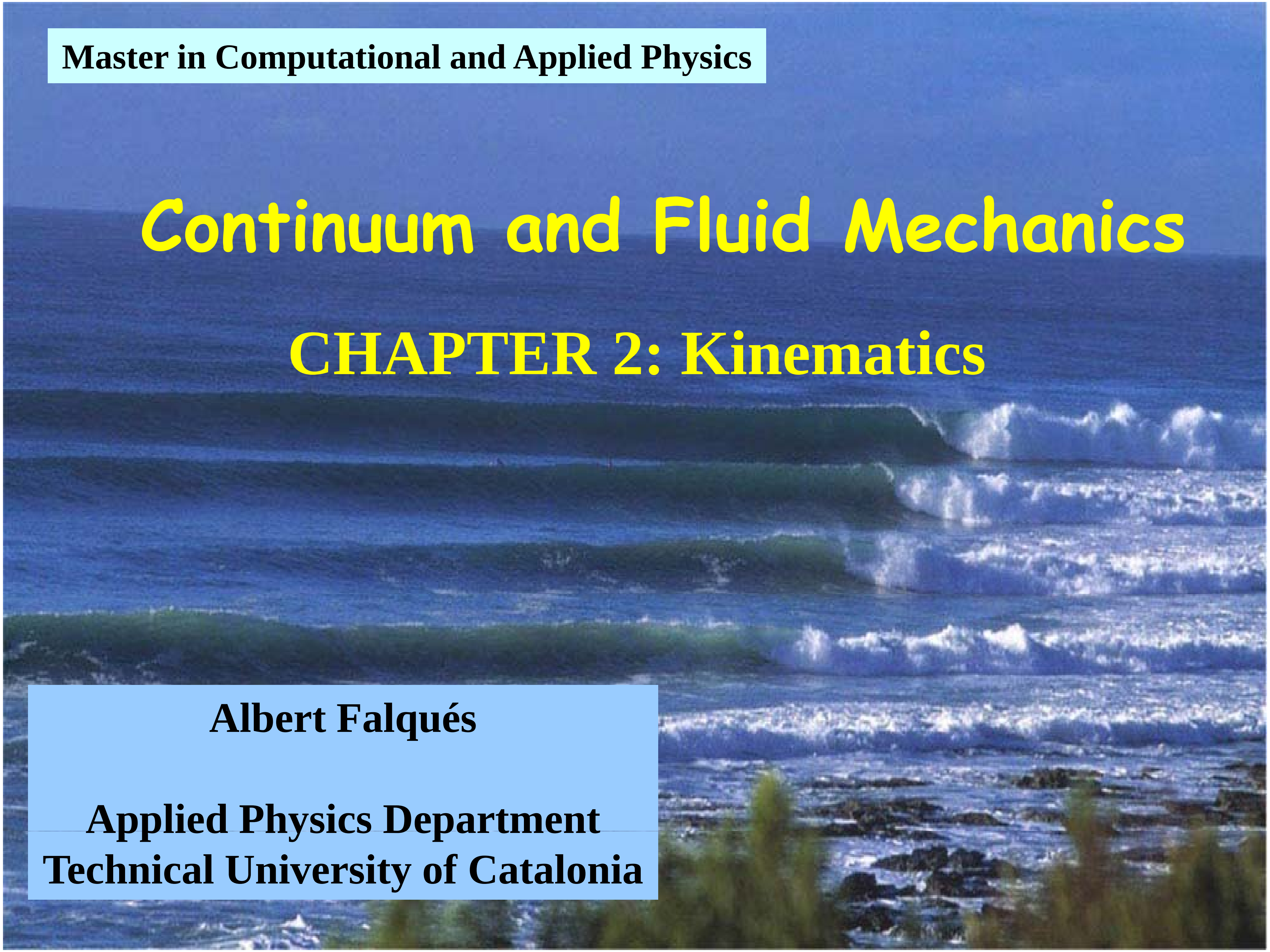


Master in Computational and Applied Physics

Continuum and Fluid Mechanics

CHAPTER 2: Kinematics

Albert Falqués

Applied Physics Department
Technical University of Catalonia

OUTLINE

1. Introduction. Continuum hypothesis.
2. Lagrangian description. Example of flow.
3. Eulerian description. Material derivative.
4. Streamlines, path lines and streamtubes
5. Material lines, surfaces and volumes

1. Introduction. Continuum hypothesis.

- ❑ **The molecular structure of matter is ignored and we deal with averages**
- ❑ **This can be done if:**

  L = length scale of motions of interest >> mean free path of molecules

For example:

- ➢ **standart air molecule:** mean free path of molecules ~ 10^{-8} m
- ➢ but, for instance, in the upper atmosphere this can be ~ 1 m

  $\Rightarrow$ standart continuum hypothesis is no longer valid unless we look at very large scales (L ~ 10 km ??)

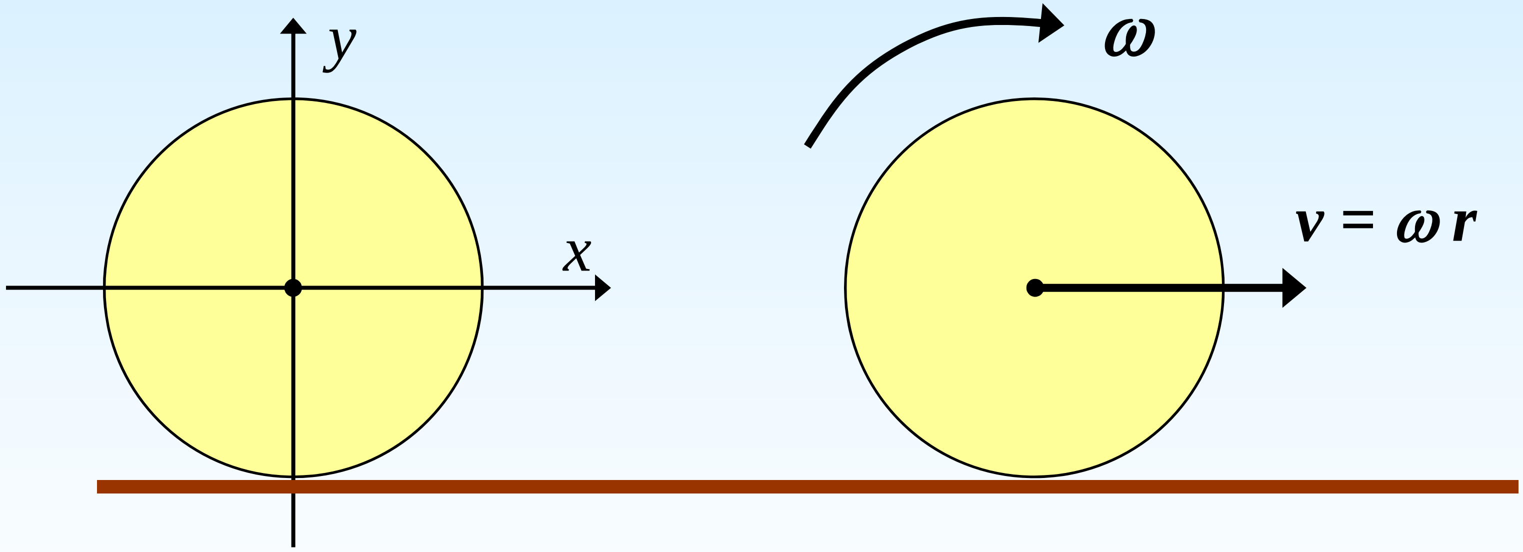
Therefore,

- ❑ we will assume that solids and fluids are continuus
- ❑ 'particles' or volume elements can theoretically be as small as we want ($dV \rightarrow 0$)
- ❑ but in practice, if we want to compare with nature, we know that there is always a lower bound which depends upon the material and may be of the order of L ~ 10^{-3} m

2. Lagrangian description.

Kinematics = description of motion without caring for its causes, i.e., forces

Example: rolling wheel without skiding

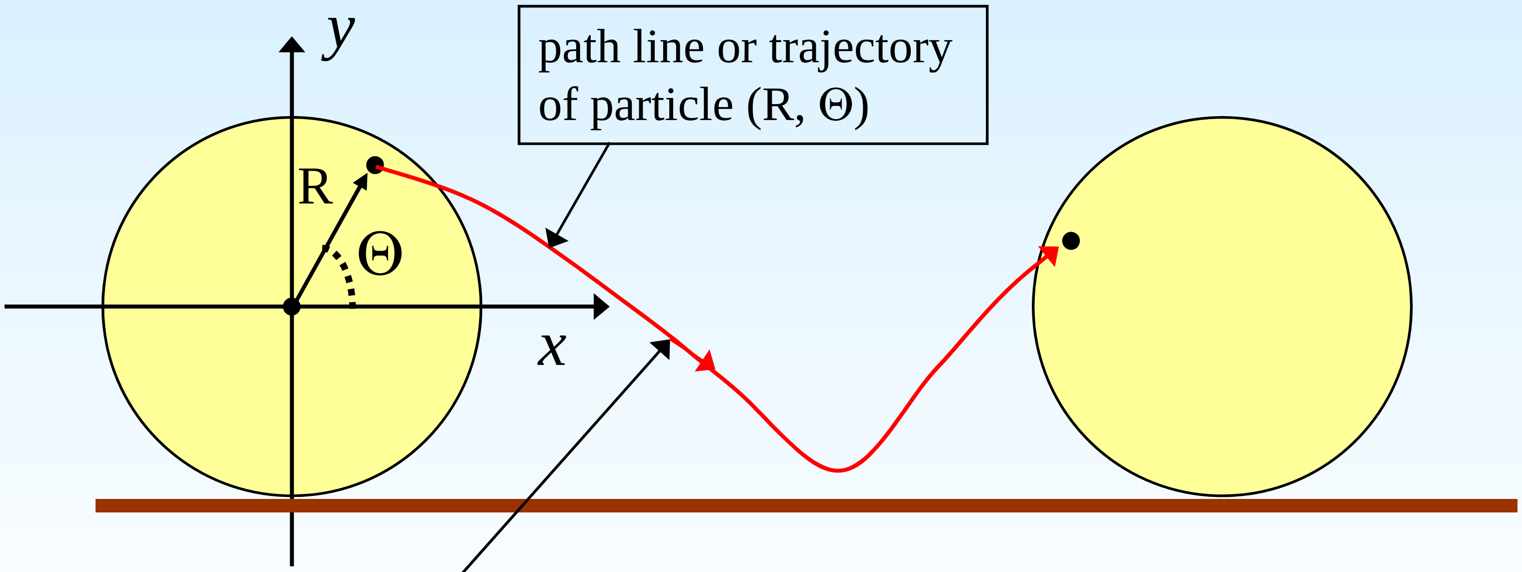


How to describe the motion of the particles?

Example: rolling wheel without skiding

How to describe the motion of the particles?

First: identify each particle (for instance with polar coordinates, R, Θ)



$$x = r\omega\ t + R\cos(\Theta - \omega\, t)$$

$$y = R\sin(\Theta - \omega\, t)$$

Lagrangian description of motion

$$\boxed{\vec{x} = F(\vec{X}, t)}$$

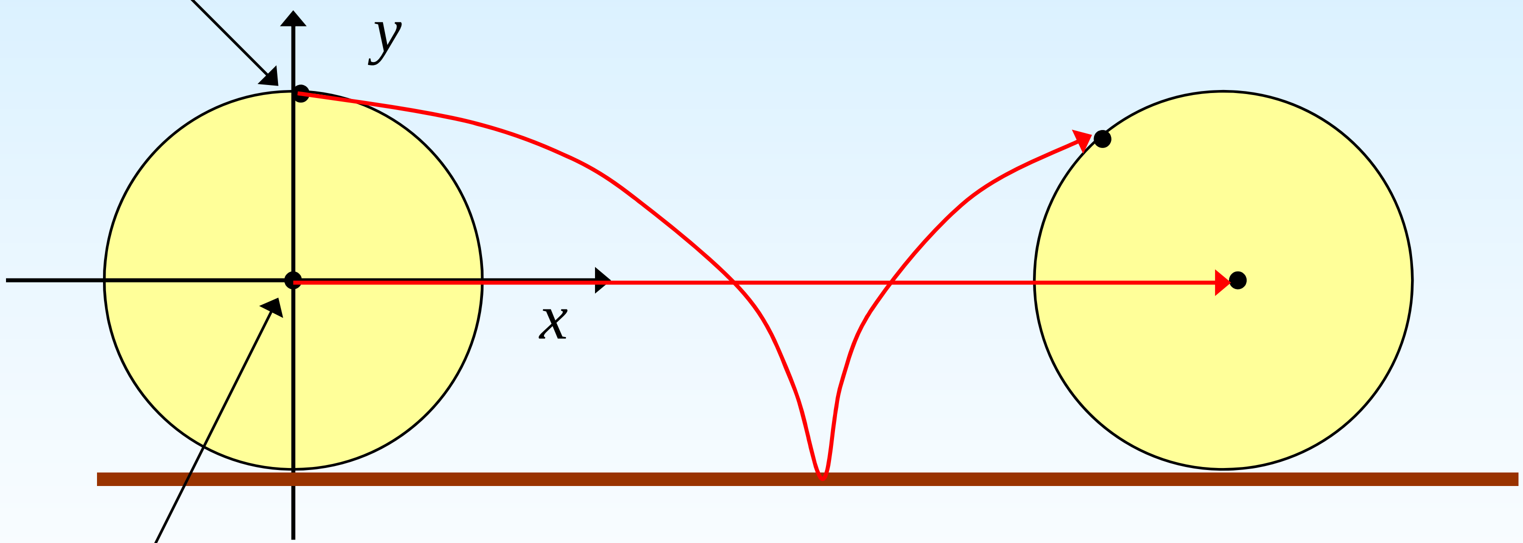
(x,y) (R, Θ)

Example: rolling wheel without skiding

particle
R = r, $\Theta = \pi/2$

$$x = r\omega\ t + R\cos(\Theta - \omega t)$$
$$y = R\sin(\Theta - \omega t)$$

$$x = r\left(\omega\ t + \cos(\pi/2 - \omega t)\right)$$
$$y = r\sin(\pi/2 - \omega t)$$



particle
R = 0, $\Theta = 0$

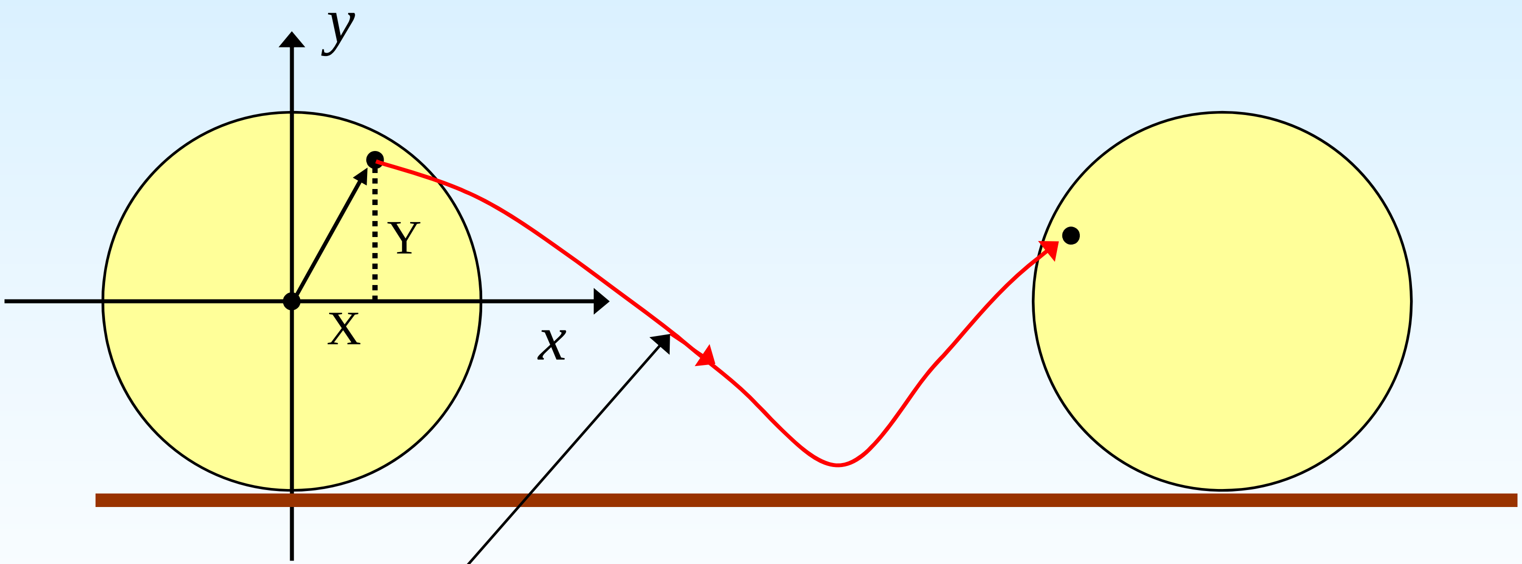
$$x = r\omega\ t + R\cos(\Theta - \omega t)$$
$$y = R\sin(\Theta - \omega t)$$

$$x = r\omega\ t$$
$$y = 0$$

Example: rolling wheel without skiding

(R, Θ) = **material or Lagrangian coordinates of the particle** = label for each particle

There are many options for the material coordinates.
For instance, cartesian coordinates



$$x = r\omega\, t + \sqrt{X^2 + Y^2}\cos\left(\tan^{-1}\left(\frac{Y}{X}\right) - \omega t\right)$$

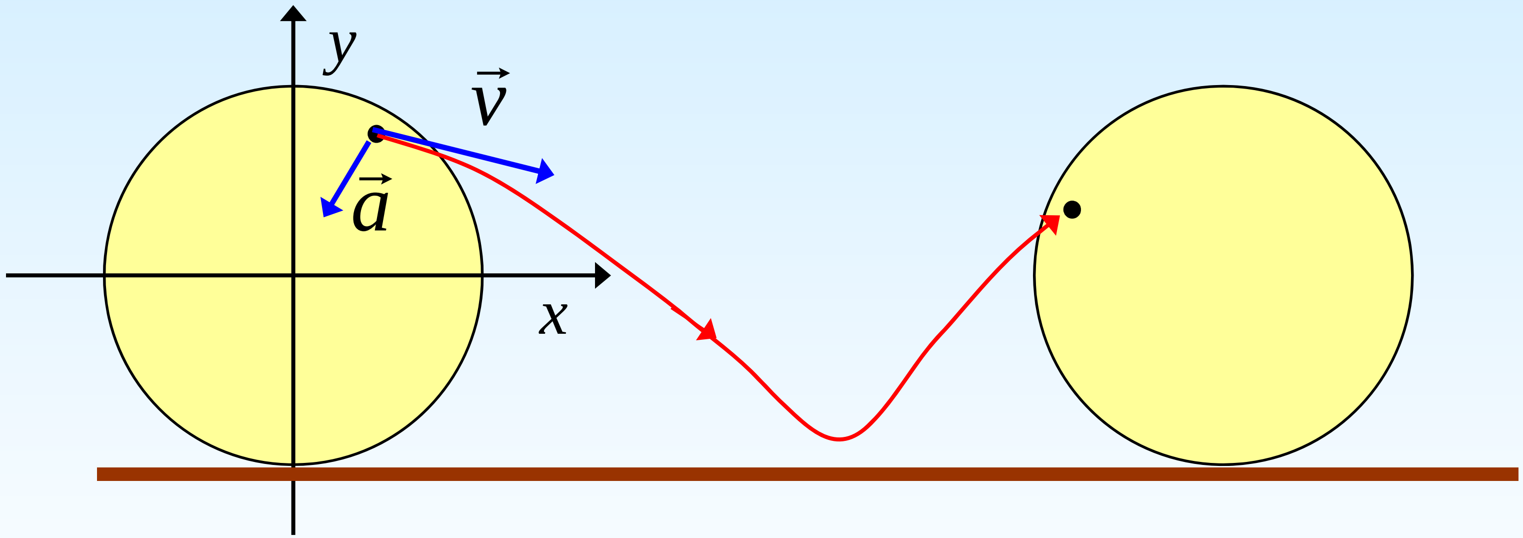
$$y = \sqrt{X^2 + Y^2}\sin\left(\tan^{-1}\left(\frac{Y}{X}\right) - \omega t\right)$$

$$\boxed{\vec{x} = F(\vec{X}, t)}$$

(x,y) (X,Y)

velocity and acceleration

$$\boxed{\vec{x} = F(\vec{X},t)} \rightarrow \boxed{\vec{v} = \frac{\partial F(\vec{X},t)}{\partial t}} \rightarrow \boxed{\vec{a} = \frac{\partial^2 F(\vec{X},t)}{\partial t^2}}$$

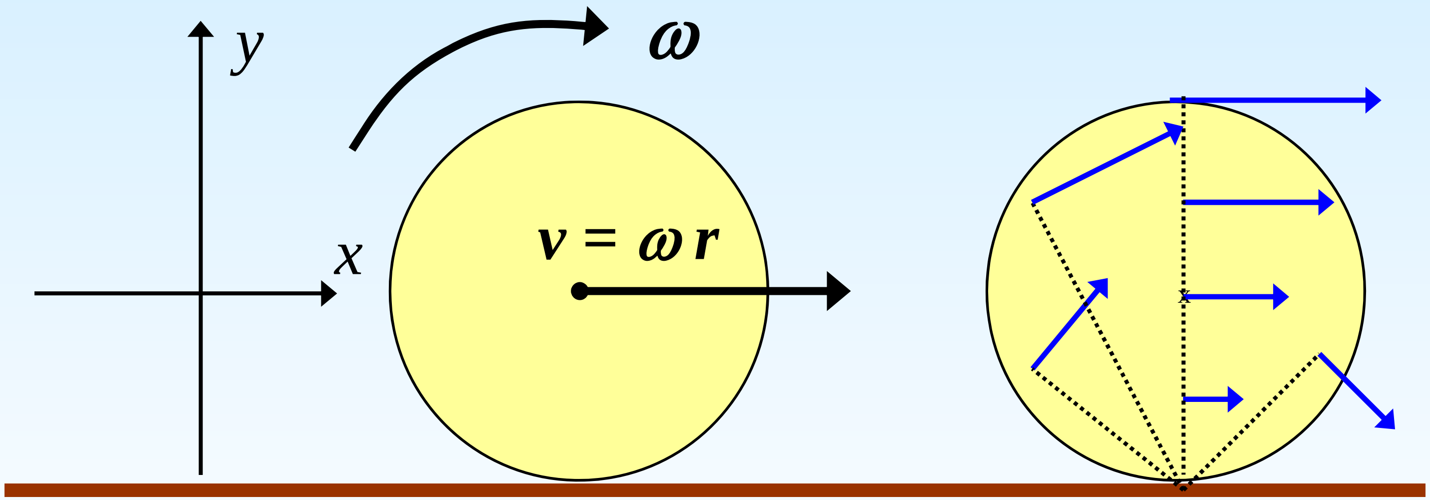


$$\begin{aligned} x &= r\omega\, t + R\cos(\Theta - \omega\, t) \\ y &= R\sin(\Theta - \omega\, t) \end{aligned} \rightarrow \begin{aligned} v_x &= r\omega + R\omega \sin(\Theta - \omega\, t) \\ v_y &= -R\omega\cos(\Theta - \omega\, t) \end{aligned} \rightarrow \begin{aligned} a_x &= -R\omega^2 \cos(\Theta - \omega\, t) \\ a_y &= -R\omega^2 \sin(\Theta - \omega\, t) \end{aligned}$$

3. Eulerian description.

Eulerian description → description of the velocity field: $\vec{v} = f(\vec{x}, t)$

Example: rolling wheel without skiding



$$\boxed{\begin{aligned} v_x &= \omega(r+y) \\ v_y &= -\omega(x - \omega r t) \end{aligned}}$$

if $(x - \omega r t)^2 + y^2 \le r^2$

$$\boxed{\begin{aligned} v_x &= 0 \\ v_y &= 0 \end{aligned}}$$

otherwise

Relationship between both descriptions

Lagrangian → Eulerian

$$\vec{x} = F(\vec{X},t)$$

$$\vec{v} = \frac{\partial F(\vec{X},t)}{\partial t}$$

$$\vec{X} = F^{-1}(\vec{x},t)$$

$$\vec{v} = f(\vec{x},t)$$

$$x = r\omega\, t + R\cos(\Theta - \omega\, t)$$
$$y = R\sin(\Theta - \omega\, t)$$

$$v_x = r\omega + R\omega\sin(\Theta - \omega\, t)$$
$$v_y = -R\omega\cos(\Theta - \omega\, t)$$

$$R = \sqrt{(x - \omega r t)^2 + y^2}$$
$$\Theta = \omega t + \tan^{-1}\left(\frac{y}{x - \omega r\, t}\right)$$

$$v_x = \omega\,(r + y)$$
$$v_y = -\omega\,(x - \omega\, r\, t)$$

Relationship between both descriptions

Eulerian → Lagrangian

$$\vec{v} = f(\vec{x}, t)$$

$$\frac{d\,\vec{x}(t)}{dt} = \vec{f}(\vec{x}(t), t)$$

system of O.D.E.

integration

$$\vec{x} = \vec{x}(t, \vec{x}_0)$$

$$\vec{x} = F(\vec{X}, t)$$

$\vec{x}_0 =$ initial conditions → $\vec{X}$

$$v_x = \omega(r + y)$$
$$v_y = -\omega(x - \omega r t)$$

$$\frac{dx}{dt} = \omega(r + y)$$
$$\frac{dy}{dt} = -\omega(x - \omega r t)$$

$$x = r\omega\, t + R\cos(\Theta - \omega t)$$
$$y = R\sin(\Theta - \omega t)$$

integration

Integration of the ODE

Particular solution of the inhomogeneous system:

$$\frac{dx}{dt} - \omega y = \omega r$$
$$\frac{dy}{dt} + \omega x = \omega^2 r t$$

$$\boxed{x = \omega r t \quad , \quad y = 0}$$

General solution of the homogeneous system:

$$\frac{dx}{dt} - \omega y = 0$$
$$\frac{dy}{dt} + \omega x = 0$$

$$(x, y) = e^{\sigma t}(a, b)$$

$$\sigma = \pm i \quad , \quad (a, b) = (1, \pm i)$$

taking real and imaginary parts:

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} \cos \omega t \\ -\sin \omega t \end{pmatrix} + B \begin{pmatrix} \sin \omega t \\ \cos \omega t \end{pmatrix} = R \begin{pmatrix} \cos(\Theta - \omega t) \\ \sin(\Theta - \omega t) \end{pmatrix}}$$

General solution:

$$\boxed{\begin{aligned} x &= r\omega \; t + R\cos(\Theta - \omega t) \\ y &= R\sin(\Theta - \omega t) \end{aligned}}$$

where R, Θ are the integration constants

Material derivative.

How can we compute the acceleration in eulerian description ?

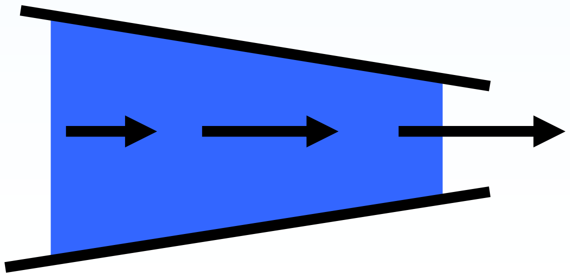
In **Lagrangian description**, the acceleration is simply the partial derivative with respect to time because this derivative is done by keeping X constant, i.e., the particle is fixed.

$$\vec{a} = \frac{\partial \vec{v}(\vec{X},t)}{\partial t} = \left.\frac{\partial \vec{v}(\vec{X},t)}{\partial t}\right|_{\vec{X}=const.}$$

In **Eulerian description**, the acceleration is not simply the partial derivative with respect to time because this derivative implies now keeping x constant, i.e., the point is fixed $\Rightarrow$ different particles go through this point as time goes.

$$\vec{a} \neq \frac{\partial \vec{v}(\vec{x},t)}{\partial t} = \left.\frac{\partial \vec{v}(\vec{x},t)}{\partial t}\right|_{\vec{x}=const.}$$

If not, for example, for a steady flow (∂ v /∂ t = 0), the acceleration would be always 0 !!



$$\frac{\partial \vec{v}(\vec{x},t)}{\partial t} = 0$$

but fluid particles have a forward acceleration !

How can we compute the acceleration in eulerian description ?

Taking into account that following the motion of a particle, $\vec{x} = F(\vec{X}, t)$

and applying the 'chain rule':

$$a_i = \frac{d}{dt} v_i(\vec{x}(\vec{X}, t), t) = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_k} \frac{\partial x_k(\vec{X}, t)}{\partial t} = \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}$$

$$\boxed{\vec{a} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}}$$

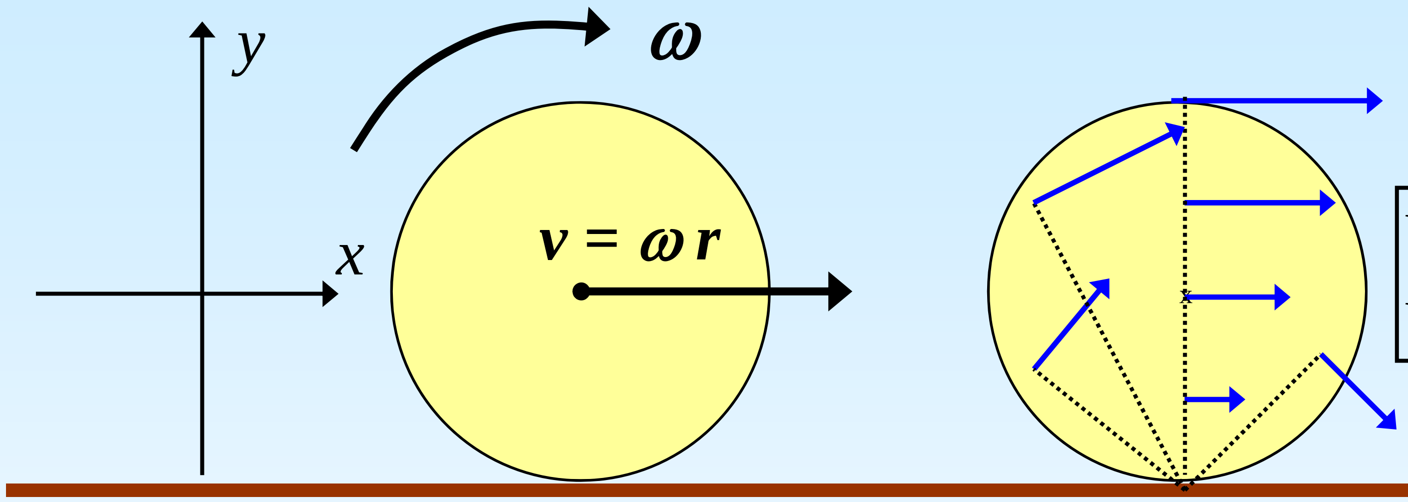
In general, given any field in Eulerian description $F(\vec{x}, t)$

the time derivative at a fixed point $\dfrac{\partial F(\vec{x}, t)}{\partial t}$ is called the **local derivative**

and the time derivative by following the material particles is called the **material derivative** and is computed as

$$\boxed{\frac{dF}{dt} = \frac{\partial F}{\partial t} + \vec{v} \cdot \nabla F}$$

Example: rolling wheel without skiding



$$v_x = \omega\,(r + y)$$
$$v_y = -\omega\,(x - \omega\, r\, t)$$

$$a_x = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = 0 + 0 - \omega^2 (x - \omega r t)$$

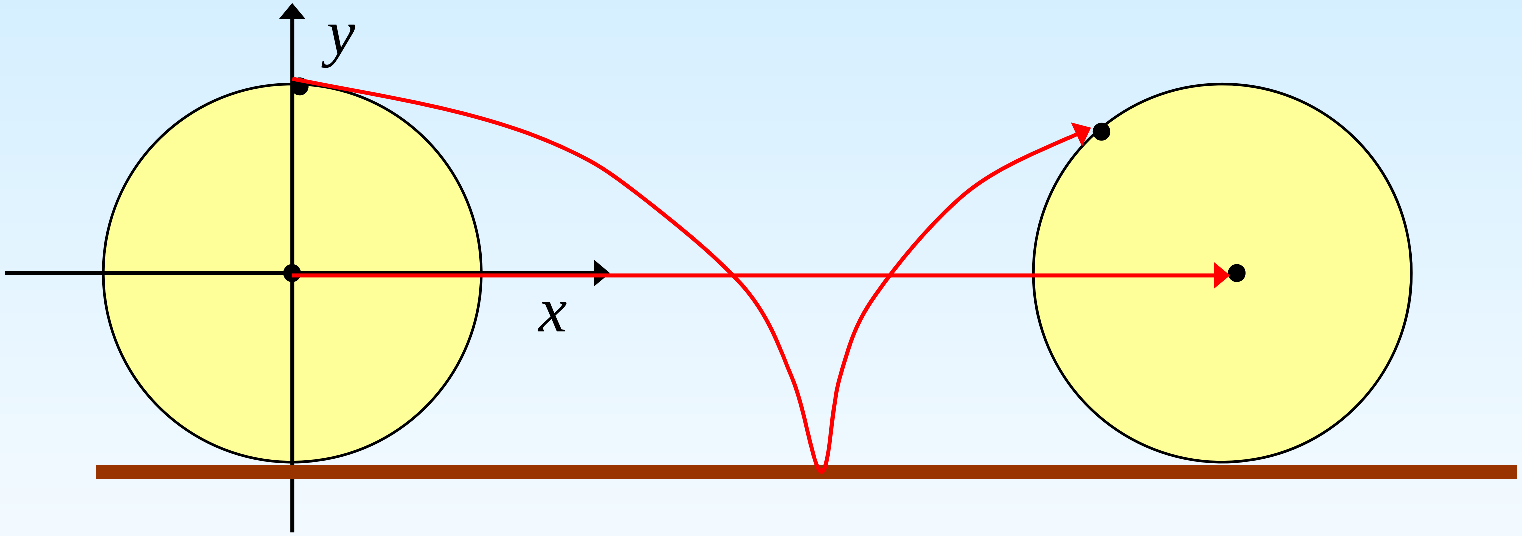
$$a_y = \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = \omega^2 r - \omega^2 (r + y) + 0$$

Lagrangian $a_x = -R\omega^2 \cos(\Theta - \omega\, t)$ $x = r\omega\; t + R\cos(\Theta - \omega\, t)$

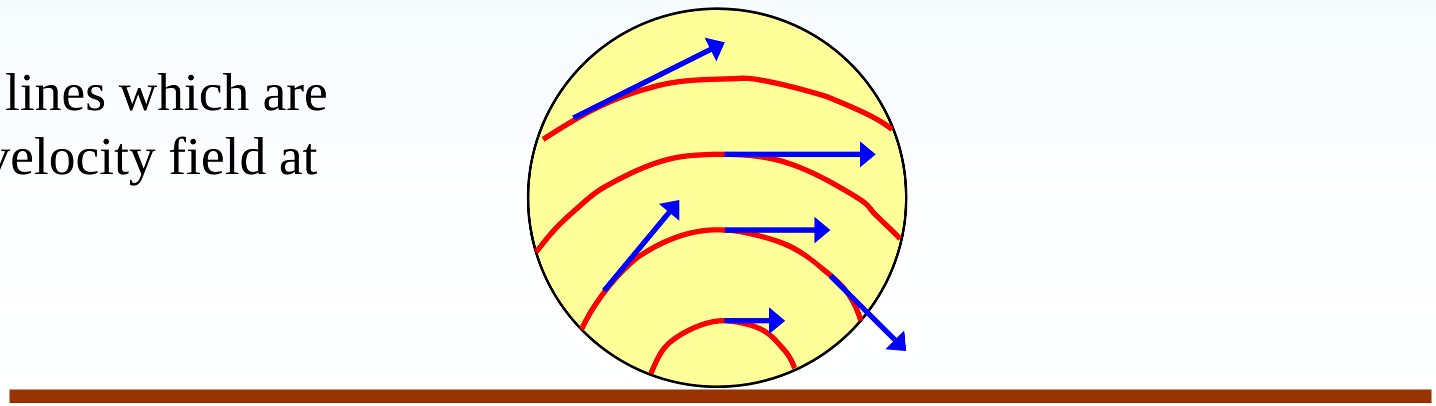
$a_y = -R\omega^2 \sin(\Theta - \omega\, t)$ $y = R\sin(\Theta - \omega\, t)$

4. Path lines, streamlines, stream tubes

Path lines = trajectories of the particles



Streamlines = lines which are tangent to the velocity field at a given time



Streamlines are found by seeking the field lines of the field $\vec{v}(\vec{x},t)$ at a given t
i.e., by solving the system of ODE in the parameter s:

$$\boxed{\frac{d\,\vec{x}}{ds}=\vec{v}(\vec{x}(s),t)} \longrightarrow \boxed{\vec{x}=\vec{x}(s,t,\vec{c})}$$

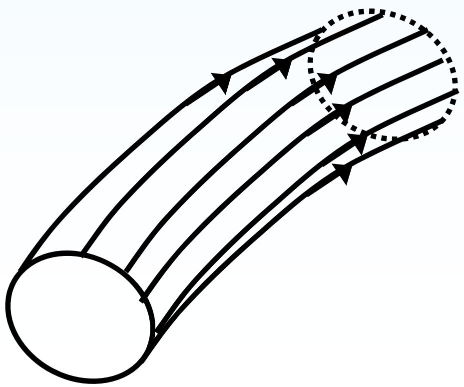
$\vec{c}=$ integration constants

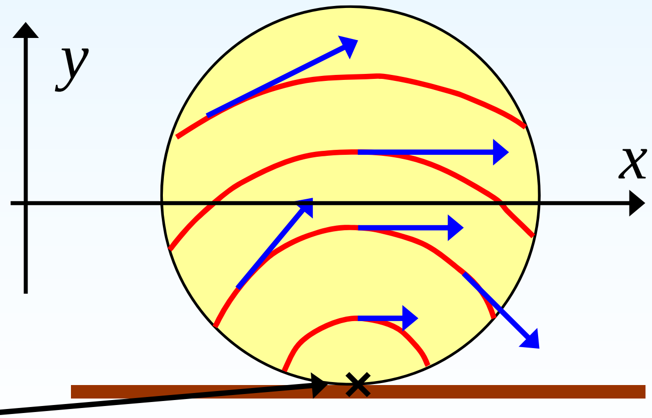
$$\boxed{\begin{aligned}\frac{dx}{ds}&=\omega\,(r+y(s))\\ \frac{dy}{ds}&=-\omega\,(x(s)-\omega\,r\,t)\end{aligned}}$$

Integration very similar to that done before, but now t is constant and s is the variable

$$\boxed{\begin{aligned}x&=r\omega\;t+R\cos(\Theta-\omega\,s)\\ y&=-r+R\sin(\Theta-\omega\,s)\end{aligned}}$$

Stream tube = streamlines passing through any closed curve





circumferences which are centered at (rωt,-r)

If the velocity field is steady, streamlines and path lines coincide

Streamlines: alternative integration

$$\boxed{\begin{aligned}\frac{dx}{ds} &= \omega\,(r + y(s))\\ \frac{dy}{ds} &= -\omega\,(x(s) - \omega\, r\, t)\end{aligned}}$$

$$\Downarrow$$

$$\frac{dy}{dx} = -\frac{x - r\omega t}{y + r} \quad \Rightarrow \quad \int \left(y + r\right) dy = -\int \left(x - r\omega t\right) dx \quad \Rightarrow$$

$$\left(y + r\right)^2 + \left(x - r\omega t\right)^2 = const.$$

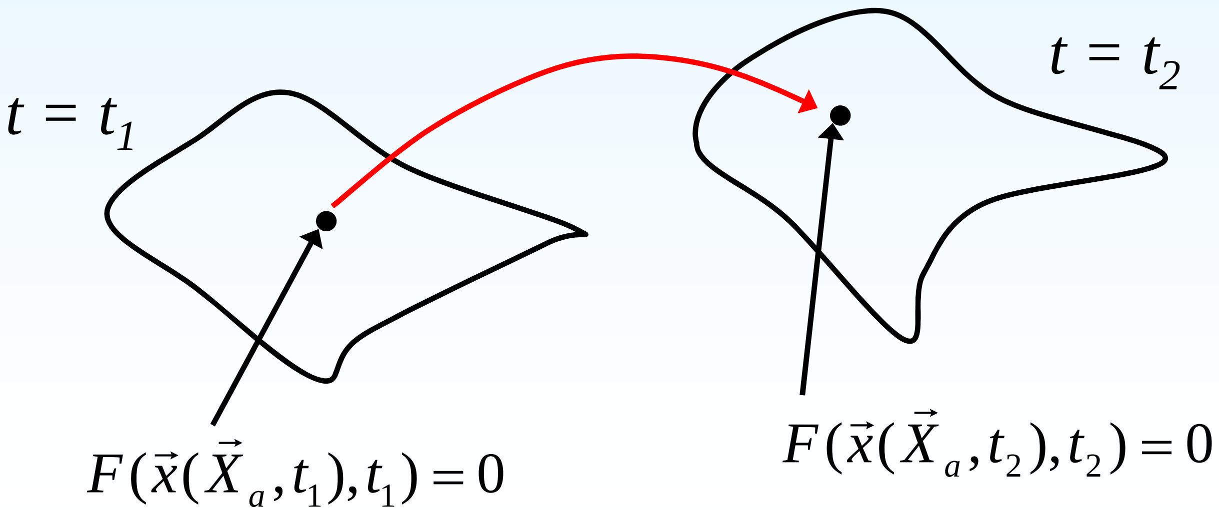
= circumferences which are centered at (rωt,-r)

5. Material volumes, surfaces and lines

= volumes, surfaces or lines which contain always the same particles

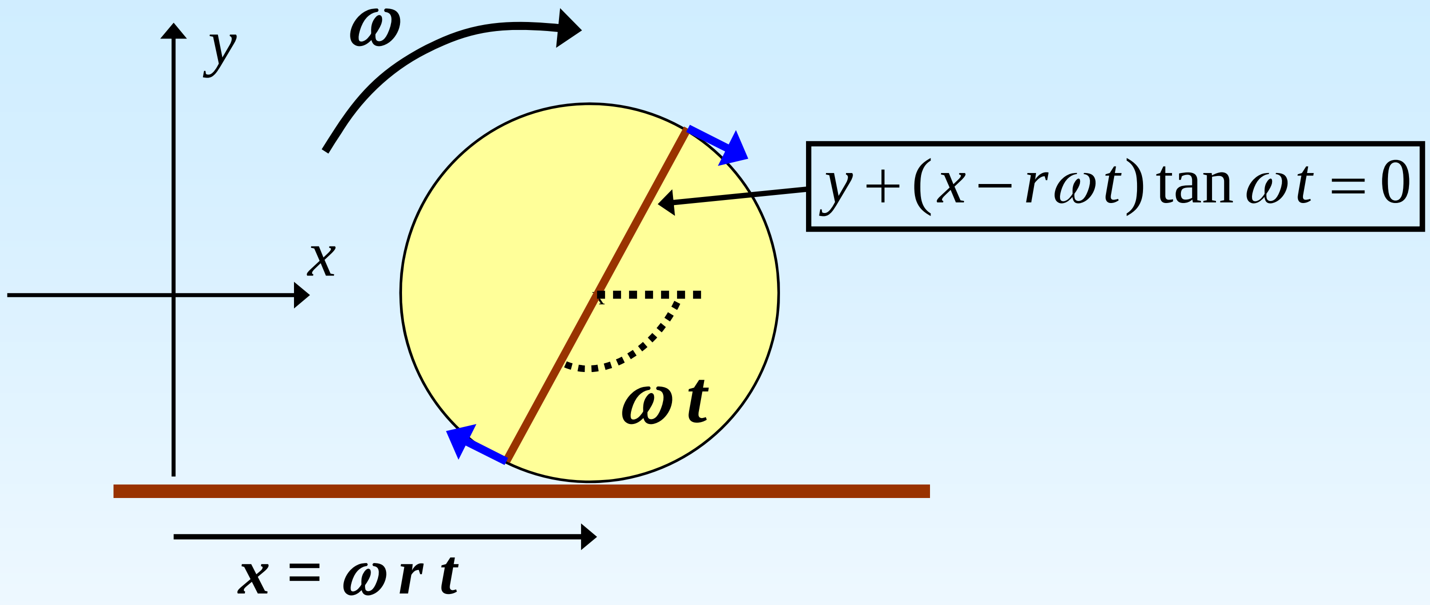
A mobile surface defined by $F(\vec{x},t)=0$ is a material surface if and only if:

$\frac{dF}{dt}=0$, that is, $\boxed{\frac{\partial F}{\partial t}+\vec{v}\cdot\nabla F=0}$



$$\Leftrightarrow \quad \frac{dF}{dt}=0$$

Example: rolling wheel without skiding. Diameter.



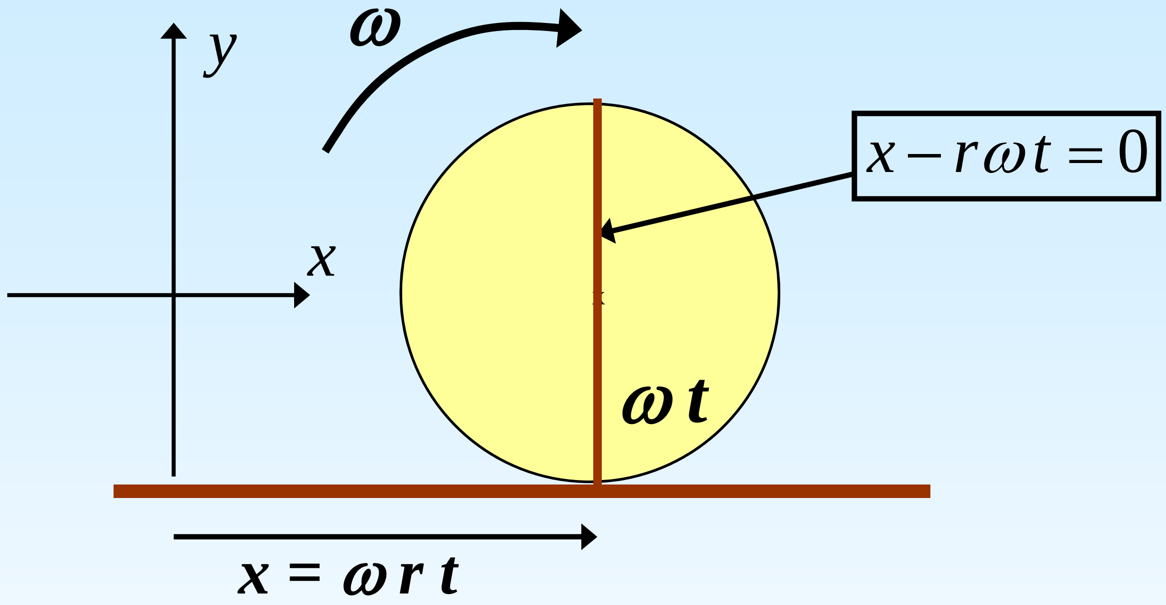
$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + v_x \frac{\partial F}{\partial x} + v_y \frac{\partial F}{\partial y} =$$

$$-\omega r \tan\omega t + \frac{\omega}{(\cos\omega t)^2}(x - r\omega t) + \omega(r + y)\tan\omega t - \omega(x - r\omega t) =$$

$$\omega\big(y + (x - r\omega t)\tan\omega t\big)\tan\omega t = 0$$

It's a material line!

Example: rolling wheel without skiding. Vertical line



$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + v_x \frac{\partial F}{\partial x} + v_y \frac{\partial F}{\partial y} =$$

$$-\omega r + \omega(r + y) - \omega(x - r\omega t)\cdot 0 = \omega\, y \neq 0$$ unless y = 0 (center of the wheel)

It's not a material line!