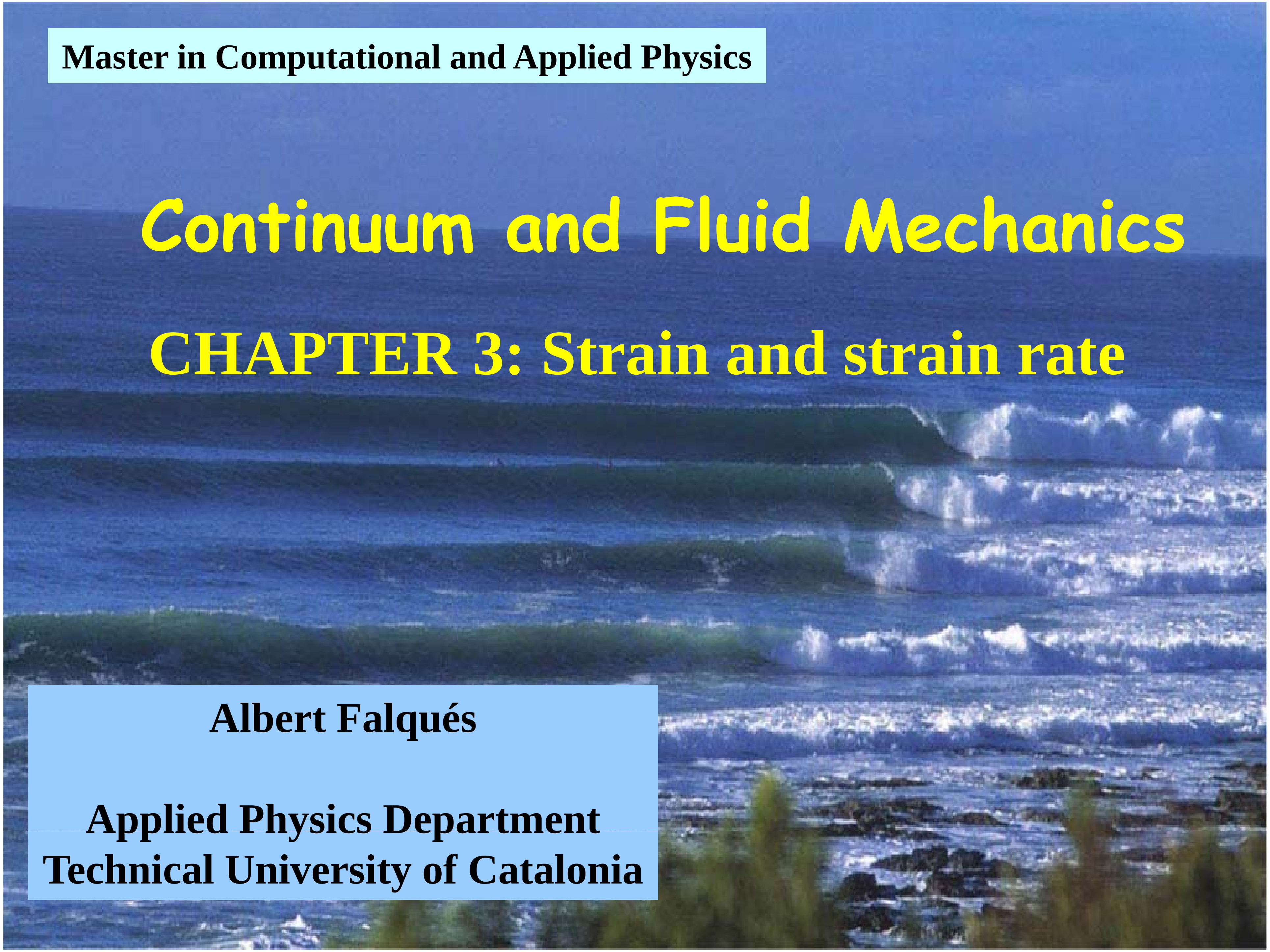

Master in Computational and Applied Physics

Continuum and Fluid Mechanics

CHAPTER 3: Strain and strain rate

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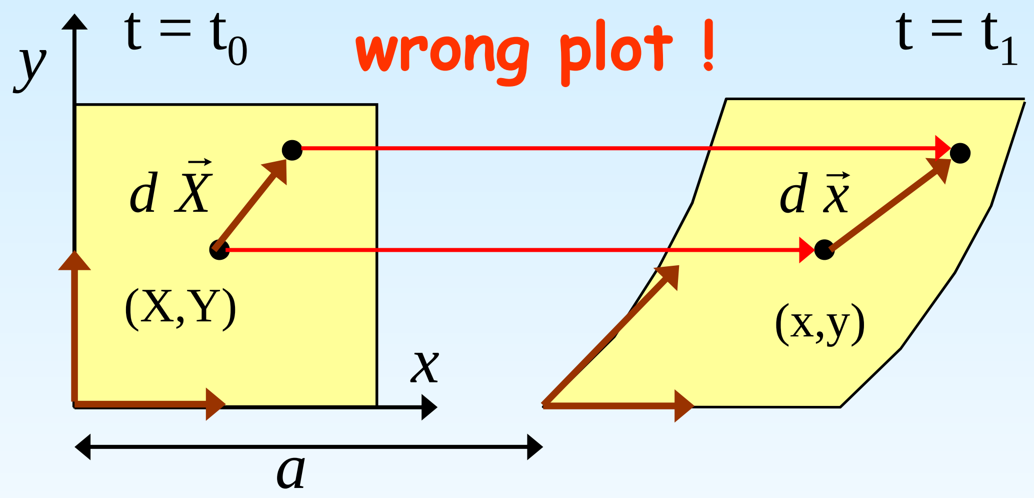
OUTLINE

1. Relative motion of neighbouring particles.
2. Strain tensor.
3. Strain and rotation.
4. Variation of volumes and areas.
5. Strain rate and vorticity.
6. Time variation of volumes, areas and lengths.

1. Relative motion of neighbouring points.

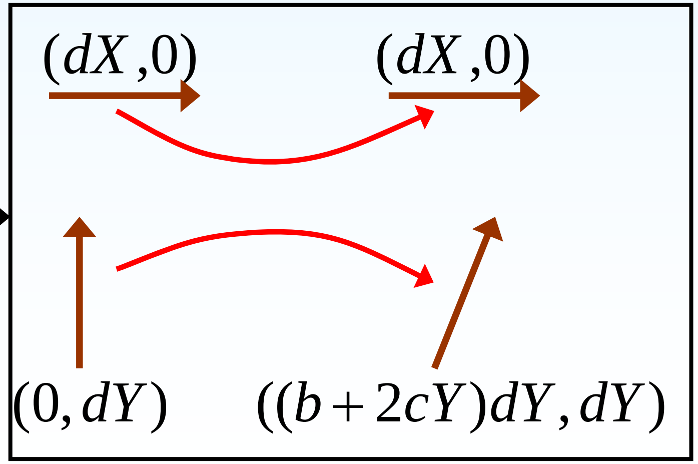
Example: shear deformation of a body between t_0 and t_1

$$
\begin{aligned}
x &= X + a + bY + cY^2 \\
y &= Y \\
z &= Z
\end{aligned}
$$

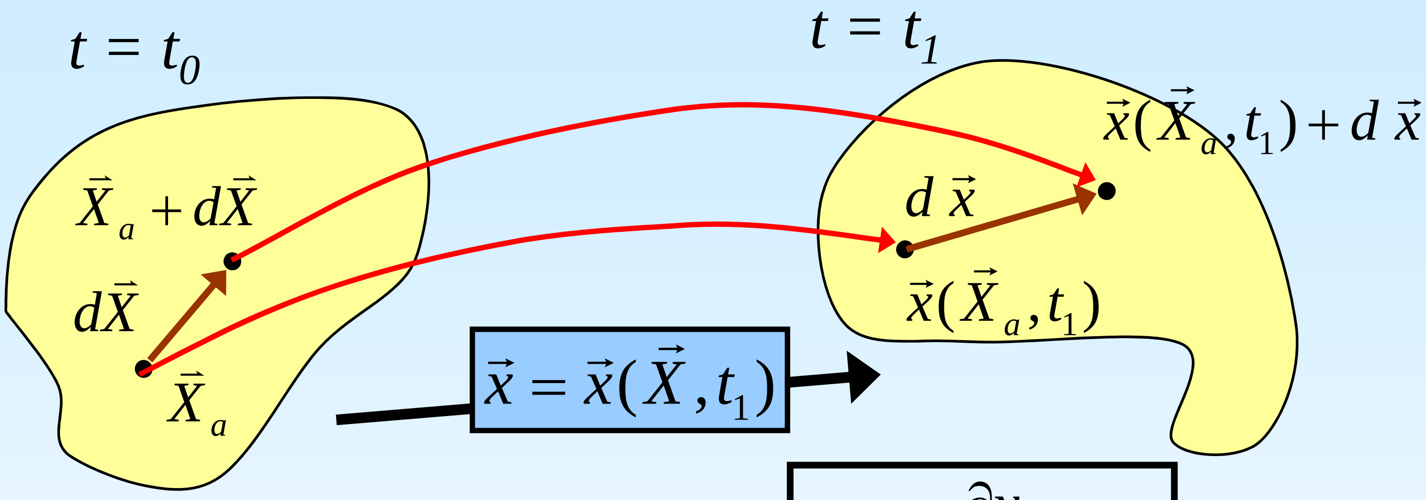


$$
dx = \frac{\partial x}{\partial X} dX + \frac{\partial x}{\partial Y} dY = dX + (b + 2cY)dY
$$

$$
dy = \frac{\partial y}{\partial X} dX + \frac{\partial y}{\partial Y} dY = dY
$$



1. Relative motion of neighbouring points.



$$dx_i = \frac{\partial x_i}{\partial X_k} dX_k$$

$$d\,\vec{x} = \mathbf{F} d\vec{X}$$

$$\mathbf{F} = \frac{\partial \vec{x}}{\partial \vec{X}} \quad , \quad F_{ik} = \frac{\partial x_i}{\partial X_k}$$

Example:

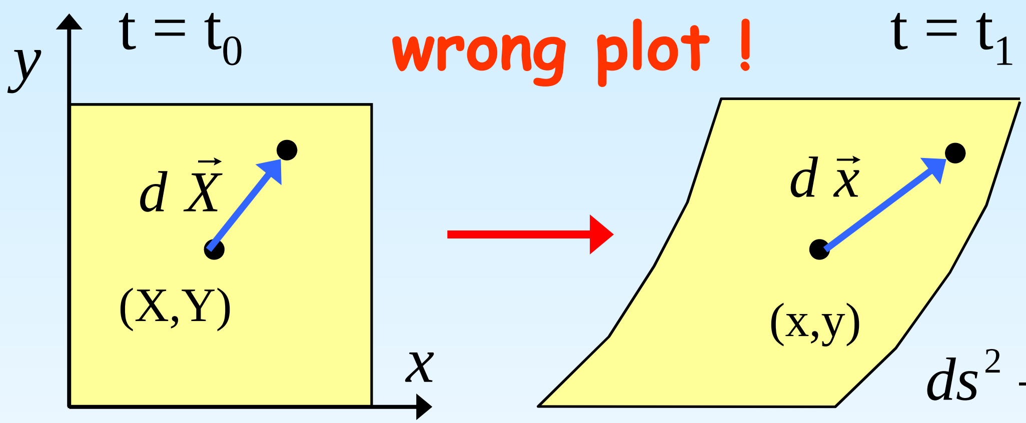
$$dx = dX + (b + 2cY)dY$$

$$dy = dY$$

$$\mathbf{F} = \begin{pmatrix} 1 & b + 2cY & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Strain tensor.

Deformation = change of the distances between particles



$$ds^2 - dS^2 = d\,x_i \cdot d\,x_i - d\,X_i \cdot d\,X_i =$$

$$\left(\frac{\partial x_i}{\partial X_j} \frac{\partial x_i}{\partial X_k} - \delta_{jk} \right) d\,X_j\, d\,X_k =$$

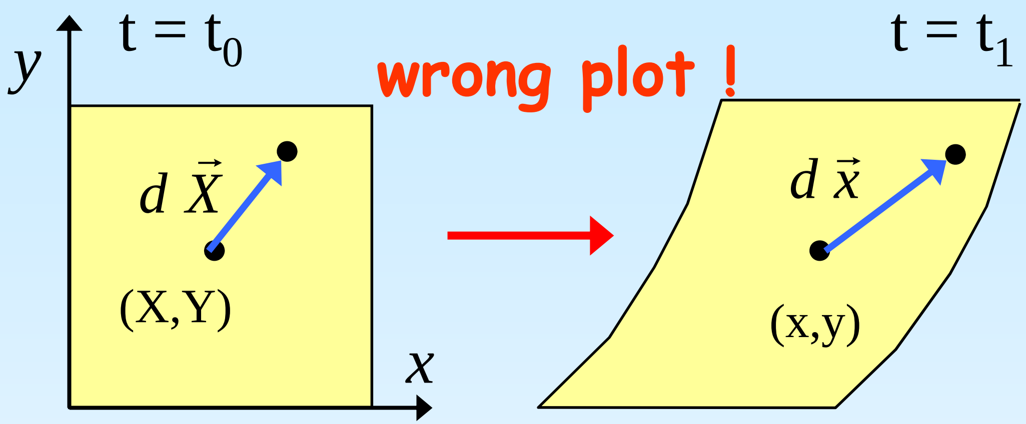
$$2E_{jk}\, d\,X_j\, d\,X_k$$

Strain tensor

$$\boxed{\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) \qquad E_{jk} = \frac{1}{2}\left(\frac{\partial x_i}{\partial X_j} \frac{\partial x_i}{\partial X_k} - \delta_{jk} \right)}$$

symmetrical

Example: shear deformation

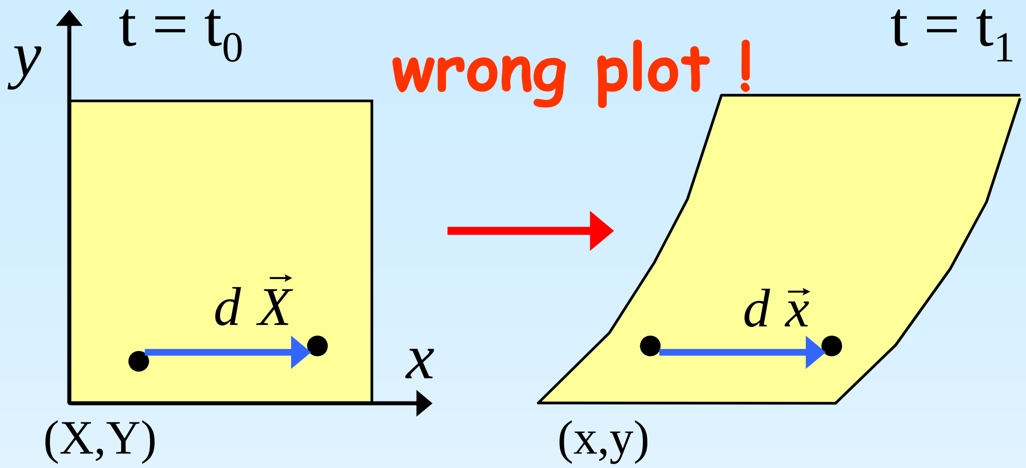


$$\mathbf{F}=\begin{pmatrix}1 & b+2cY & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\mathbf{E}=\frac{1}{2}\left(\mathbf{F}^T\mathbf{F}-\mathbf{1}\right)=\frac{1}{2}\left(\begin{pmatrix}1 & 0 & 0\\ b+2cY & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & b+2cY & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}-\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)=$$

$$\begin{pmatrix}0 & \frac{b}{2}+cY & 0\\ \frac{b}{2}+cY & \frac{1}{2}(b+2cY)^2 & 0\\ 0 & 0 & 0\end{pmatrix}$$

meaning of the components of the strain tensor?



$$d\vec{X} = (1,0,0)\,dS \longrightarrow ds^2 - dS^2 = 2E_{ij}dX_i dX_j = 2E_{11}\,dS^2$$

$\boxed{E_{11} \rightarrow \text{change of length along } x_1 \text{ axis}}$

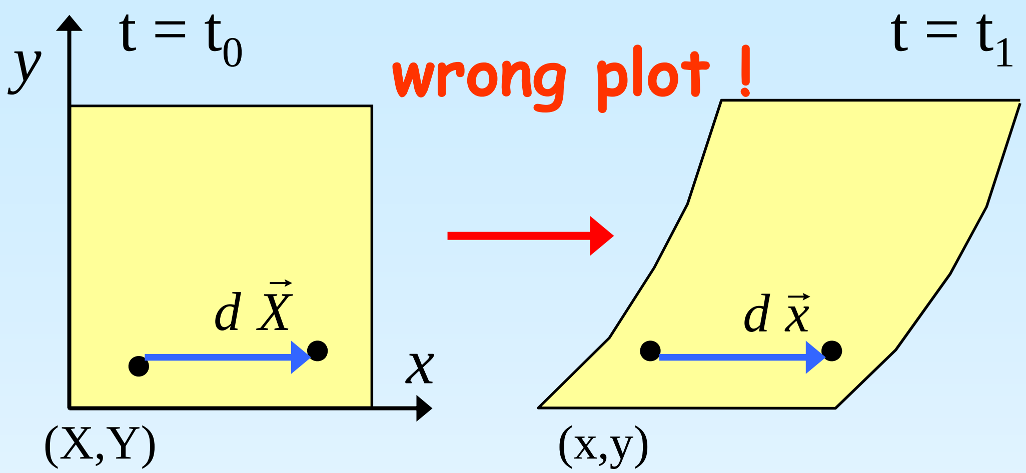
$$d\vec{X} = (0,1,0)\,dS \longrightarrow ds^2 - dS^2 = 2E_{ij}dX_i dX_j = 2E_{22}\,dS^2$$

$\boxed{E_{22} \rightarrow \text{change of length along } x_2 \text{ axis}}$

$$d\vec{X} = (0,0,1)\,dS \longrightarrow ds^2 - dS^2 = 2E_{ij}dX_i dX_j = 2E_{33}\,dS^2$$

$\boxed{E_{33} \rightarrow \text{change of length along } x_3 \text{ axis}}$

Example: shear deformation

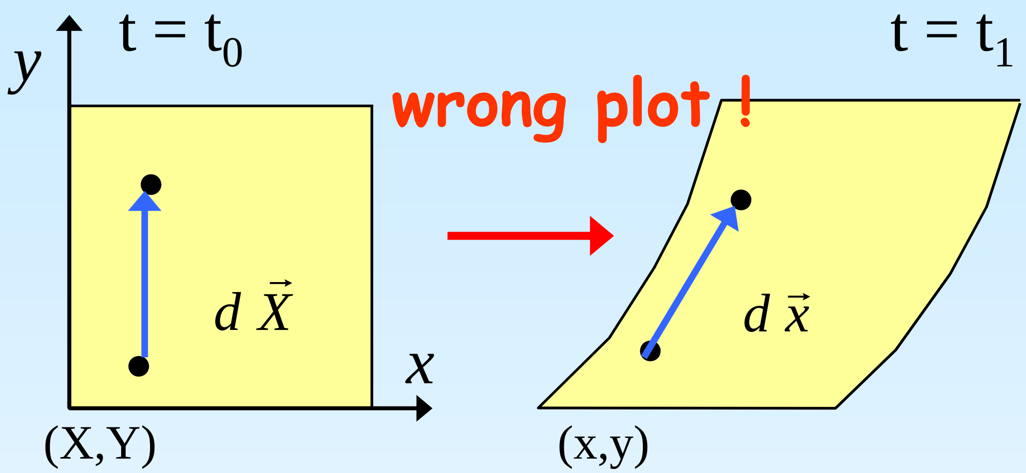


$$d\vec{X} = (1,0,0)\,dS \longrightarrow d\vec{x} = \mathbf{F}\,d\vec{X} = d\vec{X} \quad \Rightarrow \quad |d\vec{x}|^2 - |d\vec{X}|^2 = 0$$

$$\mathbf{E} = \begin{pmatrix} 0 & \frac{b}{2} + cY & 0 \\ \frac{b}{2} + cY & \frac{1}{2}(b + 2cY)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad |d\vec{x}|^2 - |d\vec{X}|^2 = 2E_{ij}dX_i dX_j = 2E_{11}dS^2 = 0$$

$\boxed{E_{11} \rightarrow \textbf{change of length along x axis}}$

Example: shear deformation



$$d\vec{X} = (0,1,0)\,dS \longrightarrow d\vec{x} = \mathbf{F}\,d\vec{X} = (b + 2cY, 1, 0)\,dS \quad \Rightarrow$$

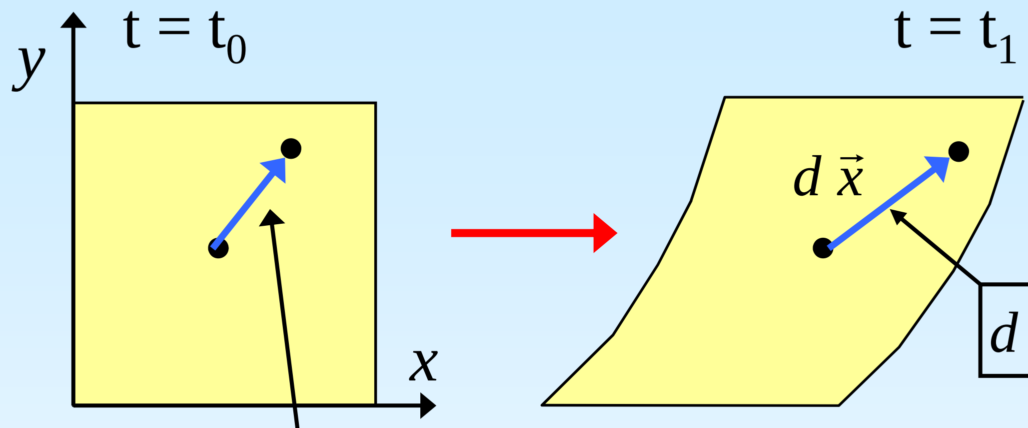
$$\left|d\vec{x}\right|^2 - \left|d\vec{X}\right|^2 = (b+2cY)^2\,dS^2$$

$$\mathbf{E} = \begin{pmatrix} 0 & \frac{b}{2} + cY & 0 \\ \frac{b}{2} + cY & \frac{1}{2}(b+2cY)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \left|d\vec{x}\right|^2 - \left|d\vec{X}\right|^2 = 2E_{ij}\,dX_i\,dX_j =$$

$$2E_{22}\,dS^2 = (b+2cY)^2\,dS^2$$

$\boxed{E_{22} \rightarrow \textbf{change of length along y axis}}$

Computation of the changes in length



$$d\,\vec{x} = \mathbf{F}\cdot d\,\vec{X} = \mathbf{F}\cdot\hat{t}\,d\,S$$

$d\,\vec{X} = \hat{t}\,dS \qquad \hat{t} =$ unit vector

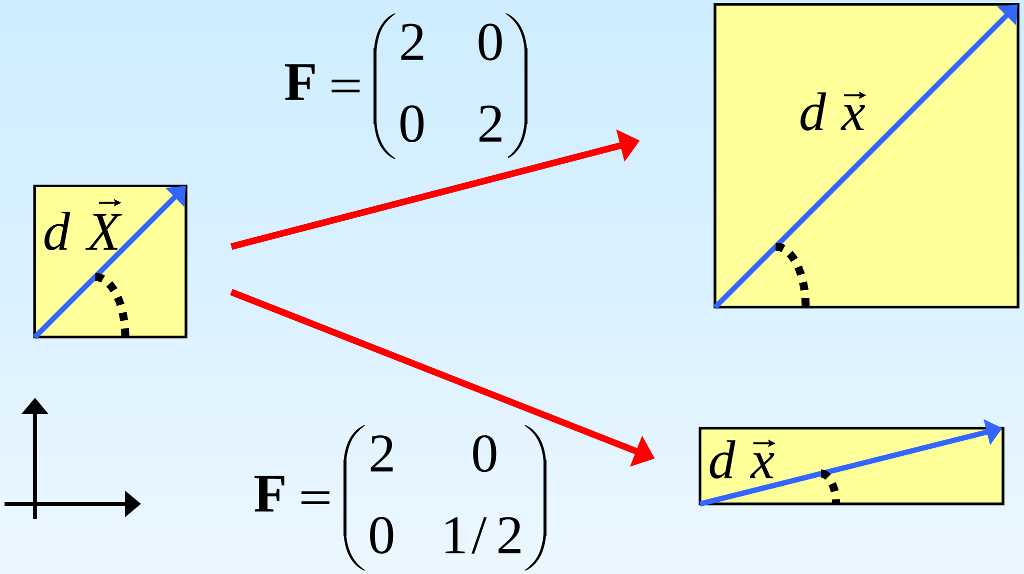
$$ds^2 - dS^2 = 2E_{jk}\,d\,X_j\,d\,X_k \quad \Rightarrow \quad \frac{ds^2 - dS^2}{dS^2} = 2E_{jk}\,t_j\,t_k$$

Change in length per unit of initial length: $$\varepsilon = \frac{ds - dS}{dS} = \sqrt{1 + 2E_{ij}t_i t_j} - 1$$

$$\hat{t} = \hat{e}_1, \hat{e}_2, \hat{e}_3 \Rightarrow \quad \varepsilon_1 = \sqrt{1+2E_{11}} - 1, \quad \varepsilon_2 = \sqrt{1+2E_{22}} - 1, \quad \varepsilon_3 = \sqrt{1+2E_{33}} - 1$$

$$|E_{ij}| << 1 \quad \Rightarrow \quad \varepsilon_1 \approx E_{11}, \quad \varepsilon_2 \approx E_{22}, \quad \varepsilon_3 \approx E_{33}$$

Changes in direction



$$\mathbf{F} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

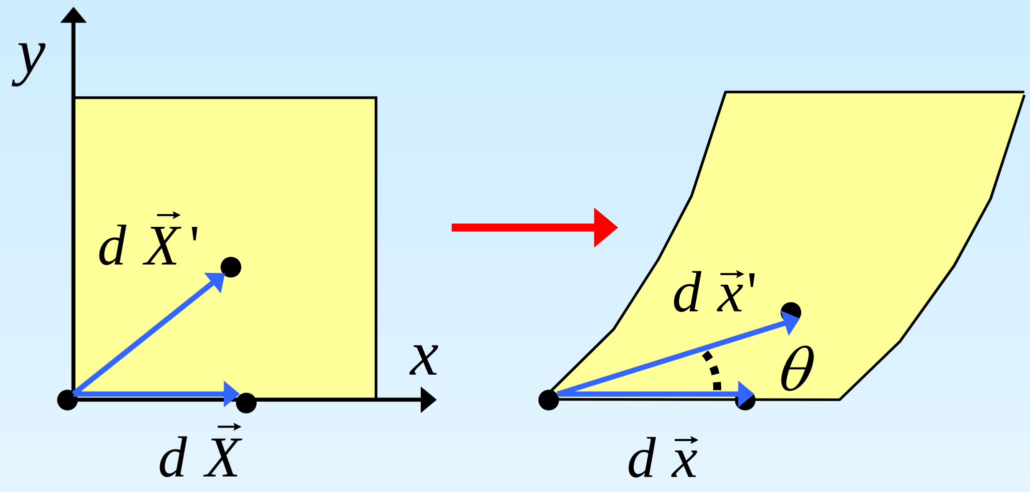
isotropic transformation $\Rightarrow$
only changes in length,
no changes in direction

anisotropic transformation $\Rightarrow$
also changes in direction

$$\mathbf{E} = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} \quad \Rightarrow \quad \varepsilon_1 = \sqrt{1+2E_{11}} - 1 = 1, \quad \varepsilon_2 = \sqrt{1+2E_{22}} - 1 = 1$$

$$\mathbf{E} = \begin{pmatrix} 3/2 & 0 \\ 0 & -3/8 \end{pmatrix} \quad \Rightarrow \quad \varepsilon_1 = \sqrt{1+2E_{11}} - 1 = 1, \quad \varepsilon_2 = \sqrt{1+2E_{22}} - 1 = -1/2$$

Computation of the changes in direction

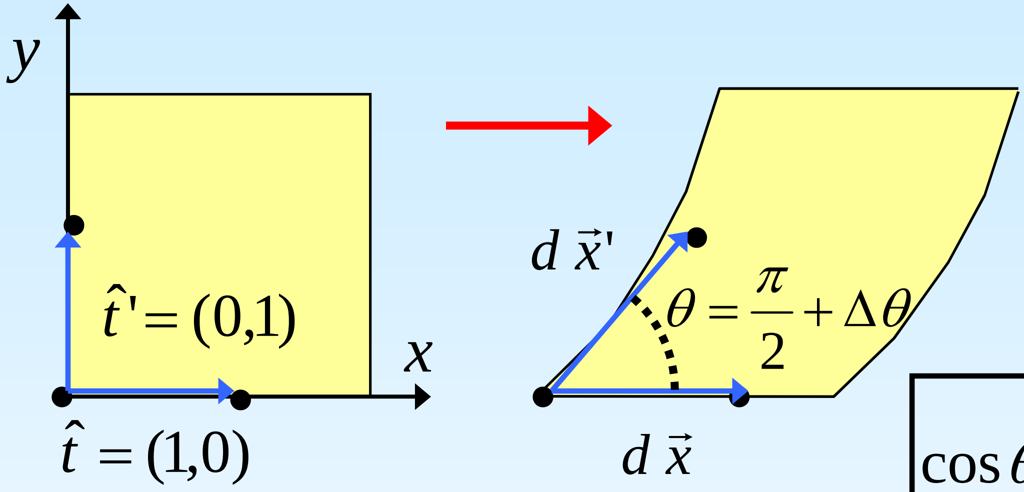


$$\boxed{\begin{aligned} & d\vec{x}\cdot d\vec{x}' = \cos\theta \, ds\, ds' = \\ & d\vec{X}\cdot \mathbf{F}^T \cdot \mathbf{F}\cdot d\vec{X}' = \\ & d\vec{X}\cdot(\mathbf{1}+2\mathbf{E})\cdot d\vec{X}' \end{aligned}}$$

$$\hat{t} = \frac{d\vec{X}}{dS} \quad , \quad \hat{t}' = \frac{d\vec{X}'}{dS'}$$

$$\boxed{\cos\theta = \frac{(\delta_{ij} + 2E_{ij})t_i t'_j}{\sqrt{1+2E_{kr}t_k t_r}\sqrt{1+2E_{nm}t'_n t'_m}}}$$

Computation of the changes in direction



$$\boxed{\cos\theta = \frac{\left(\delta_{ij} + 2E_{ij}\right)t_i t'_j}{\sqrt{1+2E_{kr}t_k t_r}\sqrt{1+2E_{nm}t'_n\, t'_m}}}$$

$$\boxed{\sin\Delta\theta = \frac{-2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}}$$

So, the non diagonal entries of *E* are related to the changes in the angles between material lines along the axes

$$|\,E_{ij}\,| << 1 \quad \Rightarrow \quad \boxed{E_{12} \approx -\frac{1}{2}\Delta\theta}$$

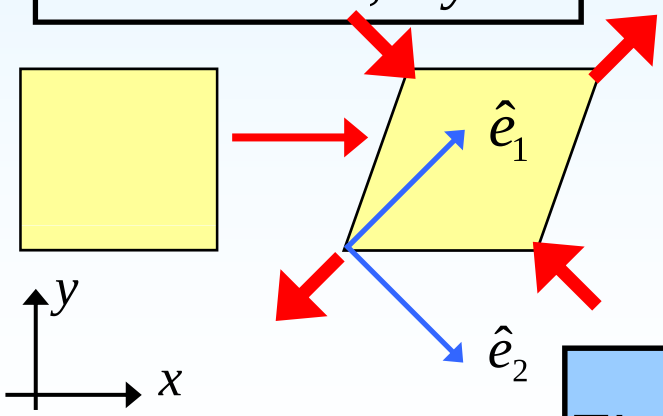
Principal axes of the strain tensor

Since ***E*** is symmetric, there always are three principal directions
with real eigenvalues $\Rightarrow$
any deformation is a stretching or compression (without rotation)
along three directions which are mutually orthogonal

using these directions as coordinate axes:

$$\mathbf{E} = \begin{pmatrix} e_{11} & 0 & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{33} \end{pmatrix}$$

$$\boxed{x = X + bY \quad , \quad y = Y} \longrightarrow \mathbf{E} = \begin{pmatrix} 0 & b/2 \\ b/2 & 0 \end{pmatrix}$$



$$\boxed{(\mathbf{E} - \lambda \mathbf{1}) \cdot \vec{e} = 0} \longrightarrow \boxed{\begin{array}{ll} \lambda = b/2\,, & \vec{e} = (1,1) \\ \lambda = -b/2\,, & \vec{e} = (1,-1) \end{array}}$$

$$\boxed{\mathbf{E}' = \begin{pmatrix} b/2 & 0 \\ 0 & -b/2 \end{pmatrix}}$$

$$\varepsilon_1 = \sqrt{1 + 2E'_{11}} - 1 = \sqrt{1+b} - 1 \approx b/2,$$

$$\varepsilon_2 = \sqrt{1 + 2E'_{22}} - 1 = \sqrt{1-b} - 1 \approx -b/2$$

if |b|<<1

3. Strain and rotation.

Polar theorem:

any linear map ***F*** can be decomposed as

$$\boxed{\mathbf{F} = \mathbf{Q} \cdot \mathbf{U}}$$

where ***Q*** is an orthogonal transformation and ***U*** is symmetric

Q and ***U*** are unique (also, ***F***=***V***·***Q***, but this is not needed now)

- Since det **F** = +1, **Q** is actually a rotation.
- Since **U** is symmetric, it is a stretching along principal directions, so that it is a deformation with

$$\boxed{\mathbf{E} = \frac{1}{2}\left(\mathbf{U}^2 - \mathbf{1}\right)}$$

Therefore, any motion of a body is locally the superposition of:
• a deformation
• a rigid rotation
• a shift (which is not captured by F)

It can be proven that:

$\mathbf{U} = \boldsymbol{1}$ everywhere in the body	$\Rightarrow$	$\mathbf{Q}$ = const., i.e., the same everywhere in the body

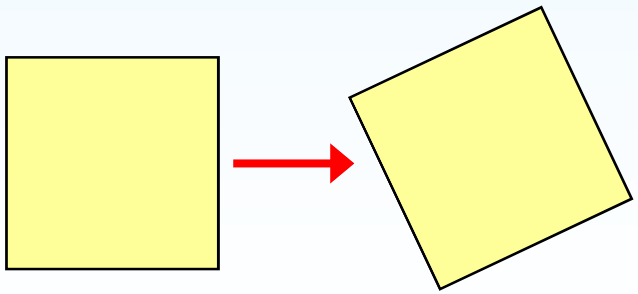
This means that if there is no deformation the rigid rotation is the same everywhere in the body $\Rightarrow$ rigid body motion

Don't confuse !

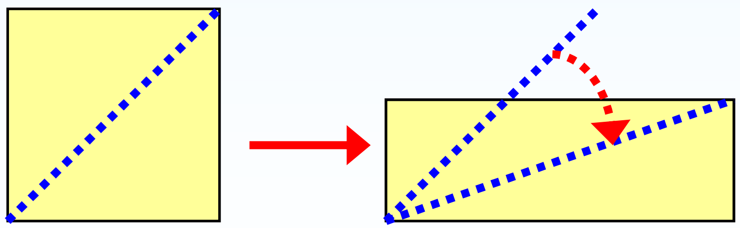
❖ the rigid body rotation which rotates all the vectors by the same angle

with

❖ the rotation associated to a strain which rotates different vectors by different angles



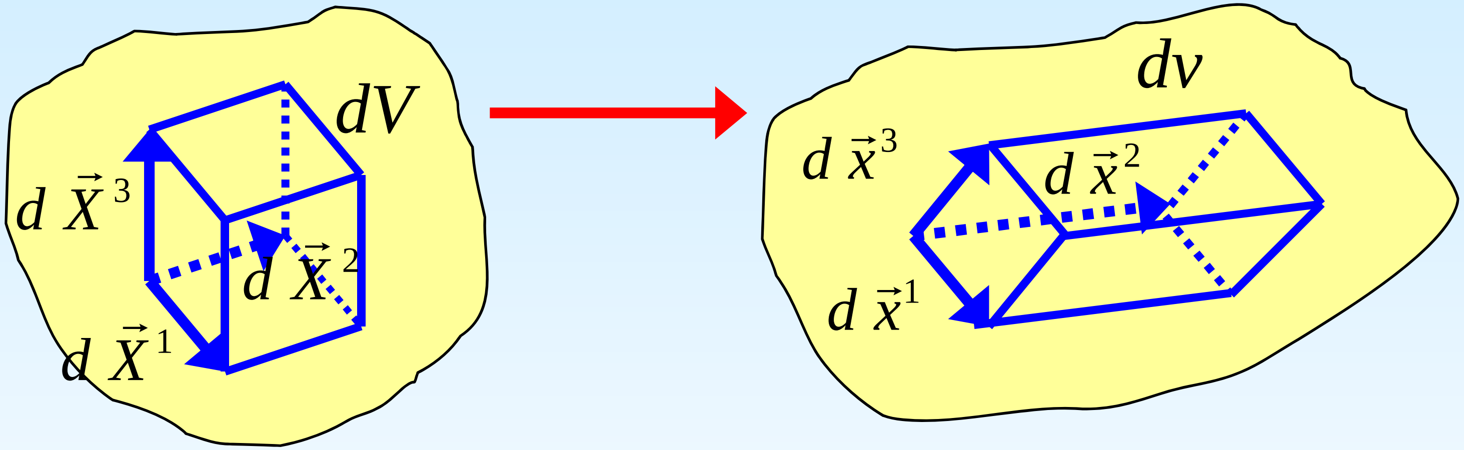
rigid body rotation, **Q**



deformation, **U** (or **E**)

4. Variation of volumes and areas.

Volume variation



$$dv = \left(d\vec{x}^1 \times d\vec{x}^2\right) \cdot d\vec{x}^3 = \det\left\{\left(dx^i\right)_j\right\} = \det\left\{F_{jk}\left(dX^i\right)_k\right\} =$$

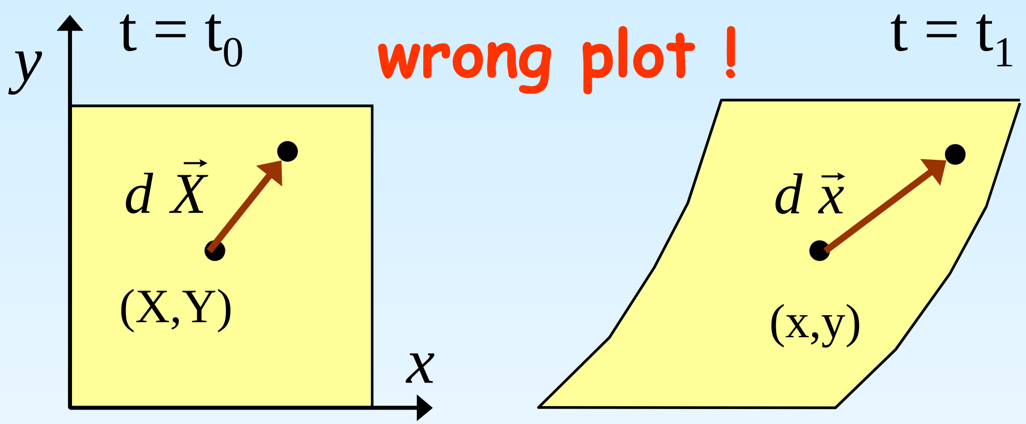
$$\det\{\mathbf{F} \cdot \mathbf{d}X\} = \det\{\mathbf{F}\}\det\{\mathbf{d}X\} = \det\{\mathbf{F}\}\,dV$$

$$\boxed{\frac{dv}{dV} = \det \mathbf{F}}$$

with $\quad F_{ij} = \dfrac{\partial x_i}{\partial X_j}$

Example:

$$\boxed{\begin{aligned} x &= X + a + bY + cY^2 \\ y &= Y \\ z &= Z \end{aligned}}$$



$$\boxed{\mathbf{F} = \begin{pmatrix} 1 & b+2cY & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \longrightarrow \boxed{\det \mathbf{F} = 1}$$

$\Rightarrow$ **no changes in volume**

Area variation

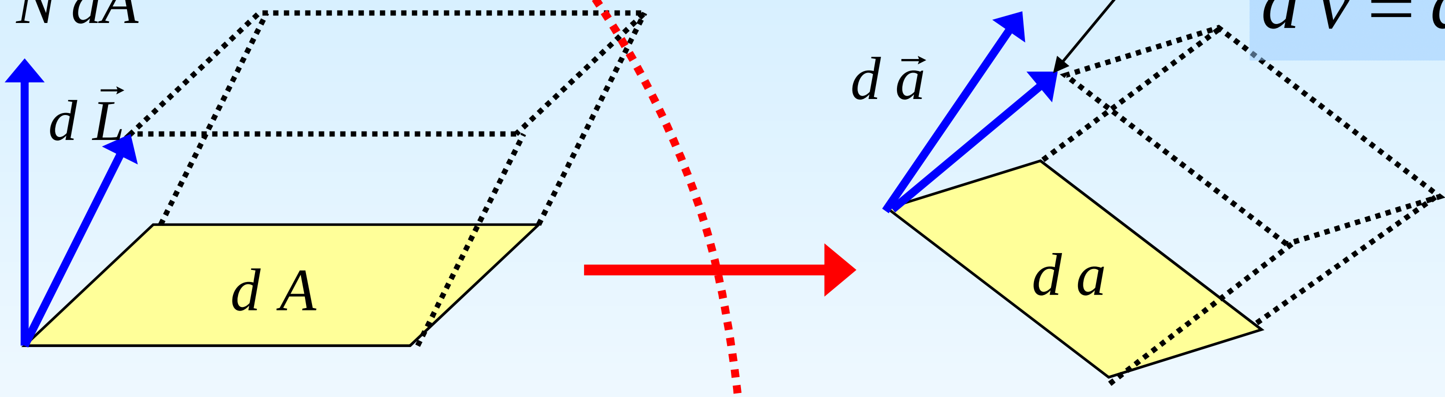
$d\vec{L}$ = arbitrary vector

$$dV = d\vec{A}\cdot d\vec{L}$$

$$d\vec{l} = \mathbf{F}\cdot d\vec{L}$$

$$dv = d\vec{a}\cdot d\vec{l}$$

$$d\vec{A} = \hat{N}\,dA$$

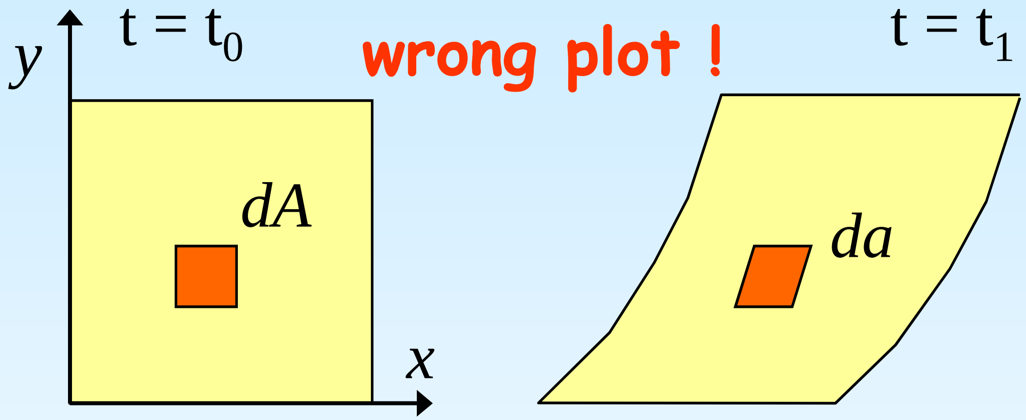


$$\left.\begin{array}{l} dv = d\vec{a}\cdot d\vec{l} = d\vec{a}\cdot\mathbf{F}\cdot d\vec{L} \\ dv = \det\mathbf{F}\;dV = \det\mathbf{F}\;d\vec{A}\cdot d\vec{L} \end{array}\right\}\quad d\vec{a}\cdot\mathbf{F} = \left(\det\mathbf{F}\right)d\vec{A} \quad\Rightarrow$$

$$\boxed{d\vec{a} = \left(\det\mathbf{F}\right)\,d\vec{A}\cdot\mathbf{F}^{-1}}$$

Example:

$$\boxed{\begin{aligned} x &= X + a + bY + cY^2 \\ y &= Y \\ z &= Z \end{aligned}}$$



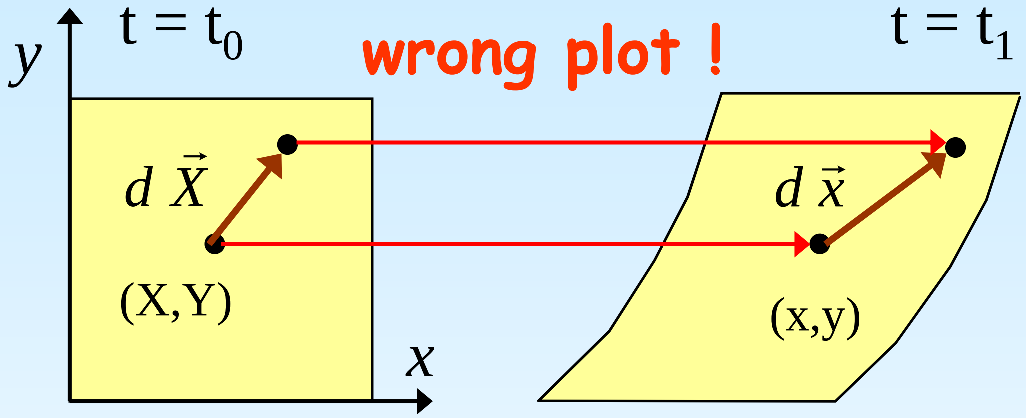
$$\boxed{\mathbf{F}^{-1} = \begin{pmatrix} 1 & -b-2cY & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \qquad \boxed{\det \mathbf{F} = 1}$$

$$\boxed{d\,\vec{a} = \left(\det \mathbf{F}\right)\, d\,\vec{A}\cdot \mathbf{F}^{-1} = (0,0,dA)\cdot \begin{pmatrix} 1 & -b-2cY & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (0,0,dA)} \Rightarrow$$

$$\boxed{d\,\vec{a} = d\,\vec{A}}$$ $\Rightarrow$ **no changes in area**

5. Strain rate and vorticity

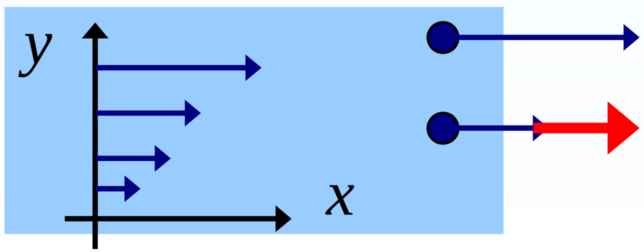
$$x = X + a + bY + cY^2$$
$$y = Y$$
$$z = Z$$



Let's now consider that t_1 is changing, that is, the time variation and let's think of the Eulerian description.
Which is the relative motion of neighbouring points?

Example: parallel shear flow

$$v_x = v(y)$$
$$v_y = 0$$
$$v_z = 0$$



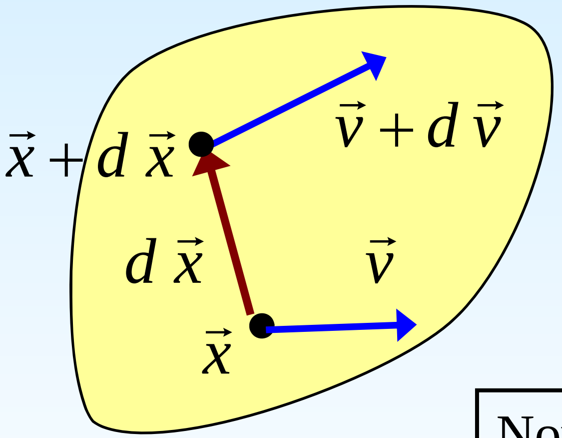
$$dv_x = v_x(y + dy) - v_x(y)$$

In general:

$$\boxed{d\,v_i = v_i\left(\vec{x}+d\,\vec{x},t\right)-v_i\left(\vec{x},t\right)=\frac{\partial v_i}{\partial x_j}dx_j}$$

$$\boxed{d\,\vec{v} = \mathbf{L}\cdot d\,\vec{x}}$$

$$L_{ij} = \frac{\partial v_i}{\partial x_j} =$$ **velocity gradient tensor**



Now:

- which part of the relative motion does correspond to strain and which part does correspond to rigid body rotation**?**
- which is the relationship with the Lagrangian description **?**

Let us consider the Lagrangian description, $\vec{x} = \vec{x}\left(\vec{X}, t\right)$:

$$\boxed{\frac{d}{dt}\left(\frac{\partial x_i(\vec{X},t)}{\partial X_j}\right) = \frac{\partial}{\partial X_j}\left(\frac{\partial x_i(\vec{X},t)}{\partial t}\right) = \frac{\partial v_i}{\partial X_j} = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j}} \longrightarrow \boxed{\frac{d\,\mathbf{F}}{dt} = \mathbf{L}\cdot\mathbf{F}}$$

Polar theorem: $\mathbf{F} = \mathbf{Q}\cdot\mathbf{U}$ $\quad\begin{cases} \mathbf{U} = \text{symmetric} \rightarrow \text{deformation} \\ \mathbf{Q} = \text{orthogonal} \rightarrow \text{rigid body rotation} \end{cases}$

$$\mathbf{L} = \frac{d\mathbf{F}}{dt}\cdot\mathbf{F}^{-1} = \frac{d\mathbf{Q}}{dt}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\frac{d\mathbf{U}}{dt}\cdot\mathbf{U}^{-1}\cdot\mathbf{Q}^T$$

Now, one can take the material coordinates as the coordinates of the particles at any fixed time, in particular, $t_0 \rightarrow t_1$, the time where the computation of **L** is done. Thus, the reference configuration can by chosen so that:

$\mathbf{Q}(t_1) \rightarrow \mathbf{1}$, $\mathbf{U}(t_1) \rightarrow \mathbf{1}$ for $t_0 \rightarrow t_1$

so that:

$$\boxed{\mathbf{L} = \left.\frac{d\mathbf{U}}{dt}\right|_{t=t_0} + \left.\frac{d\mathbf{Q}}{dt}\right|_{t=t_0} = \mathbf{D} + \mathbf{W}}$$

$$\mathbf{D} = \left.\frac{d\mathbf{U}}{dt}\right|_{\mathbf{U}=\mathbf{1}}$$

= **strain rate tensor**.
Is symmetric because U is symmetrical.
= deformation part of the relative motion

$$\mathbf{W} = \left.\frac{d\mathbf{Q}}{dt}\right|_{\mathbf{Q}=\mathbf{1}}$$

= **vorticity or rotation tensor**.
Is antisymmetric because $(dQ/dt)\cdot Q^T$ is antisymmetric
= rigid body rotation part of the relative motion

D = symmetric part of **L**, **W** = antisymmetric part of **L** $\Rightarrow$

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) \quad , \quad \mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right)$$

vector associated to the vorticity tensor $\boxed{\Omega_i = -\frac{1}{2}\varepsilon_{ijk} W_{jk}}$

$$\boxed{\mathbf{W}\cdot d\vec{X} = \vec{\Omega}\times d\vec{X}}$$

$$\boxed{\Omega_i = -\frac{1}{2}\varepsilon_{ijk} W_{jk} = -\frac{1}{4}\varepsilon_{ijk}\left(\partial_k v_j - \partial_j v_k\right) = \frac{1}{2}\varepsilon_{ijk}\partial_j v_k}$$

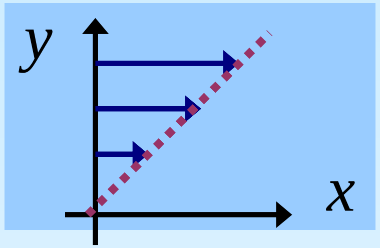
= angular velocity in case of a rigid body rotation

vorticity vector

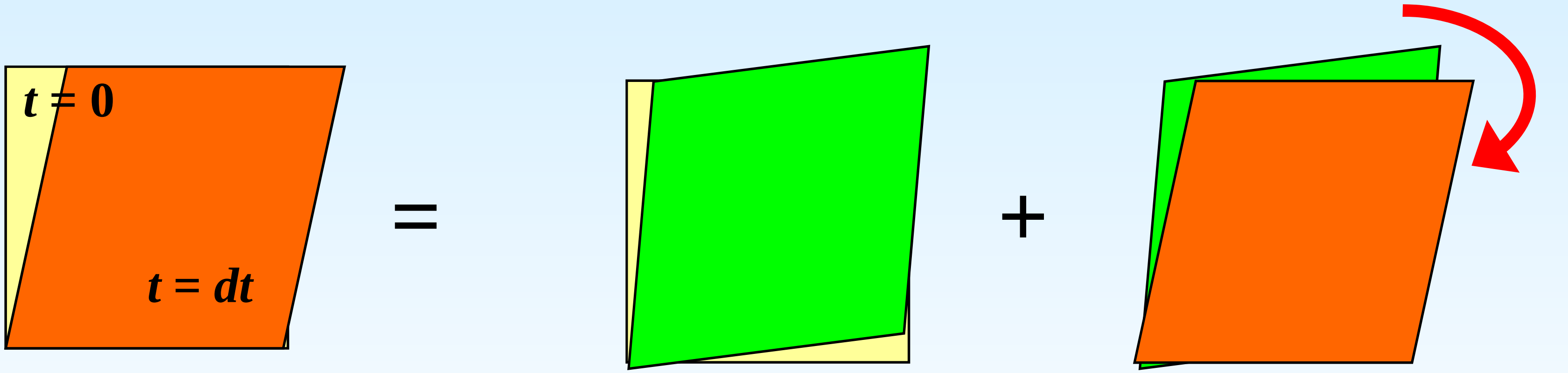
$$\vec{\omega} = \nabla \times \vec{v} = 2\vec{\Omega}$$

Example: parallel shear flow

$$\boxed{\begin{aligned} v_x &= k\,y \\ v_y &= 0 \\ v_z &= 0 \end{aligned}}$$



$$\boxed{\mathbf{L} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}}$$

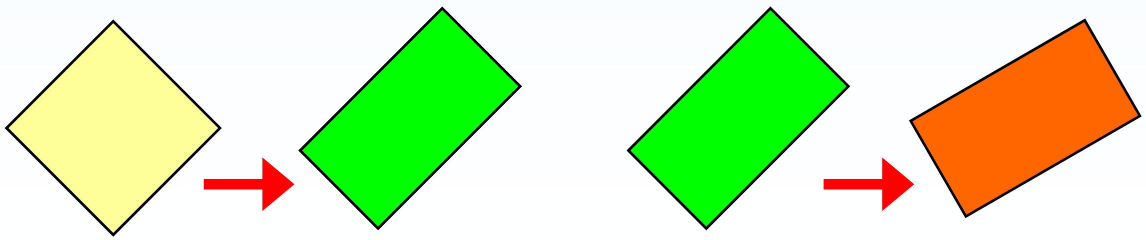


strain

$$\mathbf{D} = \begin{pmatrix} 0 & k/2 \\ k/2 & 0 \end{pmatrix}$$

rigid body rotation

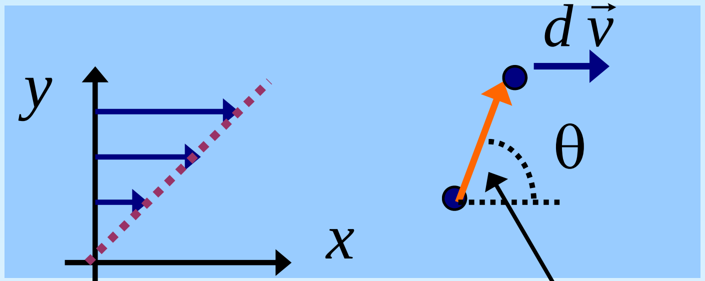
$$\mathbf{W} = \begin{pmatrix} 0 & k/2 \\ -k/2 & 0 \end{pmatrix}$$



Example: parallel shear flow. Rate of change of an arbitrary vector.

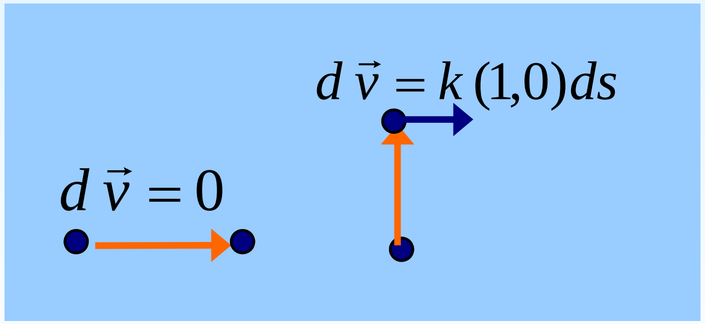
$$\boxed{v_x = k\,y\,, \quad v_y = 0}$$

$$\boxed{\mathbf{L} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}}$$



$$\boxed{d\,\vec{x} = \hat{t}\,ds = (\cos\theta, \sin\theta)\,ds}$$

$$d\,\vec{v} = \mathbf{L}\cdot d\,\vec{x} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} ds = k(\sin\theta, 0)\,ds$$



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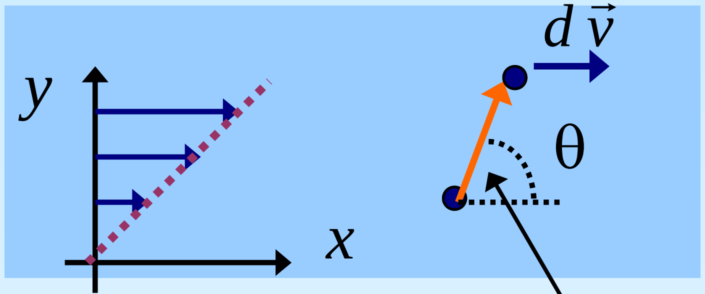
$$\boxed{\mathbf{D} = \begin{pmatrix} 0 & k/2 \\ k/2 & 0 \end{pmatrix}\,, \quad \mathbf{W} = \begin{pmatrix} 0 & k/2 \\ -k/2 & 0 \end{pmatrix}}$$

strain

$$d\,\vec{v} = \mathbf{D}\cdot d\,\vec{x} = \begin{pmatrix} 0 & k/2 \\ k/2 & 0 \end{pmatrix}\cdot\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} ds = \frac{k}{2}(\sin\theta, \cos\theta)\,ds$$



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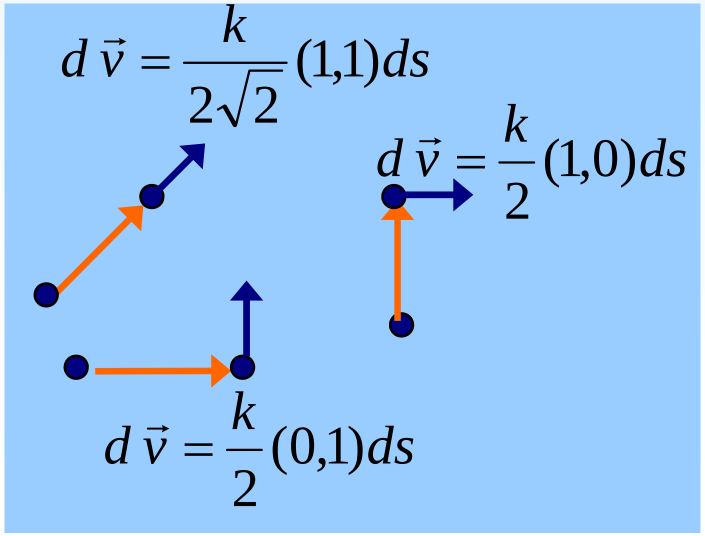

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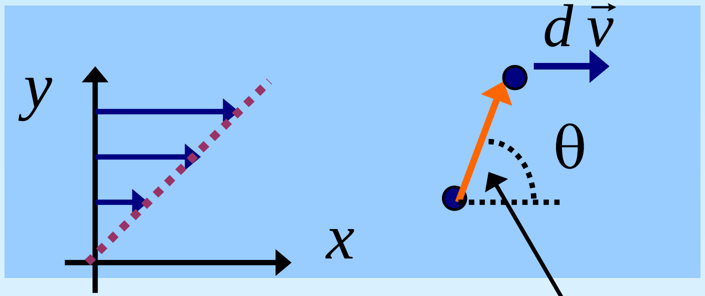
rotation

$$d\,\vec{v} = \mathbf{W}\cdot d\,\vec{x} = \begin{pmatrix} 0 & k/2 \\ -k/2 & 0 \end{pmatrix}\cdot\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} ds =$$

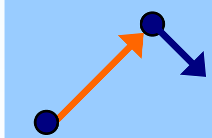
$$\frac{k}{2}\left(\sin\theta, -\cos\theta\right) ds$$

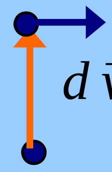
$$\boxed{\frac{d\theta}{dt} = -\frac{k}{2}}$$



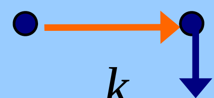
$$\boxed{d\,\vec{x} = \hat{t}\,ds = \left(\cos\theta, \sin\theta\right) ds}$$

$$d\,\vec{v} = \frac{k}{2\sqrt{2}}(1,-1)ds$$





$$d\,\vec{v} = \frac{k}{2}(1,0)ds$$



$$d\,\vec{v} = \frac{k}{2}(0,-1)ds$$

Computation of the rate of change in length

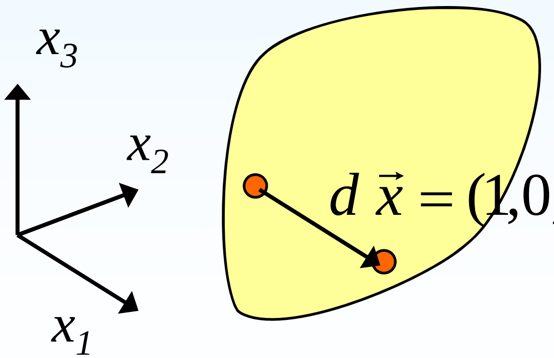
$$\frac{d}{dt}(ds^2) = \frac{d}{dt}(2\, d\vec{X} \cdot \mathbf{E} \cdot d\vec{X}) = 2\; d\vec{X} \cdot \frac{d\mathbf{E}}{dt} \cdot d\vec{X}$$

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1}\right)$$

$$\frac{d}{dt}(ds^2) = d\vec{x} \cdot \left(\mathbf{L}^T + \mathbf{L}\right) \cdot d\vec{x} = 2\, d\vec{x} \cdot \mathbf{D} \cdot d\vec{x}$$

$$d\vec{x} = \hat{t}\, ds$$

$$\frac{1}{ds}\frac{d}{dt}(ds) = \hat{t} \cdot \mathbf{D} \cdot \hat{t}$$

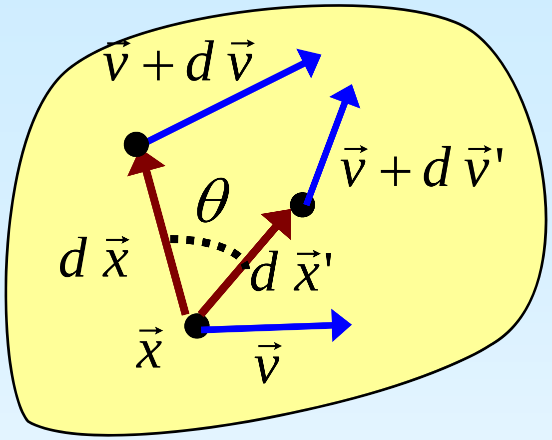


$$d\vec{x} = (1,0,0)\, ds$$

$$D_{11} = \frac{1}{ds}\frac{d}{dt}(ds)$$

and similarly for D_{22} and D_{33}

Computation of the rate of change in relative angle



$$\boxed{d\vec{x}\cdot d\vec{x}' = ds\,ds'\cos\theta}$$

$$\frac{d}{dt}\left(d\vec{x}\cdot d\vec{x}'\right) = \frac{dx_i}{dt}dx_i' + dx_i\frac{dx_i'}{dt} = dv_i dx_i' + dx_i dv_i' =$$

$$L_{ij}dx_j dx_i' + dx_i L_{ij} dx_j' = L_{ji}dx_i dx_j' + dx_i L_{ij} dx_j' =$$

$$2D_{ij}dx_i dx_j'$$

$$\frac{d}{dt}\left(ds\,ds'\cos\theta\right) = \frac{d}{dt}\left(ds\right)ds'\cos\theta + ds\frac{d}{dt}\left(ds'\right)\cos\theta - ds\,ds'\sin\theta\frac{d\theta}{dt} =$$

$$\left(D_{ij}\left(\hat{t}_i\hat{t}_j + \hat{t}_i'\hat{t}_j'\right)\cos\theta - \sin\theta\frac{d\theta}{dt}\right)ds\,ds'$$

$$\boxed{\frac{d\theta}{dt}}$$

Now, if $d\vec{x} = (1,0,0)\,ds$, $d\vec{x}' = (0,1,0)\,ds$

so that $\theta = \frac{\pi}{2}$

$$\boxed{2D_{12} = -\frac{d\theta_{12}}{dt}}$$

Principal axes of the strain rate tensor

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}$$

relative angular velocity of the axes

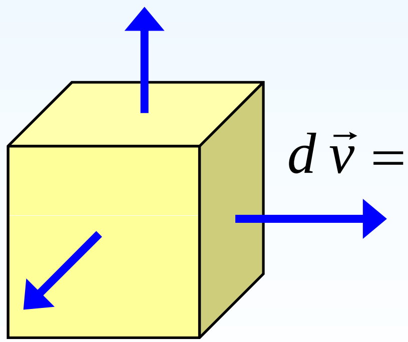
stretching rate along the axes

Since the strain rate tensor is symmetric, there exist three principal directions with respect to which

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

that is, **the deformation is just a stretching along these axes without rotation.**

$d\,\vec{v} = d_3 \hat{e}_3 \, ds$

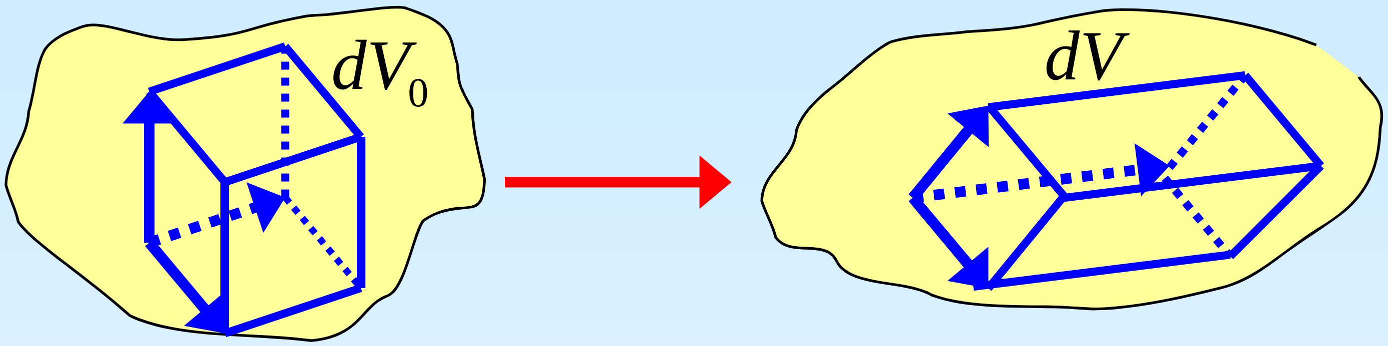


$d\,\vec{v} = d_2 \hat{e}_2 \, ds$

Warning: this is locally and at a particular time

$d\,\vec{v} = d_1 \hat{e}_1 \, ds$

6. Time variation of volumes and areas



$$\frac{dV}{dV_0} = |\mathbf{F}|$$

$$|\mathbf{F}| \equiv \det \mathbf{F}$$

$$\frac{d}{dt}(dV) = \frac{d\,|\mathbf{F}|}{dt} dV_0$$

$$\frac{d|\mathbf{F}|}{dt} = \frac{\partial|\mathbf{F}|}{\partial F_{ij}} \frac{dF_{ij}}{dt} = |\mathbf{F}| \left\{\mathbf{F}^{-1}\right\}_{ji} L_{ik} F_{kj} = |\mathbf{F}| \delta_{ki} L_{ik} = |\mathbf{F}| L_{kk}$$

$$\left\{\mathbf{F}^{-1}\right\}_{ji} = \frac{1}{|\mathbf{F}|} \frac{\partial|\mathbf{F}|}{\partial F_{ij}}$$

$$\frac{d\,\mathbf{F}}{dt} = \mathbf{L} \cdot \mathbf{F}$$

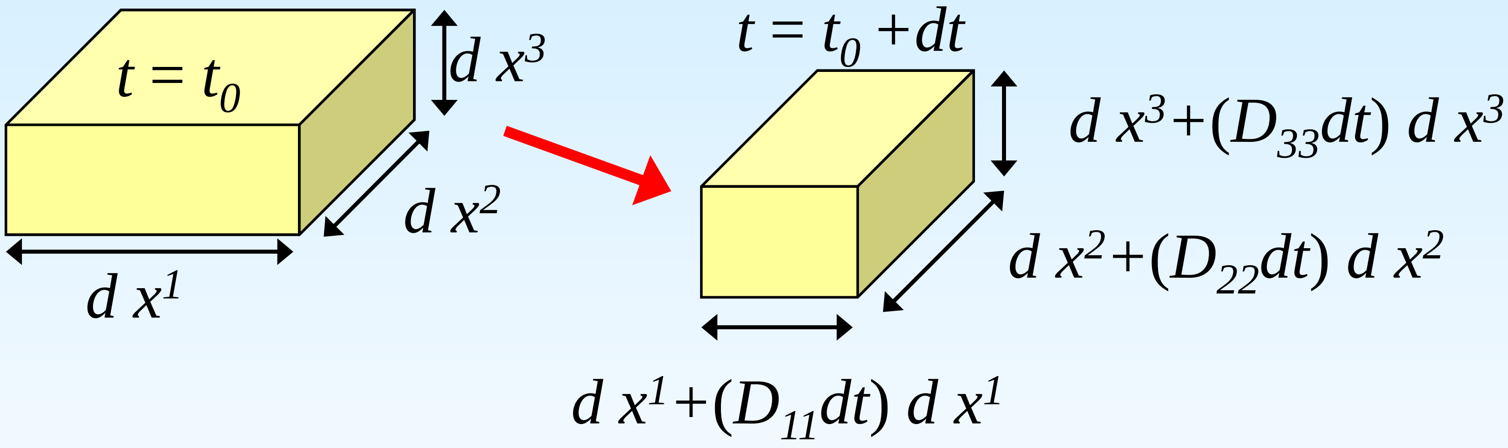
$$\frac{d}{dt}(dV) = (\nabla \cdot \vec{v}) |\mathbf{F}| dV_0$$

$$\boxed{\frac{1}{V} \frac{dV}{dt} = \nabla \cdot \vec{v}}$$

Another proof:

Let (x_1,x_2,x_3) be principal axes of **D** at a given point and time $\Rightarrow$

$$\frac{1}{dx^1}\frac{d}{dt}(dx^1) = D_{11}\ , \qquad \frac{1}{dx^2}\frac{d}{dt}(dx^2) = D_{22}\ , \qquad \frac{1}{dx^3}\frac{d}{dt}(dx^3) = D_{33}$$



$$\frac{1}{dV}\frac{d}{dt}\left(dV\right) = \frac{1}{dx^1dx^2dx^3}\frac{d}{dt}\left(dx^1dx^2dx^3\right) =$$
$$\frac{1}{dx^1}\frac{d}{dt}\left(dx^1\right) + \frac{1}{dx^2}\frac{d}{dt}\left(dx^2\right) + \frac{1}{dx^3}\frac{d}{dt}\left(dx^3\right) =$$
$$D_{11} + D_{22} + D_{33} = tr(\mathbf{D})$$

tr (**D**) is invariant $\Rightarrow$
in any coordinate system:

$$\boxed{\frac{1}{dV}\frac{d}{dt}\left(dV\right) = tr(\mathbf{D}) = \frac{\partial v_i}{\partial x_i} = \nabla \cdot \vec{v}}$$

Time variation of areas

$$d\vec{A} = |\mathbf{F}|\; d\vec{A}_0 \cdot \mathbf{F}^{-1}$$

$$\frac{d\,\mathbf{F}}{dt} = \mathbf{L}\cdot\mathbf{F}$$

$$\frac{d|\mathbf{F}|}{dt} = |\mathbf{F}|\nabla\cdot\vec{v}$$

$$\mathbf{F}\cdot\mathbf{F}^{-1} = \mathbf{1} \quad \Rightarrow$$

$$\frac{d\,\mathbf{F}}{dt}\cdot\mathbf{F}^{-1} + \mathbf{F}\cdot\frac{d\,\mathbf{F}^{-1}}{dt} = 0 \quad \Rightarrow$$

$$\frac{d\,\mathbf{F}^{-1}}{dt} = \mathbf{F}^{-1}\cdot\frac{d\,\mathbf{F}}{dt}\cdot\mathbf{F}^{-1}$$

$$\frac{d}{dt}\left(d\vec{A}\right) = \frac{d|\mathbf{F}|}{dt}\; d\vec{A}_0\cdot\mathbf{F}^{-1} + |\mathbf{F}|d\vec{A}_0\cdot\frac{d\mathbf{F}^{-1}}{dt} =$$

$$|\mathbf{F}|\nabla\cdot\vec{v}\; d\vec{A}_0\cdot\mathbf{F}^{-1} - |\mathbf{F}|d\vec{A}_0\cdot\mathbf{F}^{-1}\cdot\frac{d\mathbf{F}}{dt}\cdot\mathbf{F}^{-1} =$$

$$\nabla\cdot\vec{v}\; d\vec{A} - |\mathbf{F}|d\vec{A}_0\cdot\mathbf{F}^{-1}\cdot\mathbf{L} = \nabla\cdot\vec{v}\; d\vec{A} - d\vec{A}\cdot\mathbf{L}$$

$$\boxed{\frac{d}{dt}\left(d\vec{A}\right) = \left(\nabla\cdot\vec{v}\right)d\vec{A} - d\vec{A}\cdot\mathbf{L}}$$