^{hk}\|_Y^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|_Y^2\Big\}\\
&\quad + \frac{a_0}{2k}\Big\{\|\nabla(\phi_n - \phi_n^{hk})\|_H^2 - \|\nabla(\phi_{n-1} - \phi_{n-1}^{hk})\|_H^2\Big\}\\
&\quad + \mu_2(\operatorname{div}(\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}), e_n - e_n^{hk})_Y\\
&\leq C\Big(\|\dot{e}_n - \delta e_n\|_Y^2 + \|\nabla(\dot{\phi}_n - \delta\phi_n)\|_H^2 + \|\dot{\phi}_n - \delta\phi_n\|_Y^2 + \|e_n - r^h\|_E^2\\
&\quad + \|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 + \|\xi_n - \xi_n^{hk}\|^2 + \|\operatorname{div}(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Y^2\\
&\quad + \|e_n - e_n^{hk}\|_Y^2 + \|\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}\|_H^2 + (\delta e_n - \delta e_n^{hk}, e_n - r^h)_Y\Big). \qquad (21)
\end{aligned}
$$

Thirdly, we estimate the numerical error $\xi_n - \xi_n^{hk}$ related to the temperature speed. Subtracting variational equation (4) at time $t = t_n$ for a test function $z = z^h \in E^h \subset E$ and discrete variational equation (13), we obtain, for all $z^h \in E^h$,

$$
\begin{aligned}
&(a(\dot{\xi}_n - \delta\xi_n^{hk}) + a(\xi_n - \xi_n^{hk}), z^h)_Y + \beta_0(\operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), z^h)_Y\\
&\quad + \kappa(\nabla(\theta_n - \theta_n^{hk}), \nabla z^h)_H + \beta_1(e_n - e_n^{hk}, z^h)_Y - \kappa_1(\operatorname{div}(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), z^h)_Y = 0,
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
&(a(\dot{\xi}_n - \delta\xi_n^{hk}) + a(\xi_n - \xi_n^{hk}), \xi_n - \xi_n^{hk})_Y + \beta_0(\operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \xi_n - \xi_n^{hk})_Y\\
&\quad + \kappa(\nabla(\theta_n - \theta_n^{hk}), \nabla(\xi_n - \xi_n^{hk}))_H + \beta_1(e_n - e_n^{hk}, \xi_n - \xi_n^{hk})_Y\\
&\quad - \kappa_1(\operatorname{div}(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \xi_n - \xi_n^{hk})_Y\\
&= (a(\dot{\xi}_n - \delta\xi_n^{hk}) + a(\xi_n - \xi_n^{hk}), \xi_n - z^h)_Y + \beta_0(\operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \xi_n - z^h)_Y\\
&\quad + \kappa(\nabla(\theta_n - \theta_n^{hk}), \nabla(\xi_n - z^h))_H + \beta_1(e_n - e_n^{hk}, \xi_n - z^h)_Y\\
&\quad - \kappa_1(\operatorname{div}(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \xi_n - z^h)_Y.
\end{aligned}
$$

Now, using the following estimates

$$
\begin{aligned}
&(\dot{\xi}_n - \delta\xi_n^{hk}, \xi_n - \xi_n^{hk})_Y \geq (\dot{\xi}_n - \delta\xi_n, \xi_n - \xi_n^{hk})_Y + \frac{1}{2k}\left[\|\xi_n - \xi_n^{hk}\|_Y^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|_Y^2\right],\\
&(\nabla(\theta_n - \theta_n^{hk}), \nabla(\dot{\theta}_n - \xi_n^{hk}))_H \geq (\nabla(\theta_n - \theta_n^{hk}), \nabla(\dot{\theta}_n - \delta\theta_n))_H\\
&\quad + \frac{1}{2k}\Big[\|\nabla(\theta_n - \theta_n^{hk})\|_H^2 - \|\nabla(\theta_{n-1} - \theta_{n-1}^{hk})\|_H^2\Big],\\
&\beta_0(\operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \xi_n - \xi_n^{hk})_Y = -\beta_0(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \nabla(\xi_n - \xi_n^{hk}))_H,
\end{aligned}
$$

where the definition $\xi_n^{hk} = \delta\theta_n^{hk} = (\theta_n^{hk} - \theta_{n-1}^{hk})/k$ was used, applying again Young's inequality (19) we find that, for all $z^h \in E^h$,

$$
\begin{aligned}
&\frac{a}{2k}\left\{\|\xi_n - \xi_n^{hk}\|_Y^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|_Y^2\right\} - \beta_0(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \nabla(\xi_n - \xi_n^{hk}))_H \\
&\quad + \frac{\kappa}{2k}\Big\{\|\nabla(\theta_n - \theta_n^{hk})\|_H^2 - \|\nabla(\theta_{n-1} - \theta_{n-1}^{hk})\|_H^2\Big\} \\
&\le C\Big(\|\dot{\xi}_n - \delta\xi_n\|_Y^2 + \|\nabla(\dot{\theta}_n - \delta\theta_n)\|_H^2 + \|\dot{\theta}_n - \delta\theta_n\|_Y^2 + \|\xi_n - z^h\|_E^2 \\
&\quad + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 + \|\xi_n - \xi_n^{hk}\|^2 + \|\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Y^2 \\
&\quad + \|e_n - e_n^{hk}\|_Y^2 + \|\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}\|_H^2 + (\delta\xi_n - \delta\xi_n^{hk}, \xi_n - z^h)_Y\Big).
\end{aligned} \tag{22}
$$

Finally, we deal with the estimation of the error on the microtemperatures. Then, we subtract variational equation (4) at time $t = t_n$ for a test function $\boldsymbol{\psi} = \boldsymbol{\psi}^h \in V^h \subset V$ and discrete variational equation (14) to obtain, for all $\boldsymbol{\psi}^h \in V^h$,

$$
\begin{aligned}
&(b(\dot{\boldsymbol{M}}_n - \delta\boldsymbol{M}_n^{hk} + \boldsymbol{M}_n - \boldsymbol{M}_n^{hk}), \boldsymbol{\psi}^h)_H + \kappa_6(\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \nabla\boldsymbol{\psi}^h)_Q \\
&\quad + (\kappa_4 + \kappa_5)(\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \mathrm{div}\,\boldsymbol{\psi}^h)_Y + \kappa_2(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}, \boldsymbol{\psi}^h)_H \\
&\quad + \kappa_3(\nabla(\theta_n - \theta_n^{hk}), \boldsymbol{\psi}^h)_H + \mu_2(\nabla(e_n - e_n^{hk}), \boldsymbol{\psi}^h)_H = 0.
\end{aligned}
$$

Thus, we have, for all $\boldsymbol{\psi}^h \in V^h$,

$$
\begin{aligned}
&(b(\dot{\boldsymbol{M}}_n - \delta\boldsymbol{M}_n^{hk} + \boldsymbol{M}_n - \boldsymbol{M}_n^{hk}), \boldsymbol{M}_n - \boldsymbol{M}_n^{hk})_H + \kappa_6(\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \nabla(\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}))_Q \\
&\quad + (\kappa_4 + \kappa_5)(\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \mathrm{div}\,(\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}))_Y + \kappa_2(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}, \boldsymbol{M}_n - \boldsymbol{M}_n^{hk})_H \\
&\quad + \kappa_3(\nabla(\theta_n - \theta_n^{hk}), \boldsymbol{M}_n - \boldsymbol{M}_n^{hk})_H + \mu_2(\nabla(e_n - e_n^{hk}), \boldsymbol{M}_n - \boldsymbol{M}_n^{hk})_H \\
&= (b(\dot{\boldsymbol{M}}_n - \delta\boldsymbol{M}_n^{hk} + \boldsymbol{M}_n - \boldsymbol{M}_n^{hk}), \boldsymbol{M}_n - \boldsymbol{\psi}^h)_H + \kappa_6(\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \nabla(\boldsymbol{M}_n - \boldsymbol{\psi}^h))_Q \\
&\quad + (\kappa_4 + \kappa_5)(\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \mathrm{div}\,(\boldsymbol{M}_n - \boldsymbol{\psi}^h))_Y + \kappa_2(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}, \boldsymbol{M}_n - \boldsymbol{\psi}^h)_H \\
&\quad + \kappa_3(\nabla(\theta_n - \theta_n^{hk}), \boldsymbol{M}_n - \boldsymbol{\psi}^h)_H + \mu_2(\nabla(e_n - e_n^{hk}), \boldsymbol{M}_n - \boldsymbol{\psi}^h)_H,
\end{aligned}
$$

and keeping in mind that

$$
\begin{aligned}
&(\dot{\boldsymbol{M}}_n - \delta\boldsymbol{M}_n^{hk}, \boldsymbol{M}_n - \boldsymbol{M}_n^{hk})_H \ge (\dot{\boldsymbol{M}}_n - \delta\boldsymbol{M}_n, \boldsymbol{M}_n - \boldsymbol{M}_n^{hk})_H \\
&\qquad + \frac{1}{2k}\left[\|\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}\|_H^2 - \|\boldsymbol{M}_{n-1} - \boldsymbol{M}_{n-1}^{hk}\|_H^2\right], \\
&(\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \nabla(\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}))_Q \ge (\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \nabla(\dot{\boldsymbol{M}}_n - \delta\boldsymbol{M}_n))_Q \\
&\qquad + \frac{1}{2k}\left[\|\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{T}_{n-1} - \boldsymbol{T}_{n-1}^{hk})\|_Q^2\right], \\
&(\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \mathrm{div}\,(\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}))_Y \ge (\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \mathrm{div}\,(\dot{\boldsymbol{T}}_n - \delta\boldsymbol{T}_n))_Y \\
&\qquad + \frac{1}{2k}\left[\|\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{T}_{n-1} - \boldsymbol{T}_{n-1}^{hk})\|_Y^2\right],
\end{aligned}
$$

$$
\begin{aligned}
&(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}, \boldsymbol{M}_n - \boldsymbol{M}_n^{hk})_H \ge (\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}, \dot{\boldsymbol{T}}_n - \delta\boldsymbol{T}_n)_H \\
&\qquad + \frac{1}{2k}\left[\|\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}\|_H^2 - \|\boldsymbol{T}_{n-1} - \boldsymbol{T}_{n-1}^{hk}\|_H^2\right], \\
&\mu_2(\nabla(e_n - e_n^{hk}), \boldsymbol{M}_n - \boldsymbol{\psi}^h)_H = -\mu_2(e_n - e_n^{hk}, \mathrm{div}\,(\boldsymbol{M}_n - \boldsymbol{\psi}^h))_Y, \\
&\mu_2(\nabla(e_n - e_n^{hk}), \boldsymbol{M}_n - \boldsymbol{M}_n^{hk})_H = -\mu_2(e_n - e_n^{hk}, \mathrm{div}\,(\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}))_Y,
\end{aligned}
$$

where we recall that $\boldsymbol{M}_n^{hk} = \delta\boldsymbol{T}_n^{hk} = (\boldsymbol{T}_n^{hk} - \boldsymbol{T}_{n-1}^{hk})/k$, applying several times Young's inequality (19) we have, for all $\boldsymbol{\psi}^h \in V^h$,

$$\begin{aligned}
&\frac{b}{2k}\left[\|\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}\|_H^2 - \|\boldsymbol{M}_{n-1} - \boldsymbol{M}_{n-1}^{hk}\|_H^2\right] - \mu_2(e_n - e_n^{hk}, \mathrm{div}\,(\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}))_Y \\
&+\frac{\kappa_6}{2k}\left[\|\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{T}_{n-1} - \boldsymbol{T}_{n-1}^{hk})\|_Q^2\right] \\
&+\frac{\kappa_4+\kappa_5}{2k}\left[\|\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{T}_{n-1} - \boldsymbol{T}_{n-1}^{hk})\|_Y^2\right] \\
&+\frac{\kappa_2}{2k}\left[\|\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}\|_H^2 - \|\boldsymbol{T}_{n-1} - \boldsymbol{T}_{n-1}^{hk}\|_H^2\right] \\
\le\, & C\Big(\|\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}\|_H^2 + \|\dot{\boldsymbol{M}}_n - \delta\boldsymbol{M}_n\|_H^2 + \|\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Q^2 + \|\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}\|_H^2 \\
&+\|\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Y^2 + \|\dot{\boldsymbol{T}}_n - \delta\boldsymbol{T}_n\|_V^2 + (\delta\boldsymbol{M}_n - \delta\boldsymbol{M}_n^{hk}, \boldsymbol{M}_n - \boldsymbol{\psi}^h)_H \\
&+\|\boldsymbol{M}_n - \boldsymbol{\psi}^h\|_V^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|e_n - e_n^{hk}\|_Y^2\Big).
\end{aligned} \tag{23}$$

Combining estimates (20)-(23) it leads

$$\begin{aligned}
&\frac{\rho}{2k}\left[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\right] + \frac{\kappa}{2k}\Big\{\|\nabla(\theta_n - \theta_n^{hk})\|_H^2 - \|\nabla(\theta_{n-1} - \theta_{n-1}^{hk})\|_H^2\Big\} \\
&+\frac{\mu}{2k}\left[\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\right] \\
&+\frac{\lambda+\mu}{2k}\left[\|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\right] \\
&+\frac{J}{2k}\left\{\|e_n - e_n^{hk}\|_Y^2 - \|e_{n-1} - e_{n-1}^{hk}\|_Y^2\right\} + \frac{\xi}{2k}\Big\{\|\phi_n - \phi_n^{hk}\|_Y^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|_Y^2\Big\} \\
&+\frac{a_0}{2k}\Big\{\|\nabla(\phi_n - \phi_n^{hk})\|_H^2 - \|\nabla(\phi_{n-1} - \phi_{n-1}^{hk})\|_H^2\Big\} \\
&+\frac{a}{2k}\left\{\|\xi_n - \xi_n^{hk}\|_Y^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|_Y^2\right\} + \frac{b}{2k}\left[\|\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}\|_H^2 - \|\boldsymbol{M}_{n-1} - \boldsymbol{M}_{n-1}^{hk}\|_H^2\right] \\
&+\frac{\kappa_6}{2k}\left[\|\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{T}_{n-1} - \boldsymbol{T}_{n-1}^{hk})\|_Q^2\right] \\
&+\frac{\kappa_4+\kappa_5}{2k}\left[\|\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{T}_{n-1} - \boldsymbol{T}_{n-1}^{hk})\|_Y^2\right] \\
&+\frac{\kappa_2}{2k}\left[\|\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}\|_H^2 - \|\boldsymbol{T}_{n-1} - \boldsymbol{T}_{n-1}^{hk}\|_H^2\right] \\
\le\, & C\Big(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_H^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 \\
&+\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + \|\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n\|_V^2 + (\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H + \|\dot{\xi}_n - \delta\xi_n\|_Y^2 \\
&+\|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\xi_n - \xi_n^{hk}\|_Y^2 + \|\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Y^2 \\
&+\|\dot{e}_n - \delta e_n\|_Y^2 + \|\dot{\phi}_n - \delta\phi_n\|_E^2 + \|e_n - r^h\|_E^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 \\
&+\|e_n - e_n^{hk}\|_Y^2 + \|\boldsymbol{M}_n - \boldsymbol{M}_n^{hk}\|_H^2 + (\delta e_n - \delta e_n^{hk}, e_n - r^h)_Y \\
&+\|e_n - r^h\|_E^2 + (\delta\xi_n - \delta\xi_n^{hk}, \xi_n - z^h)_Y + \|\dot{\boldsymbol{M}}_n - \delta\boldsymbol{M}_n\|_H^2 \\
&+\|\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}\|_H^2 + \|\dot{\boldsymbol{T}}_n - \delta\boldsymbol{T}_n\|_V^2 + (\delta\boldsymbol{M}_n - \delta\boldsymbol{M}_n^{hk}, \boldsymbol{M}_n - \boldsymbol{\psi}^h)_H \\
&+\|\dot{\theta}_n - \delta\theta_n\|_E^2 + \|\xi_n - z^h\|_E^2 + \|\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Q^2 + \|\boldsymbol{M}_n - \boldsymbol{\psi}^h\|_V^2\Big).
\end{aligned}$$

Multiplying the previous estimates by k and summing up to n we find that

$$\begin{aligned}
&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|e_n-e_n^{hk}\|_Y^2\\
&\ +\|\phi_n-\phi_n^{hk}\|_Y^2+\|\nabla(\phi_n-\phi_n^{hk})\|_H^2+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2+\|\xi_n-\xi_n^{hk}\|_Y^2\\
&\ +\|\boldsymbol{M}_n-\boldsymbol{M}_n^{hk}\|_H^2+\|\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})\|_Q^2+\|\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})\|_Y^2+\|\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}\|_H^2\\
&\leq Ck\sum_{j=1}^n\Big(\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_H^2+\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\nabla(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Q^2+\|\mathrm{div}\,(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Y^2\\
&\ +\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H\\
&\ +\|\nabla(\phi_j-\phi_j^{hk})\|_H^2+\|\nabla(\theta_j-\theta_j^{hk})\|_H^2+\|\xi_j-\xi_j^{hk}\|_Y^2+\|\mathrm{div}\,(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk})\|_Y^2\\
&\ +\|\dot{e}_j-\delta e_j\|_Y^2+\|\dot{\phi}_j-\delta\phi_j\|_E^2+\|e_j-r_j^h\|_E^2+\|\theta_j-\theta_j^{hk}\|_Y^2\\
&\ +\|e_j-e_j^{hk}\|_Y^2+\|\boldsymbol{M}_j-\boldsymbol{M}_j^{hk}\|_H^2+(\delta e_j-\delta e_j^{hk},e_j-r_j^h)_Y\\
&\ +\|e_n-r_j^h\|_E^2+(\delta\xi_j-\delta\xi_j^{hk},\xi_j-z_j^h)_Y+\|\dot{\boldsymbol{M}}_j-\delta\boldsymbol{M}_j\|_H^2+\|\dot{\xi}_j-\delta\xi_j\|_Y^2\\
&\ +\|\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}\|_H^2+\|\dot{\boldsymbol{T}}_j-\delta\boldsymbol{T}_j\|_V^2+(\delta\boldsymbol{M}_j-\delta\boldsymbol{M}_j^{hk},\boldsymbol{M}_j-\boldsymbol{\psi}_j^h)_H\\
&\ +\|\xi_j-z_j^h\|_E^2+\|\dot{\theta}_j-\delta\theta_j\|_E^2+\|\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk})\|_Q^2+\|\boldsymbol{M}_j-\boldsymbol{\psi}_j^h\|_V^2\Big)\\
&\ +C\Big(\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|e_0-e_0^h\|_Y^2+\|\nabla(\theta_0-\theta_0^h)\|_H^2\\
&\ +\|\phi_0-\phi_0^h\|_E^2+\|\xi_0-\xi_0^h\|_Y^2+\|\boldsymbol{M}_0-\boldsymbol{M}_0^h\|_H^2+\|\boldsymbol{T}_0-\boldsymbol{T}_0^h\|_V^2\Big).
\end{aligned}$$

Now, taking into account that

$$\begin{aligned}
&k\sum_{j=1}^n(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H=\sum_{j=1}^n(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H\\
&\qquad=(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H+(\boldsymbol{v}_0^h-\boldsymbol{v}_0,\boldsymbol{v}_1-\boldsymbol{w}_1^h)_H\\
&\qquad\quad+\sum_{j=1}^{n-1}(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h))_H,\\
&k\sum_{j=1}^n(\delta e_j-\delta e_j^{hk}),e_j-r_j^h)_Y=\sum_{j=1}^n(e_j-e_j^{hk},e_j-r_j^h)_Y\\
&\qquad=(e_n-e_n^{hk},e_n-r_n^h)_Y+(e_0^h-e_0,e_1-r_1^h)_Y\\
&\qquad\quad+\sum_{j=1}^{n-1}(e_j-e_j^{hk},e_j-r_j^h-(e_{j+1}-r_{j+1}^h))_Y,
\end{aligned}$$

$$
\begin{aligned}
k\sum_{j=1}^{n}(\delta\xi_j-\delta\xi_j^{hk}),\xi_j-z_j^h)_Y &= \sum_{j=1}^{n}(\xi_j-\xi_j^{hk},\xi_j-z_j^h)_Y \\
&= (\xi_n-\xi_n^{hk},\xi_n-z_n^h)_Y+(\xi_0^h-\xi_0,\xi_1-z_1^h)_Y \\
&\quad +\sum_{j=1}^{n-1}(\xi_j-\xi_j^{hk},\xi_j-z_j^h-(\xi_{j+1}-z_{j+1}^h))_Y, \\
k\sum_{j=1}^{n}(\delta\boldsymbol{T}_j-\delta\boldsymbol{T}_j^{hk}),\boldsymbol{T}_j-\boldsymbol{\psi}_j^h)_H &= \sum_{j=1}^{n}(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk},\boldsymbol{T}_j-\boldsymbol{\psi}_j^h)_H \\
&= (\boldsymbol{T}_n-\boldsymbol{T}_n^{hk},\boldsymbol{T}_n-\boldsymbol{\psi}_n^h)_H+(\boldsymbol{T}_0^h-\boldsymbol{T}_0,\boldsymbol{T}_1-\boldsymbol{\psi}_1^h)_H \\
&\quad +\sum_{j=1}^{n-1}(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk},\boldsymbol{T}_j-\boldsymbol{\psi}_j^h-(\boldsymbol{T}_{j+1}-\boldsymbol{\psi}_{j+1}^h))_H,
\end{aligned}
$$

using a discrete version of Gronwall's inequality (see, for instance, [47,48]) and applying Poincaré inequality to the term involving the gradient of the temperature, it follows the a priori error estimates (18).

Estimates (18) can be used to obtain the convergence order of the algorithm. As a particular case, under suitable regularity conditions we find that the approximations given by Problem VP^{hk} are linearly convergent as we state in the following.

Corollary 1 *Let the assumptions of Theorem 2 still hold. If we assume that the solution to Problem VP has the additional regularity:*

$$
\begin{aligned}
&\boldsymbol{u},\boldsymbol{T}\in H^2(0,T;V)\cap H^3(0,T;H)\cap C^1([0,T_f];[H^2(\Omega)]^d), \\
&\theta,\phi\in H^2(0,T;E)\cap H^3(0,T;Y)\cap C^1([0,T_f];H^2(\Omega)),
\end{aligned}
\tag{24}
$$

and we use the finite element spaces V^h and E^h defined in (9) and (10), respectively, and the discrete initial conditions $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$, ϕ_0^h, e_0^h, θ_0^h, ξ_0^h, $\boldsymbol{T}_0^h$ and $\boldsymbol{M}_0^h$ given in (16), the linear convergence of the algorithm is deduced; i.e. there exists a positive constant $C>0$, independent of the discretization parameters h and k, such that

$$
\begin{aligned}
\max_{0\le n\le N}\Big\{&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|e_n-e_n^{hk}\|_Y+\|\phi_n-\phi_n^{hk}\|_E \\
&+\|\theta_n-\theta_n^{hk}\|_E+\|\xi_n-\xi_n^{hk}\|_Y+\|\boldsymbol{M}_n-\boldsymbol{M}_n^{hk}\|_H+\|\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}\|_V\Big\}\le C(h+k).
\end{aligned}
$$

Its proof is obtained using well-known results on the approximation by finite elements, the properties of the projection operators P^{1h} and P^{2h} (see [49]) and proceeding as in [48]. We omit it for the sake of readability.

4 Numerical results

In this final section, we show the numerical algorithm used to solve Problem VP^{hk} and we present some examples obtained in one and two dimensions.

Let the finite element spaces V^h and E^h defined in (9) and (10), respectively. Given $\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_{n-1}^{hk} \in V^h$, $\phi_{n-1}^{hk}, e_{n-1}^{hk}, \theta_{n-1}^{hk}, \xi_{n-1}^{hk} \in E^h$ and $\boldsymbol{T}_{n-1}^{hk}, \boldsymbol{M}_{n-1}^{hk} \in V^h$, functions $\boldsymbol{v}_n^{hk}$, e_n^{hk}, ξ_n^{hk} and $\boldsymbol{M}_n^{hk}$ are obtained from equations (11), (12), 13) and (14) as follows:

$$
\begin{aligned}
&\rho(\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + k^2\mu(\nabla \boldsymbol{v}_n^{hk}, \nabla \boldsymbol{w}^h)_Q + k^2(\lambda+\mu)(\operatorname{div} \boldsymbol{v}_n^{hk}, \operatorname{div} \boldsymbol{w}^h)_Y \\
&\quad -\mu_0 k^2(\nabla e_n^{hk}, \boldsymbol{w}^h)_H + \beta_0 k(\nabla(\tau\xi_n^{hk} + \tau k \xi_n^{hk}), \boldsymbol{w}^h)_H \\
&\quad = \rho(\boldsymbol{v}_{n-1}^{hk}, \boldsymbol{w}^h)_H - k\mu(\nabla \boldsymbol{u}_{n-1}^{hk}, \nabla \boldsymbol{w}^h)_Q - k(\lambda+\mu)(\operatorname{div} \boldsymbol{u}_{n-1}^{hk}, \operatorname{div} \boldsymbol{w}^h)_Y \\
&\quad +\mu_0 k(\nabla \phi_{n-1}^{hk}, \boldsymbol{w}^h)_H - \beta_0 k(\nabla \theta_{n-1}^{hk}, \boldsymbol{w}^h)_H + \rho k(\boldsymbol{f}_n, \boldsymbol{w}^h)_H, \\
&(Je_n^{hk}, r^h)_Y + a_0 k^2(\nabla e_n^{hk}, \nabla r^h)_H + \xi k^2(e_n^{hk}, r^h)_Y - \mu_0 k^2(\operatorname{div} \boldsymbol{v}_n^{hk}, r^h)_Y \\
&\quad -\mu_2 k(\tau \operatorname{div} \boldsymbol{M}_n^{hk} + k \operatorname{div} \boldsymbol{M}_n^{hk}, r^h)_Y - \beta_1 k(\tau\xi_n^{hk} + k\xi_n^{hk}, r^h)_Y \\
&\quad = (Je_{n-1}^{hk}, r^h)_Y - a_0 k(\nabla \phi_{n-1}^{hk}, \nabla r^h)_H - \xi k(\phi_{n-1}^{hk}, r^h)_Y - \mu_2 k(\operatorname{div} \boldsymbol{T}_{n-1}^{hk}, r^h)_Y \\
&\quad -\mu_0 k(\operatorname{div} \boldsymbol{u}_{n-1}^{hk}, r^h)_Y + \beta_1 k(\theta_{n-1}^{hk}, r^h)_Y + \rho k(l_n, r^h)_Y, \\
&(\tau a\xi_n^{hk} + ak\xi_n^{hk}, z^h)_Y + \kappa k^2(\nabla \xi_n^{hk}, \nabla z^h)_H + \beta_0 k(\operatorname{div} \boldsymbol{v}_n^{hk}, z^h)_Y + \beta_1 k(e_n^{hk}, z^h)_Y \\
&\quad -\kappa_1 k^2(\operatorname{div} \boldsymbol{M}_n^{hk}, z^h)_Y \\
&\quad = (\tau a\xi_{n-1}^{hk}, z^h)_Y - \kappa k(\nabla \theta_{n-1}^{hk}, \nabla z^h)_H + \kappa_1 k(\operatorname{div} \boldsymbol{T}_{n-1}^{hk}, z^h)_Y + \rho k(s_n, z^h)_Y, \\
&(\tau b\boldsymbol{M}_n^{hk} + b\boldsymbol{M}_n^{hk}, \boldsymbol{\psi}^h)_H + \kappa_6 k^2(\nabla \boldsymbol{M}_n^{hk}, \nabla \boldsymbol{\psi}^h)_Q + (\kappa_4+\kappa_5)k^2(\operatorname{div} \boldsymbol{M}_n^{hk}, \operatorname{div} \boldsymbol{\psi}^h)_Y \\
&\quad +(\kappa_4+\kappa_5)k^2(\operatorname{div} \boldsymbol{M}_n^{hk}, \operatorname{div} \boldsymbol{\psi}^h)_Y - \kappa_3 k^2(\nabla \xi_n^{hk}, \boldsymbol{\psi}^h)_H + \mu_2 k(\nabla e_n^{hk}, \boldsymbol{\psi}^h)_H \\
&\quad +\kappa_2 k^2(\boldsymbol{M}_n^{hk}, \boldsymbol{\psi}^h)_H \\
&\quad = (\tau b\boldsymbol{M}_{n-1}^{hk}, \boldsymbol{\psi}^h)_H - \kappa_6 k(\nabla \boldsymbol{T}_{n-1}^{hk}, \nabla \boldsymbol{\psi}^h)_Q - (\kappa_4+\kappa_5)k(\operatorname{div} \boldsymbol{T}_{n-1}^{hk}, \operatorname{div} \boldsymbol{\psi}^h)_Y \\
&\quad -\kappa_2 k(\boldsymbol{T}_{n-1}^{hk}, \boldsymbol{\psi}^h)_H - \kappa_3 k(\nabla \theta_{n-1}^{hk}, \boldsymbol{\psi}^h)_H + \rho k(\boldsymbol{G}_n, \boldsymbol{\psi}^h)_H,
\end{aligned}
$$

where discrete functions $\boldsymbol{u}_n^{hk}$, ϕ_n^{hk}, θ_n^{hk} and $\boldsymbol{T}_n^{hk}$ are updated from the relations:

$$
\boldsymbol{u}_n^{hk} = k\boldsymbol{v}_n^{hk} + \boldsymbol{u}_{n-1}^{hk}, \quad \phi_n^{hk} = ke_n^{hk} + \phi_{n-1}^{hk}, \quad \boldsymbol{T}_n^{hk} = k\boldsymbol{M}_n^{hk} + \boldsymbol{T}_{n-1}^{hk}, \quad \theta_n^{hk} = k\xi_n^{hk} + \theta_{n-1}^{hk}.
$$

This numerical scheme was implemented on a 3.2 Ghz PC using MATLAB. We note that a typical 1D run ($h = k = 0.01$) took about 0.133 seconds of CPU time, meanwhile a typical 2D run took about 3.25 seconds of CPU time.

4.1 Numerical convergence in a one-dimensional problem

As a simpler one-dimensional case, we will consider the following problem.

Problem $\mathbf{P}^{ex}$. *Find* $u : [0,1] \times [0,1] \to \mathbb{R}$, $\phi : [0,1] \times [0,1] \to \mathbb{R}$, $\theta : [0,1] \times [0,1] \to \mathbb{R}$ *and* $T : [0,1] \times [0,1] \to \mathbb{R}$ *such that*

$$
\begin{aligned}
&\ddot{u} = 5u_{xx} + 2\phi_x - (2\dot{\theta}_x + \theta_x) + f \quad \textit{in} \quad (0,1)\times(0,1),\\
&\ddot{\phi} = \phi_{xx} - 2(2\dot{T}_x + T_x) - 2u_x - 2\phi + (2\dot{\theta} + \theta) + l \quad \textit{in} \quad (0,1)\times(0,1),\\
&2\ddot{\theta} + \dot{\theta} = -\dot{u}_x - \dot{\phi} + \theta_{xx} + 2T_x + s \quad \textit{in} \quad (0,1)\times(0,1),\\
&2\ddot{T} + \dot{T} = -\dot{\phi}_x + 5T_{xx} - 5T - \theta_x + G \quad \textit{in} \quad (0,1)\times(0,1),\\
&u(0,t) = \phi(0,t) = \theta(0,t) = T(0,t) = 0 \quad \textit{for} \quad t \in (0,1),\\
&u(1,t) = \phi(1,t) = \theta(1,t) = T(1,t) = 0 \quad \textit{for} \quad t \in (0,1),\\
&u(x,0) = \dot{v}(x,0) = \phi(x,0) = \dot{\phi}(x,0) = x(x-1) \quad \textit{for a.e.} \quad x \in (0,1),\\
&\theta(x,0) = \dot{\theta}(x,0) = T(x,0) = \dot{T}(x,0) = x(x-1) \quad \textit{for a.e.} \quad x \in (0,1),
\end{aligned}
$$

where the supply forces are given by, for all $(x,t) \in (0,1)\times(0,1)$,

$$
\begin{aligned}
f(x,t) &= e^t\left(x(x-1) - 11 + 2x\right),\\
l(x,t) &= e^t\left(10x - 7\right),\\
s(x,t) &= e^t\left(4x(x-1) - 1 - 2x\right),\\
G(x,t) &= e^t\left(8x(x-1) - 12 + 4x\right).
\end{aligned}
$$

We point out that this problem is obtained from the thermomechanical problem described in Section 2 with the following data:

$$
\begin{aligned}
&\Omega = (0,1), \quad T_f = 1, \quad \rho = 1, \quad \mu = 2, \quad \lambda = 1, \quad \mu_0 = 2, \quad \beta_0 = 1, \quad \beta_1 = 1,\\
&\mu_2 = 1, \quad b = 1, \quad J = 1, \quad a_0 = 1, \quad \xi = 2, \quad \tau = 2, \quad a = 1, \quad \kappa = 1, \quad \kappa_1 = 2,\\
&\kappa_2 = 5, \quad \kappa_3 = 1, \quad \kappa_4 = 2, \quad \kappa_5 = 1, \quad \kappa_6 = 2,
\end{aligned}
$$

and the initial conditions, for all $x \in (0,1)$,

$$u_0(x) = v_0(x) = \phi_0(x) = e_0(x) = \theta_0(x) = \xi_0(x) = T_0(x) = M_0(x) = x(x-1).$$

The exact solution to Problem P^{ex} can be easily calculated and it has the form, for $(x,t) \in (0,1)\times(0,1)$,

$$u(x,t) = \phi(x,t) = \theta(x,t) = T(x,t) = e^t x(x-1).$$

The numerical errors, given by

$$
\begin{aligned}
\max_{0\le n\le N} \Big\{ &\|v_n - v_n^{hk}\|_Y + \|u_n - u_n^{hk}\|_{H^1(\Omega)} + \|e_n - e_n^{hk}\|_Y + \|\phi_n - \phi_n^{hk}\|_{H^1(\Omega)}\\
&+ \|\theta_n - \theta_n^{hk}\|_{H^1(\Omega)} + \|\xi_n - \xi_n^{hk}\|_Y + \|M_n - M_n^{hk}\|_Y + \|T_n - T_n^{hk}\|_{H^1(\Omega)} \Big\}
\end{aligned}
$$

and obtained for different discretization parameters h and k, are depicted in Table 1. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 1, is achieved.

$h \downarrow k \rightarrow$	0.01	0.005	0.0002	0.001	0.0005	0.0002	0.0001
$1/2^3$	1.069339	1.068706	1.068433	1.068360	1.068327	1.068309	1.068303
$1/2^4$	0.535232	0.534374	0.534082	0.534023	0.534000	0.533989	0.533986
$1/2^5$	0.269031	0.267533	0.267080	0.267006	0.266983	0.266974	0.266972
$1/2^6$	0.137288	0.134490	0.133656	0.133530	0.133496	0.133486	0.133484
$1/2^7$	0.073645	0.068647	0.067063	0.066824	0.066763	0.066745	0.066742
$1/2^8$	0.044652	0.036849	0.033991	0.033530	0.033411	0.033377	0.033372
$1/2^9$	0.032521	0.022370	0.017847	0.016995	0.016764	0.016698	0.016688
$1/2^{10}$	0.027688	0.016315	0.010325	0.008924	0.008497	0.008367	0.008348
$1/2^{11}$	0.025742	0.013904	0.007094	0.005164	0.004461	0.004220	0.004183
$1/2^{12}$	0.024963	0.012934	0.005797	0.003548	0.002580	0.002179	0.002108
$1/2^{13}$	0.024681	0.012546	0.005278	0.002899	0.001771	0.001207	0.001085

Table 1 Example 1: Numerical errors ($\times 10$) for some h and k.

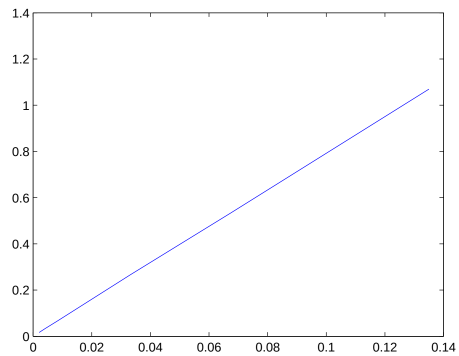

Fig. 1 Example 1: Asymptotic constant error.

If we assume now that there are not supply forces and we use the following data:

$$T_f = 20, \quad \Omega = (0,1), \quad \rho = 1, \quad \mu = 2, \quad \lambda = 1, \quad \mu_0 = 1, \quad \beta_0 = 1, \quad \beta_1 = 1,$$
$$\mu_2 = 1, \quad J = 1, \quad a_0 = 1, \quad \xi = 2, \quad \tau = 2, \quad a = 1, \quad \kappa = 5, \quad \kappa_1 = 1,$$
$$\kappa_2 = 5, \quad b = 1, \quad \kappa_3 = 1, \quad \kappa_4 = 4, \quad \kappa_5 = 1, \quad \kappa_6 = 2,$$

and the initial conditions:

$$u_0 = v_0 = \phi_0 = e_0 = T_0 = M_0, \quad \theta_0(x) = \xi_0(x) = x(x-1) \quad \text{for all } x \in (0,1),$$

taking the discretization parameters $h = k = 10^{-3}$ the evolution in time of the discrete energy E_n^{hk}, defined as

$$E_n^{hk} = \frac{1}{2}\Big\{\|v_n^{hk}\|_Y^2 + \|e_n^{hk}\|_Y^2 + \|2\xi_n^{hk} + \theta_n^{hk}\|_Y^2 + \|2M_n^{hk} + T_n^{hk}\|_Y^2 + 10\|(u_n^{hk})_x\|_Y^2$$
$$+2(u_n^{hk}, (\phi_n^{hk})_x)_Y + \|(\phi_n^{hk})_x\|_Y^2 + 2\|\phi_n^{hk}\|_Y^2 + 10\|(\theta_n^{hk})_x\|_Y^2$$
$$+10\|(T_n^{hk})_x\|_Y^2 + 5\|T_n^{hk}\|_Y^2\Big\},$$

is plotted in Fig. 2. As can be seen, it converges to zero and an exponential decay seems to be achieved.

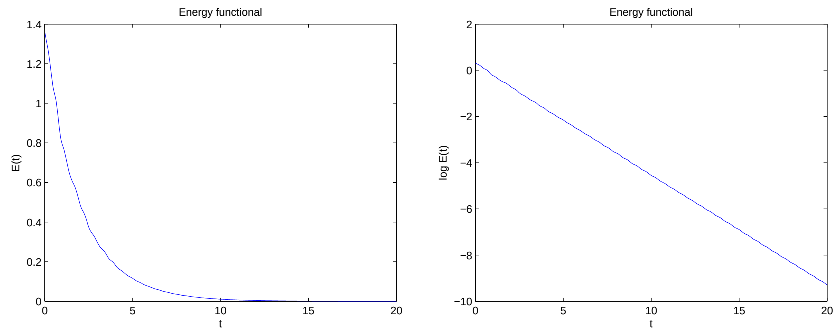

Fig. 2 Example 1: Evolution of the discrete energy in natural and semi-log scales.

4.2 Second example: comparison with the classical Fourier law

In this example, we compare the solution obtained with our Lord-Shulman model with the classical Fourier law, which corresponds to the case $\tau = 0$. We also assume that no supply forces act, so $f = s = l = G = 0$.

We have used the following data:

$$T_f = 20, \quad \Omega = (0,1), \quad \rho = 1, \quad \mu = 2, \quad \lambda = 1, \quad \mu_0 = 1, \quad \beta_0 = 1, \quad \beta_1 = 1,$$
$$\mu_2 = 1, \quad J = 1, \quad a_0 = 1, \quad \xi = 2, \quad a = 1, \quad \kappa = 1, \quad \kappa_1 = 2, \quad \kappa_2 = 5,$$
$$b = 1, \quad \kappa_3 = 1, \quad \kappa_4 = 2, \quad \kappa_5 = 1, \quad \kappa_6 = 2.$$

and the initial conditions:

$$u_0 = v_0 = \phi_0 = e_0 = T_0 = M_0, \quad \theta_0(x) = x(x-1) \quad \text{for all } x \in (0,1).$$

Moreover, we note that, since the Fourier theory does not impose any condition on the thermal acceleration (because it has a lower time derivative order), the initial condition ξ_0 is obtained from the constitutive equation $a\xi_0 = \kappa(\theta_0)_{,xx}$.

Thus, taking the discretization parameters $h = k = 10^{-3}$ in Fig. 3 we plot variables u and v at final time $T_f = 1$ for the values of the relaxation parameter $\tau = 0, 0.5, 1$. Moreover, in Fig. 4 variables ϕ and e are shown at final time for these relaxation parameters, and in Fig. 5 we plot the temperature θ and the microthermal displacement T. As we can observe, there are important differences between the solutions. Particularly, it is interesting to note that the temperature and microtemperatures seem to vanish at final time for the Fourier law.

If we consider point $x = 0.25$ the evolution in time of the temperature θ and the thermal displacement T at this point are plotted in Fig. 6 for the above values of the relaxation parameter. Again, we can observe that the solution to the Fourier law presents a great difference with the solution to the Lord-Shulman model.

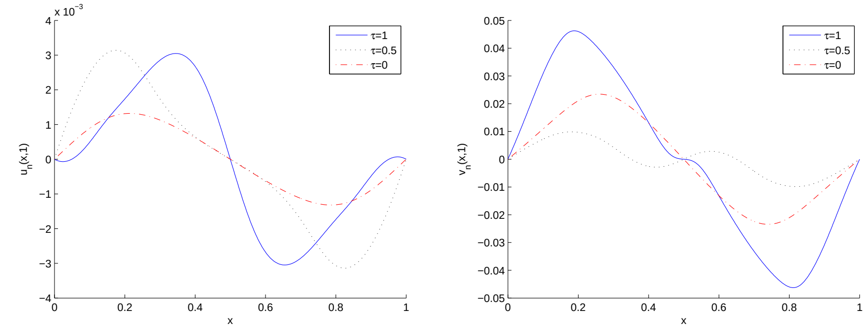

Fig. 3 Example 1D-3: Variables u (left) and v (right) at final time for relaxation parameters $\tau = 0, 0.5, 1$.

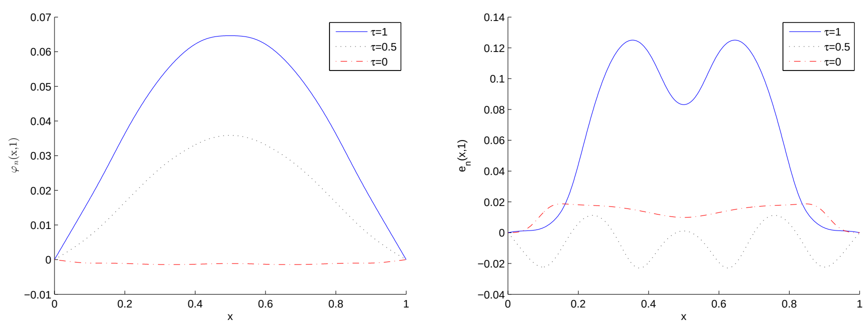

Fig. 4 Example 1D-3: Variables ϕ (left) and e (right) at final time for relaxation parameters $\tau = 0, 0.5, 1$.

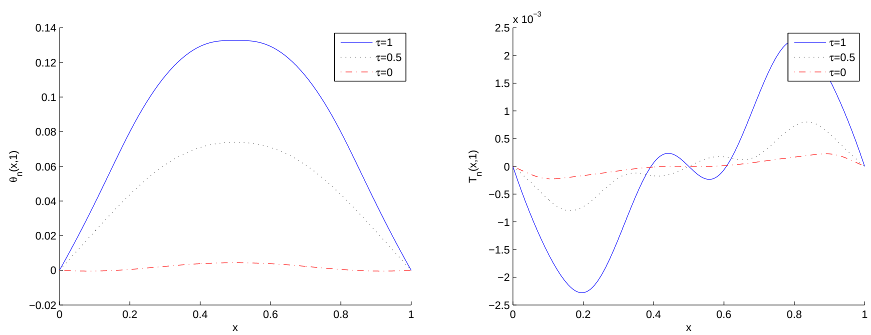

Fig. 5 Example 1D-3: Temperature θ (left) and microthermal displacement T (right) at final time for relaxation parameters $\tau = 0, 0.5, 1$.

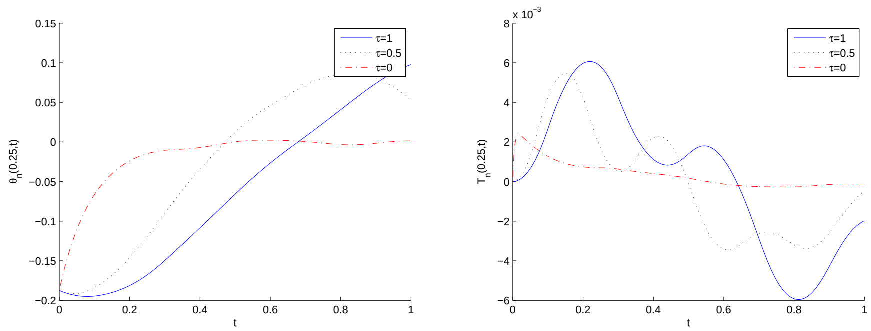

Fig. 6 Example 1D-3: Evolution in time of the temperature θ (left) and the thermal displacement T (right) at point $x = 0.25$ for relaxation parameters $\tau = 0, 0.5, 1$.

4.3 A two-dimensional problem: application of a surface force

In this two-dimensional simulation, we consider the square domain $[0,1]\times[0,1]$, which is assumed to be clamped on the left vertical boundary $\{0\} \times [0,1]$. No supply forces act supposed to act in the body, and a mechanical surface force $\boldsymbol{f}_F$ is applied on the boundary $\Gamma_F = (0,1)\times\{1\}$ with the following expression:

$$f_F(x,1,t) = (0, 0.5x^2t) \quad \text{for all} \quad x \in (0,1),\, t \in (0,1).$$

Although this mechanical condition was not considered in the previous section, the modifications are really straightforward.

The following data have been employed in this simulation:

$$\Omega = (0,1)\times(0,1), \quad T_f = 1, \quad \rho = 1, \quad \mu = 2, \quad \lambda = 1, \quad \mu_0 = 1, \quad \beta_0 = 1,$$
$$\beta_1 = 1, \quad \mu_2 = 1, \quad b = 1, \quad J = 1, \quad a_0 = 1, \quad \xi = 2, \quad \tau = 1, \quad a = 1, \quad \kappa = 5,$$
$$\kappa_1 = 1, \quad \kappa_2 = 5, \quad \kappa_3 = 1, \quad \kappa_4 = 4, \quad \kappa_5 = 1, \quad \kappa_6 = 2,$$

and the initial conditions:

$$\boldsymbol{u}_0 = \boldsymbol{v}_0 = \boldsymbol{T}_0 = \boldsymbol{M}_0 = \boldsymbol{0}, \quad \phi_0 = e_0 = \theta_0 = \xi_0 = 0.$$

In this case, we recover the actual displacement and volume fraction, taking into account the following ordinary differential equations([44]):

$$\tilde{\boldsymbol{u}} = \boldsymbol{u} + \tau\dot{\boldsymbol{u}}, \quad \tilde{\phi} = \phi + \tau\dot{\phi},$$

where $\tilde{\boldsymbol{u}}$ and $\tilde{\phi}$ are the solutions to Problem VP.

Taking the time discretization parameter $k = 0.01$, in Fig. 7 we plot the norm of both the displacement (left) and thermal displacement (right) at final time over the deformed mesh. As expected, the body bends because of the applied force. Moreover, the displacement increase along the X-axis and we observe that the thermal displacement increase in the interior area as a result of the homogeneous Dirichlet boundary condition.

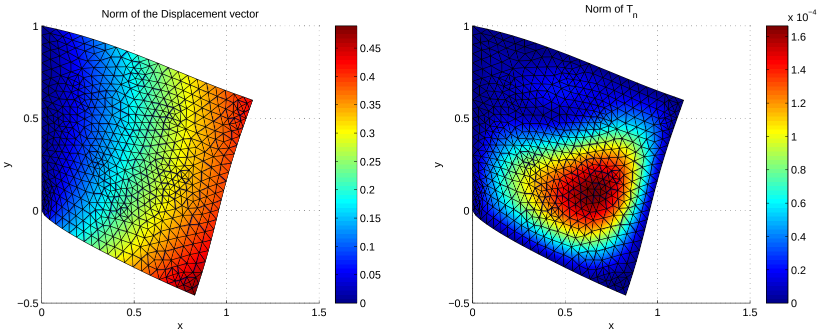

Fig. 7 Example 2D: Norms of the displacement (left) and thermal displacement (right) at final time.

Moreover, in Fig. 8 we plot the temperature (left) and volume fraction (right) at final time over the deformed mesh. They are produced due to the deformation of the body. We can observe that they both concentrate in the interior of the domain although the volume fraction seems to have an oscillating behavior.

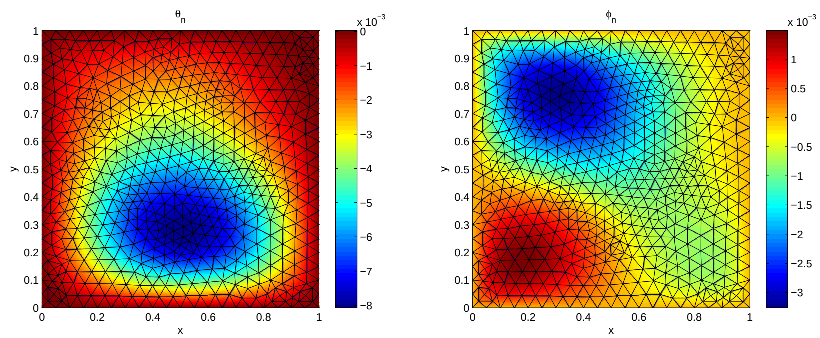

Fig. 8 Example 2D: Temperature and volume fraction at final time.

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