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# A FINITE ELEMENT FOR NONLOCAL ELASTIC ANALYSES

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**Abstract.** A nonlocal elastic behaviour of integral type is modeled assuming that the nonlocality lies in the constitutive relation. The diffusion processes of the nonlocality are governed by an integral relation containing a recently proposed symmetric spatial weight function expressed in terms of an attenuation function. Starting from the variational formulation associated with the structural boundary-value problem in the context of nonlocal elasticity, a nonlocal finite element model is proposed and a 1D example is proposed.

## 1 INTRODUCTION

It is well-known that one of the main drawbacks of local elasticity consists in the fact that many problems, such as sharp crack-tip in continuum fracture mechanics, lead to stress singularities in classical elastic theories.

A possible solution consists in considering a continuum approach in which there are information regarding the behaviour of the material microstructure by assuming that an elastic material can transmit information to neighbouring points within a certain distance. Such a distance is the internal length scale and is an essential material parameter which accounts for nonlocal effects in the continuum. Nonlocal variables turn then out to be weighted average of the corresponding local variables over the material points of the structure and the internal length controls the weighting process related to a state variable at a given point.

A continuum theory for elastic material with long range cohesive forces can be found in the pioneristic work of Kröner. A nonlocal elastic theory is presented by Eringen but a simplified and more effective nonlocal theory is contributed in [1] by assuming that nonlocality appears only in the constitutive relation. It is shown that several problems related to stress singularities in local elasticity, such as crack-tip problems, disappear by adopting the nonlocal theory. An elastic model in a geometrically linear range endowed with the nonlocal elastic material model is dealt with in [2] in which the extension to nonlocal linear elasticity of the classical principles of the total potential energy, complementary energy and mixed Hu-Washizu principle are also provided.

In the present paper, starting from the nonlocal elastic constitutive model proposed by Eringen and co-workers, the thermodynamic framework and the boundary-value problem for nonlocal elasticity are formulated and the complete set of nonlocal mixed variational principles is then provided. A recently proposed, in the context of damage mechanics, symmetric spatial weight function which preserves constant fields is considered. A firm variational basis to the nonlocal model is provided. A consistent symmetric nonlocal finite element procedure is then derived starting from the nonlocal counterpart of the displacement-based variational formulation. A piecewise homogeneous bar is solved by the recourse to the proposed nonlocal finite element method for an imposed displacement and different attenuation functions. The solutions obtained are in a good agreement each other and no pathological behaviours at the boundary are present.

## 2 NONLOCAL ELASTICITY

The nonlocal elastic model is based on the idea that the long range forces arising in a homogeneous isotropic elastic structure are described by the following constitutive relation [3,4]:

$$\overline{\boldsymbol{\sigma}}(\boldsymbol{x}) = (R\boldsymbol{\sigma})(\boldsymbol{x}) = (R\mathbf{E}\boldsymbol{\varepsilon})(\boldsymbol{x}) = \int_{\Omega} W(\boldsymbol{x},\boldsymbol{y})\mathbf{E}(\boldsymbol{\varepsilon}(\boldsymbol{y}))d\boldsymbol{y} \tag{1}$$

The linear regularization operator $R$ transforms the local stress field $\boldsymbol{\sigma}$ into the related nonlocal stress $\overline{\boldsymbol{\sigma}}$ since its value at the point $\boldsymbol{x}$ of the body $\Omega$ depends on the entire field $\boldsymbol{\sigma}$. In linear isotropic elasticity, the elastic operator is $\mathbf{E}=K(\mathbf{1}\otimes\mathbf{1})+2G\mathbf{I}^*$ where $K$ and $G$ denote the bulk and the shear moduli respectively, $\mathbf{I}^*=\mathbf{I}-1/3(\mathbf{1}\otimes\mathbf{1})$ is the fourth-order deviatoric projection tensor being $\mathbf{I}$ and $\mathbf{1}$ the fourth-order and the second-order identity tensors respectively.

From a mechanical standpoint, the space weight function $W$ describes the mutual long-range elastic interaction. The function $W$ is positive, have its maximum for $\boldsymbol{x}=\boldsymbol{y}$ and decreases monotonically and rapidly to zero approaching the boundary of the interaction zone. The space weight function $W$ vanishes, or it approaches to zero, for $\|\boldsymbol{x}-\boldsymbol{y}\|\geq r$ where $r$ is the chosen influence distance,

A nonlocal behavior is present for high space variation of the local stress $\boldsymbol{\sigma}$ so that it results $R=I$ for uniform fields $\boldsymbol{\sigma}$ being $I$ the identity operator. Accordingly the weight function $W$ must fulfill the normalizing condition:

$$\int_{\Omega} W(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y} = 1 \tag{2}$$

for any $\boldsymbol{x}$ in $\Omega$. In order to impose such a condition, also for points close to the boundary of the body in which the interaction zone is deprived of a contribution, the following expression is considered in the sequel:

$$W(\boldsymbol{x},\boldsymbol{y}) = \left[1-\alpha\frac{V(\boldsymbol{x})}{V_{\infty}}\right]\delta(\boldsymbol{x},\boldsymbol{y}) + \frac{\alpha}{V_{\infty}}g(\boldsymbol{x},\boldsymbol{y}) \tag{3}$$

which is similar to the one proposed in [5] within the context of nonlocal damage. In the equation (3), $V$ is the representative volume:

$$V(\boldsymbol{x}) = \int_{\Omega} g(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y}\,, \tag{4}$$

$V_{\infty}$ is the value assumed by the representative volume $V$ for an unbounded body, the symbol $\delta(\boldsymbol{x},\boldsymbol{y})$ denotes the Dirac delta distribution and $\alpha$ is an adimensional scalar parameter. The scalar function $g(\boldsymbol{x},\boldsymbol{y})$ is a symmetric attenuation function.

Due to the symmetry of the weight function *W*, the regularization operator is self-adjoint, i.e. $R=R'$ where $R'$ denotes the dual operator.

## 3 THERMODYNAMIC FRAMEWORK

Let us analyze the thermodynamic framework for the nonlocal elastic model. The first principle of thermodynamics (see e.g. [6]) for a nonlocal behaviour can be formulated as follows:

$$\int_\Omega \dot{e}(\boldsymbol{x})d\boldsymbol{x} = \langle \bar{\boldsymbol{\sigma}}, \dot{\boldsymbol{\varepsilon}} \rangle + \int_\Omega \dot{Q}(\boldsymbol{x})d\boldsymbol{x} \tag{5}$$

where *e* is the internal energy density depending on strain $\boldsymbol{\varepsilon}$ and entropy *s*. The heat supplied to an element of volume is $dQ/dt = -\mathrm{div}\mathbf{q}$ being $\mathbf{q}$ the heat flux and "div" is the divergence operator.

The relation (5) can be written pointwise as follows:

$$\dot{e}(\boldsymbol{x}) = \bar{\boldsymbol{\sigma}}(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + P(\boldsymbol{x}) + \dot{Q}(\boldsymbol{x}) \tag{6}$$

where the nonlocality residual function *P* takes into account the energy exchanges between neighbor particles [7]. The residual *P* fulfils the insulation condition:

$$\int_\Omega P(\boldsymbol{x})d\boldsymbol{x} = 0 \tag{7}$$

since the body is a thermodynamically isolated system with reference to energy exchanges due to nonlocality.

The second principle of thermodynamics is enforced in its classical pointwise form:

$$\dot{s}(\boldsymbol{x})T(\boldsymbol{x}) + \mathrm{div}\mathbf{q}(\boldsymbol{x}) - \nabla T(\boldsymbol{x}) * \frac{\mathbf{q}(\boldsymbol{x})}{T(\boldsymbol{x})} \geq 0 \tag{8}$$

everywhere in Ω where $ds/dt$ is the internal entropy production rate per unit volume and *T* is the absolute temperature. The symbol ∇ denotes the gradient operator.

The thermodynamic laws (6) and (8) yield the non-negative dissipation at a given point of the body:

$$D(\boldsymbol{x}) = \bar{\boldsymbol{\sigma}}(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) - \dot{\phi}(\boldsymbol{x}) + P(\boldsymbol{x}) - s(\boldsymbol{x})\dot{T}(\boldsymbol{x}) - \nabla T(\boldsymbol{x}) * \frac{\mathbf{q}(\boldsymbol{x})}{T(\boldsymbol{x})} \geq 0 \tag{9}$$

where $\phi = e - sT$ is the free energy.

Considering isothermal processes, the inequality (9) becomes:

$$D(\boldsymbol{x}) = \bar{\boldsymbol{\sigma}}(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) - \dot{\phi}(\boldsymbol{x}) + P(\boldsymbol{x}) \geq 0 . \tag{10}$$

The free energy function at a point $\boldsymbol{x}$ of the body Ω is defined according to the relation:

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) = \frac{1}{2}(R\mathbf{E}\boldsymbol{\varepsilon})(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) . \tag{11}$$

A direct evaluation shows that, for a piecewise homogeneous material, the operators *R* and **E** commute with respect to the scalar product in $L^2(\Omega)$ so that the following equality holds $\langle R\mathbf{E}\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \rangle = \langle \boldsymbol{\varepsilon}_1, R\mathbf{E}\boldsymbol{\varepsilon}_2 \rangle$ for any strain $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$.

The body energy dissipation *L* follows from the integration of (10) to get:

$$L = \langle \bar{\boldsymbol{\sigma}}, \dot{\boldsymbol{\varepsilon}} \rangle - \int_\Omega \dot{\phi}(\boldsymbol{x})d\boldsymbol{x} \geq 0 . \tag{12}$$

Note that the global free energy is the functional of the strain $\boldsymbol{\varepsilon}$ obtained by integrating the specific free energy (11) over the whole domain of the body:

$$\Phi(\boldsymbol{\varepsilon}) = \int_{\Omega} \phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) d\boldsymbol{x} = \frac{1}{2}\langle R\mathbf{E}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} \rangle \tag{13}$$

and the complementary potential turns then out to be the quadratic functional:

$$\Phi^*(\overline{\boldsymbol{\sigma}}) = \frac{1}{2}\left\langle \overline{\boldsymbol{\sigma}}, (R\mathbf{E})^{-1} \overline{\boldsymbol{\sigma}} \right\rangle . \tag{14}$$

Recalling the equality (13), the body energy dissipation (12) becomes:

$$\langle \overline{\boldsymbol{\sigma}}, \dot{\boldsymbol{\varepsilon}} \rangle - \langle R\mathbf{E}\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}} \rangle \geq 0 . \tag{15}$$

The relation (15) must hold for any admissible deformation mechanism so that, following widely used arguments [6], the following state law is obtained:

$$\overline{\boldsymbol{\sigma}}(\boldsymbol{x}) = (R\mathbf{E}\boldsymbol{\varepsilon})(\boldsymbol{x}) = d\phi(\boldsymbol{\varepsilon}(\boldsymbol{x})) . \tag{16}$$

It is then apparent that the relation (15) holds as an equality. Moreover the dissipation (10) can be viewed as the integrand of (15) so that the inequality (10) vanishes according to the reversible nature of the model:

$$D(\boldsymbol{x}) = \overline{\boldsymbol{\sigma}}(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) - \dot{\phi}(\boldsymbol{x}) + P(\boldsymbol{x}) = 0 \tag{17}$$

and the explicit expression for the nonlocality residual function at a given point of the body is given by:

$$P(\boldsymbol{x}) = \dot{\phi}(\boldsymbol{x}) - \overline{\boldsymbol{\sigma}}(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) = \frac{1}{2}(R\mathbf{E}\dot{\boldsymbol{\varepsilon}})(\boldsymbol{x}) * \boldsymbol{\varepsilon}(\boldsymbol{x}) - \frac{1}{2}(R\mathbf{E}\boldsymbol{\varepsilon})(\boldsymbol{x}) * \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) . \tag{18}$$

Finally the nonlocal elastic relation can be expressed in the following equivalent forms in terms of the free energy functional Φ and its conjugate Φ*:

$$\overline{\boldsymbol{\sigma}} = d\Phi(\boldsymbol{\varepsilon}) \Leftrightarrow \boldsymbol{\varepsilon} = d\Phi^*(\overline{\boldsymbol{\sigma}}) \Leftrightarrow \Phi(\boldsymbol{\varepsilon}) + \Phi^*(\overline{\boldsymbol{\sigma}}) = \langle \overline{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon} \rangle \tag{19}$$

where the last equality represents the Fenchel's relation.

## 4. THE NONLOCAL ELASTIC STRUCTURAL PROBLEM

In order to develop the structural model, it is convenient to formulate the constitutive relations in a global form, i.e. in terms of quantities pertaining to the whole structure. In the sequel such quantities will be referred to as fields. For a continuous model such fields are functionals defined in the domain Ω occupied by the body and belong to suitable functional spaces.

Local subdifferential (or differential) relations, enforced almost everywhere in Ω, can be equivalently expressed in global form by integrating the relevant functions over the domain Ω.

It can be proved that if the local function is convex (concave), the corresponding global one turn out to be convex (concave) in the corresponding fields. An analogous definition holds for any other functional to be defined over the whole body .

Let $\mathbf{u} \in U$ be the displacement field which is square integrable in Ω together with its distributional derivatives up to the order $m$ [8]. Conforming displacement fields fulfill linear constraint conditions and belong to a closed linear subspace $L \subset U$.

The kinematic operator **B** is a bounded linear operator from $U$ to the Hilbert space of square integrable strain fields $\boldsymbol{\varepsilon} \in D$.

Denoting by $F$ the subspace of external forces, which is dual of $U$, the continuous operator

$\mathbf{B}'$ is the equilibrium operator and is dual of $\mathbf{B}$. Let $\ell=\{\mathbf{t},\mathbf{b}\}$, belonging to $F$, be the load functional where $\mathbf{t}$ and $\mathbf{b}$ denote the tractions and the body forces.

In a geometrically linear range, the compatibility condition and the equilibrium equation are given by:

$$\boldsymbol{\varepsilon} = \mathbf{B}(\mathbf{u}+\mathbf{w}), \qquad \mathbf{f} = \mathbf{B}'\boldsymbol{\sigma}. \tag{20}$$

The external relation between reactions $\mathbf{r}\in F$ and displacements $\mathbf{u}\in U$ can be given in terms of two conjugate concave functionals Y and Y* by means of the following equivalent relations:

$$\mathbf{r} \in \partial \mathrm{Y}(\mathbf{u}) \Leftrightarrow \mathbf{u} \in \partial \mathrm{Y}^*(\mathbf{r}) \Leftrightarrow \mathrm{Y}(\mathbf{u}) + \mathrm{Y}^*(\mathbf{r}) = \langle \mathbf{r},\mathbf{u}\rangle \tag{21}$$

where the symbol $\partial$ denotes the superdifferential of concave functionals [9]. The last equality represents the Fenchel's relation.

In the case of external frictionless bilateral constraints with non-homogeneous boundary conditions, the admissible set of displacements is given by the subspace $L=\mathbf{w}+L_o$ where $L_o$ collects conforming displacements which satisfy the homogeneous boundary conditions. Then the functional Y turns out to be the indicator of $L_o$, i.e. Y($\mathbf{u}$)=0 if $\mathbf{u}$-$\mathbf{w}$ belongs to $L_o$ and $-\infty$ otherwise.

The relations governing the nonlocal elastic structural problem for a given load history $\ell(t)$ are:

$$\begin{cases} \mathbf{B}'\boldsymbol{\sigma} = \ell + \mathbf{r} \\ \mathbf{B}(\mathbf{u}+\mathbf{w}) = \boldsymbol{\varepsilon} \\ \overline{\boldsymbol{\sigma}} = d\Phi(\boldsymbol{\varepsilon}) \\ \mathbf{u} \in \partial \mathrm{Y}^*(\mathbf{r}) \end{cases} \tag{22}$$

The variational formulation in the complete set of the state variables is provided by the next statement.

***Proposition 1*** - *The set of state variables* $(\mathbf{u},\overline{\boldsymbol{\sigma}},\boldsymbol{\varepsilon},\mathbf{r})$ *is a solution of the saddle problem:*

$$\min_{\boldsymbol{\varepsilon}} \max_{\mathbf{r}} \operatorname*{stat}_{\mathbf{u},\overline{\boldsymbol{\sigma}}} M(\mathbf{u},\overline{\boldsymbol{\sigma}},\boldsymbol{\varepsilon},\mathbf{r}) \tag{23}$$

*where:*

$$M(\mathbf{u},\overline{\boldsymbol{\sigma}},\boldsymbol{\varepsilon},\mathbf{r}) = \Phi(\boldsymbol{\varepsilon}) + \mathrm{Y}^*(\mathbf{r}) + \langle \overline{\boldsymbol{\sigma}}, \mathbf{B}(\mathbf{u}+\mathbf{w}) - \boldsymbol{\varepsilon}\rangle - \langle \ell + \mathbf{r},\mathbf{u}\rangle \tag{24}$$

*if and only if it is a solution of the nonlocal elastic structural problem (22).*

The variational formulation in terms of displacements $\mathbf{u}$ is obtained by enforcing the external constraint relation $(22)_4$ in terms of the Fenchel's relation $(21)_3$ and the compatibility condition $(22)_2$ in the expression of the functional $M$ to get:

***Proposition 2*** - *The displacement* $\mathbf{u}$ *is a solution of the convex optimization problem:*

$$\min_{\mathbf{u}} P(\mathbf{u}) \tag{25}$$

*where:*

$$P(\mathbf{u}) = \Phi(\mathbf{B}(\mathbf{u}+\mathbf{w})) - \mathrm{Y}(\mathbf{u}) - \langle \ell,\mathbf{u}\rangle \tag{26}$$

*if and only if it is a solution of the nonlocal elastic structural problem (22).*

The potential $P$ is the nonlocal counterpart of the classical total potential energy in elasticity.

## 5. A NONLOCAL FINITE ELEMENT

The nonlocal total potential energy functional $P$ can be adopted in order to develop a finite element procedure for the proposed nonlocal elastic model. Using a conforming finite element discretization, let $\Omega_e$ (e=1,…,$N$) be the domain decomposition induced by the meshing of the domain $\Omega$. The unknown displacement field $\mathbf{v}(\boldsymbol{x})$ is given, for each element, in the interpolated form $\mathbf{v}_h^e(\boldsymbol{x})=\mathbf{N}_e(\boldsymbol{x})\mathbf{q}_e$ with $\boldsymbol{x}\in \Omega_e$ where $\mathbf{q}_e$ is the vector collecting the nodal displacement of the $e$-th finite element and $\mathbf{N}_e(\boldsymbol{x})$ is the chosen shape-function matrix.

The conforming displacement field $\mathbf{v}_h=\{\mathbf{v}_h^1, \mathbf{v}_h^2,\ldots, \mathbf{v}_h^N\}$ satisfies the homogeneous boundary conditions and the interelement continuity conditions. The rigid-body displacements are ruled out by imposing the conformity requirement. The displacement parameters $\mathbf{q}_e$ can be expressed in terms of nodal parameters $\mathbf{q}$ by means of the standard assembly operator $A_e$ according to the parametric expression $\mathbf{q}_e=A_e\mathbf{q}$. The interpolated counterpart of the nonlocal total potential energy $P$ can be obtained by adding up the contributions of each non-assembly element and imposing the conforming requirement to the interpolating displacement to get:

$$P_h(\mathbf{v}_h)=\frac{1}{2}\left\langle R\mathbf{E}\mathbf{B}(\mathbf{v}_h+\mathbf{w}_h),\mathbf{B}(\mathbf{v}_h+\mathbf{w}_h)\right\rangle-\left\langle \ell,\mathbf{u}\right\rangle, \tag{27}$$

with $\mathbf{v}_h\in L_o$. The interpolated nonlocal total potential energy $P_h$ can then be explicitly written as follows:

$$\begin{aligned} P_h(\mathbf{v}_h)&=\frac{1}{2}\sum_{e=1}^{N}\int_{\Omega_e}\left(1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right)\mathbf{E}_e\mathbf{B}(\mathbf{v}_h^e+\mathbf{w}_h^e)(\boldsymbol{x})*\mathbf{B}(\mathbf{v}_h^e+\mathbf{w}_h^e)(\boldsymbol{x})d\boldsymbol{x}+\\ &+\frac{\alpha}{V_\infty}\sum_{e=1}^{N}\sum_{m=1}^{N}\int_{\Omega_e}\int_{\Omega_m}g(\boldsymbol{x},\boldsymbol{y})\mathbf{E}_m\mathbf{B}(\mathbf{v}_h^m+\mathbf{w}_h^m)(\boldsymbol{y})*\mathbf{B}(\mathbf{v}_h^e+\mathbf{w}_h^e)(\boldsymbol{x})d\boldsymbol{y}d\boldsymbol{x}+\\ &-\sum_{e=1}^{N}\int_{\Omega_e}\mathbf{b}(\boldsymbol{x})*\mathbf{v}_h^e(\boldsymbol{x})d\boldsymbol{x}-\sum_{e=1}^{N}\int_{S_e}\mathbf{t}(\boldsymbol{x})*\Gamma\mathbf{v}_h^e(\boldsymbol{x})d\boldsymbol{x} \end{aligned} \tag{28}$$

where $S_e=\partial\Omega\cap\partial\Omega_e$.

The matrix form of the discrete problem is obtained by imposing the stationarity of $P_h$ with respect to $\mathbf{v}_h$ which is given by:

$$\begin{aligned} &\sum_{e=1}^{N}\int_{\Omega_e}\left(1-\alpha\frac{V(\boldsymbol{x})}{V_\infty}\right)\mathbf{E}_e\mathbf{B}(\mathbf{v}_h^e+\mathbf{w}_h^e)(\boldsymbol{x})*\mathbf{B}\delta\mathbf{v}_h^e(\boldsymbol{x})d\boldsymbol{x}+\\ &+\frac{\alpha}{V_\infty}\sum_{e=1}^{N}\sum_{m=1}^{N}\int_{\Omega_e}\int_{\Omega_m}g(\boldsymbol{x},\boldsymbol{y})\mathbf{E}_m\mathbf{B}(\mathbf{v}_h^m+\mathbf{w}_h^m)(\boldsymbol{y})*\mathbf{B}\delta\mathbf{v}_h^e(\boldsymbol{x})d\boldsymbol{y}d\boldsymbol{x}=\\ &=\sum_{e=1}^{N}\int_{\Omega_e}\mathbf{b}(\boldsymbol{x})*\delta\mathbf{v}_h^e(\boldsymbol{x})d\boldsymbol{x}+\sum_{e=1}^{N}\int_{S_e}\mathbf{t}(\boldsymbol{x})*\Gamma\delta\mathbf{v}_h^e(\boldsymbol{x})d\boldsymbol{x} \end{aligned} \tag{29}$$

for any $\delta\mathbf{v}_h^e\in L_o$.

Defining the component submatrices and subvectors in the form:

$$\mathbf{K}_{ee}^{l} = \int_{\Omega_e} (\mathbf{BN}_e)^T(\boldsymbol{x})\mathbf{E}_e(\mathbf{BN}_e)(\boldsymbol{x})d\boldsymbol{x}$$

$$\mathbf{K}_{ee}^{nl} = -\frac{\alpha}{V_\infty}\int_{\Omega_e} (\mathbf{BN}_e)^T(\boldsymbol{x})V(\boldsymbol{x})\mathbf{E}_e(\mathbf{BN}_e)(\boldsymbol{x})d\boldsymbol{x}$$

$$\mathbf{K}_{ee} = \mathbf{K}_{ee}^{l} + \mathbf{K}_{ee}^{nl} \tag{30}$$

$$\mathbf{K}_{em}^{nl} = \frac{\alpha}{V_\infty}\int_{\Omega_e}\int_{\Omega_m} (\mathbf{BN}_e)^T(\boldsymbol{x})g(\boldsymbol{x},\boldsymbol{y})\mathbf{E}_m(\mathbf{BN}_m)(\boldsymbol{y})d\boldsymbol{y}d\boldsymbol{x}$$

and

$$\mathbf{f}_e^{l} = \int_{\Omega_e}\mathbf{N}_e^T(\boldsymbol{x})\mathbf{b}(\boldsymbol{x})d\boldsymbol{x} + \int_{S_e}\mathbf{N}_e^T(\boldsymbol{x})\mathbf{t}(\boldsymbol{x})d\boldsymbol{x} - \int_{\Omega_e}(\mathbf{BN}_e)^T(\boldsymbol{x})\mathbf{E}_e(\mathbf{BN}_e)(\boldsymbol{x})d\boldsymbol{x}\boldsymbol{w}$$

$$\mathbf{f}_e^{nl} = \frac{\alpha}{V_\infty}\int_{\Omega_e}(\mathbf{BN}_e)^T(\boldsymbol{x})V(\boldsymbol{x})\mathbf{E}_e(\mathbf{BN}_e)(\boldsymbol{x})d\boldsymbol{x}\,\boldsymbol{w} + $$

$$-\frac{\alpha}{V_\infty}\int_{\Omega_e}\int_{\Omega_m}(\mathbf{BN}_e)^T(\boldsymbol{x})g(\boldsymbol{x},\boldsymbol{y})\mathbf{E}_m(\mathbf{BN}_m)(\boldsymbol{y})d\boldsymbol{y}d\boldsymbol{x}\boldsymbol{w} \tag{31}$$

$$\mathbf{f}_{\mathrm{e}} = \mathbf{f}_e^{l} + \mathbf{f}_e^{nl}$$

the matrix form of the discrete problem is:

$$\sum_{e=1}^{N} A_e^T\mathbf{K}_{ee}A_e\mathbf{q} + \sum_{e=1}^{N}\sum_{m=1}^{N} A_e^T\mathbf{K}_{em}^{nl}A_m\mathbf{q} = \sum_{e=1}^{N} A_e^T\mathbf{f}_e\ . \tag{32}$$

The integration appearing in $(30)_1$ is performed elementwise so that $\mathbf{K}_{ee}{}^{l}$ turns out to be the standard stiffness matrix while $\mathbf{K}_{ee}{}^{nl}$ and $\mathbf{K}_{em}{}^{nl}$ in (30) turn out to be the nonlocal symmetric stiffness matrices reflecting the nonlocality of the model. The elements of the matrix $\mathbf{K}_{em}{}^{nl}$ vanish if the related elements are too far with respect to the influence distance *r*. Accordingly the matrix $\mathbf{K}_{em}{}^{nl}$ is banded with a band width larger than in the standard stiffness matrix.

Hence the solving linear equation system follows from (29) and is given by:

$$\mathbf{Kq} = (\mathbf{K}^{l} + \mathbf{K}^{nl})\mathbf{q} = \mathbf{f} \tag{33}$$

where the global stiffness matrix **K** is symmetric and positive definite.

In the case of a local elastic behaviour, the nonlocal terms disappear and the solving equation system reduces to the standard local finite element method given by $\mathbf{K}^{l}\mathbf{q}=\mathbf{f}^{l}$.

## 6. A COMPUTATIONAL EXAMPLE

The elastic bar reported in Fig. 1 is solved by the proposed nonlocal finite element approach.

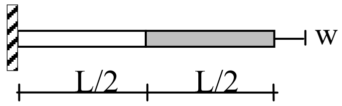



**Figure 1**: A one-dimensional bar in tension.

The bar has a unit cross-section and a length *L=100 cm*. It is clamped at the end *x=0* and is subjected to a given displacement *w* at the other end *x=L*. The bar is piecewise homogeneous and the Young modulus has the following expression $E(x)= \beta Eo$ for $0\leq x\leq L/2$ and $E(x)=E_o=21\times 10^4$ $MPa$ for $L/2\leq x\leq L$ with the parameter $\beta$ varying from $0$ to $1$. Three different weight functions $W_1$, $W_2$ and $W_3$ of the type (3) are employed in which the attenuation function $g$ is, respectively, given by the Gauss-like function:

$$g_1(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{l\sqrt{2\pi}}\exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{2l^2}\right) \tag{34}$$

the bi-exponential function:

$$g_2(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2l}\exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{2l}\right) \tag{35}$$

and the bell-shaped polynomial function:

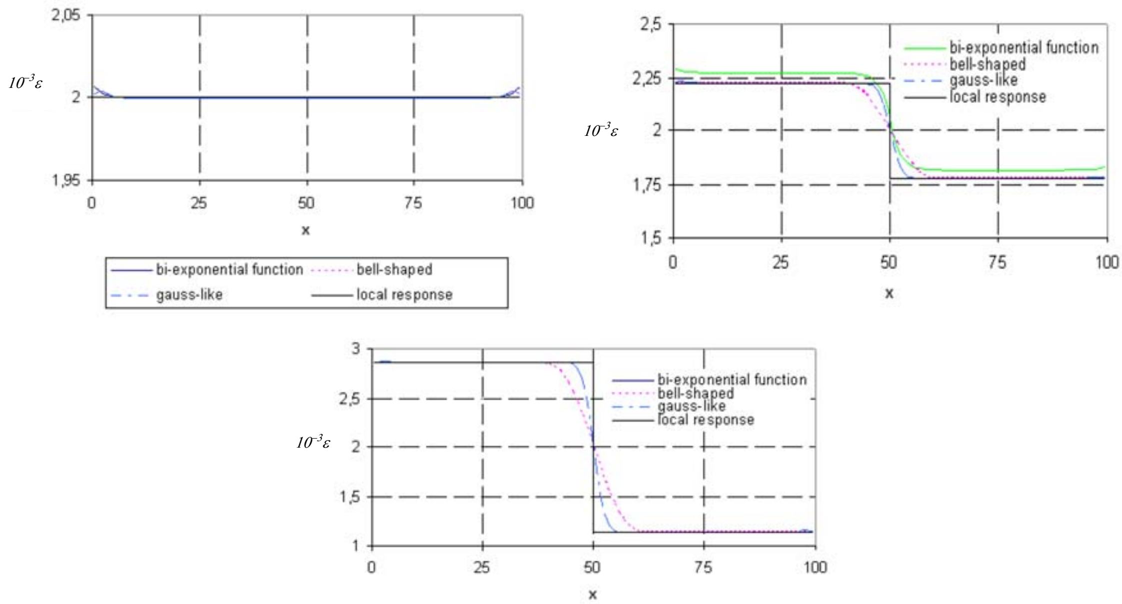
$$g_3(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \dfrac{1}{l\sqrt{2\pi}}\exp\left(-\dfrac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{2l^2}\right) & \text{if } \|\boldsymbol{x}-\boldsymbol{y}\| \leq r \\ 0 & \text{if } \|\boldsymbol{x}-\boldsymbol{y}\| > r \end{cases} \tag{36}$$

where it is assumed $l=r/6$. The internal length is $l=2$ $cm$, the influence distance is $r=12$ cm and the material parameter is $\alpha=-1$. The imposed displacement at the end $x=L$ is $w=0.2$ $cm$.

A series of computations have been accomplished by using the above data, the three space weight functions $W_1$, $W_2$, $W_3$ and different jumps of the elastic modulus provided by the parameter $\beta$.

In the case of a homogeneous material, i.e. for $\beta=1$, strains and stresses coincide to the classical solution for homogeneous media independently of the internal length.

The strain plot $\boldsymbol{\varepsilon}$ are provided in Figs. 2 for different values of $\beta$.




(a) (c) (b)

**Figure 2**: Strain plots of the bar in tension for different attenuation functions $g$.

The relations (34)-(36) of the attenuation function $g$ are considered in the expressions of the symmetric function $W_1$, $W_2$, $W_3$ and different values of the ratio $\beta$ are also addressed. In Fig. 2a the homogeneous bar with $E_1 = E_2$ ($\beta = 1$) is considered. In Fig. 2b the piecewise homogeneous bar with $E_1 = 0.8E_o$ and $E_2 = E_o$ ($\beta = 0.8$) is solved and in Fig. 2c the piecewise homogeneous bar with $E_1 = 0.4E_o$ and $E_2 = E_o$ ($\beta = 0.4$) is considered.

As expected, the solution for $\beta=1$ reported in Fig. 2(a) coincides to the local one and the value $\boldsymbol{\varepsilon} = w_h/L = 2\times10^{-3}$ is attained independently of the choice of the attenuation function $g$ in the expression of the space weight function $W$. On comparing the nonlocal behaviour with the local one in the Figs. 2(b) and 2(c) for different values of the ratio $\beta$, it is apparent the presence in the nonlocal response of a narrow layer around the middle section of the bar in which the strain $\boldsymbol{\varepsilon}$ smoothly varies with more or less slope depending on the considered attenuation function $g$.

The comparison shows that the use of the bell-shaped and Gauss-like attenuation functions in the expression of the spatial weight function $W$ provides the best fit of the constant strain for different values of $\beta$. Moreover, the Gauss-like function presents a narrow layer around the middle section of the bar with a sharper slope than the one corresponding to the bell-shaped attenuation function.

The displacement profiles corresponding to the considered values of $\beta$ are reported in Fig. 3 in which the discontinuity in the middle section of the bar is apparent in the case of nonhomogeneity.

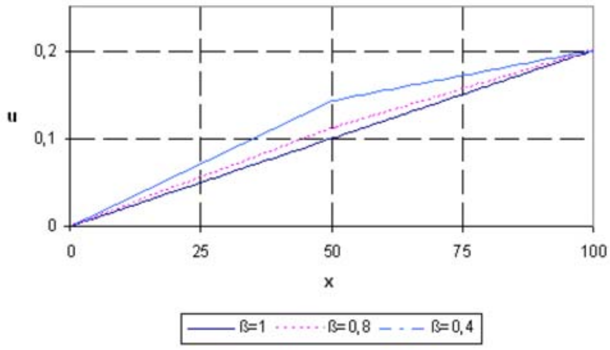



**Figure 3**: Displacement plots of the bar in tension for different values of the ratio $\beta$.

The stress plots are reported in Fig. 4.

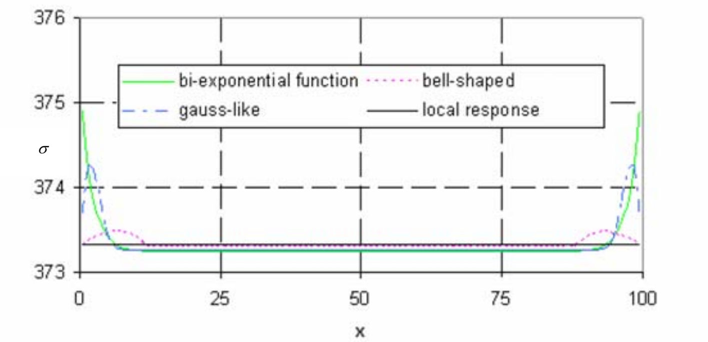



**Figure 4**: Stress plots of the bar in tension for different attenuation functions $g$.

The stress is evaluated for a piecewise homogeneous bar with $E_1 = 0.8E_o$ and $E_2 = E_o$ ($\beta = 0.8$) considering the attenuation functions *g* (34)-(36) in the expression of the symmetric function *W*.

No boundary effects are present in the stress field in a layer near the end cross-sections.

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