

**Master in Computational and Applied Physics**

# **Continuum and Fluid Mechanics**

## **CHAPTER 4: Conservation Laws**

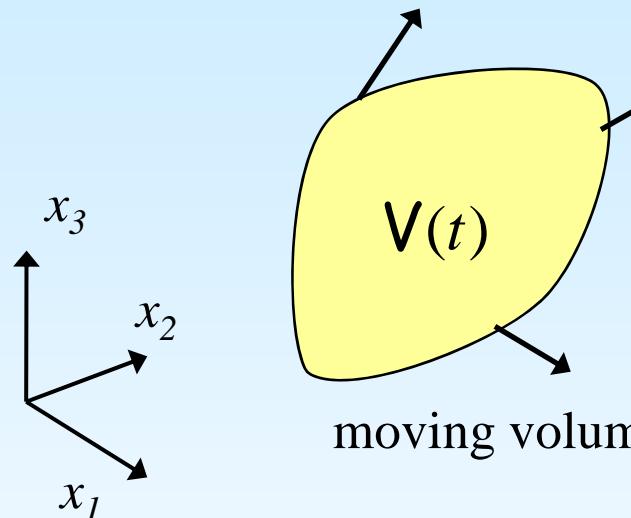
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# OUTLINE

1. Time derivative of volume, area and line integrals
2. Mass conservation. Continuity equation
3. General form of conservation laws
4. Diffusion equations
5. Momentum conservation
6. Angular momentum conservation
7. Mechanical energy balance
8. First Law of Thermodynamics
9. Second Law of Thermodynamics

# 1. Time derivative of volume, area and line integrals



$$\Psi(t) = \int_{V(t)} \psi(\vec{x}, t) dV$$

$$\frac{d\Psi}{dt} = ?$$

Trivial case: the volume is at rest  $V(t) = \text{const.}$

$$\frac{d\Psi(t)}{dt} = \int_V \frac{\partial \psi}{\partial t} dV$$

# Material volume

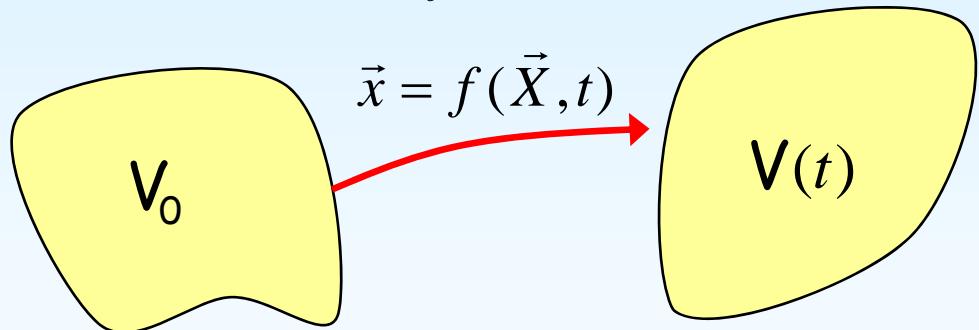
Since the integration domain is variable, one cannot simply interchange the integral and the time derivative

$$\frac{d\Psi(t)}{dt} = \frac{d}{dt} \int_{V(t)} \psi(\vec{x}, t) dV = ?$$

but we can use the map defined by the motion, to map the integration domain  $V(t)$  into the corresponding volume in the reference configuration,  $V_0$  and do the integration over  $V_0$

$$\boxed{\int_{V(t)} \psi(\vec{x}, t) dV = \int_{V_0} \psi(\vec{X}, t) |\mathbf{F}| dV_0}$$

$$\mathbf{F} = \frac{\partial \vec{x}}{\partial \vec{X}} \quad |\mathbf{F}| = \det \mathbf{F}$$



since now the integration domain,  $V_0$ , is constant:

$$\frac{d}{dt} \int_{V_0} \psi(\vec{X}, t) |\mathbf{F}| dV_0 = \int_{V_0} \frac{\partial}{\partial t} (\psi(\vec{X}, t) |\mathbf{F}|) dV_0 = \int_{V_0} \left( \frac{\partial \psi}{\partial t} |\mathbf{F}| + \psi \frac{\partial |\mathbf{F}|}{\partial t} \right) dV_0$$

$$\frac{d|\mathbf{F}|}{dt} = |\mathbf{F}| \nabla \cdot \vec{v}$$

$$\int_{V_0} \left( \frac{\partial \psi}{\partial t} |\mathbf{F}| + \psi \frac{\partial |\mathbf{F}|}{\partial t} \right) dV_0 \downarrow \int_{V_0} \left( \frac{\partial \psi}{\partial t} + \psi \nabla \cdot \vec{v} \right) |\mathbf{F}| dV_0$$

coming back to the present configuration,  $V(t)$ :

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \vec{v} \cdot \nabla \psi$$

$$\int_{V_0} \left( \frac{\partial \psi}{\partial t} + \psi \nabla \cdot \vec{v} \right) |\mathbf{F}| dV_0 = \int_{V(t)} \left( \frac{d\psi}{dt} + \psi \nabla \cdot \vec{v} \right) dV \downarrow \int_{V(t)} \left( \frac{\partial \psi}{\partial t} + \nabla \cdot (\psi \vec{v}) \right) dV$$

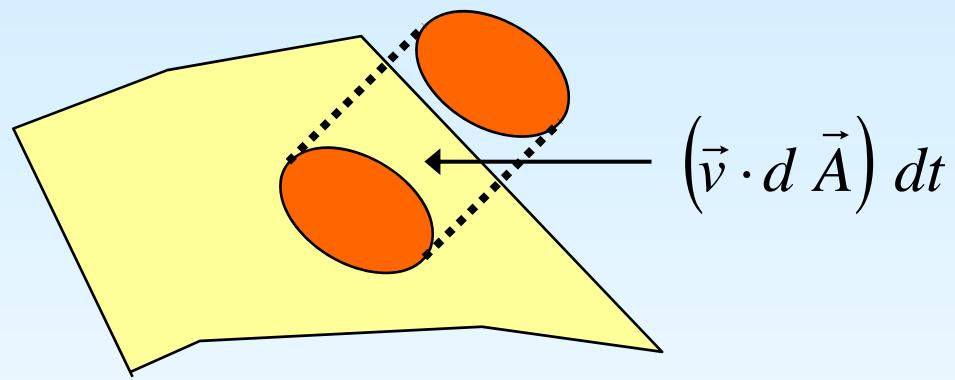
application of Gauss theorem:

$$\Rightarrow \frac{d}{dt} \int_{V(t)} \psi(\vec{x}, t) dV = \int_{V(t)} \frac{\partial \psi}{\partial t} dV + \int_{\partial V(t)} \psi \vec{v} \cdot d\vec{A}$$

**variation due to  
the local time  
dependence of  $\psi$**

**variation due to  
the time dependence  
of the integration  
domain**

$$\frac{d}{dt} \int_{V(t)} \psi(\vec{x}, t) dV = \int_{V(t)} \frac{\partial \psi}{\partial t} dV + \int_{\partial V(t)} \psi \vec{v} \cdot d\vec{A}$$



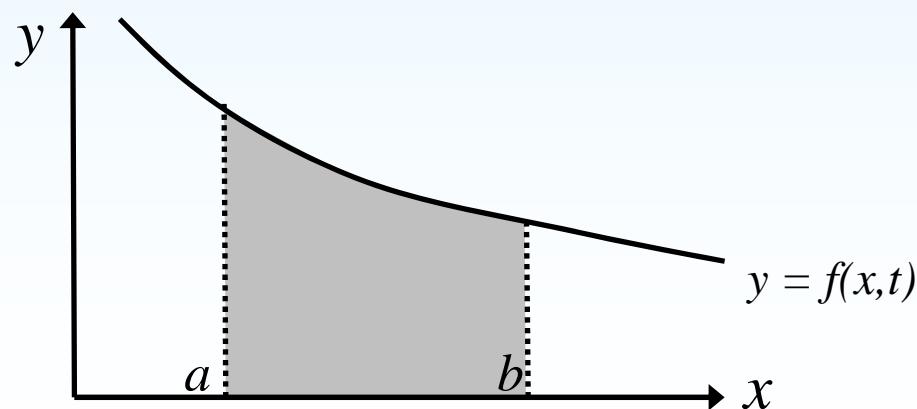
$$\frac{d}{dt} \int_{V(t)} \psi(\vec{x}, t) dV = \int_{V(t)} \frac{\partial \psi}{\partial t} dV + \int_{\partial V(t)} \psi \vec{v} \cdot d\vec{A}$$

totally similar to the ‘Leibnitz rule’ (integral over a straight line):

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = \underbrace{\int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx}_{\text{variation due to the time dependence of } f} + f(b(t), t) \frac{db}{dt} - f(a(t), t) \frac{da}{dt}$$

**variation due to the time dependence of  $f$**

**variation due to the time dependence of the integration domain**



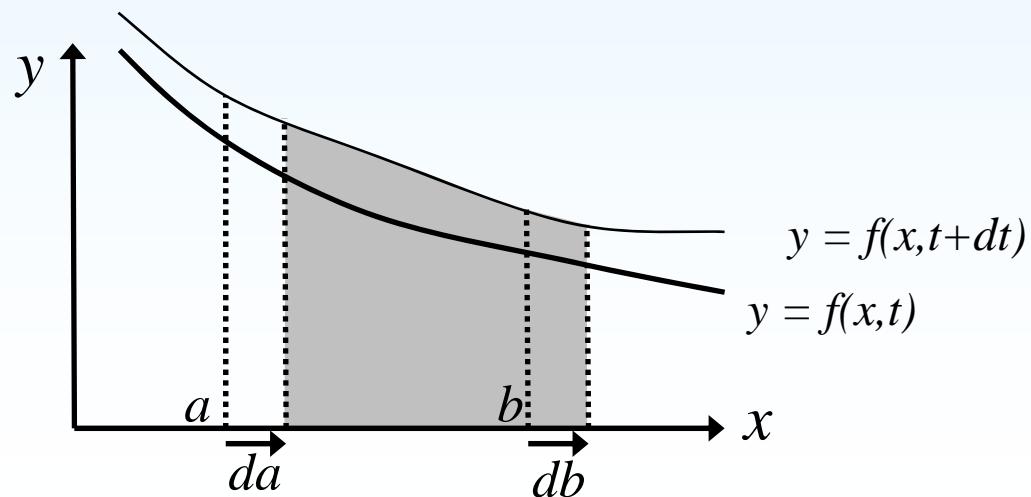
$$\frac{d}{dt} \int_{V(t)} \psi(\vec{x}, t) dV = \int_{V(t)} \frac{\partial \psi}{\partial t} dV + \int_{\partial V(t)} \psi \vec{v} \cdot d\vec{A}$$

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**variation due to the time dependence of  $f$**

**variation due to the time dependence of the integration domain**



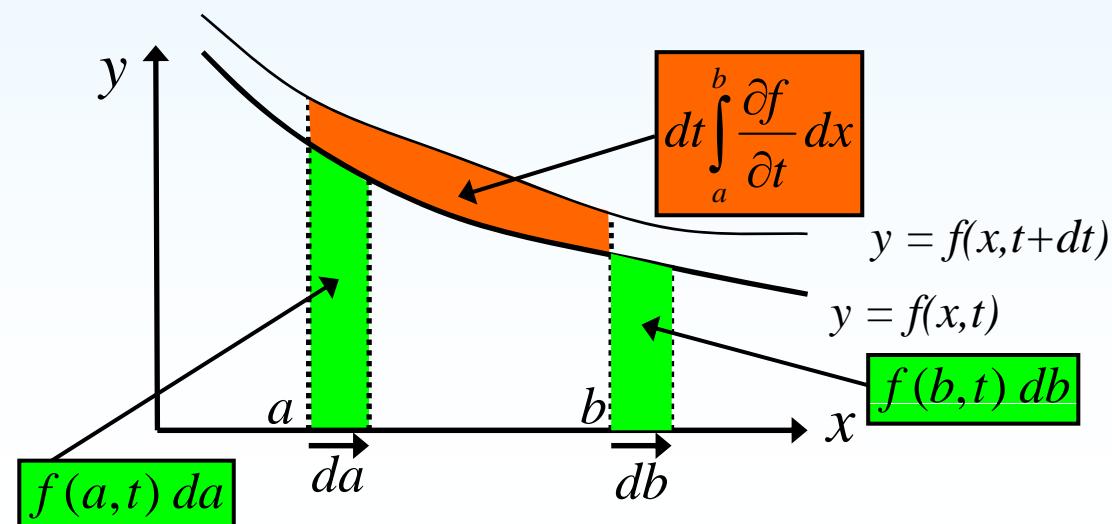
$$\frac{d}{dt} \int_{V(t)} \psi(\vec{x}, t) dV = \int_{V(t)} \frac{\partial \psi}{\partial t} dV + \int_{\partial V(t)} \psi \vec{v} \cdot d\vec{A}$$

totally similar to the ‘Leibnitz rule’ (integral over a straight line):

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = \underbrace{\int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx}_{\text{variation due to the time dependence of } f} + f(b(t), t) \frac{db}{dt} - f(a(t), t) \frac{da}{dt}$$

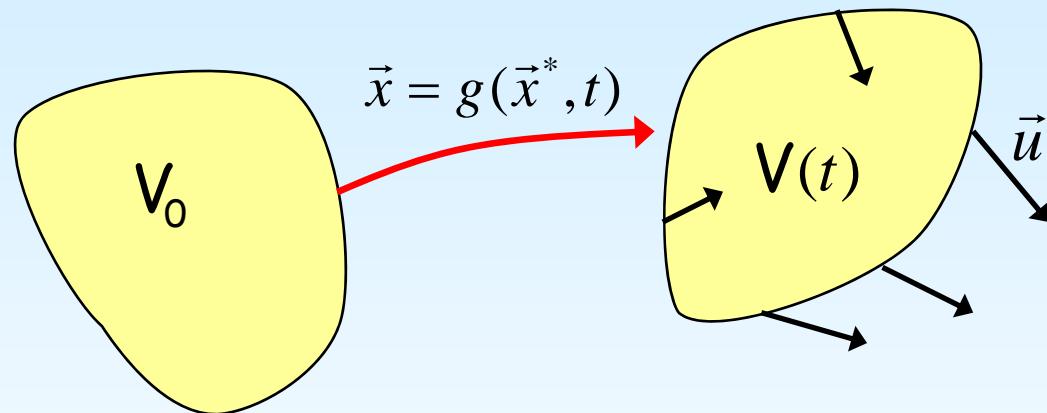
**variation due to the time dependence of  $f$**

**variation due to the time dependence of the integration domain**



## Arbitrary volume

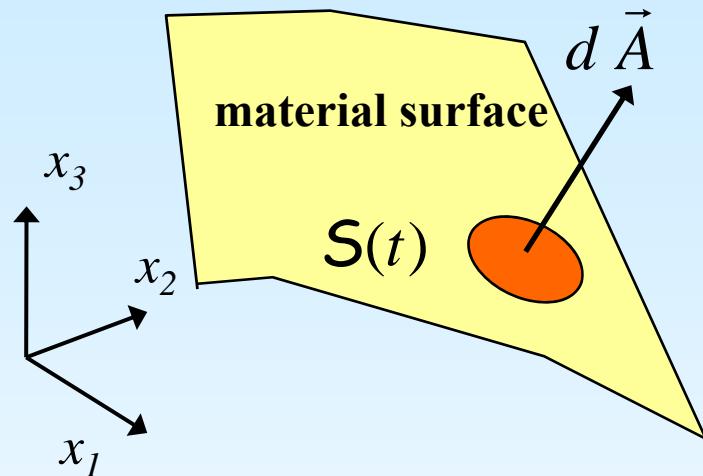
In case of an arbitrary volume, one can proceed in a similar way by seeking a transformation which maps the moving volume into a fixed volume in space.



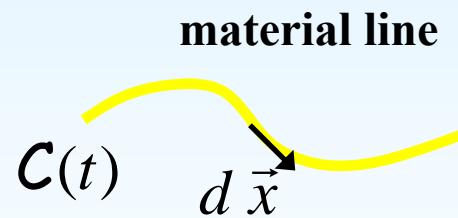
$$\frac{d}{dt} \int_{V(t)} \psi(\vec{x}, t) dV = \int_{V(t)} \frac{\partial \psi}{\partial t} dV + \int_{\partial V(t)} \psi \vec{u} \cdot d\vec{A}$$

velocity of the bounding surface  
not velocity of the particles !

# Material surfaces and material lines



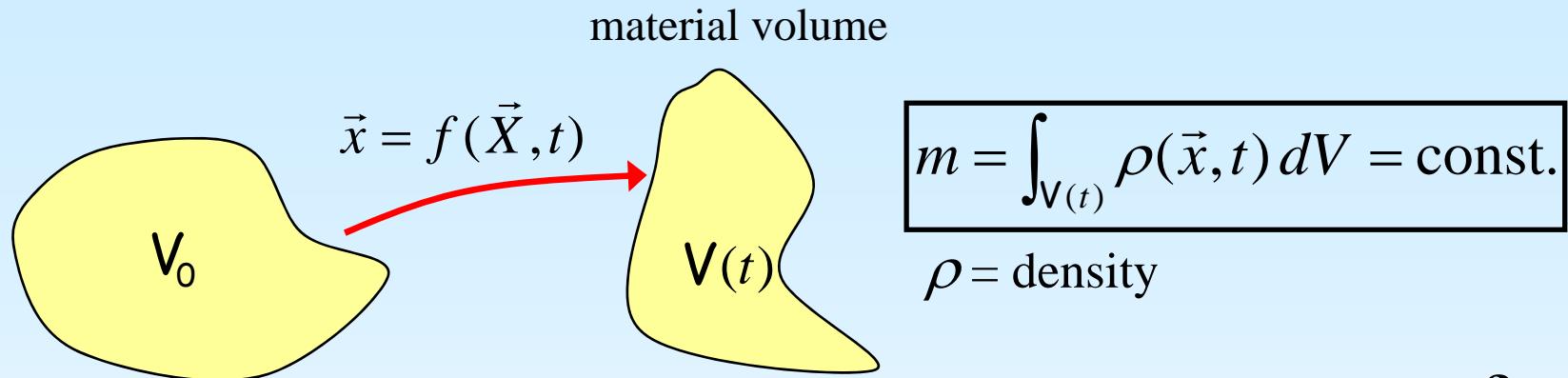
$$\frac{d}{dt} \int_{S(t)} \psi(\vec{x}, t) dA_i = \int_{S(t)} \left( \frac{d\psi}{dt} + \psi \frac{\partial v_k}{\partial x_k} \right) dA_i - \int_{S(t)} \left( \psi \frac{\partial v_k}{\partial x_i} \right) dA_k$$



$$\frac{d}{dt} \int_{C(t)} \psi(\vec{x}, t) dx_i = \int_{C(t)} \frac{d\psi}{dt} dx_i + \int_{C(t)} \psi \frac{\partial v_i}{\partial x_k} dx_k$$

Proofs: J.M. Massaguer & A. Falqués, 'Mecánica del Continuo: Geometría y Dinámica', eds. UPC, 1994. pag. 185.

## 2. Mass conservation. Continuity equation.



local formulation ?

$$\frac{dm}{dt} = \frac{d}{dt} \int_{V(t)} \rho(\vec{x}, t) dV = \int_{V(t)} \left( \frac{d\rho}{dt} + \rho \nabla \cdot \vec{v} \right) dV = 0$$

since  $V(t)$  may be any material volume  
within the body, it follows that

$$\frac{d\rho}{dt} + \rho \nabla \cdot \vec{v} = 0$$

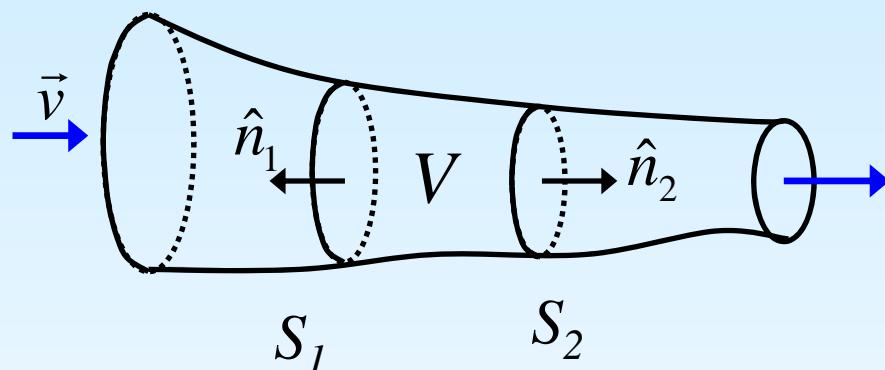
continuity equation

**Warning:** in case the derivatives are not continuous,  
the local form is not valid (e.g., shock waves)

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

The meaning of the continuity equation can be illustrated by considering a control volume fixed in space



$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$m$  = mass inside  $V$  at any time

$$\frac{dm}{dt} = \frac{\partial}{\partial t} \int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV = - \int_V \nabla \cdot (\rho \vec{v}) dV = - \underbrace{\int_{\partial V} \rho \vec{v} \cdot d\vec{A}}$$

net mass that goes in through the boundary

$$- \int_{\partial V} \rho \vec{v} \cdot d\vec{A} = - \int_{S_2} \rho \vec{v} \cdot \hat{n}_2 dA - \int_{S_1} \rho \vec{v} \cdot \hat{n}_1 dA = \int_{S_1} \rho v_n dA - \int_{S_2} \rho v_n dA$$

discharge (cabal, caudal) in Kg/s

in

out

**Another expression of the continuity equation  
that is considered as the Lagrangian version**

$$\begin{array}{c}
 \boxed{\frac{d\rho}{dt} + \rho \nabla \cdot \vec{v} = 0} \xrightarrow{} \boxed{|\mathbf{F}| \frac{d\rho}{dt} + \rho \frac{d|\mathbf{F}|}{dt} = 0} \xrightarrow{} \boxed{\frac{d}{dt}(\rho|\mathbf{F}|) = 0} \\
 \uparrow \\
 \boxed{\nabla \cdot \vec{v} = \frac{1}{|\mathbf{F}|} \frac{d|\mathbf{F}|}{dt}}
 \end{array}$$

### Incompressibility condition

$$\begin{array}{c}
 \text{Eulerian description} \\
 \boxed{\frac{d\rho}{dt} + \rho \nabla \cdot \vec{v} = 0} \xrightarrow{} \boxed{\nabla \cdot \vec{v} = 0} \\
 \text{Lagrangian description} \\
 \boxed{\frac{d}{dt}(\rho|\mathbf{F}|) = 0} \xrightarrow{} \boxed{\frac{d|\mathbf{F}|}{dt} = 0}
 \end{array}$$

# Reynolds transport theorem

Very often, the function to integrate is a quantity times the density,  
(e.g., the momentum is density  $\times$  velocity) and we have

$$\psi = \rho f$$

In this case, the identity  $\frac{d}{dt} \int_{V(t)} \psi(\vec{x}, t) dV = \int_{V(t)} \left( \frac{d\psi}{dt} + \psi \nabla \cdot \vec{v} \right) dV$

reads

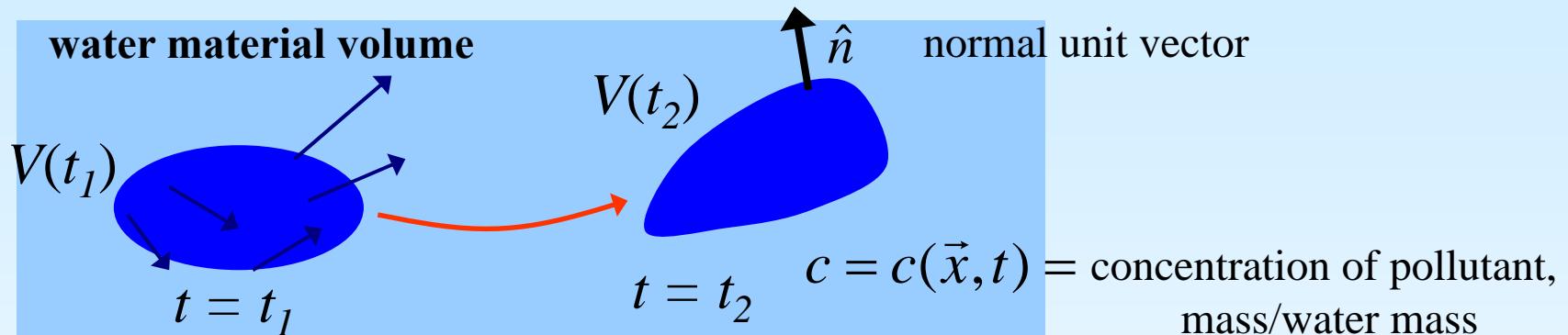
$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \rho(\vec{x}, t) f(\vec{x}, t) dV &= \int_{V(t)} \left( \frac{d}{dt} (\rho f) + \rho f \nabla \cdot \vec{v} \right) dV = \\ &\int_{V(t)} \left\{ f \underbrace{\left( \frac{d\rho}{dt} + \rho \nabla \cdot (\vec{v}) \right)}_{\text{mass}} + \rho \frac{df}{dt} \right\} dV \\ &\xrightarrow{\text{mass conservation}} = 0 \end{aligned}$$

$$\frac{d}{dt} \int_{V(t)} \rho(\vec{x}, t) f(\vec{x}, t) dV = \int_{V(t)} \rho \frac{df}{dt} dV$$

like if  $\frac{d}{dt}(dm) = \frac{d}{dt}(\rho dV) = 0$

### 3. General form of conservation laws

**Example: transport of a pollutant in the sea**



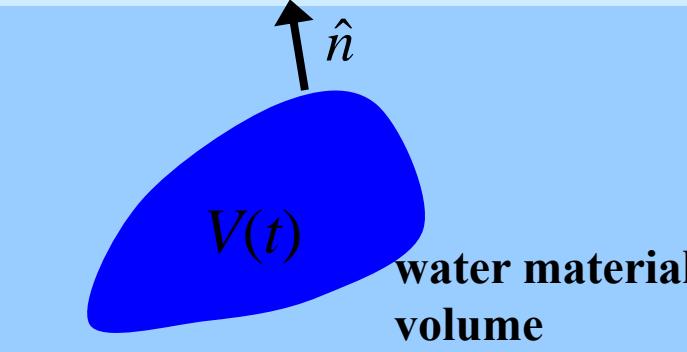
$$\text{total mass of pollutant in } V(t) = \int_{V(t)} \rho(\vec{x}, t) c(\vec{x}, t) dV$$

$$\frac{d}{dt} \int_{V(t)} \rho(\vec{x}, t) c(\vec{x}, t) dV = \boxed{\text{pollutant created inside } V(t)} - \boxed{\text{pollutant quitting through the boundary of } V(t)}$$

$$\frac{d}{dt} \int_{V(t)} \rho c dV = \int_{V(t)} \rho K dV - \int_{\partial V(t)} \hat{n} \cdot \vec{J} dA$$

$K$  = mass of pollutant created per time unit and per water mass unit

$\vec{J}$  = flux = mass of pollutant that crosses the area unit per time unit



$$\frac{d}{dt} \int_{V(t)} \rho c dV = \int_{V(t)} \rho K dV - \int_{\partial V(t)} \hat{n} \cdot \vec{J} dA$$

**Reynolds transport theorem**

$$\frac{d}{dt} \int_{V(t)} \rho c dV = \int_{V(t)} \rho \frac{dc}{dt} dV$$

**Gauss theorem**

$$\int_{\partial V(t)} \hat{n} \cdot \vec{J} dA = \int_{V(t)} \nabla \cdot \vec{J} dV$$

$$\int_{V(t)} \left( \rho \frac{dc}{dt} + \nabla \cdot \vec{J} - \rho K \right) dV = 0$$

Since  $V(t)$  is an arbitrary material volume

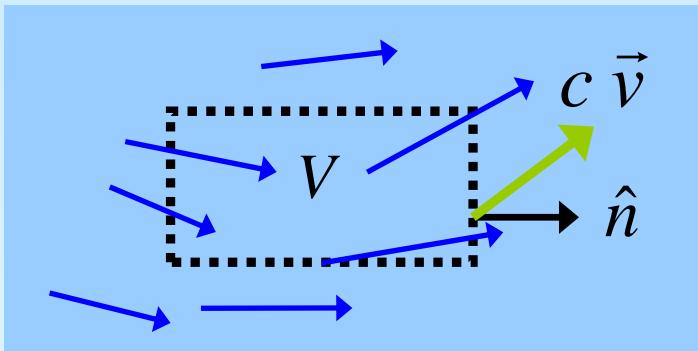
$\Rightarrow$

$$\rho \frac{dc}{dt} + \nabla \cdot \vec{J} = \rho K$$

or

$$\rho \left( \frac{\partial c}{\partial t} + \vec{v} \cdot \nabla c \right) + \nabla \cdot \vec{J} = \rho K$$

## Balance of pollutant amount in a fixed control volume



$$\rho \left( \frac{\partial c}{\partial t} + \vec{v} \cdot \nabla c \right) + \nabla \cdot \vec{J} = \rho K$$

$$\frac{\partial}{\partial t} (\rho c) + \nabla \cdot (\rho c \vec{v}) + \nabla \cdot \vec{J} = \rho K$$

integration over  $V$  + Gauss Theorem

$$\frac{d}{dt} \int_V \rho c dV = \int_V \rho K dV - \int_{\partial V} \hat{n} \cdot \vec{J} dA - \int_{\partial V} \rho c \hat{n} \cdot \vec{v} dA$$

pollutant created  
inside  $V$

pollutant quitting  
because of flux  
out of a material  
volume

pollutant quitting  $V$   
because of flow  
advection

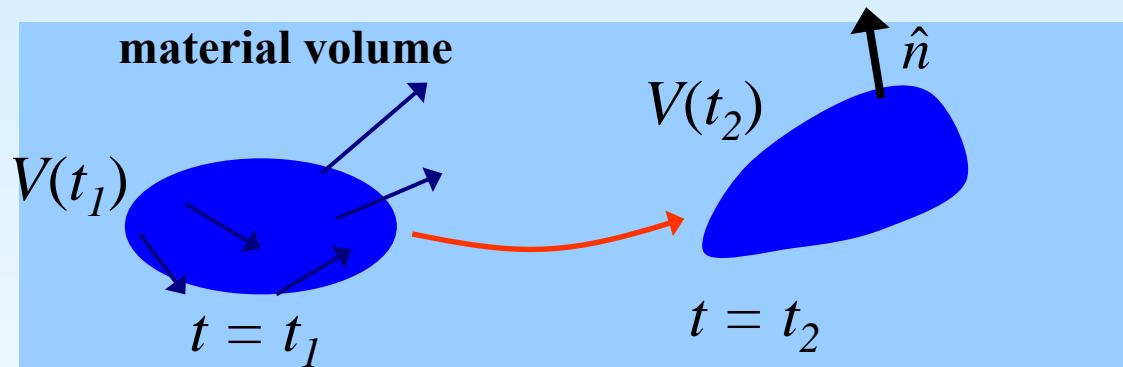
in a material volume

$$\frac{d}{dt} \int_{V(t)} \rho c dV = \int_{V(t)} \rho K dV - \int_{\partial V(t)} \hat{n} \cdot \vec{J} dA$$

## General case

$f(\vec{x}, t)$  = density of a property per body mass unit

total amount of the property in  $V(t) = \int_{V(t)} \rho(\vec{x}, t) f(\vec{x}, t) dV$



e.g.  $\vec{p} = m\vec{v}$   
velocity is momentum  
per mass unit

$$\vec{p} = \int_V \rho \vec{v} dV$$

= total momentum  
in volume  $V$

$$\frac{d}{dt} \int_{V(t)} \rho(\vec{x}, t) f(\vec{x}, t) dV = \boxed{\text{amount created inside } V(t)} - \boxed{\text{amount quitting through the boundary of } V(t)}$$

$$\frac{d}{dt} \int_{V(t)} \rho f dV = \int_{V(t)} \rho K_f dV - \int_{\partial V(t)} \hat{n} \cdot J_f dA$$

$K_f$  = amount of the property created per time unit and per water mass unit

$J_f$  = flux = amount of the property that crosses the area unit per time unit

## Local balance law:

$$\rho \frac{df}{dt} + \nabla \cdot \mathbf{J}_f = \rho K_f$$

or

$$\rho \left( \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f \right) + \nabla \cdot \mathbf{J}_f = \rho K_f$$

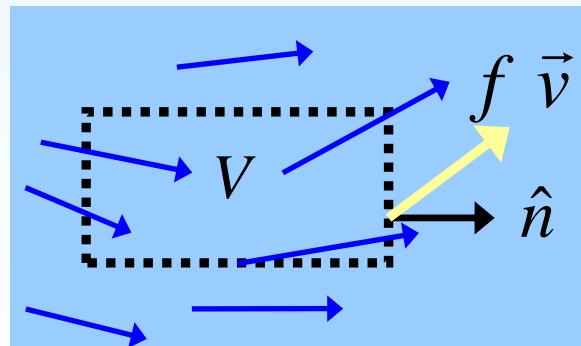
If  $f$  is a scalar (like the concentration of pollutant)

$$K_f = \text{scalar} \quad \vec{J}_f = \text{vector}$$

If  $f$  is a vector (like the velocity or momentum density)

$$K_f = \text{vector} \quad \mathbf{J}_f = \text{second order tensor}$$

## Balance in a fixed volume



$$\frac{d}{dt} \int_V \rho f \, dV = \int_V \rho K_f \, dV - \int_{\partial V} \hat{n} \cdot \mathbf{J}_f \, dA - \int_{\partial V} \rho f \hat{n} \cdot \vec{v} \, dA$$

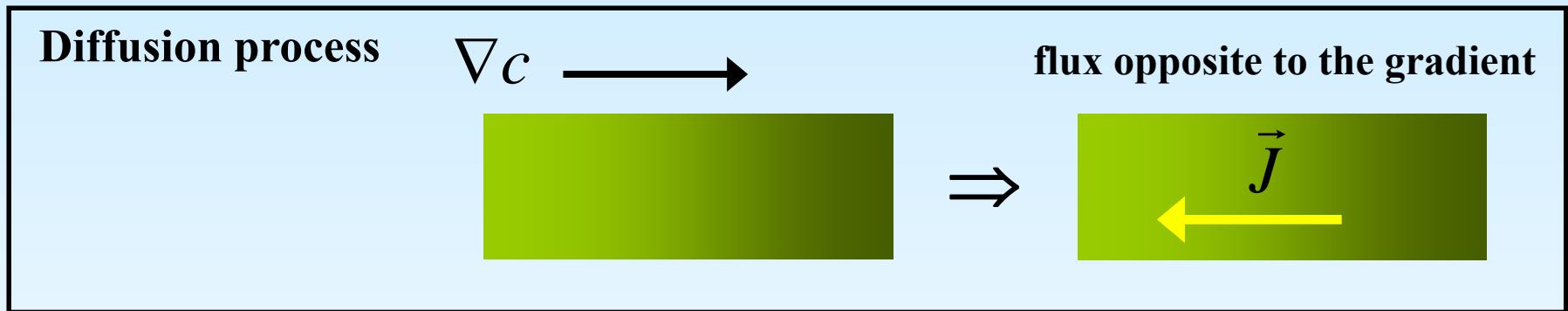
amount created  
inside  $V$

amount quitting  
because of flux  
out of a material  
volume

amount quitting  
because of flow  
advection

## 4. Diffusion equations

Example: transport of a pollutant in the sea



Fick law: the flux is proportional to the gradient

$$\vec{J} = -\gamma \nabla c$$

proportionality constant =  $\gamma$  = **diffusion coefficient**

$$\rho \frac{\partial c}{\partial t} = \rho K - \nabla \cdot \vec{J} - \rho \vec{v} \cdot \nabla c$$

Diagram illustrating the components of the diffusion equation:

- Upward arrow pointing to  $\rho \frac{\partial c}{\partial t}$ : production
- Upward arrow pointing to  $-\nabla \cdot \vec{J}$ : flux out of a material volume
- Upward arrow pointing to  $-\rho \vec{v} \cdot \nabla c$ : flux due to advection
- Rightward arrow pointing from the equation to the right: total flux

$$\frac{\partial c}{\partial t} + \vec{v} \cdot \nabla c = K + \frac{1}{\rho} \nabla \cdot (\gamma \nabla c)$$

**advection-diffusion equation**

if  $\rho = \text{constant}$

$$\frac{\partial c}{\partial t} + \vec{v} \cdot \nabla c = K + \nabla \cdot (D \nabla c)$$

**diffusivity**  

$$D = \frac{\gamma}{\rho}$$

**diffusion equation**

in case  $\vec{v} = 0$

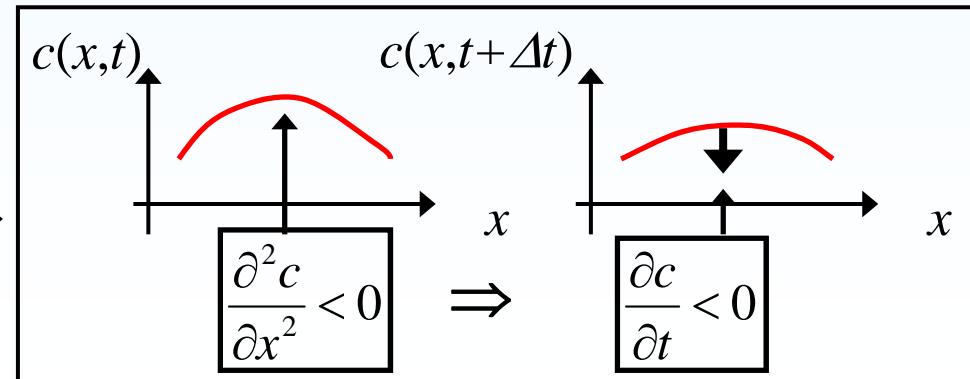
$$\frac{\partial c}{\partial t} = \nabla \cdot (D \nabla c) + K$$

simplest case:  $D - \text{const.}, K - 0$

$$\frac{\partial c}{\partial t} = D \nabla^2 c$$

The diffusion equation describes how gradients tend to smooth out with time

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$



# 5. Momentum conservation

**Types of forces in continuum mechanics:**

**Body forces: forces from ‘action at a distance’ without contact.**

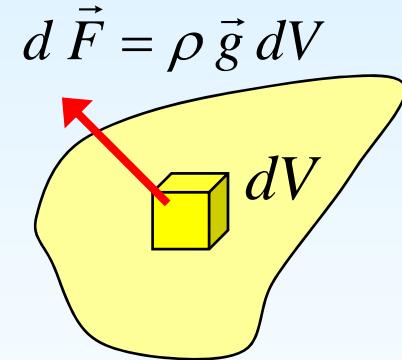
- due to a certain force field (usually originated externally)  
– e.g. gravity, electromagnetic, etc.

- distributed throughout the body  
and defined per mass unit

$$\vec{g} = \lim_{dm \rightarrow 0} \frac{d\vec{F}}{dm}$$

- total force over a volume  $V$ :

$$\vec{F} = \int_V \rho \vec{g} dV$$



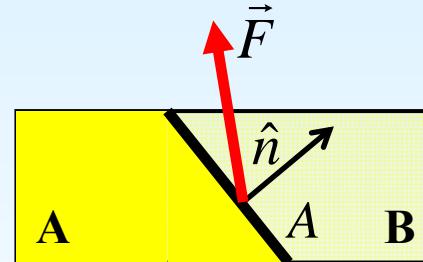
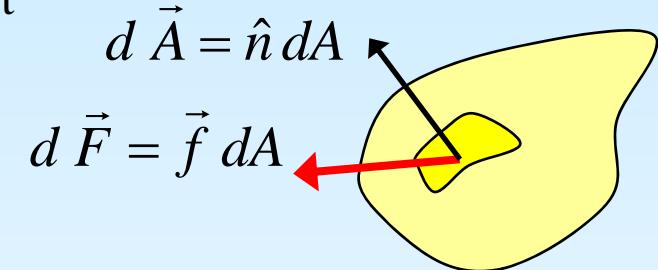
## Surface forces: contact forces either external or internal exerted through a surface.

- proportional to the area and defined per area unit

$$\vec{f} = \lim_{dA \rightarrow 0} \frac{d\vec{F}}{dA}$$

- total force over a surface is  $\vec{F} = \int_S \vec{f} dA$

- force of part B over part A  $\Rightarrow$   
the normal is outwards from A



- force per area unit is a linear function of the unit normal vector to the surface

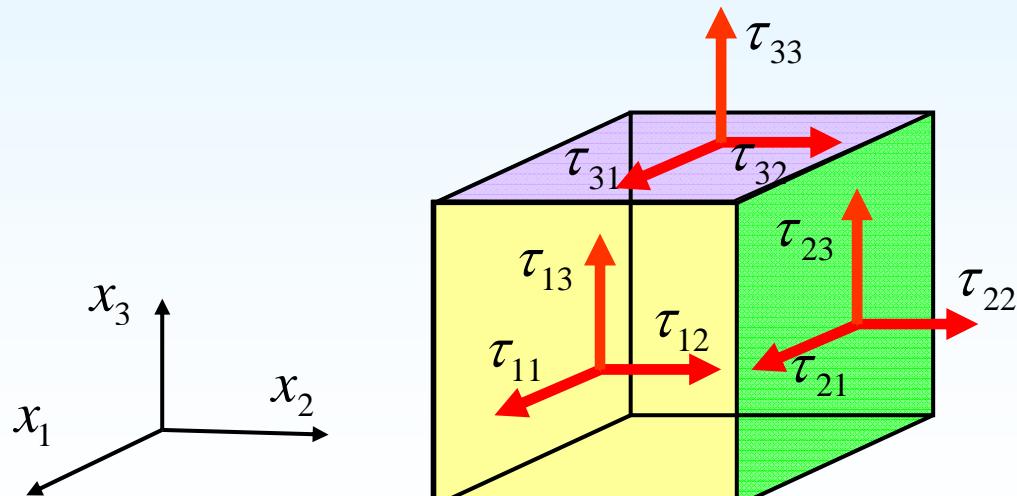
$$\vec{f} = \hat{n} \cdot \tau \quad \tau = \text{stress tensor}$$

# Revisiting stress tensor

$$\vec{f} = n_1 \vec{\tau}_1 + n_2 \vec{\tau}_2 + n_3 \vec{\tau}_3$$

$$(f_1, f_2, f_3) = (n_1, n_2, n_3) \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

stress tensor matrix



$\tau_{11} > 0$  : pulling ;  $\tau_{11} < 0$  : pushing (compression)

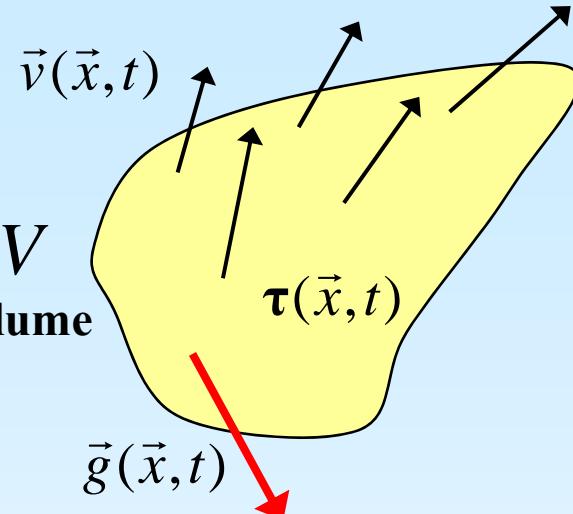
meaning of the components

$\tau_{11}, \tau_{22}, \tau_{33} = \text{normal stresses}$   
 $\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \dots = \text{shear stresses}$

# Momentum balance

$$\frac{d}{dt} \left( \sum_i m_i \vec{v}_i \right) = \sum \vec{F}$$

**V**  
material volume



$$\frac{d}{dt} \int_{V(t)} \rho \vec{v} dV = \int_{V(t)} \rho \vec{g} dV + \int_{\partial V(t)} \hat{n} \cdot \boldsymbol{\tau} dA$$

Reynolds transport theorem

$$\frac{d}{dt} \int_{V(t)} \rho \vec{v} dV = \int_{V(t)} \rho \frac{d\vec{v}}{dt} dV$$

Gauss theorem

$$\int_{\partial V(t)} \hat{n} \cdot \boldsymbol{\tau} dA = \int_{V(t)} \nabla \cdot \boldsymbol{\tau} dV$$

$$\int_{V(t)} \left( \rho \frac{d\vec{v}}{dt} - \rho \vec{g} - \nabla \cdot \boldsymbol{\tau} \right) dV = 0$$

since  $V(t)$  is an arbitrary material volume

$$\rho \frac{d\vec{v}}{dt} = \nabla \cdot \boldsymbol{\tau} + \rho \vec{g}$$

**Cauchy Equation**

$$\rho \frac{dv_i}{dt} = \frac{\partial \tau_{ji}}{\partial x_j} + \rho g_i$$

$$\rho \frac{d\vec{v}}{dt} = \nabla \cdot \boldsymbol{\tau} + \rho \vec{g}$$

↓

$$\rho \frac{df}{dt} + \nabla \cdot J_f = \rho K_f$$

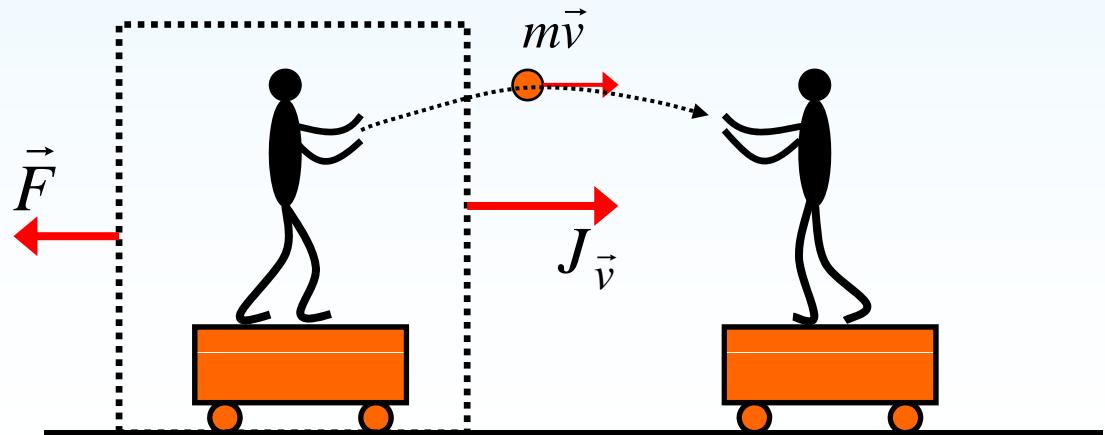
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Comparison with the general form of the conservation laws:

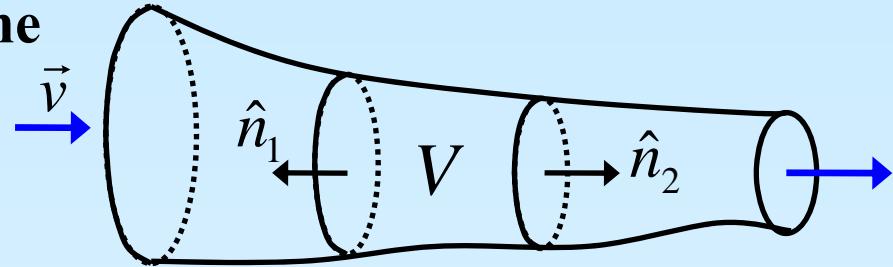
density of momentum:  $f = \vec{v}$

flux of momentum:  $J_{\vec{v}} = -\boldsymbol{\tau}$  stress tensor

production of momentum:  $K_{\vec{v}} = \vec{g}$  body force



## Momentum balance in a fixed volume

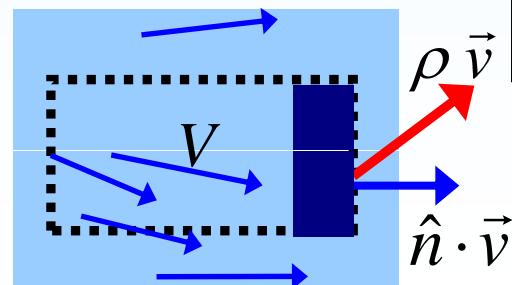


From the general form of the balance in a fixed volume

$$\frac{d}{dt} \int_V \rho f \, dV = \int_V \rho K_f \, dV - \int_{\partial V} \hat{n} \cdot J_f \, dA - \int_{\partial V} \rho f \hat{n} \cdot \vec{v} \, dA$$

$$f = \vec{v} \quad J_{\vec{v}} = -\tau \quad K_{\vec{v}} = \vec{g}$$

$$\frac{d}{dt} \int_V \rho \vec{v} \, dV = \underbrace{\int_V \rho \vec{g} \, dV}_{\text{rate of change of momentum}} + \underbrace{\int_{\partial V} \hat{n} \cdot \tau \, dA}_{\text{forces exerted on } V} - \underbrace{\int_{\partial V} \rho (\hat{n} \cdot \vec{v}) \vec{v} \, dA}_{\text{rate of momentum quitting } V}$$

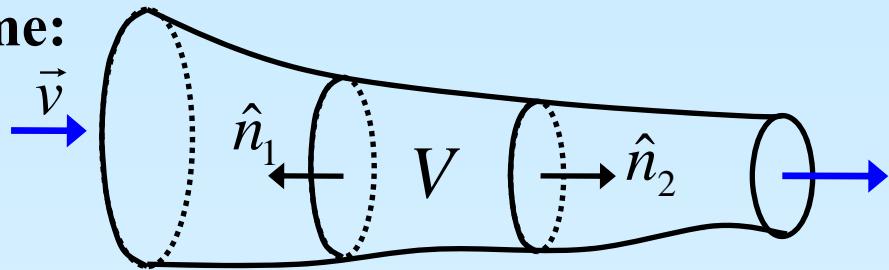


rate of change of momentum

forces exerted on V

$\int_{\partial V} (\hat{n} \cdot \vec{v})(\rho \vec{v}) \, dA =$   
rate of momentum quitting V

## Momentum balance in a fixed volume: direct proof



$$\frac{d}{dt} \int_V \rho \vec{v} dV = \int_V \frac{\partial}{\partial t} (\rho \vec{v}) dV = \int_V \left( \rho \frac{\partial \vec{v}}{\partial t} + \frac{\partial \rho}{\partial t} \vec{v} \right) dV =$$

$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$

$$\int_V \rho \left( \frac{d\vec{v}}{dt} - \vec{v} \cdot \nabla \vec{v} \right) dV - \int_V \nabla \cdot (\rho \vec{v}) \vec{v} dV$$

$\boxed{\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}}$



$$\frac{d}{dt} \int_V \rho v_i dV = \int_V \left( \rho \frac{dv_i}{dt} - \rho v_j \frac{\partial v_i}{\partial x_j} - \frac{\partial}{\partial x_j} (\rho v_j) v_i \right) dV =$$

$\boxed{\rho \frac{d\vec{v}}{dt} = \nabla \cdot \tau + \rho \vec{g}}$

$$\int_V \left( \frac{\partial \tau_{ji}}{\partial x_j} + \rho g_i - \rho v_j \frac{\partial v_i}{\partial x_j} - \frac{\partial}{\partial x_j} (\rho v_j) v_i \right) dV =$$

$$\int_V \rho g_i dV - \int_V \frac{\partial}{\partial x_j} (\rho v_i v_j - \tau_{ij}) dV$$

Gauss theorem:

$$\int_V \frac{\partial}{\partial x_j} (\rho v_i v_j - \tau_{ij}) dV = \int_{\partial V} n_j (\rho v_i v_j - \tau_{ij}) dA$$



$$\frac{d}{dt} \int_V \rho \vec{v} dV = \int_V \rho \vec{g} dV + \int_{\partial V} \hat{n} \cdot \boldsymbol{\tau} dA - \int_{\partial V} \hat{n} \cdot (\rho \vec{v} \otimes \vec{v}) dA$$

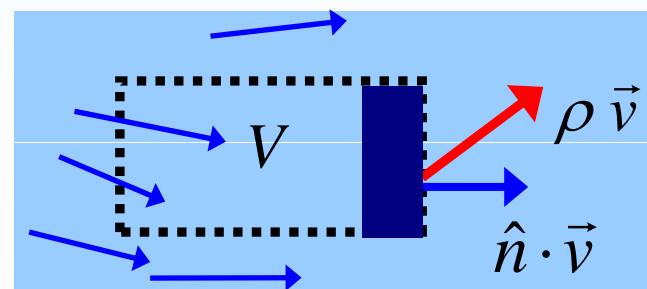
or

$$\underbrace{\frac{d}{dt} \int_V \rho \vec{v} dV}_{\text{rate of change of momentum}} = \underbrace{\int_V \rho \vec{g} dV + \int_{\partial V} \hat{n} \cdot \boldsymbol{\tau} dA}_{\text{forces exerted on } V} - \underbrace{\int_{\partial V} \rho (\hat{n} \cdot \vec{v}) \vec{v} dA}_{\int_{\partial V} (\hat{n} \cdot \vec{v}) (\rho \vec{v}) dA}$$

rate of change  
of momentum

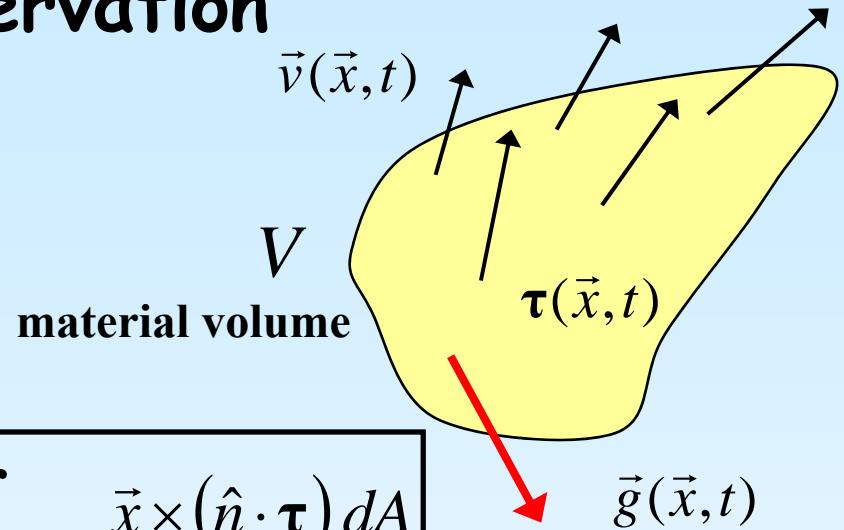
forces exerted on  $V$

$\int_{\partial V} (\hat{n} \cdot \vec{v}) (\rho \vec{v}) dA =$   
rate of momentum  
quitting  $V$



## 6. Angular momentum conservation

$$\frac{d}{dt} \left( \sum_i m_i \vec{x}_i \times \vec{v}_i \right) = \sum_i \vec{x}_i \times \vec{F}$$



$$\frac{d}{dt} \int_{V(t)} \vec{x} \times (\rho \vec{v}) dV = \int_{V(t)} \vec{x} \times (\rho \vec{g}) dV + \int_{\partial V(t)} \vec{x} \times (\hat{n} \cdot \boldsymbol{\tau}) dA$$

Reynolds transport theorem  $\Rightarrow$

$$\frac{d}{dt} \int_{V(t)} \vec{x} \times (\rho \vec{v}) dV = \int_{V(t)} \rho \vec{x} \times \frac{d\vec{v}}{dt} dV \quad \vec{a} = \frac{d\vec{v}}{dt}$$

$$\int_{\partial V(t)} \vec{x} \times (\hat{n} \cdot \boldsymbol{\tau}) dA = \int_{V(t)} \rho \vec{x} \times (\vec{a} - \vec{g}) dV$$

momentum balance

$$\rho \vec{a} = \nabla \cdot \boldsymbol{\tau} + \rho \vec{g}$$

$$\int_{\partial V(t)} \vec{x} \times (\hat{n} \cdot \boldsymbol{\tau}) dA = \int_{V(t)} \vec{x} \times \nabla \cdot \boldsymbol{\tau} dV$$

$$\int_{\partial V(t)} \vec{x} \times (\hat{n} \cdot \boldsymbol{\tau}) dA = \int_{V(t)} \vec{x} \times \nabla \cdot \boldsymbol{\tau} dV$$



$$\int_{\partial V(t)} \epsilon_{ijk} x_j n_r \tau_{rk} dA = \int_{V(t)} \epsilon_{ijk} x_j \frac{\partial \tau_{rk}}{\partial x_r} dV$$

Gauss theorem

$$\int_{\partial V(t)} \epsilon_{ijk} x_j n_r \tau_{rk} dA = \int_{V(t)} \frac{\partial}{\partial x_r} (\epsilon_{ijk} x_j \tau_{rk}) dV = \int_{V(t)} \epsilon_{ijk} \delta_{jr} \tau_{rk} dV + \int_{V(t)} \epsilon_{ijk} x_j \frac{\partial \tau_{rk}}{\partial x_r} dV$$



$$\int_{V(t)} \epsilon_{ijk} \delta_{jr} \tau_{rk} dV = \int_{V(t)} \epsilon_{ijk} \tau_{jk} dV = 0$$

since  $V$  is any material volume

$$\epsilon_{ijk} \tau_{jk} = 0$$

$$\epsilon_{123} \tau_{23} + \epsilon_{132} \tau_{32} = 0 \Rightarrow \tau_{23} - \tau_{32} = 0$$

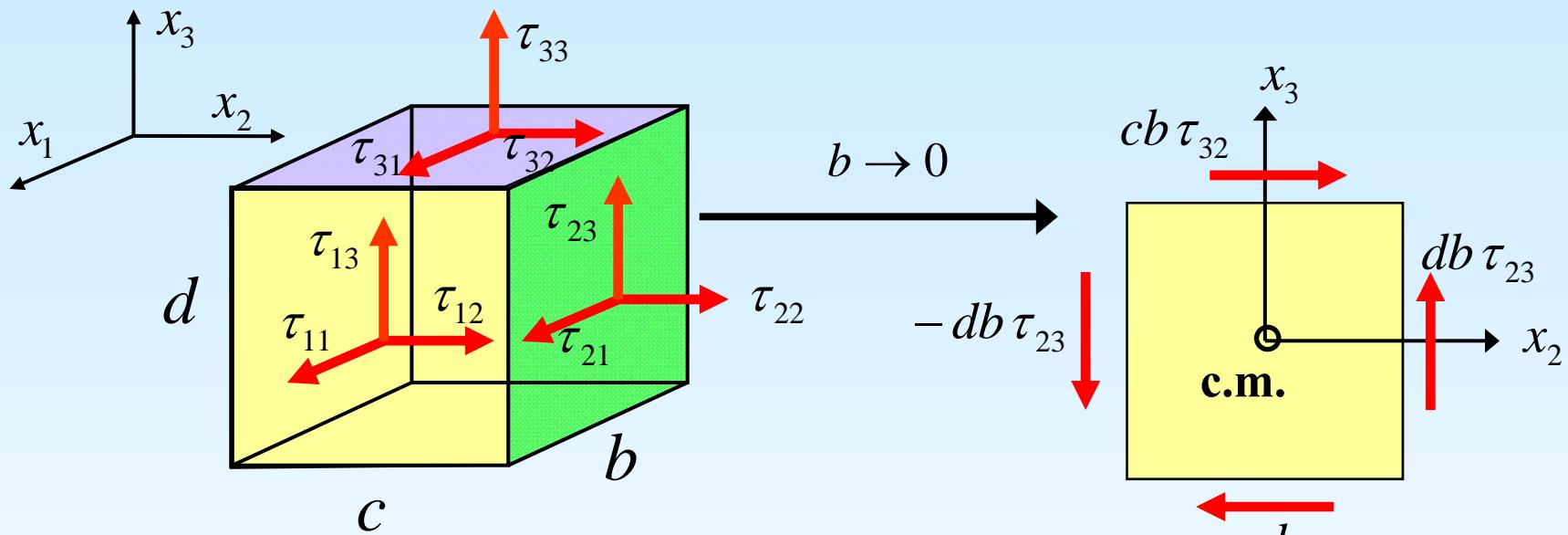
$$\tau_{31} - \tau_{13} = 0$$

$$\tau_{12} - \tau_{21} = 0$$

**angular momentum conservation  $\Rightarrow$  symmetry of the stress tensor**

another way:

$$\int_{\partial V(t)} \vec{x} \times (\hat{n} \cdot \boldsymbol{\tau}) dA = \int_{V(t)} \rho \vec{x} \times (\vec{a} - \vec{g}) dV$$



$$(db \tau_{23})c - (cb \tau_{32})d = \int_V \epsilon_{ijk} x_j (a_k - g_k) \rho dV$$

$$|\tau_{23} - \tau_{32}| = \frac{1}{bcd} \left| \int_V \epsilon_{ijk} x_j (a_k - g_k) \rho dV \right| \leq \underbrace{\frac{1}{bcd} bcd \sqrt{b^2 + c^2 + d^2}}_{0 \text{ for } V \rightarrow 0} \max |\rho(\vec{a} - \vec{g})|$$

$$\tau_{23} - \tau_{32} = 0$$

## Angular momentum balance in a fixed volume

$$\begin{aligned} \frac{d}{dt} \int_V \rho \vec{x} \times \vec{v} dV &= \int_V \frac{\partial}{\partial t} (\rho \vec{x} \times \vec{v}) dV = \int_V \vec{x} \times \left( \rho \frac{\partial \vec{v}}{\partial t} + \frac{\partial \rho}{\partial t} \vec{v} \right) dV = \\ &= \int_V \rho \vec{x} \times \left( \frac{d\vec{v}}{dt} - \vec{v} \cdot \nabla \vec{v} \right) dV - \int_V (\nabla \cdot (\rho \vec{v})) \vec{x} \times \vec{v} dV \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \int_V \rho \epsilon_{ijk} x_j v_k dV &= \int_V \left( \rho \epsilon_{ijk} x_j \frac{dv_k}{dt} - \rho \epsilon_{ijk} x_j v_r \frac{\partial v_k}{\partial x_r} - \epsilon_{ijk} x_j \frac{\partial}{\partial x_r} (\rho v_r) v_k \right) dV = \\ &\quad \int_V \epsilon_{ijk} x_j \left( \frac{\partial \tau_{rk}}{\partial x_r} + \rho g_k - \rho v_r \frac{\partial v_k}{\partial x_r} - \frac{\partial}{\partial x_r} (\rho v_r) v_k \right) dV = \\ &\quad \int_V \rho \epsilon_{ijk} x_j g_k dV - \int_V \epsilon_{ijk} x_j \frac{\partial}{\partial x_r} (\rho v_k v_r - \tau_{kr}) dV = \\ &\quad \int_V \rho \epsilon_{ijk} x_j g_k dV - \int_V \frac{\partial}{\partial x_r} (\epsilon_{ijk} x_j (\rho v_k v_r - \tau_{kr})) dV = \\ &\quad \int_V \rho \epsilon_{ijk} x_j g_k dV - \int_{\partial V} \epsilon_{ijk} x_j (\rho v_k v_r - \tau_{kr}) n_r dA \end{aligned}$$

similarly to  
momentum  
balance (Reynolds  
transport  
theorem, Gauss  
theorem, etc.)

$$\frac{d}{dt} \int_V \rho \vec{x} \times \vec{v} dV = \int_V \vec{x} \times \vec{g} \rho dV + \int_{\partial V} \vec{x} \times (\boldsymbol{\tau} \cdot \hat{n}) dA - \int_{\partial V} (\rho \vec{x} \times \vec{v}) \vec{v} \cdot \hat{n} dA$$

rate of change  
of angular  
momentum

torques exerted on  $V$

rate of angular  
momentum  
quitting  $V$

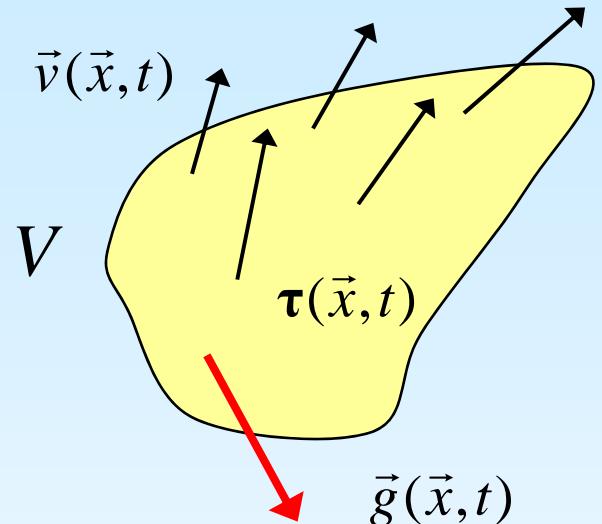
# 7. Mechanical energy balance

**momentum balance**

$$\rho \frac{d\vec{v}}{dt} = \nabla \cdot \boldsymbol{\tau} + \rho \vec{g}$$

$$\rho v_k \frac{dv_k}{dt} = v_k \frac{\partial \tau_{jk}}{\partial x_j} + \rho g_k v_k$$

$$\frac{1}{2} \rho \frac{d}{dt} (v_k v_k) = \frac{\partial}{\partial x_j} (v_k \tau_{jk}) - \tau_{jk} \frac{\partial v_k}{\partial x_j} + \rho g_k v_k$$



$$\tau_{jk} \frac{\partial v_k}{\partial x_j} = \tau_{jk} (D_{kj} + W_{kj}) = \tau_{jk} D_{kj} = \tau_{jk} D_{jk}$$

$$\frac{1}{2} \rho \frac{d}{dt} (v^2) = \nabla \cdot (\boldsymbol{\tau} \cdot \vec{v}) - \boldsymbol{\tau} : \mathbf{D} + \rho \vec{g} \cdot \vec{v}$$

integration over  $V$   
(material volume)

$$\frac{d}{dt} \int_{V(t)} \frac{1}{2} \rho v^2 dV = \int_{V(t)} \rho \vec{g} \cdot \vec{v} dV + \int_{\partial V(t)} \hat{n} \cdot \boldsymbol{\tau} \cdot \vec{v} dA - \int_{V(t)} \boldsymbol{\tau} : \mathbf{D} dV$$

rate of change  
of kinetic energy

external work  
done by the  
body forces

external work  
done by the stresses  
on the surface

internal work  
done by  
the stresses

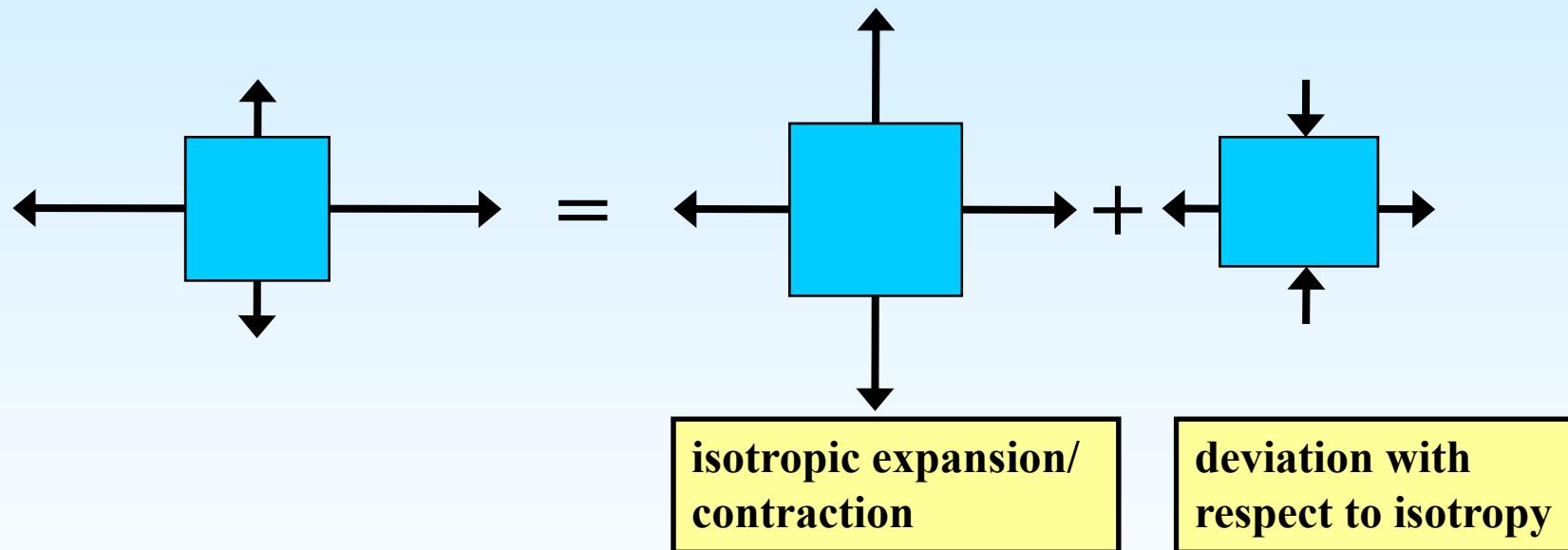
**if there is  
deformation**

$$\frac{d}{dt} \left( \sum_i \frac{1}{2} m_i v_i^2 \right) = \dot{W}^{\text{ext}} + \dot{W}^{\text{int}}$$

**interpretation**

# 1) Isotropic and deviatoric strain rate tensors

$$\mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



general case:

$$\mathbf{D} = \frac{1}{3} \operatorname{tr}(\mathbf{D}) \mathbf{1} + \mathbf{D}'$$

or

$$\mathbf{D} = \frac{1}{3} (\nabla \cdot \vec{v}) \mathbf{1} + \mathbf{D}'$$

with  $\operatorname{tr}(\mathbf{D}') = 0$

## 2) Isotropic and deviatoric stress tensors

similarly ....

$$\tau = \begin{pmatrix} -3 & 1 & -1 \\ 1 & 5 & 0 \\ -1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} -5 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

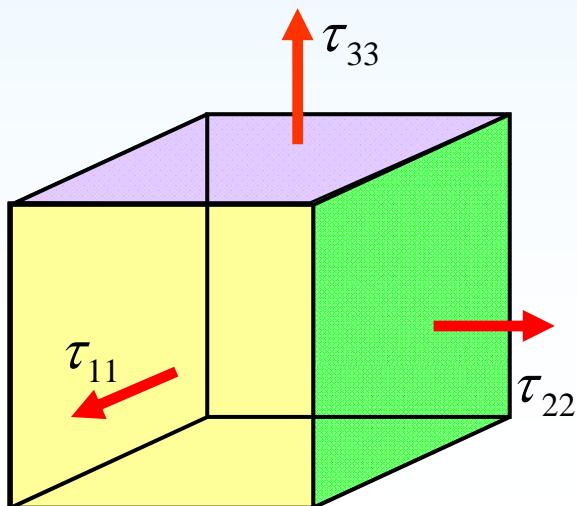
general case:

$$\tau = \frac{1}{3} \text{tr}(\tau) \mathbf{1} + \tau'$$

with  $\text{tr}(\tau') = 0$

isotropic  
stress

deviation  
with respect to isotropy



**Isotropic stress =**

isotropic tension ( $\tau > 0$ ) or compression ( $\tau < 0$ )

- normal stresses are the same along any direction
- no shear stresses

$$\tau_{11} = \tau_{22} = \tau_{33} = \tau$$

In case of a **fluid in equilibrium**,  
we will see that the stresses are isotropic and negative (compression):

$$\tau_{11} = \tau_{22} = \tau_{33} = \tau = -p \quad p > 0 \equiv \text{pressure}$$

therefore, **in general, the pressure is defined as – the mean normal stress**

$$p \equiv -\frac{1}{3} \operatorname{tr}(\boldsymbol{\tau})$$



$$\boldsymbol{\tau} = -p \mathbf{1} + \boldsymbol{\tau}'$$

where  $p$  can be  $> 0$  (compression) or  $< 0$  (tension)

If  $\tau_1 \tau_2 \tau_3$  are the principal stresses, then

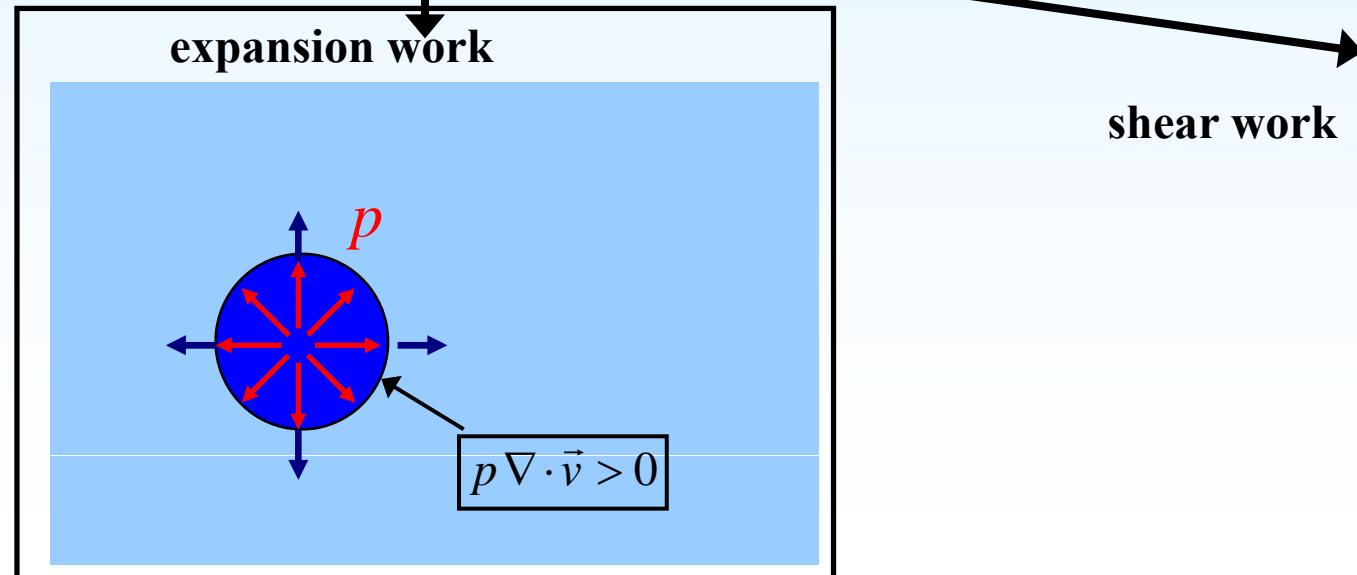
$$p = -\frac{1}{3} (\tau_1 + \tau_2 + \tau_3) = -\bar{\tau}$$

### 3) Internal deformation work

$$\boldsymbol{\tau} : \mathbf{D} = \tau_{ij} D_{ij} = \left( -p\delta_{ij} + \tau'_{ij} \right) \left( \frac{1}{3} (\nabla \cdot \vec{v}) \delta_{ij} + D'_{ij} \right) = -p \nabla \cdot \vec{v} + \tau'_{ij} D'_{ij}$$

$$\frac{1}{2} \rho \frac{d}{dt} (v^2) = -\boldsymbol{\tau} : \mathbf{D} = p \nabla \cdot \vec{v} - \boldsymbol{\tau}' : \mathbf{D}'$$

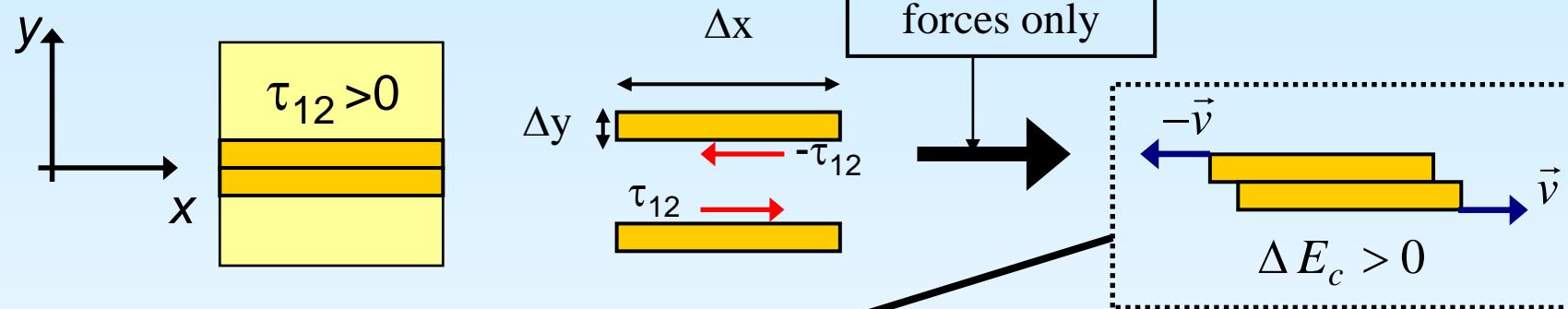
$$-\boldsymbol{\tau} : \mathbf{D} = p \nabla \cdot \vec{v} - \boldsymbol{\tau}' : \mathbf{D}'$$



## shear work

$$\frac{1}{2} \rho \frac{d}{dt} (v^2) = -\boldsymbol{\tau} : \mathbf{D} = p \nabla \cdot \vec{v} - \boldsymbol{\tau}' : \mathbf{D}'$$

motion caused by the internal forces only



$$\frac{dE_c}{dt} = 2(\tau_{12} \Delta x \Delta z)v$$

$$v_x = -k y, \quad v_y = v_z = 0$$

$$D = \begin{pmatrix} 0 & -k/2 & 0 \\ -k/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\tau}' : \mathbf{D}' = \tau_{12} D_{12} + \tau_{21} D_{21} = -k \tau_{21}$$

$$\begin{aligned} -\boldsymbol{\tau}' : \mathbf{D}' (\Delta x (2\Delta y) \Delta z) &= \\ 2k \tau_{21} \Delta x \Delta y \Delta z &= \\ 2 \frac{v}{\Delta y} \tau_{21} \Delta x \Delta y \Delta z &= 2v \tau_{21} \Delta x \Delta z \end{aligned}$$

## Energy balance for a fixed volume

$$\frac{d}{dt} \int_V \frac{1}{2} \rho v^2 dV = \int_V \rho \vec{g} \cdot \vec{v} dV - \int_V \boldsymbol{\tau} : \mathbf{D} dV + \int_{\partial V} \hat{n} \cdot \boldsymbol{\tau} \cdot \vec{v} dA - \int_{\partial V} \frac{1}{2} \rho (\hat{n} \cdot \vec{v}) v^2 dA$$

rate of change  
of kinetic energy

external work  
done by the  
body forces

internal work  
done by  
the stresses

external work  
done by the  
stresses on  
the surface

rate of energy  
quitting  $V$

# Energy balance for a fixed volume

$$\rho \frac{d}{dt} \left( \frac{1}{2} v^2 \right) - \nabla \cdot (\boldsymbol{\tau} \cdot \vec{v}) = -\boldsymbol{\tau} : \mathbf{D} + \rho \vec{g} \cdot \vec{v}$$

$$\rho \frac{df}{dt} + \nabla \cdot J_f = \rho K_f$$

Comparison with the general form of the conservation laws:

density of energy:  $f = v^2 / 2$

flux of energy:  $J_{v^2/2} = -\boldsymbol{\tau} \cdot \vec{v}$

production of energy:  $K_{v^2/2} = \vec{g} \cdot \vec{v} - \boldsymbol{\tau} : \mathbf{D} / \rho$

$$\frac{d}{dt} \int_V \frac{1}{2} \rho v^2 dV = \int_V \rho \vec{g} \cdot \vec{v} dV - \int_V \boldsymbol{\tau} : \mathbf{D} dV + \int_{\partial V} \hat{n} \cdot \boldsymbol{\tau} \cdot \vec{v} dA - \int_{\partial V} \frac{1}{2} \rho (\hat{n} \cdot \vec{v}) v^2 dA$$

rate of change  
of kinetic energy

external work  
done by the  
body forces

internal work  
done by  
the stresses

external work  
done by the  
stresses on  
the surface

rate of energy  
quitting  $V$

## 8. First Law of Thermodynamics

$$\frac{d}{dt} \int_{V(t)} \frac{1}{2} \rho v^2 dV = \int_{V(t)} \rho \vec{g} \cdot \vec{v} dV + \int_{\partial V(t)} \hat{n} \cdot \boldsymbol{\tau} \cdot \vec{v} dA - \int_{V(t)} \boldsymbol{\tau} : \mathbf{D} dV$$

rate of change  
of kinetical energy

external work

internal work



Where does the energy transferred by the internal work go ?  
or where does it come from ?

$$\int_{V(t)} \rho \vec{g} \cdot \vec{v} dV$$

**inflow of energy  
by external work**

$$\int_{\partial V(t)} \hat{n} \cdot \boldsymbol{\tau} \cdot \vec{v} dA$$

**kinetic energy**

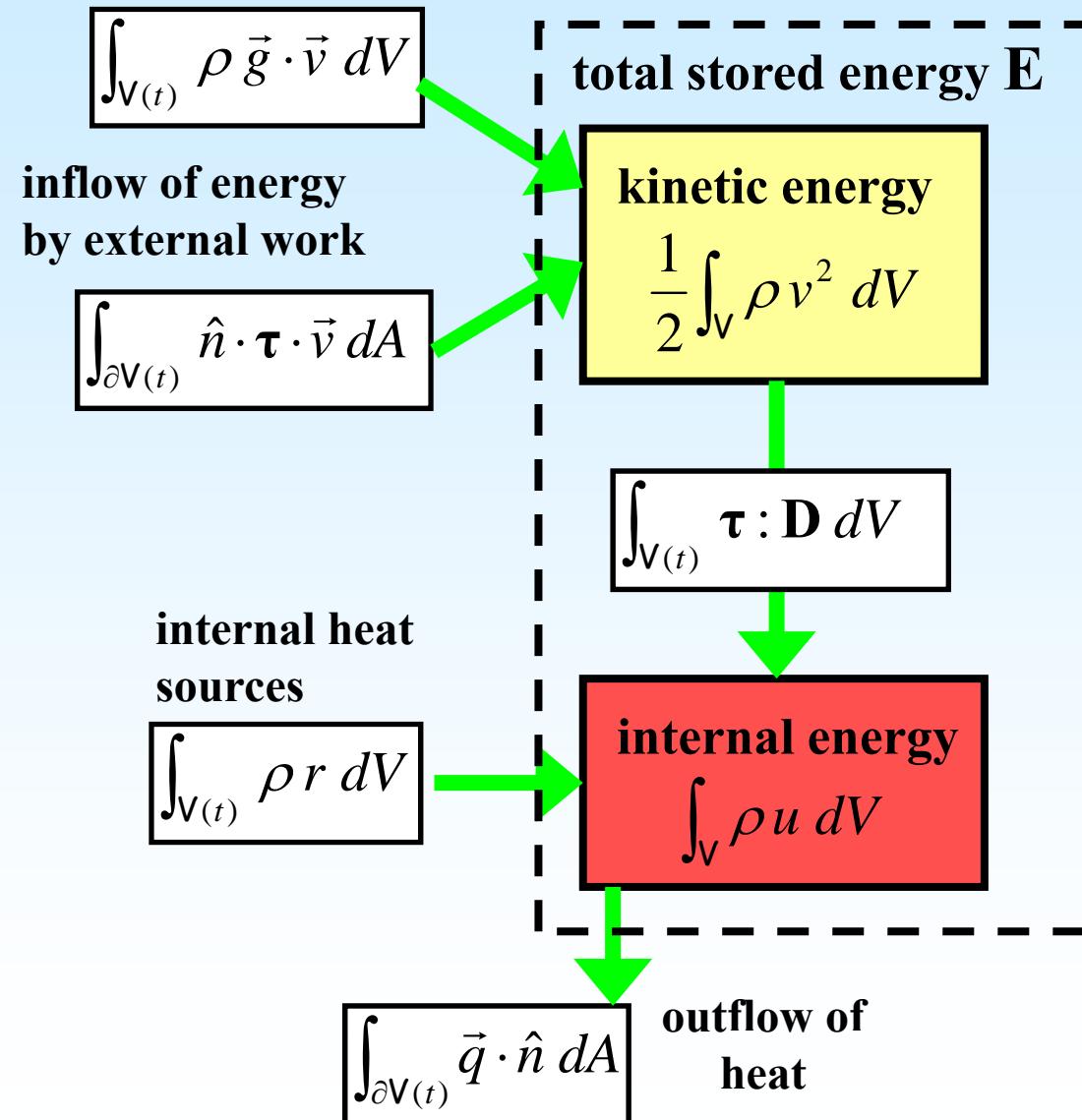
$$\frac{1}{2} \int_V \rho v^2 dV$$

$$\int_{V(t)} \boldsymbol{\tau} : \mathbf{D} dV$$

**internal  
deformation  
work**

?

## First Law of Thermodynamics: energy conservation principle



## First Law of Thermodynamics: **energy conservation principle**

There exists a ‘stored energy function’  $E$  such that:

$$\frac{dE}{dt} = \int_{V(t)} \rho \vec{g} \cdot \vec{v} dV + \int_{\partial V(t)} \hat{n} \cdot \boldsymbol{\tau} \cdot \vec{v} dA - \int_{\partial V(t)} \vec{q} \cdot \hat{n} dA + \int_{V(t)} \rho r dV$$

Energy transfer to the body by the  
external work

Heat inflow  
to the body  
through the surface

Heat  
production  
inside  
the body

$\vec{q}$  = heat flux  
per area unit

$r$  = heat produced  
inside the body  
per mass unit

$$E = \int_V \left( \frac{1}{2} \rho v^2 + \rho u \right) dV$$

$u$  = **internal energy**

(molecular or ‘thermal’ energy,  
elastic potential energy, ...)

by subtracting:

$$\frac{d}{dt} \int_{V(t)} \left( \frac{1}{2} \rho v^2 + \rho u \right) dV = \int_{V(t)} \rho \vec{g} \cdot \vec{v} dV + \int_{\partial V(t)} \hat{n} \cdot \boldsymbol{\tau} \cdot \vec{v} dA - \int_{\partial V(t)} \vec{q} \cdot \hat{n} dA + \int_{V(t)} \rho r dV$$

$$\frac{d}{dt} \int_{V(t)} \frac{1}{2} \rho v^2 dV = \int_{V(t)} \rho \vec{g} \cdot \vec{v} dV + \int_{\partial V(t)} \hat{n} \cdot \boldsymbol{\tau} \cdot \vec{v} dA - \int_{V(t)} \boldsymbol{\tau} : \mathbf{D} dV$$



$$\frac{d}{dt} \int_{V(t)} \rho u dV = \int_{V(t)} \boldsymbol{\tau} : \mathbf{D} dV - \int_{\partial V(t)} \vec{q} \cdot \hat{n} dA + \int_{V(t)} \rho r dV$$

rate of increase  
of internal energy

internal  
deformation  
work

heat inflow  
to the body  
through the surface

heat  
production  
inside  
the body

or, in local form

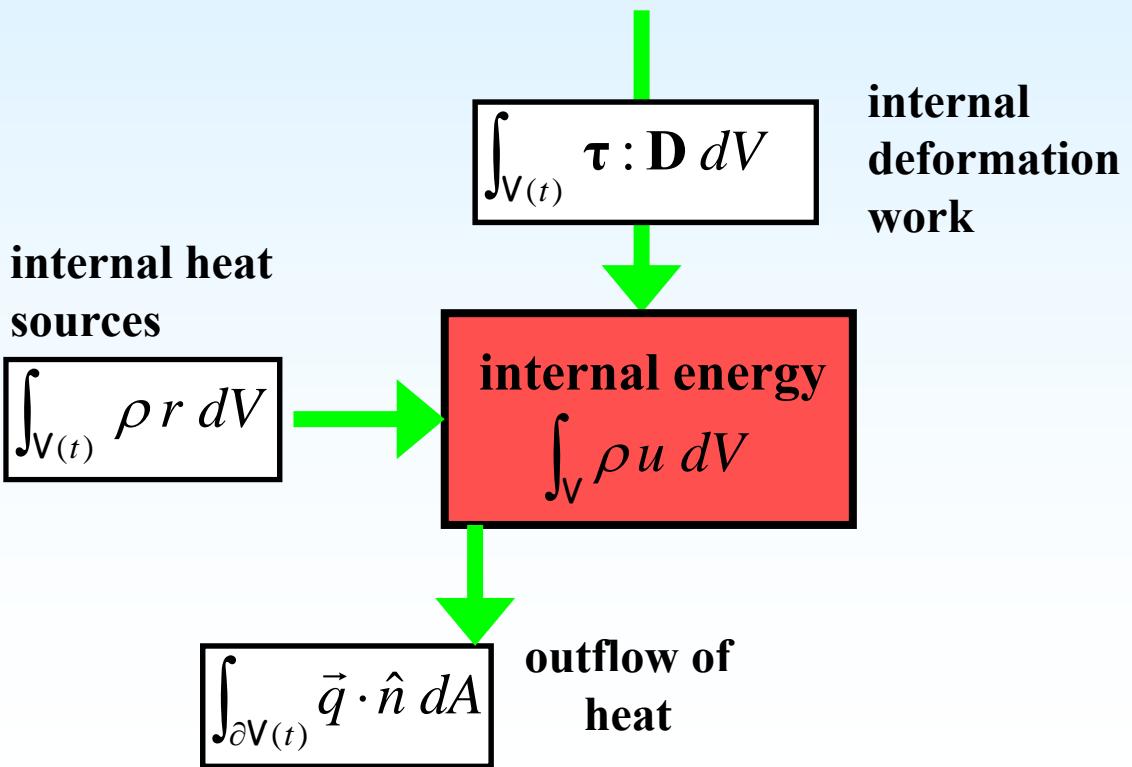
$$\rho \frac{du}{dt} = \boldsymbol{\tau} : \mathbf{D} - \nabla \cdot \vec{q} + \rho r$$

Classical Thermodynamics  
of homogeneous systems:

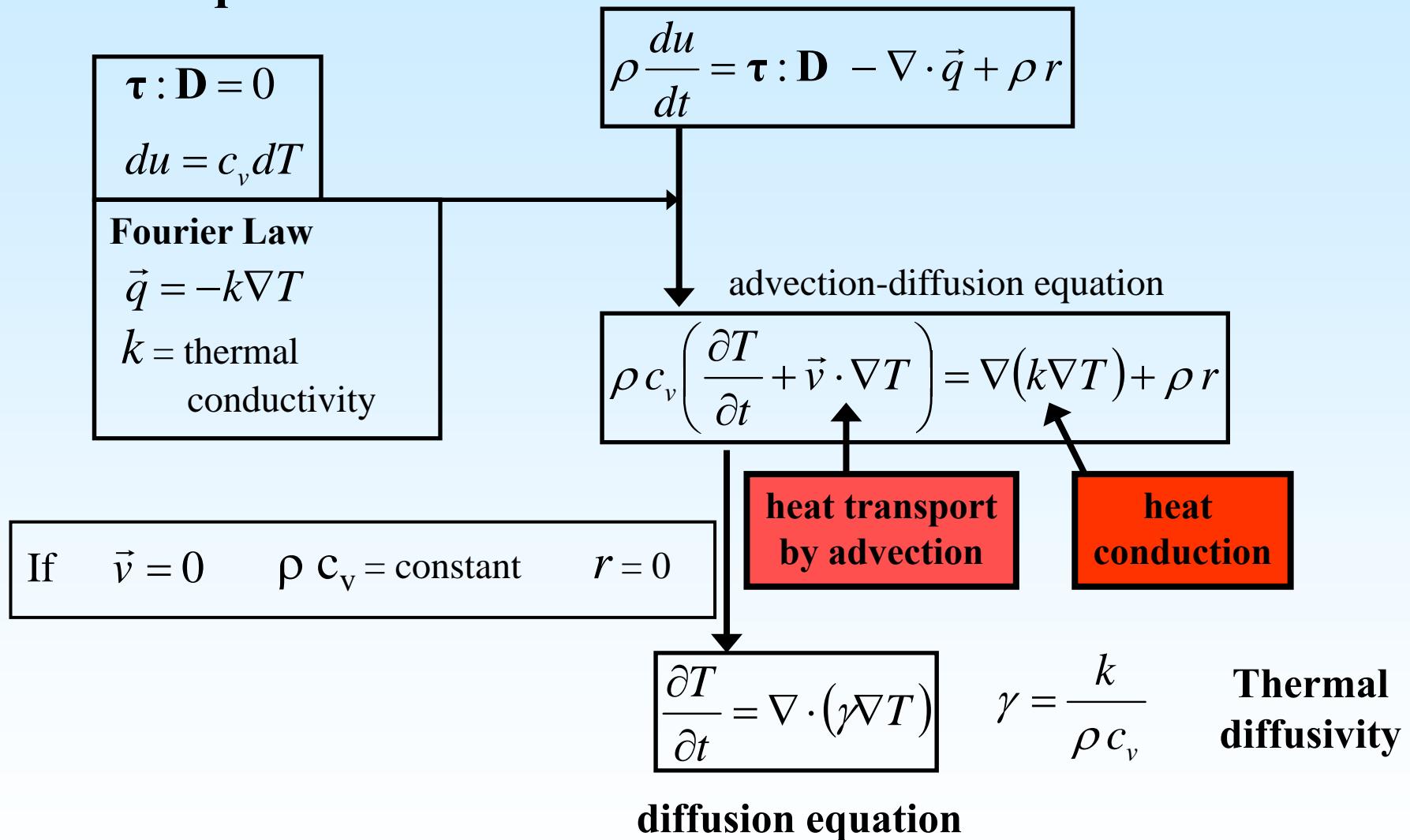
$$\Delta U = -W + Q$$

$W$  = work out  
of the system

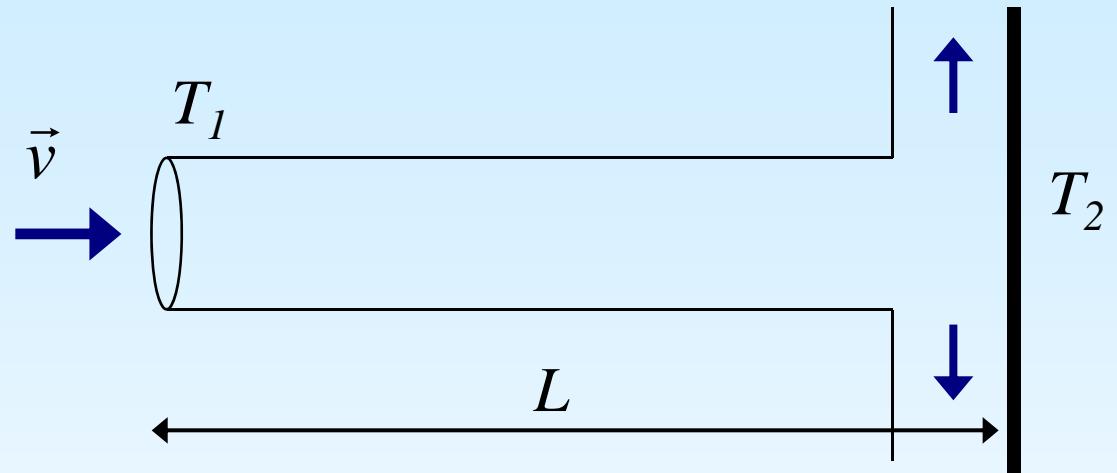
$Q$  = heat into the system



# heat equation



## Example of application of the advection-diffusion heat equation



Initially, all the water in the pipe is at temperature  $T_2$ .

After  $t=0$ , water at temperature  $T_1$  flows into the pipe.

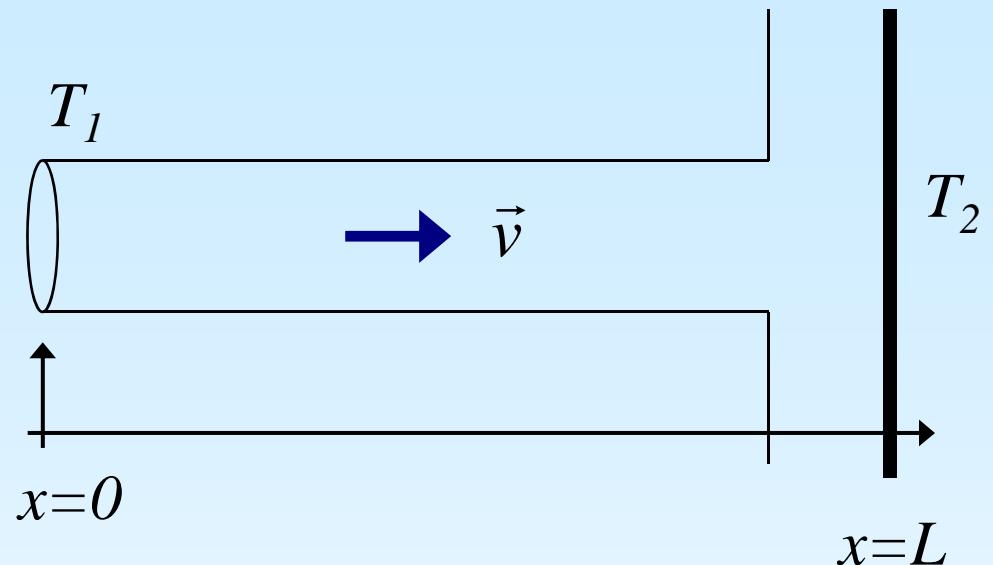
Which is the final temperature distribution in steady conditions?

How is the transient behaviour?

## mathematical problem

we assume a one-dimensional problem

$$T = T(x, t)$$



$$\left\{ \begin{array}{ll} \frac{\partial T}{\partial t} + v \frac{\partial T}{\partial x} = D \frac{\partial^2 T}{\partial x^2} & 0 < x < L \\ T(0, t) = T_1 \quad , \quad T(L, t) = T_2 & \text{boundary conditions} \\ T(x, 0) = T_2 \quad x > 0 & \text{initial conditions} \end{array} \right.$$

thermal  
diffusivity  
of the fluid

$$D = \frac{k}{\rho c_v}$$

## Final equilibrium solution:

$$\tau = \frac{T}{T_1} , \quad \alpha = \frac{D}{vL} , \quad y = \frac{x}{L}$$

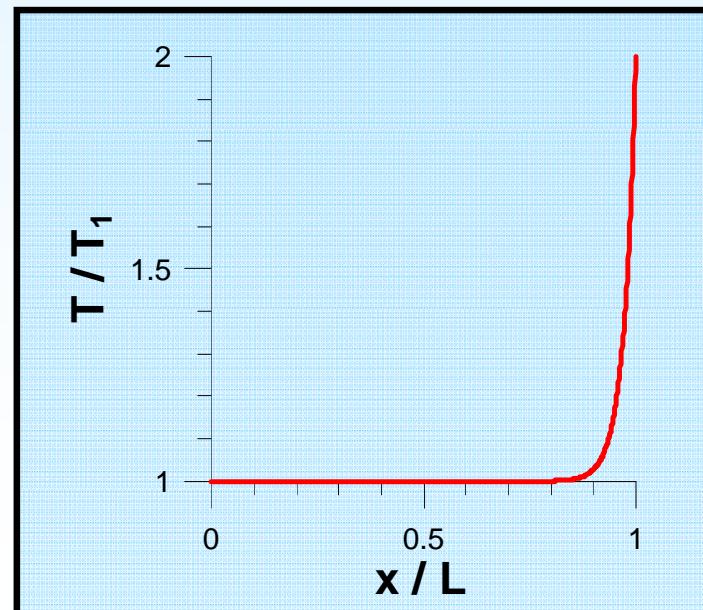
$$D = \frac{k}{\rho c_v}$$

$$T_2 = 2 T_1 , \quad v = 0.1 \text{ mm/s} , \quad L = 5 \text{ cm}$$

water:  $D = 1.4 \times 10^{-7} \text{ m}^2/\text{s}$

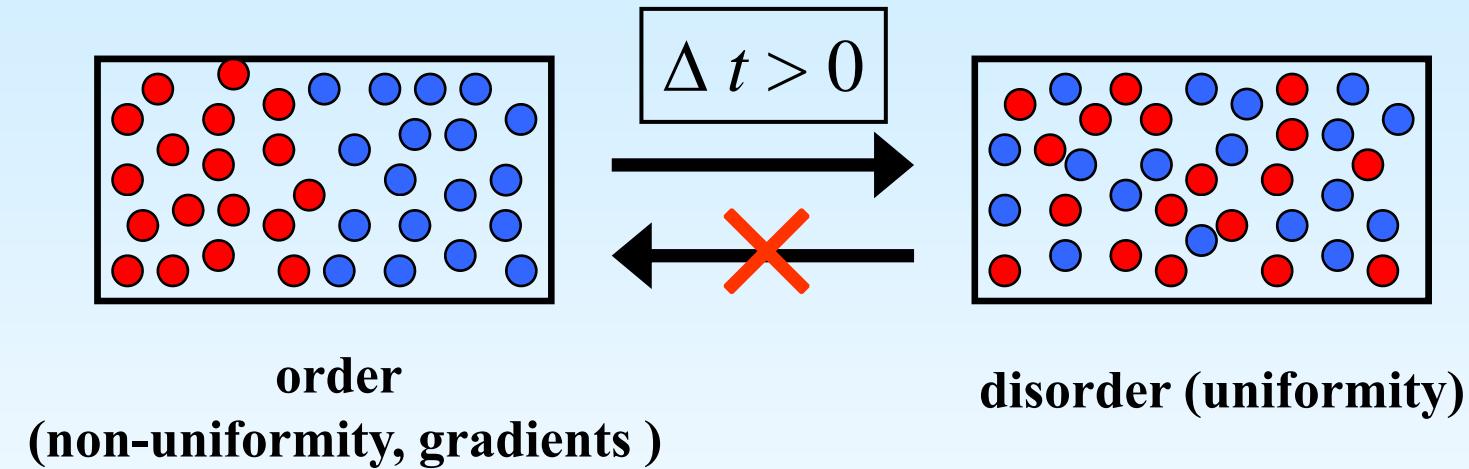
$$\tau = \frac{1 + (\tau_2 - 1) e^{(y-1)/\alpha} - \tau_2 e^{-1/\alpha}}{1 - e^{-1/\alpha}}$$

$$\alpha^{-1} = 35.7$$



## 9. Second Law of Thermodynamics

The disorder in any isolated system increases with time



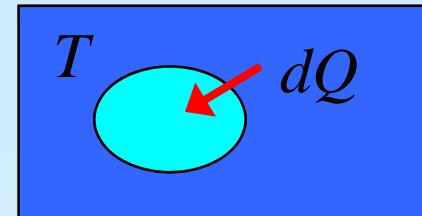
**entropy** = thermodynamic variable which is a measure of the disorder or randomness

**Second Law:**  
In any isolated system, the entropy must increase with time

## Classical Thermodynamics of homogeneous systems:

$$S = \text{entropy defined by} \quad dS = \frac{dQ_{rev}}{T}$$

Second Law:  $dS \geq 0$  for any real processes in any isolated system



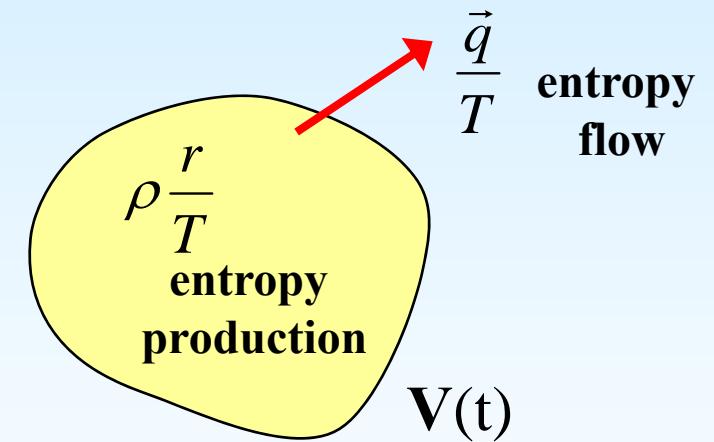
## Continuum Thermomechanics

Second Law:

$$\frac{dS}{dt} \geq \int_{V(t)} \rho \frac{r}{T} dV - \int_{\partial V(t)} \frac{\vec{q}}{T} \cdot \hat{n} dA$$

$$\begin{matrix} \text{entropy} \\ \text{increase} \\ \text{in any} \\ \text{process} \end{matrix} \geq \begin{matrix} \text{entropy} \\ \text{supply} \\ \text{inside the} \\ \text{body} \end{matrix} + \begin{matrix} \text{entropy flow} \\ \text{from outside} \end{matrix}$$

Clausius inequality



$$S = \int_{V(t)} \rho s(\vec{x}, t) dV$$

$$\rho \frac{ds}{dt} - \rho \frac{r}{T} + \nabla \cdot \left( \frac{\vec{q}}{T} \right) \geq 0$$

**local form**

The Second Law poses restrictions on the properties of the solids and fluids

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### Example: Fourier Law of heat conduction

Second Law  $\Rightarrow$  thermal conductivity is positive,  $\gamma > 0$   
i.e., heat flows from hot to cold

$$\vec{q} = -\gamma \nabla T$$

Free Helmholtz energy:  $\psi \equiv u - Ts$

$$\rho \frac{du}{dt} = \boldsymbol{\tau} : \mathbf{D} - \nabla \cdot \vec{q} + \rho r$$

$$\rho T \frac{ds}{dt} = \rho \frac{du}{dt} - \rho \frac{d\psi}{dt} - \rho s \frac{dT}{dt} = \boldsymbol{\tau} : \mathbf{D} - \nabla \cdot \vec{q} + \rho r - \rho \frac{d\psi}{dt} - \rho s \frac{dT}{dt}$$

$$\boxed{\rho \frac{ds}{dt} - \rho \frac{r}{T} + \nabla \cdot \left( \frac{\vec{q}}{T} \right) \geq 0}$$

$$\boxed{\boldsymbol{\tau} : \mathbf{D} - \rho \frac{d\psi}{dt} - \rho s \frac{dT}{dt} - \frac{\vec{q}}{T} \cdot \nabla T \geq 0}$$

$$\tau : \mathbf{D} - \rho \frac{d\psi}{dt} - \rho s \frac{dT}{dt} - \frac{\vec{q}}{T} \cdot \nabla T \geq 0 \rightarrow -p \nabla \cdot \vec{v} + \tau' : \mathbf{D}' - \rho \frac{d\psi}{dt} - \rho s \frac{dT}{dt} - \frac{\vec{q}}{T} \cdot \nabla T \geq 0$$

Let's assume  $\psi = \psi(T, \rho)$   $\rightarrow$

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial T} \frac{dT}{dt} + \frac{\partial \psi}{\partial \rho} \frac{d\rho}{dt} = \frac{\partial \psi}{\partial T} \frac{dT}{dt} - \rho \frac{\partial \psi}{\partial \rho} \nabla \cdot \vec{v}$$

$$\vec{q} = -\gamma \nabla T$$

$$\left( \rho^2 \frac{\partial \psi}{\partial \rho} - p \right) \nabla \cdot \vec{v} + \tau' : \mathbf{D}' - \left( \frac{\partial \psi}{\partial T} + s \right) \frac{dT}{dt} + \gamma |\nabla T|^2 \geq 0$$

This inequality must be verified for any process, i.e., for any value of

$$\nabla \cdot \vec{v}, \quad \tau' : \mathbf{D}', \quad \frac{dT}{dt}, \quad \nabla T$$

$\Rightarrow$

$$p = \rho^2 \frac{\partial \psi}{\partial \rho}$$

$$s = -\frac{\partial \psi}{\partial T}$$

$$\tau' : \mathbf{D}' \geq 0$$

warning: if  $\psi$   
does not depend on  $\mathbf{E}$

$$\gamma \geq 0$$

## **Meaning of the Helmholtz free energy:**

$$d\Psi = dU - TdS - SdT = dQ - dW - TdS - SdT$$

For an isothermical process ( $dT=0$ ):

$$d\Psi = dQ - dW - TdS$$

Second Law:  $T dS \geq d Q$

$$dW \leq -d\Psi \quad (= \text{in case of reversible process})$$

**Therefore, the Helmholtz free energy is the maximum work that the system can do in an isothermical process**