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ONSET OF TWO-DIMENSIONAL TURBULENCE WITH HIGH REYNOLDS NUMBERS IN THE NAVIER-STOKES EQUATIONS.

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Abstract. Even though turbulence is a tri-dimensional phenomenon, two-dimensional flows at high Reynolds numbers Re give some clues of transition to real turbulence, mainly through the vorticity. This transition is shown here with flows that are obtained computationally from the unsteady Navier-Stokes equations in stream function and vorticity variables on the well known un-regularized driven cavity problem. The work covers the range of $5000 \leq Re \leq 31000$; the results are obtained with a numerical scheme based on a fixed point iterative process applied to the elliptic nonlinear system that results after time discretization, it was reported in [1] for lower Re; it started since WCCM V, 2002, [2]-[3], and we are still working on it. The scheme has the ability to start from rest, initially, regardless of Re and has shown to be robust enough to handle high Reynolds numbers, which is not an easy task to deal with.

1 INTRODUCTION

The main goal of this paper is to present numerical results for high Reynolds numbers in the range of $5000 \leq Re \leq 31000$. Actually, in [2] the range $10000 \leq Re \leq 20000$ is covered, in [3], with primitive variables, the range $25000 \leq Re \leq 40000$, and in [1] the range $400 \leq Re \leq 5000$ to capture the steady state flow and the range $10000 \leq Re \leq 20000$ for time-dependent flows; all the cases of time-dependent flows at high Reynolds numbers are displayed at bigger times than the one considered here as well as on different meshes. The results are obtained using a simple numerical scheme for the unsteady Navier-Stokes equations in stream function and vorticity variables. They give us some clues of transition to real tri-dimensional turbulence, based on the fact that the Navier-Stokes equations is one of the three classical approaches to turbulence; the other two being: the dynamical systems approach and the conventional statical theory of turbulence, [4]. The numerical scheme is based mainly on a fixed point iterative process applied to the steady subproblem that results after a convenient time discretization is applied, [1].

At moderate Reynolds numbers, say for instance $Re \leq 7500$, the flow approaches to an asymptotic steady state as t tends to ∞. For higher Reynolds numbers, like the ones reported here, as time elapses the flow does not seem to be "stationed" somewhere, indicating that the solution is time-dependent.

The flows are obtained from the well known un-regularized driven (or lid-driven) cavity problem which originates recirculation phenomena due to the nonzero velocity boundary condition on the top wall: the recirculation is originated by the fluid flow coming from the upstream top corner, and then hitting the downstream top corner.

To get the results, unlike in [3] where very coarse meshes are used since an up-winding effect is considered, here no stabilization process is used; then the meshes in this work follow the size dictated by the thickness of the boundary layer (of order of $Re^{-\frac{1}{2}}$) and no refining on the mesh is required near the boundary. The results clearly show that as the Reynolds number increases the mesh has to be refined and this in turn leads to decrease the time step: numerically, by stability matters and physically, to capture the fast dynamics of the flow. We have already pointed out in earlier works that to get the right vorticity contours, say the ones given by the values in [5], which are supposed to be correct is more difficult than to get the right streamlines of the stream function; this the reason that some published works do no report the iso-vorticity contours, at best due to oscillations on the top right corner of the cavity for insufficient mesh refining, [6]; for instance, the result in [7] for $Re = 10000$, with a mesh $\frac{1}{128}$, shows a reasonable streamlines but awful iso-vorticity contours, due to high oscillations, (not reported by them but computed by us using the same mesh and agreeing with them with the streamlines flow). Actually, concerning the mesh size for our results we have chosen, at this stage, the one for which such oscillations are reduced to a minimum, smaller than the ones in [6].

Last but not least, as it will be seen in the Numerical Results section, from some Reynolds number on the iso-vorticity contours resemble the bi-dimensional view we see in the real 3D hurricane pictures, close to the hurricane's eye; it may be due to the recirculation that is originated by the fast fluid water flow hitting the big waves, thermal effects; and high wind velocities interacting with the ocean nonlinearly, and viceversa, [8].

2 THE CONTINUOUS PROBLEM AND THE NUMERICAL METHOD

Let $\Omega \subset R^N$ (N=2,3) be the region of the flow of a viscous incompressible fluid, and Γ its boundary. It is well known that this kind of unsteady flow is governed by the non-dimensional Navier-Stokes equations given by

$$\left\{\begin{array}{lll} \mathbf{u}_t - \frac{1}{Re}\Delta\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} & = \mathbf{f} \;\; in\ \Omega, t > 0, & (a) \\ \nabla\cdot\mathbf{u} & = 0 \;\; in\ \Omega, t > 0, & (b) \end{array}\right. \quad (1)$$

where $\mathbf{u}$, and p are the velocity and pressure of the flow, respectively. The parameter Re is the Reynolds number. The momentum equation $(1a)$ must be supplemented with appropriate initial condition and boundary condition, for instance $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ in Ω $(\nabla\cdot\mathbf{u}_0 = 0)$ and $\mathbf{u} = \mathbf{f_1}$ on Γ, $t \geq 0$ $(\int \mathbf{f_1}\cdot\mathbf{n}d\Gamma = 0)$ respectively.

Restricting the domain for equations $(1a) - (1b)$ to the two dimensional case, taking the curl on both sides of (1a) and taking into account the relations

$$u_1 = \frac{\partial\psi}{\partial y}, \quad u_2 = -\frac{\partial\psi}{\partial x}, \qquad (2)$$

which follow from $(1b)$, with ψ the stream function, and $(u_1, u_2) = \mathbf{u}$; the component in the direction of $\mathbf{k}$ gives the scalar system

$$\left\{\begin{array}{lll} \nabla^2\psi & = -\omega & (a) \\ \omega_t - \nu\nabla^2\omega + \mathbf{u}\cdot\nabla\omega & = 0 & (b) \end{array}\right. \quad (3)$$

where $\frac{1}{Re}$ has been replaced by the viscosity parameter ν, and ω is the vorticity, which from $\omega\mathbf{k} = \nabla\times\mathbf{u} = -\nabla^2\psi\mathbf{k}$, is given by

$$\omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \quad (4)$$

System (3) is the Navier-Stokes equations in stream function-vorticity variables associated with the one in primitive variables (1), considering the external force given by $\mathbf{f} = \mathbf{0}$. It should be noted that because of the relations (2) the incompressibility condition $(1b)$ is automatically satisfied in Ω, an advantage against the disadvantage that no boundary condition is given for the vorticity. Actually in [9], a procedure is given to get the boundary condition for ω in general domains.

For two dimensional rectangular domains, equations (3) are set in the domain $\Omega = (0, a)\times(0, b)$; with $a, b > 0$. The motion boundary condition in terms of the primitive variable $\mathbf{u}$ is defined by $\mathbf{u} = (1, 0)$ at the moving boundary (the top one $y = b$) and $\mathbf{u} = (0, 0)$ elsewhere.

A translation of the boundary condition in terms of the velocity primitive variable $\mathbf{u}$ to the $\psi - \omega$ variables has to be performed. Following [1], ψ is a constant function on solid and fixed walls; at the moving wall $y = b$, a constant function for ψ is also obtained, then $\psi = 0$ is chosen in Γ. By Taylor expansion of $(3a)$ on the boundary, with h_x and h_y the space steps, one obtains

$$\begin{cases} \omega(0,y,t) = -\frac{1}{2h_x^2}[8\psi(h_x,y,t) - \psi(2h_x,y,t)] + O(h_x^2) \\ \omega(a,y,t) = -\frac{1}{2h_x^2}[8\psi(a-h_x,y,t) - \psi(a-2h_x,y,t)] + O(h_x^2) \\ \omega(x,0,t) = -\frac{1}{2h_y^2}[8\psi(x,h_y,t) - \psi(x,2h_y,t)] + O(h_y^2) \\ \omega(x,b,t) = -\frac{1}{2h_y^2}[8\psi(x,b-h_y,t) - \psi(x,b-2h_y,t)] - \frac{3}{h_y} + O(h_y^2). \end{cases} \tag{5}$$

About time discretization, the time derivative ω_t is approximated by the second-order scheme

$$f_t(\mathbf{x},(n+1)\Delta t) \approx \frac{3f^{n+1}-4f^n+f^{n-1}}{2\Delta t}, \tag{6}$$

where $n \geq 1$, $\mathbf{x} \,\epsilon\, \Omega$, $\Delta t > 0$ the time step.

Then, at each time level the following nonlinear elliptic system is obtained

$$\begin{cases} \nabla^2\psi = -\omega, \quad \psi|_\Gamma = 0; & (a) \\ \alpha\omega - \nu\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = f_\omega, \quad \omega|_\Gamma = \omega_{bc}\,, & (b) \end{cases} \tag{7}$$

where $\alpha = \frac{3}{2\Delta t}$ and $f_\omega = \frac{4\omega^n - \omega^{n-1}}{2\Delta t}$. To obtain (ψ^1, ω^1), any second order strategy using a combination of one step can be applied and systems of the form (7) are also obtained.

Taking into account that the elliptic system (7) in addition to be nonlinear is of non-potential (or transport) type, a fixed point iterative process is used to solve it. A distinctive aspect here is that the iterative process is extended until the boundary to handle the ω boundary conditions given implicitly by unknown interior values of ψ.

Denoting

$$R_\omega(\omega,\psi) \equiv \alpha\omega - \nu\nabla^2\omega + \mathbf{u}\cdot\nabla\omega - f_\omega \quad in\ \Omega\,,$$

system (7) is equivalent to

$$\begin{cases} \nabla^2\psi & = -\omega \quad in\ \Omega\,, \quad \psi = 0 \quad on\ \ \Gamma \\ R_\omega(\omega,\psi) & = \quad 0 \quad in\ \Omega \quad \omega|_\Gamma = \omega_{bc} \end{cases} \tag{8}$$

Then, (8) is solved at time level (n+1), by the fixed point iterative process:
Given ω^0 and θ^0 *solve until convergence in* ω *and* θ

$$\begin{cases} \nabla^2\psi^{m+1} = -\omega^m \quad in\ \Omega\,, \\ \psi^{m+1} = 0\ on\ \Gamma \\ \omega^{m+1} = \omega^m - \rho_\omega(\alpha I - \nu\nabla^2)^{-1}R_\omega(\omega^m,\psi^{m+1})\ in\ \Omega\,, \\ \omega^{m+1} = \omega_{bc}^{m+1}\ on\ \Gamma\,,\ \rho_\omega > 0\,; \end{cases} \tag{9}$$

and then, take $(\omega^{n+1},\psi^{n+1},\theta^{n+1}) = (\omega^{m+1},\psi^{m+1},\theta^{m+1})$.

Finally, system (9), with the corresponding ω^0, is equivalent to

$$\begin{cases} \nabla^2\psi^{m+1} = -\omega^m \quad in\ \Omega, \\ \psi^{m+1} = 0\ on\ \Gamma \\ (\alpha I - \nu\nabla^2)\omega^{m+1} = (\alpha I - \nu\nabla^2)\omega^m - \rho_\omega R_\omega(\omega^m, \psi^{m+1})\ in\ \Omega, \\ \omega^{m+1} = \omega_{bc}^{m+1}\ on\ \Gamma,\ \rho_\omega > 0. \end{cases} \tag{10}$$

Two uncoupled elliptic linear problems associated with the operators $-\Delta$, $\alpha I - \nu\Delta$ have to be solved. Therefore, the solution of the original system, at each iteration of each time level, leads to the solution of standard symmetric linear elliptic operators.

It is well known that for the space discretization of problems like (10), either finite differences or finite elements may be used, as far as rectangular domains are concerned; it is also known that in either case very efficient solvers exist. For the specific results in this work, the second order approximation of the Fishpack solver [10] has been used, where the linear systems are solved through an efficient cyclic reduction iterative method; then, such second order approximation in space, combined with the second order one for vorticity boundary conditions and the second order approximation in time (6) imply that the whole approximate problem is second order.

3 Numerical Experiments.

The experiments take place on the well known un-regularized driven cavity problem. Then, the problem is set in the region $\Omega = (0,1) \times (0,1)$, with boundary given by the four walls of the cavity; the top one is moving with a nonzero velocity given by $(1,0)$ and the other are solid and fixed, the velocity (by viscocity) is given by (0,0). The range that is considered for the Reynolds numbers is $5000 \leq Re \leq 31000$. The results are reported through the iso-vorticity contours; the size mesh and the time step are denoted respectively by h and Δt, and they will be specified by each case under study. All the flows that are reported are flows at time $t = 5$.

A) Figures 1, 2, and 3 picture the iso-vorticity contours for $Re = 5000$, 10000, and 20000 with mesh size given by $h = 1/256$, 3/384, and 1/640 respectively, and time step given by $\Delta t = 0.0025$ for all of them. B) Figures 4, 5 and 7 display the iso-vorticity contours for $Re = 25000$, 30000 and 31000, with mesh size $h = 1/768$ for the first two, 1/512 for third one, and time step given by $\Delta t = 0.00025$ for all of them. C) In connection with the flow that is showed in Figure 5, Figure 6 displays the profile for the vorticity along the line $y = x$ which clearly shows that the great variation, positive and negative, occurs close to the top right corner which is in concordance with what Figure 5 shows. Actually, for $Re = 30000$ in Figure 5 we are talking about the Max/min values for the vorticity are given by Max/min=$2.845 \times 10^3/{-2.1812} \times 10^3$ and they occur at $(x,y) = (0.999, 1)/(x,y) = (0.999, 0.00195)$.

Some discussion follows: **1**) It is known that the flow for $Re = 5000$, Figure 1 in A), arrives at a steady state, we have reported this in earlier works as a validation matter. It is displayed here just to compare it, at the same time $t = 5$, with the others which are supposed to be time-dependent flows (they do not arrive at some steady state); this flow at this short time shows that the vorticity goes to accumulate around the walls in some uniform manner. As Re increases, Figures 2 and 3 in A), the vorticty tends to spread in all the cavity and the uniform manner tends to disappear, Figure 3. **2**) Surprisingly, from $Re = 25000$, Figure 4, the structure changes drastically, a "turbulent" part appears; then, the end of the "turbulent" part tends to go down as long as Re increases; Figures 5 and 6; which is a consequence that the fluid motion is faster. In this connection, the flow for $Re = 31000$ in Figure 7, with mesh size coarser than the one in Figure 5, shows a bigger oscillation in the top right corner; nevertheless, it is not a significant one since it shows that the "turbulent" end is below the previous one for $Re = 30000$ in Figure 5. **3**) Outside the "turbulent" part in pictures 4, 5, and 7, the vorticity is zero which could be in disagreement with the position Max/min given just above but it is not: it can be verified, for instance with the corresponding 3D vorticity picture, that close to the bottom left corner there appear the lowest values but they are so few that the plotter does not take care of them in Figure 6.

4 Conclusions

We have presented fluid flows at high Reynolds numbers using the Navier-Stokes equations approach to two-dimensional turbulence through the vorticity which is caused by recirculation in driven cavity problem, solving the stream function-vorticity formulation through a simple numerical scheme. For the range of Reynolds numbers considered $5000 \leq Re \leq 31000$, our results, at the fixed time $t = 5$, distingue three stages of Reynolds number sizes: a low one that is supposed to reach its steady state; two which are supposed to be time-dependent flows; and four for which a "turbulent" part appears close to the downstream top corner. Preliminary calculations for higher Reynolds numbers, $Re \geq 40000$, with finer mesh than 1/768 and Δt smaller than 0.00025, show that the "turbulent" part fills more the cavity downwards, and more at a bigger time than $t = 5$. These results will be reported elsewhere. It must be taken into consideration that these kind of two-dimensional fluid flow views, mainly for $Re \geq 25000$, are nowhere realised in nature or the laboratory but only in computer simulations, [11]; however, as we have been already pointed out in the Introduction, for this specific recirculation problem these views have to do, at least qualitatively, with the hurricanes phenomenon.

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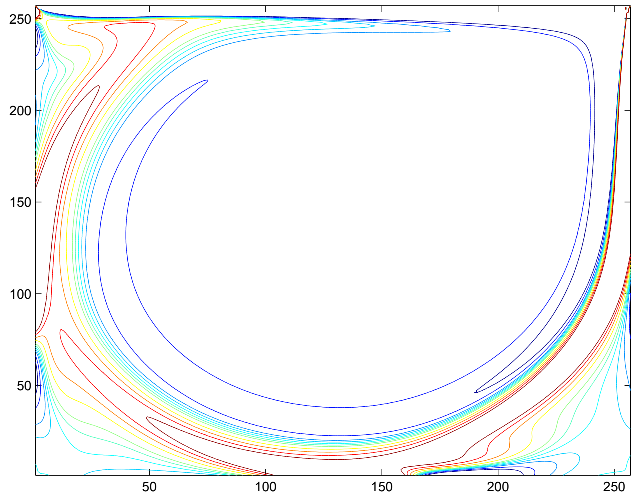

Figure 1: Vorticity: $Re = 5000$, $h = 1/256$, $dt = 0.0025$; $t = 5$

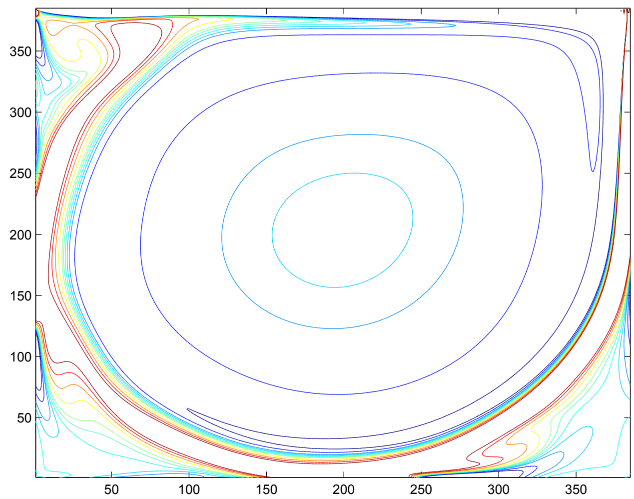

Figure 2: Vorticity: $Re = 10000$, $h = 1/384$, $dt = 0.0025$; $t = 5$

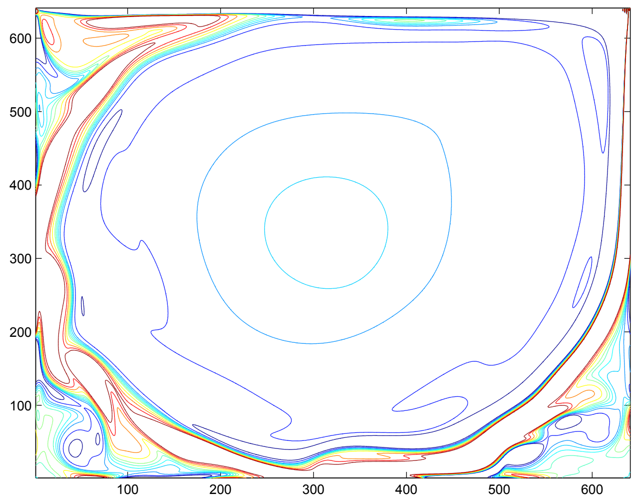

Figure 3: Vorticity: $Re = 20000$, $h = 1/640$, $dt = 0.0025$; $t = 5$

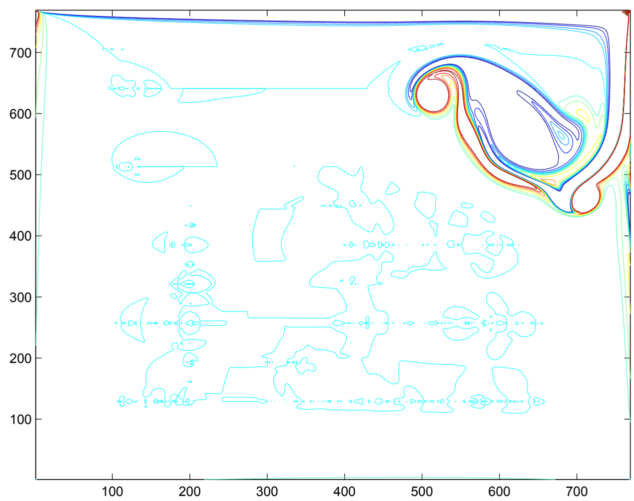

Figure 4: Vorticity: $Re = 25000$, $h = 1/768$, $dt = 0.00025$; $t = 5$

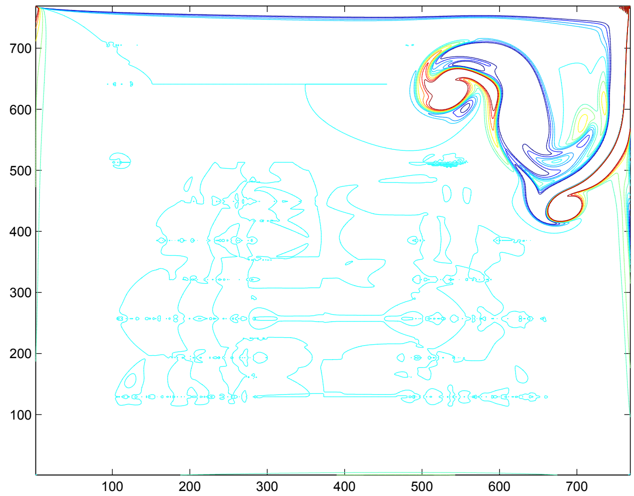

Figure 5: Vorticity: $Re = 30000$, $h = 1/768$, $dt = 0.00025$; $t = 5$

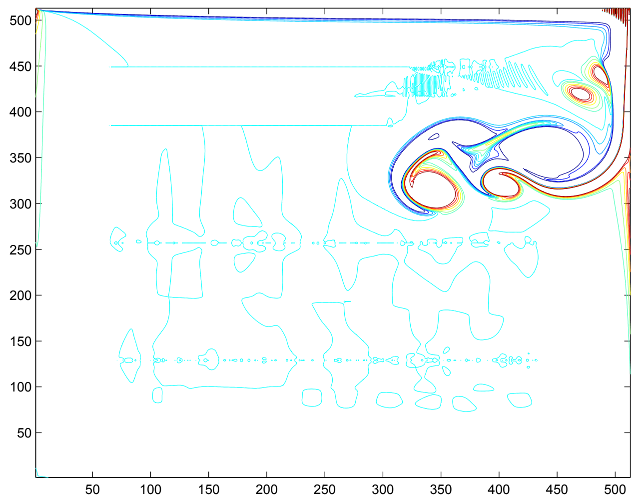

Figure 6: Vorticity: $Re = 31000$, $h = 1/512$, $dt = 0.00025$; $t = 5$

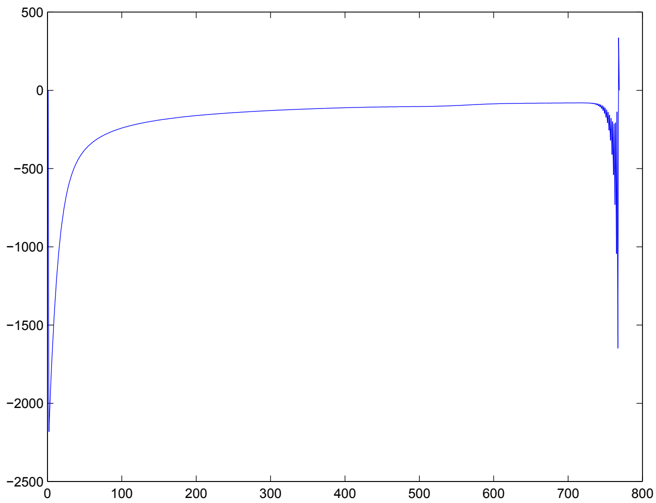

Figure 7: Vorticity profile along $y = x$ for $Re = 30000$: $h = 1/768$, $dt = 0.00025$; $t = 5$