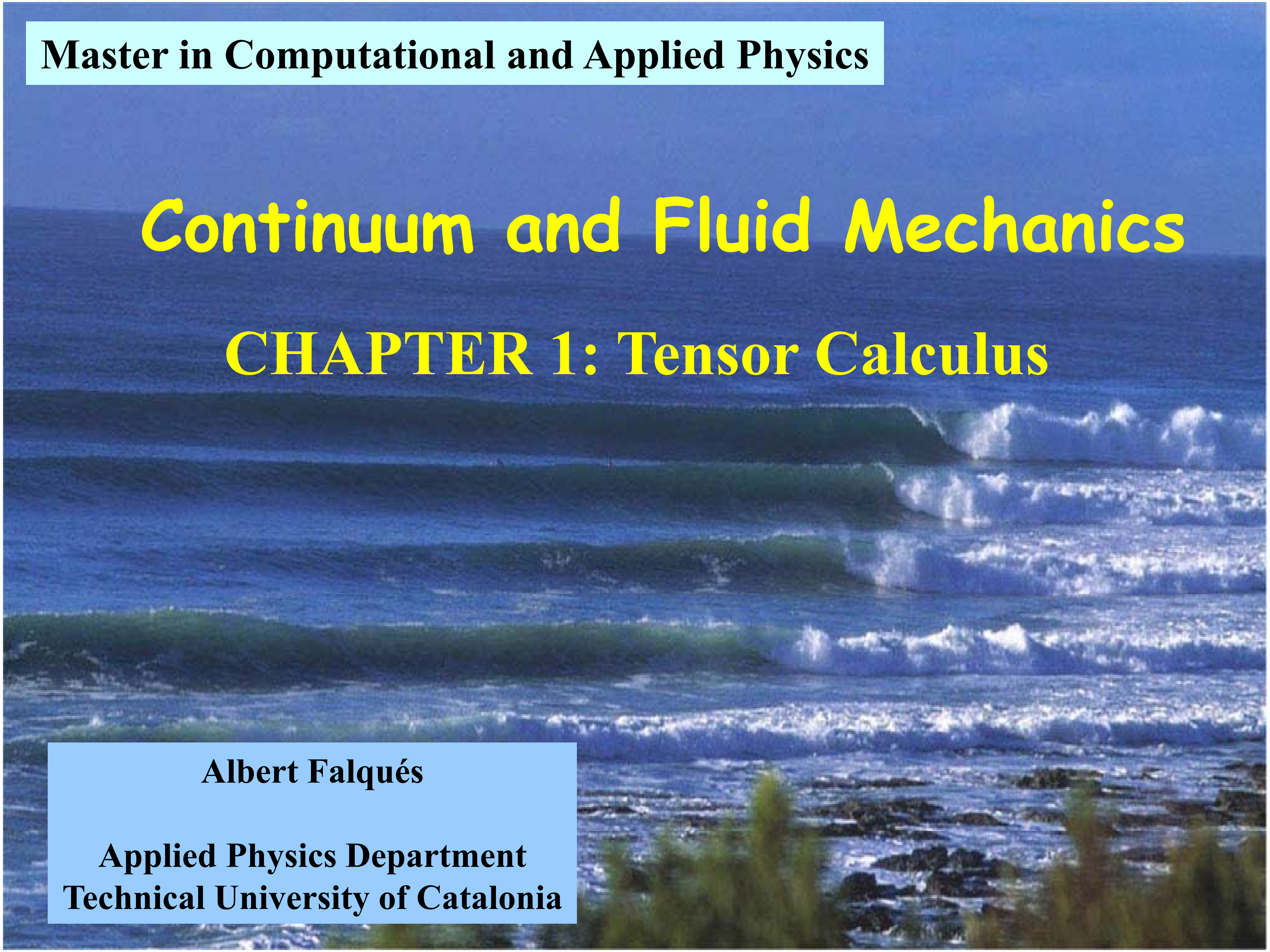

Master in Computational and Applied Physics

Continuum and Fluid Mechanics

CHAPTER 1: Tensor Calculus

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OUTLINE

1. Scalars, vectors and tensors. Cartesian basis. Rotation of axes.
2. Example: stress tensor.
3. Matrix algebra. Multiplication and contraction.
4. Isotropic tensors.
5. Algebraic properties of symmetric second order tensors. Eigenvalues and eigenvectors.
6. Gradient operator, divergence and curl. Gauss and Stokes theorems. Vector identities.

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1. Scalars, vectors and tensors. Cartesian basis. Rotation of axes.

- **Scalar:** quantity which is defined by its magnitude only.

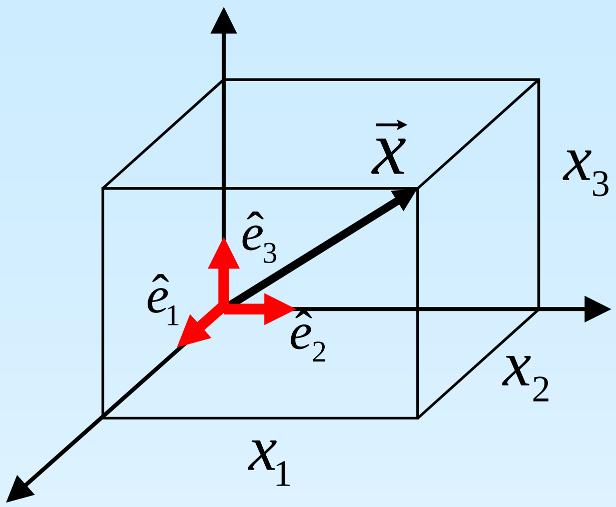
  Example: temperature, density, kinetic energy

- **Vector:** quantity which is defined by its magnitude and direction. Given a coordinate axis, a vector is defined by three components.

  Example: force, velocity, acceleration, momentum, torque

- **Tensor:** linear or multilinear map between vectors (or vectors and scalars). Given a coordinate axes system, a tensor is defined by, at least, nine components.

  Example: stress tensor (force per area unit across any section of a body at a point), strain tensor (deformation rate of a body in any direction at a point)

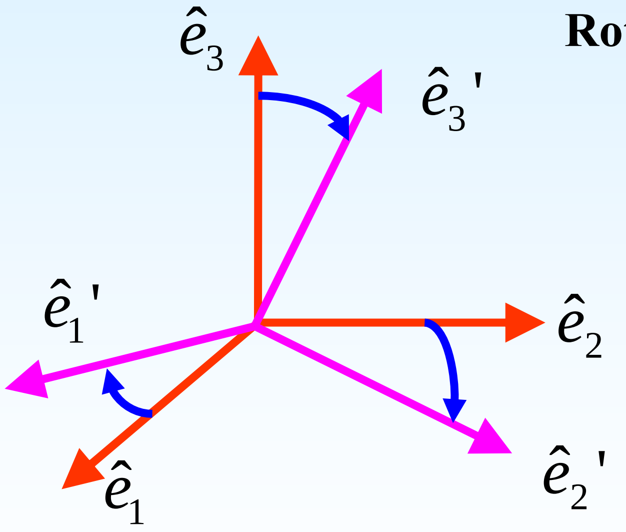


Cartesian axes

cartesian coordinates: $x_1\,,x_2\,,x_3$

ortonormal basis: $\hat{e}_1=(1,0,0)\,,\,\hat{e}_2=(0,1,0)\,,\,\hat{e}_3=(0,0,1)$

position vector: $\vec{x}=\sum_{i=1}^{3}x_i\hat{e}_i$



Rotation of axis

$$\begin{cases}\hat{e}_1' = C_{11}\hat{e}_1 + C_{21}\hat{e}_2 + C_{31}\hat{e}_3\\ \hat{e}_2' = C_{12}\hat{e}_1 + C_{22}\hat{e}_2 + C_{32}\hat{e}_3\\ \hat{e}_3' = C_{13}\hat{e}_1 + C_{23}\hat{e}_2 + C_{33}\hat{e}_3\end{cases}$$

$$\hat{e}_i' = \sum_{j=1}^{3} C_{ji}\hat{e}_j\ , \quad i=1,2,3$$

i = free index; j = dummy index

Summation convention over repeated indexes

- ❖ In any product of terms a repeated index is held to be summed over 1,2,3.
- ❖ An index not repeated in any product can take any of the values 1,2,3.

Examples:

$$\hat{e}_i{}' = C_{ji}\hat{e}_j \quad \Leftrightarrow \quad \hat{e}_i{}' = \sum_{j=1}^{3} C_{ji}\hat{e}_j \, , \quad i = 1,2,3$$

$$a_i b_{ijk} c_{kn} = R_{kjnk} \quad \Leftrightarrow \quad \sum_{i,k=1}^{3} a_i b_{ijk} c_{kn} = \sum_{k=1}^{3} R_{kjnk} \qquad j,n = 1,2,3$$

Warning: an index should not be repeated more than twice.

Example:

$a_{111} + a_{222} + a_{333} = 0$ should be abbreviated as $\sum_{i=1}^{3} a_{iii} = 0$

but not as $a_{iii} = 0$ that would mean: $a_{111} = a_{222} = a_{333} = 0$

Warning: same free indexes at both sides of an equation or for all the terms in a sum.

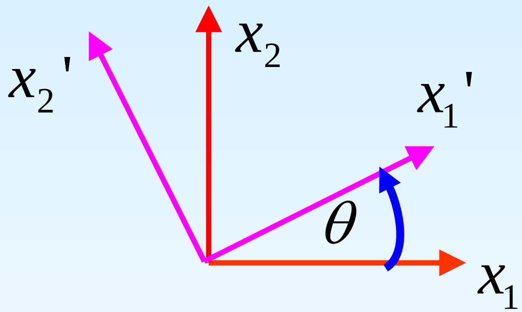
Example:

$$a_{ij} - b_i = c_{jk} a_{kn} \quad \Leftrightarrow \quad ???$$

Rotation matrix. Properties

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

$(C_{1i}, C_{2i}, C_{3i}) =$ components of e_i' on the basis e_1, e_2, e_3
= cosinus of the angles between the new axis x_i' and the old axes x_1', x_2', x_3'



$$C_{11} = \cos\theta\,, \quad C_{21} = \cos(\pi/2 - \theta)$$
$$C_{12} = \cos(\pi/2 + \theta), \quad C_{22} = \cos\theta$$

The transpose matrix:

$$\mathbf{C}^{\mathbf{T}} = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

verifies: $\mathbf{C}\cdot\mathbf{C}^{\mathbf{T}} = \mathbf{1}$ (and also $\mathbf{C}^{\mathbf{T}}\cdot\mathbf{C} = \mathbf{1}$)

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is the ‘orthogonality condition’ and implies that: $\mathbf{C}^{-1} = \mathbf{C}^{\mathbf{T}}$

Change of the coordinates of a point.

Given that the basis' vectors change according to: $\hat{e}_i' = C_{ji}\hat{e}_j$

How do the coordinates of any point change?

$$\vec{x} = x_j \hat{e}_j = x_k' \hat{e}_k' = x_k' C_{jk} \hat{e}_j \quad \Rightarrow$$

$$x_j = x_k' C_{jk} \quad \Rightarrow \quad x_j' = x_k \left\{C^{-1}\right\}_{jk}$$

and since $\mathbf{C}^{-1} = \mathbf{C}^{\mathbf{T}}$ the coordinates do change with the same matrix **C**

$$x_j' = C_{kj} x_k \quad \text{or} \quad \mathbf{x}' = \mathbf{C}^T \cdot \mathbf{x}$$

Orthogonal transformations ….. **det** C = ±1 …..

Formal definition of a vector and a scalar.

A physical quantity b is said to be a **scalar** if it is invariant under axis rotations, i.e.,

$$b' = b$$

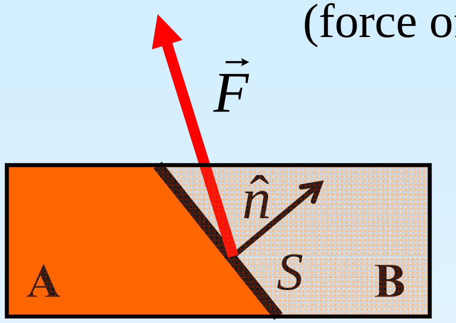
Three physical quantities (a_1, a_2, a_3) are said to define a **vector** $\vec{a}$
if they change as the coordinates of a point under axis rotations, i.e.,

$$a_i' = C_{ki} a_k$$

Tensors arise as linear and multilinear maps between vectors (or vectors and scalars).
Given a coordinate system, a tensor is defined by 3^n components where n is its **order**.
A scalar can be considered a tensor of order 0 and a vector a tensor of order 1.
A formal definition based on the transformation of their components will be given later on.
We will first introduce a particular tensor as an example: the **stress tensor**

2. Stress tensor

Force per unit area across a section of a body
(force of part B on part A: then the normal is outwards from A):

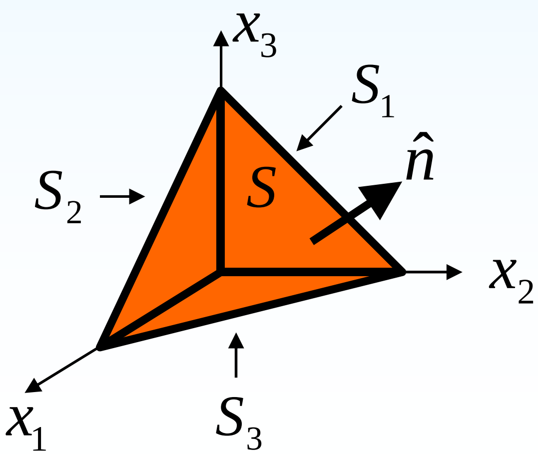


$$\vec{f} = \lim_{S \to 0} \frac{\vec{F}}{S}$$

$\vec{f}$ is a vector that depends on the orientation of the surface, i.e., another vector, $\hat{n}$

How is this relation ?

2on Newton Law applied to the tetrahedron when S → 0 :

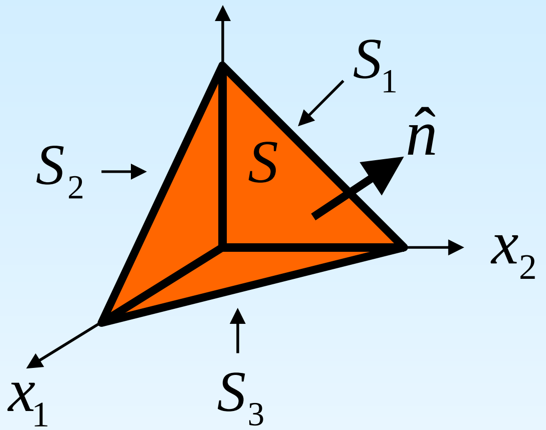


$$S\,\vec{f} + S_1\,\vec{f}_1 + S_2\,\vec{f}_2 + S_3\,\vec{f}_3 = \frac{1}{3}\rho S h \vec{a} \quad \Rightarrow$$

$$\vec{f} + \frac{S_1}{S}\,\vec{f}_1 + \frac{S_2}{S}\,\vec{f}_2 + \frac{S_3}{S}\,\vec{f}_3 = \frac{1}{3}\rho h \vec{a} \to 0$$

$\vec{f}_i$ = external force on S_i
so corresponding to a normal $\vec{n} = -\hat{e}_i$

$$\vec{f} = -\frac{S_1}{S}\vec{f}_1 - \frac{S_2}{S}\vec{f}_2 - \frac{S_3}{S}\vec{f}_3$$



- S_i are the projections of $S \Rightarrow \quad \frac{S_i}{S} = \cos\theta_i = n_i$

- $\vec{f}_i =$ external force on S_i
  so corresponding to a normal $\vec{n} = -\hat{e}_i$

$-\vec{f}_i = \vec{\tau}_i =$ force per unit area corresponding to the normals $\vec{n} = +\hat{e}_i$

$$\vec{f} = n_1\vec{\tau}_1 + n_2\vec{\tau}_2 + n_3\vec{\tau}_3$$

therefore, $\vec{f}$ is a linear function of $\hat{n}$

$$\vec{f} = \boldsymbol{\tau}(\hat{n})$$

$\boldsymbol{\tau} =$ **stress tensor**

Matrix expression

$$\vec{f} = n_1\vec{\tau}_1 + n_2\vec{\tau}_2 + n_3\vec{\tau}_3$$

$(\tau_{i1}, \tau_{i2}, \tau_{i3}) =$ components of $\vec{\tau}_i$

Symmetry of the stress tensor !!

$$\tau_{ij} = \tau_{ji}$$

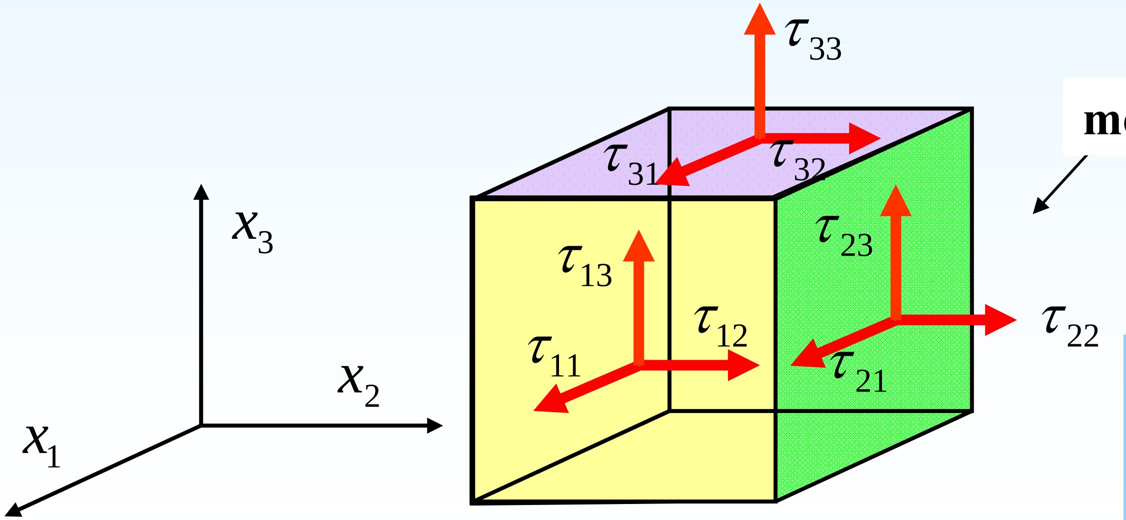
$$(f_1, f_2, f_3) = (n_1, n_2, n_3)\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

stress tensor matrix

$\tau_{ij} =$ components of the stress tensor

3^2 components ⇒ second order tensor

meaning of the components



$\tau_{11}, \tau_{22}, \tau_{33} =$ **normal stresses**

$\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \ldots =$ **shear stresses**

$\tau_{11} > 0$: pulling ; $\tau_{11} < 0$: pushing (compression)

How do the components of the stress tensor change under axis rotation?

$$f_i = n_j \tau_{ji}$$

$$f_i' = C_{ji} f_j \,,\; n_i' = C_{ji} n_j$$

$$\Updownarrow$$

$$f_i = C_{ij} f_j' \,,\; n_i = C_{ij} n_j'$$

$$C_{ij} f_j' = C_{jk} n_k' \tau_{ji}$$

$$f_n' = \left\{\mathbf{C}^{-1}\right\}_{ni} \left(C_{jk} n_k' \tau_{ji}\right) = \left\{\mathbf{C}^{\mathrm{T}}\right\}_{ni} \left(C_{jk} n_k' \tau_{ji}\right) = C_{in} C_{jk} n_k' \tau_{ji}$$

$$f_n' = n_k' \left(C_{jk} \tau_{ji} C_{in}\right) = n_k' \tau_{kn}'$$

$$\tau_{kn}' = C_{jk} C_{in} \tau_{ji} \quad \text{or} \quad \boldsymbol{\tau}' = \mathbf{C}^{\mathrm{T}} \cdot \boldsymbol{\tau} \cdot \mathbf{C}$$

Formal definition of a tensor.

Nine physical quantities $\{T_{ij}\}$ are said to define a **second order tensor** $\mathbf{T}$ if they change as the stress tensor under axis rotations, i.e.,

$$\boxed{T_{kn}' = C_{jk} C_{in} T_{ji}}$$

3^n physical quantities $\{T_{i_1 \ldots i_n}\}$ are said to define a **n-order tensor** $\mathbf{T}$ if they change under axis rotations as,

$$\boxed{T_{i_1 \ldots i_n}' = C_{j_1 i_1} C_{j_2 i_2} \ldots C_{j_n i_n} T_{j_1 \ldots j_n}}$$

3. Matrix algebra. Multiplication and contraction.

Matrix algebra can be expressed in several ways:

a) Matricial
b) Components
c) Intrinsic or symbolic

Example:

a) $$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) $C_{ik} C_{jk} = \delta_{ij} \longleftarrow$ entries of the identity matrix: $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

c) $\mathbf{C} \cdot \mathbf{C}^{\mathrm{T}} = \mathbf{1}$

Tensor multiplication of vectors and tensors.

Tensor product $\otimes$ of two vectors

Any pair of vectors, $\vec{a}, \vec{b}$, define a linear map $\vec{a} \otimes \vec{b}$ between vectors:

$$\vec{v} \xrightarrow{\vec{a} \otimes \vec{b}} (\vec{a} \otimes \vec{b})\vec{v} \equiv (\vec{b} \cdot \vec{v})\,\vec{a}$$

The components of tensor $\vec{a} \otimes \vec{b}$ are simply $a_i b_j$

Tensor product of vectors and tensors

In general, by multiplying the components of a n-order tensor and those of an m–order tensor, an (n+m)-order tensor is defined through its components.

Examples:

- ❑ From a_i and T_{ij} the third order tensor: $Q_{ijk}=a_i T_{jk}$ may be defined
- ❑ From two 2nd order tensors, A_{ij} , B_{ij}, the 4th order tensor $P_{ijkl}=A_{ij}B_{kl}$ may be defined

Quotient rule.

➢ Let $A_{i_1 \ldots i_n}$, $X_{j_1 \ldots j_m}$ be 3^n+3^m physical quantities and let $B_{k_1 \ldots k_{n+m}}$ be their product

$$B_{i_1 \ldots i_n j_1 \ldots j_m} = A_{i_1 \ldots i_n} X_{j_1 \ldots j_m}$$

➢ Assume that **A** and **B** are tensors

Then, $X_{j_1 \ldots j_m}$ are the components of an m-order tensor

Namely;

$$\underbrace{A_{i_1 \ldots i_n}}_{\textbf{tensor}} X_{j_1 \ldots j_m} = \underbrace{B_{i_1 \ldots i_n j_1 \ldots j_m}}_{\textbf{tensor}} \quad \Rightarrow \quad \mathbf{X} = \textbf{tensor}$$

Contraction.

Given a n-order tensor, **contraction** is a procedure to obtain a lower order tensor. Two indices are equated and a summation is performed over these repeated indices.

Examples:

- From the components of a 2nd order tensor, T_{ij} , the only possible contraction is

$$T_{ii} = T_{11} + T_{22} + T_{33}$$

  which is a scalar

- From a third order tensor, T_{ijk} , three different contractions are possible but all of them give a vector:

$$a_i = T_{ijj} \quad , \quad b_j = T_{iji} \quad , \quad c_k = T_{iik}$$

- From two 2nd order tensors, A_{ij} , B_{ij} , the four order tensor $Q_{ijkl} = A_{ij} B_{kl}$ may be defined by multiplying. Four 2nd order tensors may be then obtained by contracting:

$$A_{ij} B_{jl} = \left(\mathbf{A} \cdot \mathbf{B}\right)_{il} \qquad A_{ij} B_{kj} = \left(\mathbf{A} \cdot \mathbf{B}^{\mathbf{T}}\right)_{ik} \qquad A_{ij} B_{ik} = \left(\mathbf{A}^{\mathbf{T}} \cdot \mathbf{B}\right)_{jk} \qquad A_{ij} B_{ki} = \left(\mathbf{B} \cdot \mathbf{A}\right)_{kj}$$

  The components of all of them can be computed from standard matrix product

- A second contraction may be applied to these 2nd order tensors and a scalar is obtained in two possible ways $A_{ij} B_{ji}$ or $A_{ij} B_{ij}$. They are indicated by **A**:**B** or by **A**:**B**$^{\mathbf{T}}$

4. Isotropic tensors.

An **isotropic tensor** is one whose components are invariant under axes rotations. i.e.

$$T'_{i_1 i_2 \dots i_n} = T_{i_1 i_2 \dots i_n} \quad \forall\, i_1 i_2 \dots i_n = 1,2,3$$

Isotropic tensor are associated to geometric invariance under rotation.

0-order isotropic tensors: all 0-order tensors are isotropic as they are scalars

1st-order isotropic tensors: there are none

2nd-order isotropic tensors: only one, the **identity tensor**, whose components are given by the **Kronecker delta** in any basis:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \qquad \boldsymbol{\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Very common use of δ_{ij} is that in any expression where it appears with index ‘i’ being contracted, it can be dropped out by substituting ‘i’ by ‘j’ in the expression.

Examples:

$$\delta_{ij} A_{nim} = A_{njm} \quad , \quad \delta_{ij}\delta_{kn} B_{ij} C_{kl} = B_{ii} C_{nl}$$

3rd-order isotropic tensors: only one, the **alternating tensor**, whose components are

$$\varepsilon_{ijk} = \begin{cases} 1 \text{ if } \; i,j,k = \textit{even permutation of 123 (i.e., 123 or 231 or 312)} \\ 0 \text{ if } \; \textit{two or three indices are equal} \\ -1 \text{ if } \; i,j,k = \textit{odd permutation of 123 (i.e., 132 or 213 or 321)} \end{cases}$$

Properties:

- $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$ even permutations

  $\varepsilon_{ijk} = -\varepsilon_{ikj}$, $\varepsilon_{ijk} = -\varepsilon_{kji}$, $\varepsilon_{ijk} = -\varepsilon_{jik}$ odd permutations

- $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

- The cross product of two vectors, reads $\vec{a} \times \vec{b} = \varepsilon_{ijk} a_j b_k \hat{e}_i$

- Given any matrix **A**, det **A** = $\varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$

4th-order isotropic tensors: there are three and their linear combinations. So the most general is:

$$\lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) + \nu(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})$$

where λ, μ, ν are arbitrary numbers.

5. Symmetric and antisymmetric second order tensors. Eigenvalues and eigenvectors

A 2nd order tensor **B** is called

- ❖ **symmetric** if $B_{ij} = B_{ji}$
- ❖ **antisymmetric** if $B_{ij} = -B_{ji}$

Properties:

- ❑ Any 2nd order tensor can be represented as the sum of a symmetric part and an antisymmetric one:

$$B_{ij} = \underbrace{\frac{1}{2}\left(B_{ij} + B_{ji}\right)}_{\textbf{symmetric}} + \underbrace{\frac{1}{2}\left(B_{ij} - B_{ji}\right)}_{\textbf{antisymmetric}}$$

- ❑ If A_{ij} is antisymmetric and B_{ij} is symmetric, then $A_{ij}B_{ij} = 0$

- ❑ A symmetric tensor has only 6 independent components

- ❑ An antisymmetric tensor has zero diagonal components and has only 3 independent components. These 3 components are associated with a vector ➞

Properties (continued):

- Every antisymmetric tensor can be associated with a vector and vice versa

$$\mathbf{R} \longrightarrow \omega_k = \frac{1}{2}\varepsilon_{kij}R_{ij}$$

$$\vec{\omega} \longrightarrow R_{ij} = \varepsilon_{ijk}\omega_k$$

$$\mathbf{R} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

Eigenvectors and eigenvalues

Eigenproblem:

Given a 2nd order tensor, **A**, are there vectors $\vec{v}$ and numbers λ , such that

$$\boxed{\mathbf{A}\cdot\vec{v} = \lambda\,\vec{v}} \quad \textbf{?}$$

$\vec{v}$ = **eigenvector ,** λ = **eigenvalue**

- λ are the solutions of the third order equation (characteristic equation)

$$\det\left[\mathbf{A}-\lambda\boldsymbol{\delta}\right]=0$$

- for each λ, $\vec{v}$ is a solution of $\left(\mathbf{A}-\lambda\boldsymbol{\delta}\right)\cdot\vec{v}=0$
- the eigenvalues and the direction of the eigenvectors are invariant under axes rotations
- the characteristic equation reads: $\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$
  where

$$I_1 = A_{ii} = \lambda_1+\lambda_2+\lambda_3$$

$$I_2 = \frac{1}{2}\left(A_{ij}A_{ij} - \left(A_{ii}\right)^2\right) = \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1$$

$$I_3 = \det\mathbf{A} = \lambda_1\lambda_2\lambda_3$$

are invariant under axes rotations

If **A** is symmetric:

- there are three eigenvalues that are real, λ_1, λ_2, λ_3 (not necessarily distinct)
- associated to them, there are three eigenvectors which are mutually orthogonal
- if the coordinate system is rotated as to coincide with the eigenvectors, matrix **A** takes a diagonal form:

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

- extremal property: the components A_{ij} change with the coordinate axes, but the diagonal elements cannot be larger than the largest λ and smaller than the smallest λ

6. Gradient operator, divergence and curl. Gauss and Stokes theorems. Vector identities

Given a scalar field, $\phi(x_1,x_2,x_3)$, its **gradient** is defined by

$$\nabla\phi = \frac{\partial\phi}{\partial x_i}\hat{e}_i$$

- $\nabla\phi$ is a vector
- $\nabla\phi$ gives the direction of maximum increase of ϕ
- $|\nabla\phi|$ = magnitude of the derivative of ϕ along this direction
- $\nabla\phi$ is perpendicular to the surfaces of $\phi(x_1,x_2,x_3)$ = const.
- The derivative of ϕ along a direction associated to $\hat{n}$ is $\frac{\partial\phi}{\partial n} = \hat{n}\cdot\nabla\phi$

Since $\nabla\phi$ is a vector and ϕ is a scalar, the operator ∇ itself can be considered a vector (quotient rule):

$$\vec{\nabla} = \frac{\partial}{\partial x_i}\hat{e}_i \equiv \partial_i\hat{e}_i = \textbf{ nabla operator}$$

(we will omit the arrow for simplicity, $\vec{\nabla} = \nabla$)

The **gradient of a vector field** is a second order tensor:

$$\left(\nabla \vec{v}\right)_{ij} = \partial_i v_j$$

such that multiplied by a unit vector $\hat{n}$ gives the derivative of such a vector along the direction of $\hat{n}$:

$$\frac{d\,\vec{v}}{dn} = \hat{n} \cdot \nabla \vec{v} = \left(n_i \partial_i v_j\right) \hat{e}_j$$

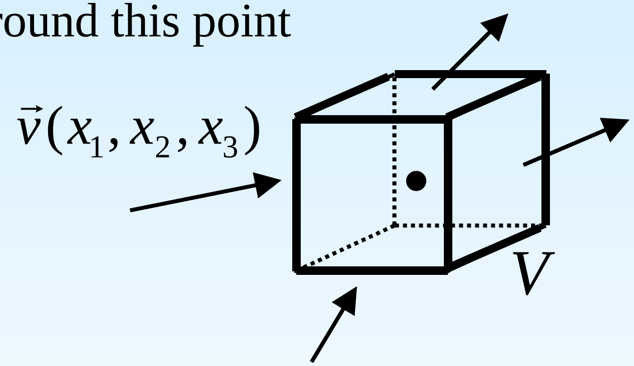
Similarly, the gradient of a tensor field of order tensor n is a n+1 order tensor with similar meaning.

The **divergence of a vector field** is defined as the contraction of its gradient:

$$\nabla \cdot \vec{v} = \partial_i v_i = \frac{\partial v_i}{\partial x_i}$$

(or dot product of ∇ and $\vec{v}$)

The divergence of a vector field at a point is associated to its flux going in or out a small surface around this point



$$\nabla \cdot \vec{v} = \lim_{V \to 0} \frac{1}{V} \iint \vec{v} \cdot \hat{n}\, dS$$

The divergence of a tensor field of n-order may be defined trough a contraction of its gradient and it is a (n-1)-order tensor.
There are however several options depending on which contraction is performed.
For a 2nd order tensor:

$$(\nabla \mathbf{T})_{ijk} = \partial_i T_{jk}$$

$$\partial_i T_{ik} = (\nabla \cdot \mathbf{T})_k$$

$$\partial_i T_{ji} = (\mathbf{T} \cdot \nabla)_j$$

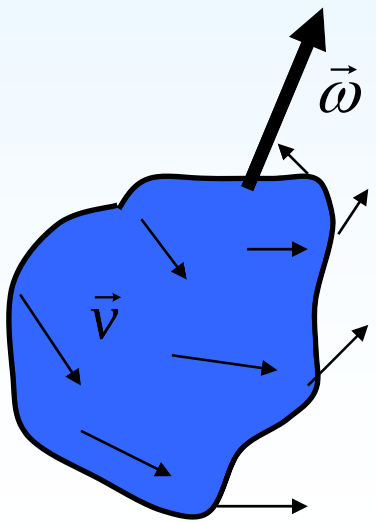
The **curl of a vector field** is defined as the cross product of vector ∇ and $\vec{v}$

$$\nabla \times \vec{v} = \varepsilon_{ijk} \partial_j v_k \hat{e}_i$$

The curl of a vector field is associated with the rotation of the vector field.
Example: velocity field of a rigid body:

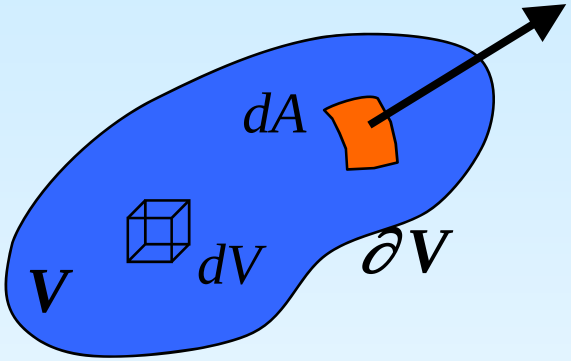
$$\vec{v} = \vec{\omega} \times \vec{x} \quad \Rightarrow \quad \nabla \times \vec{v} = 2\vec{\omega}$$

So, the curl of the velocity field is proportional to the angular velocity of the body



Gauss Theorem (divergence theorem)

$$\overrightarrow{dA} = \hat{n}\, dA$$



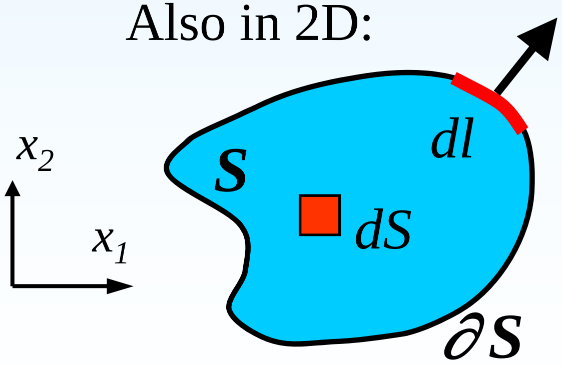
$$\iiint_V \nabla \cdot \vec{u}\; dV = \oiint_{\partial V} \vec{u} \cdot \hat{n}\; dA$$

$$\iiint_V \frac{\partial u_i}{\partial x_i}\, dV = \oiint_{\partial V} u_i n_i\; dA$$

V = volume
∂V = surface, boundary of V

Also in 2D:

$$\overrightarrow{dl} = \hat{n}\, dl$$

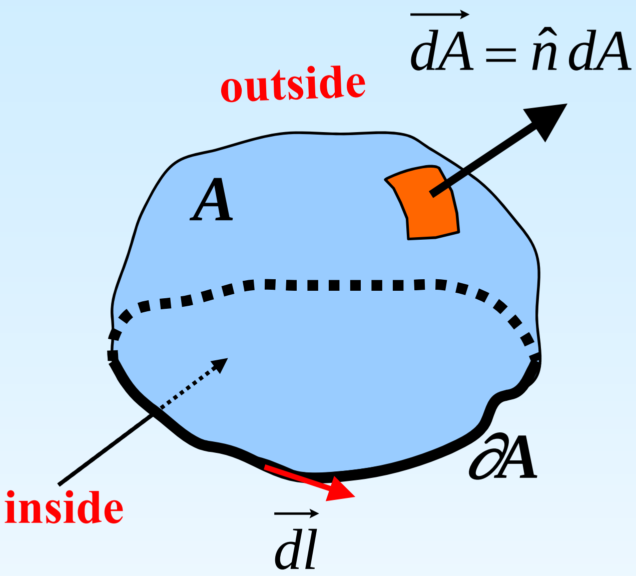


$$\iint_S \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) dS = \oint_{\partial S} \left(u_1 n_1 + u_2 n_2 \right) dl$$

$V \rightarrow S$ surface
∂S = curve, boundary of S

(in general in n-dimensions, n>1)

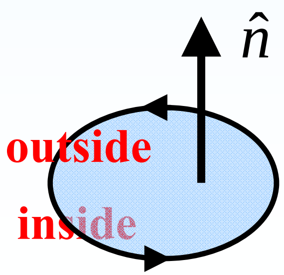
Stokes Theorem



$$\iint_A (\nabla \times \vec{u}) \cdot \hat{n}\, dA = \oint_{\partial A} \vec{u} \cdot \overrightarrow{dl}$$

A = surface

∂A = curve, boundary of A



sign: choose one of both sides of the surface and define it as the outside, the normal vector pointing from inside to outside. Then, the positive direction of $\overrightarrow{dl}$ is the anticlockwise one looking from the outside

Vector identities

$\vec{a}, \vec{b} =$ vector fields $\qquad f\,,\, g =$ scalar fields

$$\nabla\cdot\left(\nabla\times\vec{a}\right)=0 \qquad\qquad \nabla\times\left(\nabla f\right)=0$$

$$\nabla\times\left(\nabla\times\vec{a}\right)=\nabla\left(\nabla\cdot\vec{a}\right)-\nabla^2\vec{a}$$

$$\nabla^2\left(fg\right)= f\,\nabla^2 g+g\,\nabla^2 f+2\nabla f\cdot\nabla g$$

$$\nabla\times\left(f\vec{a}\right)= f\,\nabla\times\vec{a}-\vec{a}\times\nabla f$$

$$\nabla\left(\vec{a}\cdot\vec{b}\right)=\vec{a}\times\left(\nabla\times\vec{b}\right)+\vec{b}\times\left(\nabla\times\vec{a}\right)+\vec{a}\cdot\nabla\vec{b}+\vec{b}\cdot\nabla\vec{a}$$

$$\nabla\cdot\left(\vec{a}\times\vec{b}\right)=\vec{b}\cdot\left(\nabla\times\vec{a}\right)-\vec{a}\cdot\left(\nabla\times\vec{b}\right)$$

$$\nabla\times\left(\vec{a}\times\vec{b}\right)=\left(\nabla\cdot\vec{b}\right)\vec{a}-\left(\nabla\cdot\vec{a}\right)\vec{b}+\vec{b}\cdot\nabla\vec{a}-\vec{a}\cdot\nabla\vec{b}$$

$$\vec{a}\cdot\nabla\vec{a}=\left(\nabla\times\vec{a}\right)\times\vec{a}+\frac{1}{2}\nabla\left(\vec{a}\cdot\vec{a}\right)$$

Laplacian operator (scalar): $\quad \nabla^2\equiv\nabla\cdot\nabla=\partial_i\partial_i$