Master in Computational and Applied Physics

Continuum and Fluid Mechanics

CHAPTER 5:
Basic equations of Fluid Mechanics

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OUTLINE

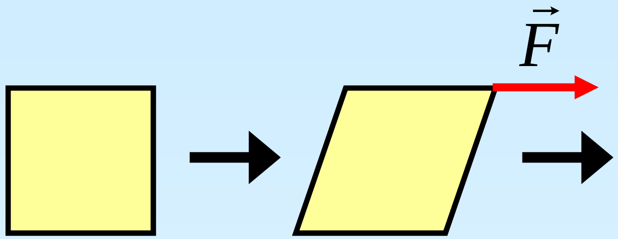
1. Solids and fluids
2. Fluid Statics
3. Perfect gas
4. Newtonian fluids
5. Navier-Stokes equations
6. Rotating frames
7. Bernouilli equation

1. Solids and fluids

Solid:

❖ has a preferred shape

❖ it takes another (constant) shape under the action of (constant) external forces

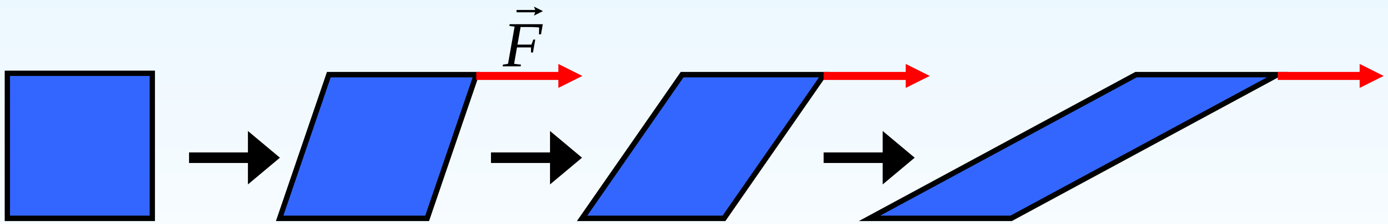


❖ it relaxes to that shape when the external forces are removed

Fluid:

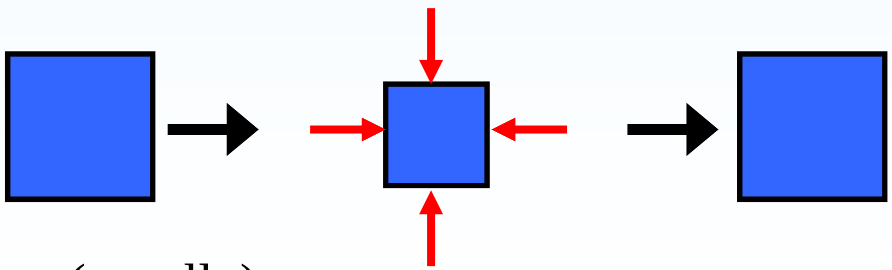
❖ has not any preferred shape

❖ it changes shape continuously under the action of fixed external forces



This is so for shear stresses, but:

➢ for normal stresses fluids and solids behave similarly



❖ normal stresses can only be a compression (usually)

Fluids:

- ❖**Gases:** always tend to expand and occupy the entire volume of any container
- ❖**Liquids:** the volume does not change very much

The distinction between solids and fluids apply well to many materials under normal conditions but:

- ➢ solids under very strong shear stresses may behave partially as fluids (plasticity)
- ➢ some fluids (like jelly, paint, polymer solutions, …) may behave partially as solids and have a partial memory of a ‘preferred shape’ (viscoelasticity)

2. Fluid statics (or Hydrostatics)

Fluid at rest:

It is empirically found that stresses are isotropic:

$$\boxed{\boldsymbol{\tau} = -\, p \ \mathbf{1}}$$

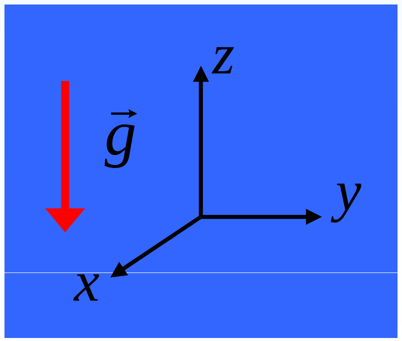
where p is > 0 (compression) and is called the **pressure**

$$\boxed{\rho \frac{d\vec{v}}{dt} = \nabla \cdot \boldsymbol{\tau} + \rho\, \vec{g}}$$

momentum equation

$$\boxed{\nabla p = \rho\, \vec{g}}$$

In case of gravity:



$\vec{g}\ =$ gravity acceleration

$$\boxed{\frac{\partial p}{\partial x} = 0\,, \quad \frac{\partial p}{\partial y} = 0\,, \quad \frac{\partial p}{\partial z} = -\rho\, g}$$

if g = const. ,
ρ = const.

$$\boxed{p = -\rho\, g\, z + p_0}$$

3. Perfect gas

The density of fluids depends on pressure and temperature.

$\boxed{\rho = \rho(p, T)}$ is called the equation of state

most gasses in normal conditions obey the following equation of state:

$$\boxed{p = \rho R T}$$

T = absolute temperature

$R = \dfrac{R_U}{m_m}$

R_U = universal constant = 8.314 J mol^{-1} K^{-1}

m_m = molecular mass. For dry air= 28.97 Kg/kmol

Perfect gas = the limit case when this equation is verified exactly

Thermodynamic properties of perfect gasses

Specific heat:

$$C_V = \frac{1}{m}\left(\frac{dQ}{dT}\right)_{V=\text{const.}} \qquad C_p = \frac{1}{m}\left(\frac{dQ}{dT}\right)_{p=\text{const.}} \qquad C_p - C_v = R_U$$

$$\gamma = \frac{C_p}{C_V}$$

for air at ordinary temperatures:
γ=1.4 and C_p=1005 J Kg^{-1} K^{-1}

For an adiabatic (no heat transfer) process: $$\frac{p}{\rho^{\gamma}} = \text{const.}$$

Thermal expansion coefficient $$\alpha = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{p=\text{const.}} = \frac{1}{T}$$

The specific internal energy is a function of the temperature only: $U = U(T)$

4. Newtonian fluids

Problem of fluid motions

Equation of motion: Cauchy or momentum equations

$$\boxed{\rho \frac{d\vec{v}}{dt} = \nabla \cdot \boldsymbol{\tau} + \rho\, \vec{g}} \qquad \left(m\vec{a} = \vec{F} \right)$$

Usually, the body forces $\vec{g}$ are known

- but, we need to know $\boxed{\boldsymbol{\tau}(\vec{v})}$ in order to solve for the motion

constitutive equations

Fluid at rest:

$$\boxed{\boldsymbol{\tau} = -p\ \mathbf{1}}$$

p = called thermodynamic pressure

Fluid in motion:

$$\boxed{\boldsymbol{\tau} = -p\ \mathbf{1} + \boldsymbol{\sigma}}$$

additional stresses which are proportional to the strain rate:

$$\sigma_{ij} = c_{ijmn} D_{mn}$$

If the fluid is isotropic:

$$c_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu \delta_{im} \delta_{jn} + \gamma \delta_{in} \delta_{jm}$$

σ_{ij} = symmetric $\Rightarrow$ $\mu = \gamma$ $\Rightarrow$ $\sigma_{ij} = 2\mu D_{ij} + \lambda D_{kk} \delta_{ij}$

$$\boxed{\tau_{ij} = -p\delta_{ij} + \lambda D_{kk} \delta_{ij} + 2\mu D_{ij}}$$

But the pressure was already defined from the mean normal stress $\bar{p} = -\frac{1}{3}\tau_{ii}$

this is called the **mean pressure** and it is different from the thermodynamic one

$$\boxed{\tau_{ij} = -p\delta_{ij} + \lambda D_{kk}\delta_{ij} + 2\mu D_{ij}} \Rightarrow \boxed{\bar{p} = p - \left(\lambda + \frac{2}{3}\mu\right) D_{ii}}$$

$$\Downarrow$$

$$\boxed{p - \bar{p} = \left(\lambda + \frac{2}{3}\mu\right)\nabla\cdot\vec{v}}$$

For **incompressible fluid**: $\boxed{\nabla\cdot\vec{v} = 0} \Rightarrow \boxed{p = \bar{p}} \Rightarrow \boxed{\tau_{ij} = -p\delta_{ij} + 2\mu D_{ij}}$

In general, bulk viscosity: $\kappa = \lambda + \frac{2}{3}\mu$ is found to be ≈ 0

Newtonian fluid: $\boxed{\kappa = \lambda + \frac{2}{3}\mu = 0} \longrightarrow \boxed{\tau_{ij} = -\left(p + \frac{2}{3}\mu\nabla\cdot\vec{v}\right)\delta_{ij} + 2\mu D_{ij}}$

Air and water obey very well the Newtonian fluid model

Examples of non Newtonian fluids:

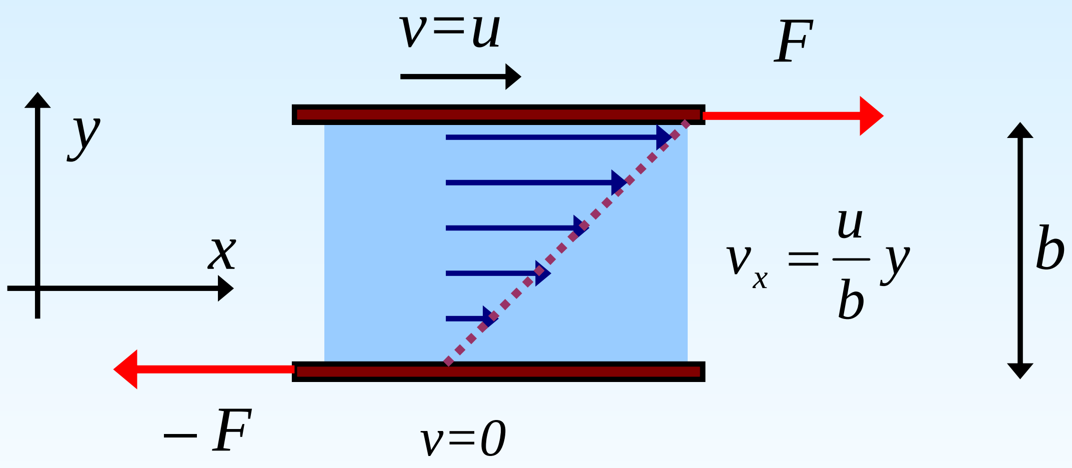
❖ solutions containing polymer molecules
❖ blood
❖ water with clay

→ Stresses are nonlinear functions of strain rates
→ Stresses depend not only on instantaneous values of strain rate but also on its history → material with memory → viscoelastic

μ = **viscosity coefficient**

it depends on the thermodynamic properties (T, ρ)

Meaning: shear experiment



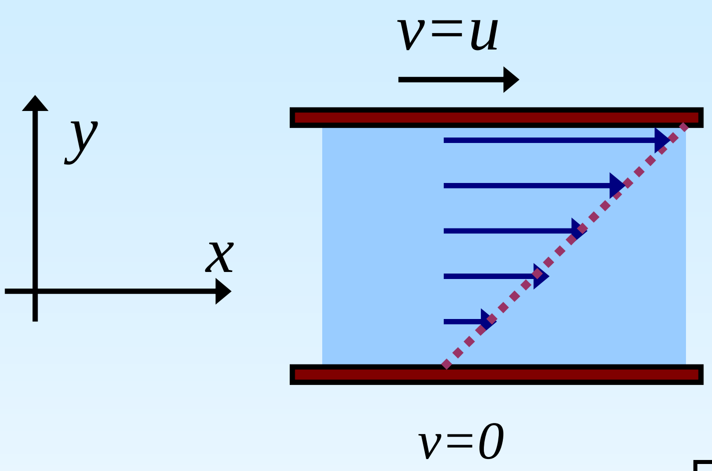
$$v_x = \frac{u}{b}\,y$$

$$D = \begin{pmatrix} 0 & u/2b & 0 \\ u/2b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\tau_{ij} = -\left(p + \frac{2}{3}\mu\nabla\cdot\vec{v}\right)\delta_{ij} + 2\mu D_{ij}} \Rightarrow \tau_{21} = 2\mu D_{21} \Rightarrow \frac{F}{A} = 2\mu\frac{u}{2b}$$

$$\boxed{\mu = \frac{F/A}{u/b}}$$

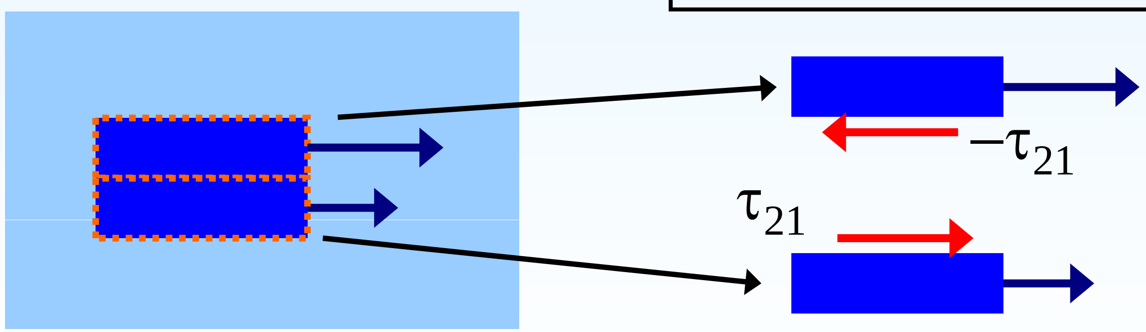
Meaning: the viscosity tends to smooth out the gradients in velocity



$$v_x = \frac{u}{b} y$$

$$D = \begin{pmatrix} 0 & u/2b & 0 \\ u/2b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\tau_{21} = 2\mu D_{21} = \mu \frac{u}{b} > 0}$$



faster particles are slowed down

slower particles are accelerated

Viscous dissipation

$$\boldsymbol{\tau} : \mathbf{D} \quad = \quad -p\nabla\cdot\vec{v} \quad + \quad \boldsymbol{\tau}' : \mathbf{D}'$$

deformation work

work associated to changes in volume

shear work

$$\tau_{ij} = -\left(p + \frac{2}{3}\mu\nabla\cdot\vec{v}\right)\delta_{ij} + 2\mu D_{ij} \longrightarrow \tau'_{ij} = 2\mu D'_{ij}$$

$$\tau'_{ij}\, D'_{ij} = 2\mu D'_{ij}\, D'_{ij}$$

Second law of thermodynamics

$\boldsymbol{\tau}' : \mathbf{D}' \geq 0$

$\mu \geq 0$

$2\mu D'_{ij}\, D'_{ij} \geq 0$ → **viscous dissipation work**

- the viscosity is always positive
- the viscosity always dissipates mechanical energy

5. Navier-Stokes equations

Problem of fluid motions

Equation of motion

$$\boxed{\rho \frac{dv_i}{dt} = \frac{\partial \tau_{ji}}{\partial x_j} + \rho\, g_i}$$

Newtonian fluid

$$\boxed{\tau_{ij} = -\left(p + \frac{2}{3}\mu \nabla \cdot \vec{v} \right)\delta_{ij} + 2\mu\, D_{ij}}$$

$$\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho\, g_i + \frac{\partial}{\partial x_j}\left(\mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2\mu}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij} \right)$$

μ = const. $\longrightarrow$

$$\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho\, g_i + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\mu}{3}\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j} \right)$$

Navier-Stokes equations

$$\boxed{\rho \frac{d\vec{v}}{dt} = -\nabla p + \rho\, \vec{g} + \mu \nabla^2 \vec{v} + \frac{\mu}{3}\nabla\left(\nabla \cdot \vec{v} \right)}$$

$$\boxed{\frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}}$$

but the pressure and the density changes are unknown → ρ and *p* are also variables

↓

continuity equation and
state equation
are involved:

$$\boxed{\frac{d\rho}{dt} + \rho \, \nabla \cdot \vec{v} = 0}$$

$$\boxed{\rho = \rho(p, T)}$$

$\Rightarrow$ T is also a variable **!**

equation for the temperature is needed: first law of Thermodynamics

$$\boxed{\rho \frac{du}{dt} = \boldsymbol{\tau} : \mathbf{D} \; - \nabla \cdot \vec{q} + \rho \, r}$$

$$u = u(T, \rho)$$

Therefore, the dynamical problem involves at least (in general):

6 variables: velocity, pressure, density, temperature:

$$\vec{v}(\vec{x},t),\ p(\vec{x},t), \rho(\vec{x},t), T(\vec{x},t)$$

6 equations:

$$\begin{cases} \rho \dfrac{d\vec{v}}{dt} = -\nabla p + \rho\,\vec{g} + \mu \nabla^2 \vec{v} + \dfrac{\mu}{3}\nabla\left(\nabla \cdot \vec{v}\right) \\ \dfrac{d\rho}{dt} + \rho\,\nabla \cdot \vec{v} = 0 \\ \rho = \rho(p,T) \\ \rho \dfrac{du}{dt} = \boldsymbol{\tau} : \mathbf{D} \; - \nabla \cdot \vec{q} + \rho\, r \qquad\qquad u = u(T,\rho) \end{cases}$$

If viscosity can be neglected:

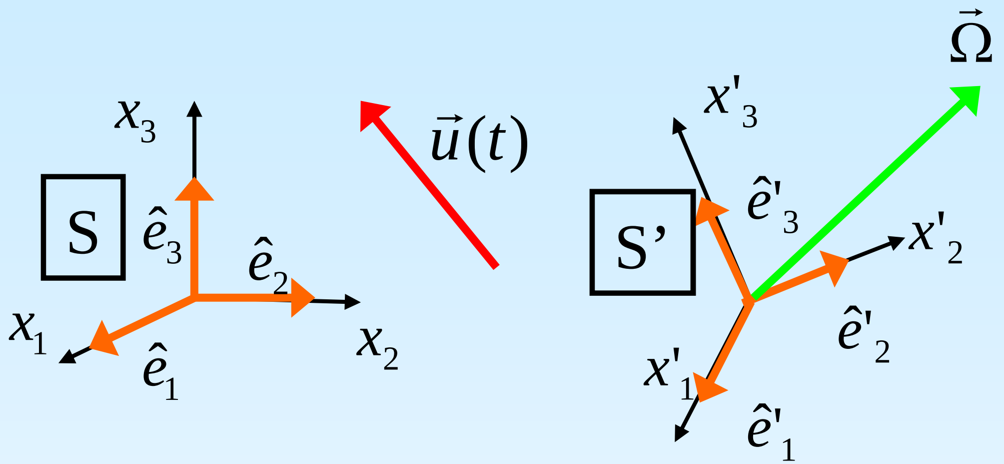
$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \rho \vec{g}$$

Euler equations

$$\Updownarrow$$

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i$$

6. Rotating frames



time derivative in S

$$\left(\frac{d\vec{u}}{dt}\right)_S = \frac{du_i}{dt}\hat{e}_i$$

time derivative in S'

$$\left(\frac{d\vec{u}}{dt}\right)_{S'} = \frac{du'_i}{dt}\hat{e}'_i$$

$$\vec{u} = u_i\hat{e}_i = u'_i\,\hat{e}'_i \quad \Rightarrow \quad \left(\frac{d\vec{u}}{dt}\right)_S = \frac{du'_i}{dt}\hat{e}'_i + u'_i\left(\frac{d\hat{e}'_i}{dt}\right)_S = \left(\frac{d\vec{u}}{dt}\right)_{S'} + u'_i\left(\frac{d\hat{e}'_i}{dt}\right)_S$$

$$\left(\frac{d\hat{e}'_i}{dt}\right)_S = ? \qquad \hat{e}'_i = Q_{ji}\hat{e}_j \quad \Rightarrow \quad \frac{d\hat{e}'_i}{dt} = \frac{dQ_{ji}}{dt}\hat{e}_j = \frac{dQ_{ji}}{dt}Q_{jk}\hat{e}'_k = W_{ki}\hat{e}'_k$$

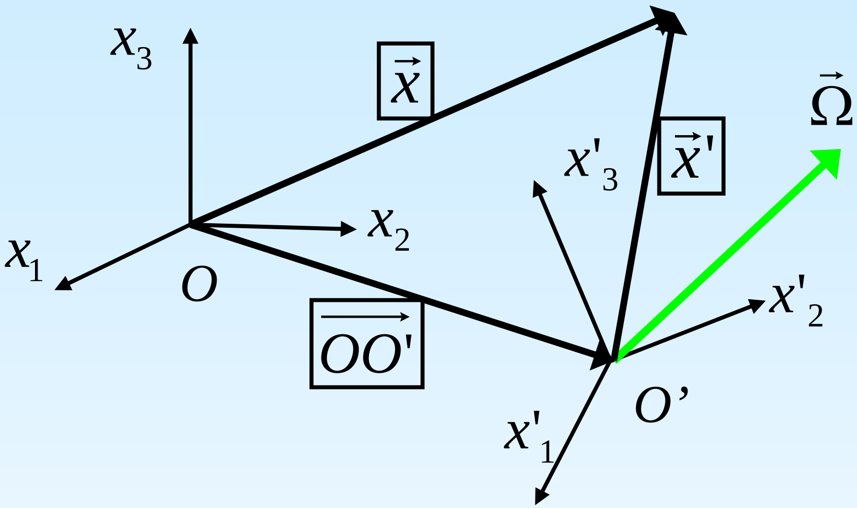
antisymmetric

$$\boxed{\Omega_i = -\frac{1}{2}\varepsilon_{ijk}W_{jk}} \rightarrow \boxed{\left(\frac{d\hat{e}'_i}{dt}\right)_S = \vec{\Omega}\times\hat{e}'_i}$$

$\vec{\Omega} =$ **angular velocity of frame S' relative to frame S**

Euler equations

$$\boxed{\left(\frac{d\vec{u}}{dt}\right)_S = \left(\frac{d\vec{u}}{dt}\right)_{S'} + \vec{\Omega}\times\vec{u}}$$



$$\vec{x} = \overrightarrow{OO'} + \vec{x}'$$

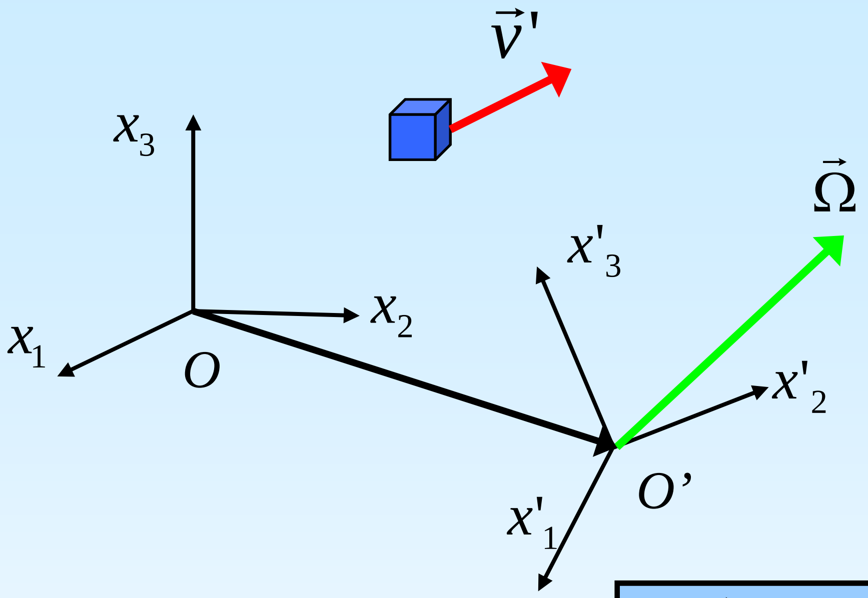
$$\boxed{\left(\frac{d\vec{x}'}{dt}\right)_S = \left(\frac{d\vec{x}'}{dt}\right)_{S'} + \vec{\Omega}\times\vec{x}'}$$

$$\boxed{\vec{v} = \vec{v}_{OO'} + \vec{v}' + \vec{\Omega}\times\vec{x}'}$$

$$\boxed{\left(\frac{d\vec{u}}{dt}\right)_S = \left(\frac{d\vec{u}}{dt}\right)_{S'} + \vec{\Omega}\times\vec{u}}$$

$$\boxed{\vec{a} = \vec{a}_{OO'} + \vec{a}' + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{x}\right) + 2\,\vec{\Omega}\times\vec{v}' + \frac{d\vec{\Omega}}{dt}\times\vec{x}'}$$

centripetal acceleration	Coriolis acceleration



Navier Stokes equations in the rotating frame

$$\boxed{\rho \frac{d\vec{v}\,'}{dt} = -\nabla p + \rho\, \vec{g}\,' + \mu \nabla^2 \vec{v} + \frac{\mu}{3} \nabla\left(\nabla \cdot \vec{v}\,\right) \underbrace{- 2\rho\, \vec{\Omega} \times \vec{v}\,'}_{\textbf{Coriolis force}}}$$

with

$$\boxed{\vec{g}\,' = \vec{g} \underbrace{- \vec{\Omega} \times \left(\vec{\Omega} \times \vec{x}\,'\right)}_{\textbf{centrifugal force / }\rho} - \vec{a}_{O'O} - \frac{d\vec{\Omega}}{dt} \times \vec{x}\,'}$$

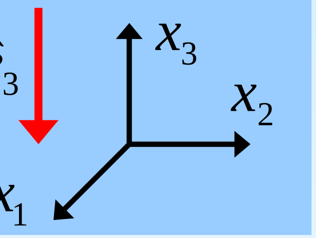
7. Bernouilli equation

Simplified expression of the Euler equations
(equations of motion for inviscid fluid, i.e., viscosity is neglected)

Assumptions:

- no viscosity, μ = 0
- barotropic flow, i.e.,ρ = ρ(*p*)
- body force is gravity (= const.)

$\vec{g} = -g\hat{e}_3$



Euler equations: $$\boxed{\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla p + \vec{g}}$$

- $\vec{g} = -\nabla\left(g x_3\right)$
- $\vec{v}\cdot\nabla\vec{v} = \vec{\omega}\times\vec{v} + \nabla\left(\frac{1}{2}v^2\right)$ vector identity already proven
- $F(p) \equiv \int \frac{dp}{\rho(p)} \quad \Rightarrow \quad \nabla F = \frac{dF}{dp}\nabla p = \frac{1}{\rho}\nabla p$

$$\boxed{\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \vec{g}}$$

- $\vec{g} = -\nabla\left(g x_3\right) \equiv -\nabla\left(g z\right)$
- $\vec{v} \cdot \nabla \vec{v} = \vec{\omega} \times \vec{v} + \nabla\left(\frac{1}{2} v^2\right)$
- $F(p) \equiv \int \frac{dp}{\rho(p)} \quad \Rightarrow \quad \nabla F = \frac{dF}{dp} \nabla p = \frac{1}{\rho} \nabla p$

$$\boxed{\frac{\partial \vec{v}}{\partial t} + \nabla \Theta = \vec{v} \times \vec{\omega}} \quad \text{with} \quad \boxed{\Theta = \frac{1}{2} v^2 + g\, z + \int \frac{dp}{\rho(p)}}$$

Bernouilli function

This equation is specially useful in two particular cases:

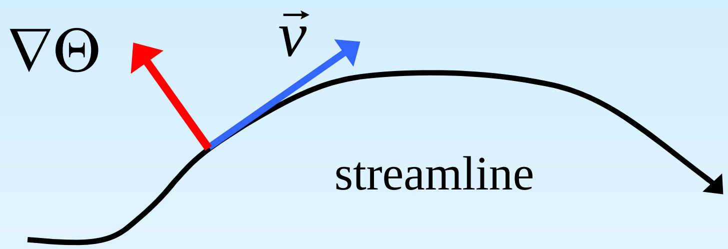
1. Steady flow
2. Irrotational flow

Bernouilli equation for steady flow

$$\boxed{\frac{\partial \vec{v}}{\partial t} + \nabla\Theta = \vec{v} \times \vec{\omega}}$$

$$\Downarrow$$

$$\boxed{\nabla\Theta = \vec{v} \times \vec{\omega}} \quad \Rightarrow \quad \boxed{\nabla\Theta \perp \vec{v}}$$



$$\Downarrow$$

$$\boxed{\begin{aligned} \Theta &= \frac{1}{2}v^2 + g\,z + \int \frac{dp}{\rho(p)} \\ &= \text{constant along the streamlines} \end{aligned}}$$

The constant may be different for different streamlines

$\vec{\omega} = 0 \quad \Rightarrow \quad$ the constant is the same everywhere in the flow

Bernouilli equation for irrotational flow

Irrotational flow: $\boxed{\vec{\omega} = \nabla \times \vec{v} = 0}$ $\Leftrightarrow$ $\boxed{\exists \phi \;\; \Big| \;\; \vec{v} = \nabla \phi}$

potential flow

$$\boxed{\frac{\partial \vec{v}}{\partial t} + \nabla \Theta = \vec{v} \times \vec{\omega}} \longrightarrow \boxed{\frac{\partial}{\partial t}\left(\nabla \phi\right) + \nabla \Theta = 0}$$

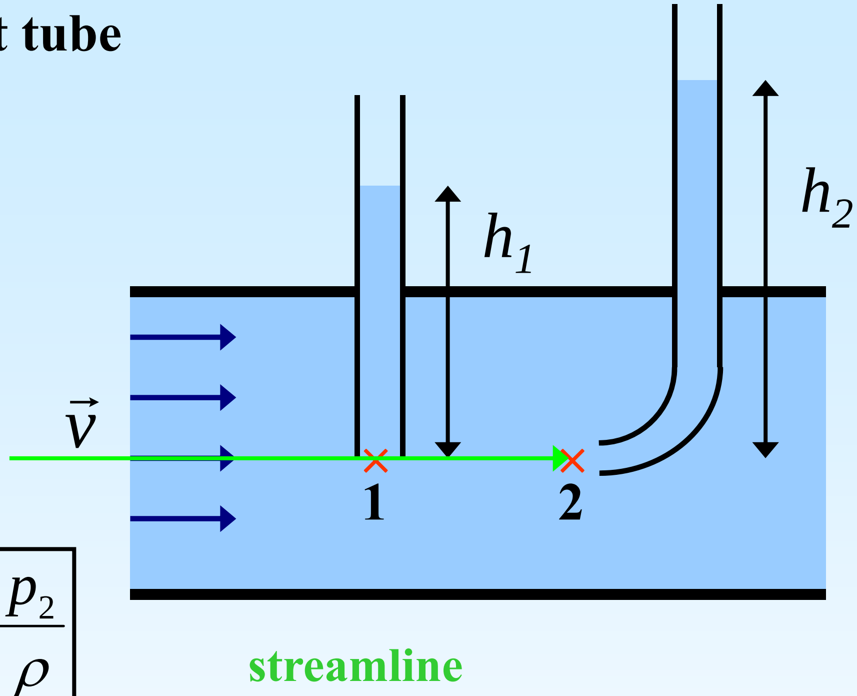
$$\boxed{\nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + g\, z + \int \frac{dp}{\rho(p)} \right) = 0}$$

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + g\, z + \int \frac{dp}{\rho(p)} = F(t)}$$

Application of Bernouilli: Pitot tube

device to measure a flow velocity

- steady flow
- viscosity is neglected
- ρ = const.



$$\boxed{\frac{1}{2}v_1^2 + gz_1 + \frac{p_1}{\rho} = \frac{1}{2}v_2^2 + gz_2 + \frac{p_2}{\rho}}$$

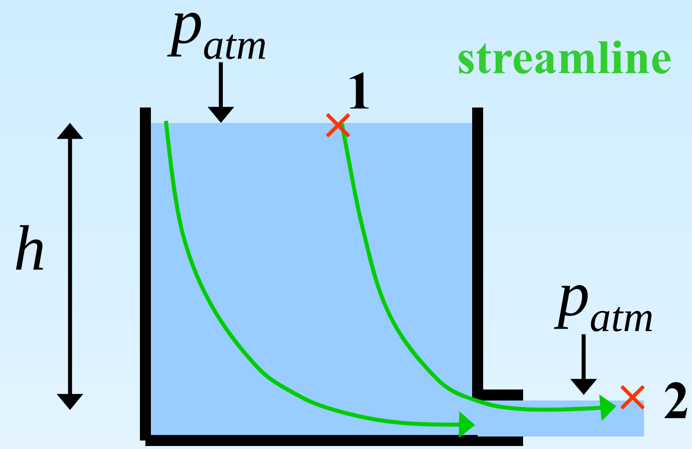
$$\boxed{\frac{1}{2}v_1^2 + \frac{p_1}{\rho} = \frac{p_2}{\rho}}$$

$$\boxed{v_1 = \sqrt{2g\left(h_2 - h_1\right)}}$$

inside the vertical tubes there is hydrostatic balance:
$p_1 = p_{atm} + \rho g h_1$, $p_2 = p_{atm} + \rho g h_2$

Application of Bernouilli: orifice in a tank

- orifice small $\Rightarrow$ flow approximately steady
- viscosity is neglected
- ρ = const.



$$\boxed{\frac{1}{2}v_1^2 + gz_1 + \frac{p_1}{\rho} = \frac{1}{2}v_2^2 + gz_2 + \frac{p_2}{\rho}}$$

$$\boxed{gh = \frac{1}{2}v_2^2} \longrightarrow \boxed{v_2 = \sqrt{2gh}} \longrightarrow \boxed{\frac{dm}{dt} = \rho A\sqrt{2gh}}$$

Actually, the function

$$\Theta = \frac{1}{2}v^2 + gz + \frac{p}{\rho}$$

has the same value everywhere since it has the same value at any point on the free surface $\Rightarrow$ **flow is irrotational**

$$\frac{\partial \vec{v}}{\partial t} + \nabla\Theta = \vec{v} \times \vec{\omega}$$

$$\Downarrow$$

$$\vec{v} \times \vec{\omega} = 0 \quad \Rightarrow \quad \vec{\omega} = 0$$