

GEOMETRIC HAMILTON–JACOBI THEORY FOR NONHOLONOMIC DYNAMICAL SYSTEMS

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Abstract

The geometric formulation of Hamilton–Jacobi theory for systems with nonholonomic constraints is developed, following the ideas of the authors in previous papers. The relation between the solutions of the Hamilton–Jacobi problem with the symplectic structure defined from the Lagrangian function and the constraints is studied. The concept of complete solutions and their relationship with constants of motion, are also studied in detail. Local expressions using quasivelocities are provided. As an example, the nonholonomic free particle is considered.

Key words: Hamilton–Jacobi equation, nonholonomic Lagrangian system, quasivelocity, symplectic manifold, constant of motion, complete integral

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1 Introduction

In classical mechanics, Hamilton–Jacobi theory tries to integrate a Hamiltonian system of differential equations through an appropriate canonical transformation [3, 18]. The equation to be satisfied by the generating function of this transformation is a partial differential equation, and having enough solutions to it finally leads to the integration of the system. The Hamilton–Jacobi equation is also very close, from the classical side, to the Schrödinger equation of quantum mechanics —see for instance [28]. For these reasons, Hamilton–Jacobi theory has been a matter of continuous interest.

From the viewpoint of geometric mechanics, the intrinsic formulation of Hamilton–Jacobi equation is also clear [1, 25, 27]. Nevertheless, in a recent paper [8] we presented a new geometric framework for the Hamilton–Jacobi theory. The motivation for this work was that the usual formulation of the Hamilton–Jacobi equation heavily relies on the symplectic structure of the phase space. However, there are interesting integrable systems that have alternative Lagrangian (and Hamiltonian) formulations; two different Lagrangians for the same dynamics may lead to two different symplectic structures, and therefore one may wonder about the relevance of a concrete symplectic structure and its relation with the solutions of the Hamilton–Jacobi problem. Following this program, we formulated the Hamilton–Jacobi equation both in the Lagrangian and in the Hamiltonian formalisms of time-independent mechanics, and studied the relations between the solutions of the Hamilton–Jacobi equation and the symplectic form; we recovered the usual Hamilton–Jacobi equation as a special case in our generalised framework. Additional details on the relationship between Hamilton–Jacobi equation and the geometric structures of mechanics have recently been presented in [9].

Within the Lagrangian formulation, dynamics is described by a second-order vector field Γ defined on the tangent bundle TQ of the configuration manifold Q . The first step in our formulation is to describe the integral curves of Γ as the canonical liftings of the integral curves of a family of vector fields X_λ on Q . From a geometrical viewpoint, this is pretty simple: each of these vector fields has to be X_λ -related to Γ . The usual formulation of Hamilton–Jacobi equation corresponds to the case where the image of X_λ is a Lagrangian submanifold of TQ with respect to the symplectic form ω_L . With some changes, the same formulation can be given in the Hamiltonian framework; properly speaking, it is in this case that we recover the usual Hamilton–Jacobi theory.

Our work [8] was mainly devoted to regular autonomous Lagrangians. However, we also considered the time-dependent case (through the so-called homogeneous formalism) as well as a special instance of singular Lagrangians: those not yielding Lagrangian constraints. It was clear that more general situations could be given a similar description, and it is the purpose of this paper to consider the very important issue of mechanical systems with nonholonomic constraints —that is, non-integrable constraints depending on the velocities.

Nonholonomic mechanical systems have been discussed since long ago. There are many papers dealing with geometric aspects of such systems, beginning with [35], and including different viewpoints as [4, 14, 16, 20, 24, 26, 30] —see also [5, 11, 19, 31, 34]. When a nonholonomic system is regular, at the end, there is a well-defined dynamics on the submanifold $\mathcal{D} \subset TQ$ defined by the constraints. Therefore, it seems quite straightforward to apply our previous framework developed in [8] for the Hamilton–Jacobi theory to the case of nonholonomic mechanical systems, and, in fact, this has been done in some recent papers as [17, 21, 32], where the Hamiltonian case and some applications are analyzed in deep, as well as in other instances like classical field theo-

ries [22]. Note, however, that the relation with the symplectic structure is not so much clear, and this is one of the points we address in the present paper, where this new geometric perspective for the Hamilton–Jacobi problem is performed under the Lagrangian formalism. In this sense, our approach could be considered as complementary to that developed in [17, 21, 32]. As in our previous work [8], we state the standard classical nonholonomic Hamilton–Jacobi problem as a particular case of a more general one. Furthermore, we consider two Lagrangian frameworks for this: a plain formulation on the velocity space and also an intrinsic formulation on the constraint submanifold (the so-called distributional approach to nonholonomic mechanics). Finally, in the same lines of our previous paper, we discuss complete solutions for the Hamilton–Jacobi problem and their relationship with constants of motion.

The paper is organised as follows. In section 2 we give a short account of nonholonomic mechanics. The Hamilton–Jacobi problem for Lagrangian nonholonomic systems is presented in section 3 in both the general and the restricted (standard) versions. Section 4 is devoted to the study of the same problem in an intrinsic formulation. Local coordinate expressions are given in section 5 by using quasivelocities. Complete solutions are studied in section 6. Finally, a detailed example, the nonholonomic free particle, is presented in section 7.

2 Nonholonomic Lagrangian systems

We consider an n -dimensional manifold Q , its tangent bundle $\tau_Q: TQ \rightarrow Q$, and a constraint submanifold, which we assume to be a vector subbundle $\mathcal{D} \subset TQ$ of rank r . We consider the annihilator $\mathcal{D}^\circ \subset T^*Q$ and the set $\widetilde{\mathcal{D}}^\circ \subset T^*(TQ)$ defined by $\widetilde{\mathcal{D}}^\circ = \{ \alpha \circ T\tau_Q \in T^*(TQ) \mid \alpha \in \mathcal{D}^\circ \}$; this is a vector bundle over TQ , whose fibre at a point $v \in TQ$, such that $\tau_Q(v) = q$, is more explicitly described as

$$\widetilde{\mathcal{D}}_v^\circ = \{ \lambda_v \in T_v^*(TQ) \mid \text{there exists } \alpha_q \in \mathcal{D}_q^\circ \text{ such that } \lambda_v = \alpha_q \circ T_v\tau_Q \}.$$

Given a Lagrangian function $L \in C^\infty(TQ)$, we consider the nonholonomic system defined by the Lagrangian L and the linear constraints given by \mathcal{D} , that is, only velocities in \mathcal{D} are admissible. The Lagrange–d’Alembert principle states that the dynamics of the system is given by the integral curves (with initial condition in \mathcal{D}) of the vector fields $\Gamma \in \mathfrak{X}(TQ)$ tangent to \mathcal{D} that satisfy the second-order condition and the Lagrange–d’Alembert equation (see for instance [24])

$$(i_\Gamma \omega_L - dE_L)|_{\mathcal{D}} \in \text{Sec}(\widetilde{\mathcal{D}}^\circ), \tag{1}$$

where ω_L is the Lagrange 2-form associated with L . This expression means that, on the points of \mathcal{D} , the 1-form $i_\Gamma \omega_L - dE_L$ takes its values in the codistribution $\widetilde{\mathcal{D}}^\circ$.

From now on we assume that L is a regular Lagrangian, which means either that its fibre derivative (Legendre transformation) $\mathcal{F}L: TQ \rightarrow T^*Q$ is a local diffeomorphism, that the Lagrange 2-form ω_L is a symplectic form, or that its fibre Hessian $\mathcal{F}^2L = G^L: TQ \rightarrow T^*Q \otimes T^*Q$ is everywhere a nondegenerate bilinear form. Given $u, v, w \in T_qQ$, the fibre Hessian of the Lagrangian can also be expressed as $G_u^L(v, w) = \omega_L(\tilde{v}, w_u^V)$, where $\tilde{v} \in T_uTQ$ is any vector which projects onto v , and w_u^V is the vertical lift of w on the point u .

The nonholonomic system (L, \mathcal{D}) is said to be *regular* if there is a unique solution to Lagrange–d’Alembert equation. Here uniqueness must be understood as follows: two solutions are considered equal if they coincide when restricted to \mathcal{D} .

There are several equivalent ways to ensure regularity of the constrained system. We define the subbundle $\mathcal{T}^{\mathcal{D}}\mathcal{D} \subset T\mathcal{D} \rightarrow \mathcal{D}$ by

$$\mathcal{T}^{\mathcal{D}}\mathcal{D} = \{ V \in T\mathcal{D} \mid T\tau_Q(V) \in \mathcal{D} \} .$$

We also consider the restriction $G^{L\mathcal{D}}$ of the fibre Hessian G^L to the distribution \mathcal{D} . Then (see for instance [12]):

Theorem 1 *The following properties are equivalent:*

1. *The constrained Lagrangian system (L, \mathcal{D}) is regular,*
2. $\text{Ker } G^{L\mathcal{D}} = \{0\}$.
3. $TTQ|_{\mathcal{D}} = \mathcal{T}^{\mathcal{D}}\mathcal{D} \oplus (\mathcal{T}^{\mathcal{D}}\mathcal{D})^{\perp}$,

where $(\mathcal{T}^{\mathcal{D}}\mathcal{D})^{\perp}$ denotes the orthogonal complement of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ with respect to the symplectic form ω_L .

In the regular case, the constrained dynamics can be found by projection of the free dynamics according to the decomposition given in item 3. It follows that the dynamical vector field is a SODE on \mathcal{D} , that is, Γ is tangent to \mathcal{D} and $T\tau_Q(\Gamma(v)) = v$ for every $v \in \mathcal{D}$.

3 The Lagrangian Hamilton–Jacobi problem for nonholonomic systems

As in our previous paper [8], we decompose the study of the Hamilton–Jacobi problem for a nonholonomic Lagrangian system in two pieces: first, we consider a general setting to describe the solutions of the nonholonomic dynamics Γ on \mathcal{D} in terms of the solutions of a family of first-order differential equations; second, we study the interplay of these first-order vector fields with the corresponding symplectic structure, and impose additional conditions on them in order to simplify the problem. All this is performed in the Lagrangian formalism —the case of Hamiltonian formalism can be developed in quite a similar way.

3.1 General Lagrangian nonholonomic Hamilton–Jacobi problem

Following the same lines as in [8], we formulate the Hamilton–Jacobi problem in this way:

Statement 1 (General Lagrangian nonholonomic Hamilton–Jacobi problem) *Given a regular nonholonomic Lagrangian system (L, \mathcal{D}) , with dynamics given by a SODE vector field $\Gamma \in \mathfrak{X}(\mathcal{D})$, the general Lagrangian nonholonomic Hamilton–Jacobi problem consists in finding the vector fields $X: Q \rightarrow TQ$ such that, if $\gamma: \mathbb{R} \rightarrow Q$ is an integral curve of X , then $\dot{\gamma}: \mathbb{R} \rightarrow TQ$ takes values in $\mathcal{D} \subset TQ$ and it is an integral curve of Γ ; that is,*

$$X \circ \gamma = \dot{\gamma} \implies \Gamma \circ \dot{\gamma} = \overline{X \circ \gamma} \quad \text{and} \quad \dot{\gamma}(t) \in \mathcal{D} \text{ for each } t \in \mathbb{R} .$$

Any of such X is said to be a solution to the general Lagrangian nonholonomic Hamilton–Jacobi problem.

Theorem 2 *A vector field $X \in \mathfrak{X}(Q)$ is a solution to the general Lagrangian nonholonomic Hamilton–Jacobi problem if, and only if, $X \in \text{Sec}(\mathcal{D})$ and $\Gamma \circ X = TX \circ X$.*

Proof Let $X \in \mathfrak{X}(Q)$ be a solution to the general nonholonomic Hamilton–Jacobi problem. For every $q \in Q$, let γ be the integral curve of X starting at q ; that is, $\dot{\gamma} = X \circ \gamma$ and $\gamma(0) = q$. Then $\eta = \dot{\gamma}$ is a solution to the constrained problem; that is, $\eta(0) \in \mathcal{D}$ and $\dot{\eta} = \Gamma \circ \eta$. From the first one we have that $X(q) = X(\gamma(0)) = \dot{\gamma}(0) = \eta(0) \in \mathcal{D}$. As q is arbitrary, it follows that X takes values in \mathcal{D} . Moreover,

$$\begin{aligned} (\Gamma \circ X)(q) &= (\Gamma \circ X \circ \gamma)(0) = (\Gamma \circ \eta)(0) = \dot{\eta}(0) = \frac{d\dot{\gamma}}{dt}(0) = \frac{d}{dt}(X \circ \gamma)(0) \\ &= (TX \circ \dot{\gamma})(0) = (TX \circ X \circ \gamma)(0) = (TX \circ X)(q), \end{aligned}$$

from which it follows that $\Gamma \circ X = TX \circ X$.

Conversely, let X be a vector field taking values in \mathcal{D} such that $\Gamma \circ X = TX \circ X$. If γ is an integral curve of X then $\eta = X \circ \gamma$ is an integral curve of Γ :

$$\Gamma \circ \eta = \Gamma \circ X \circ \gamma = TX \circ X \circ \gamma = TX \circ \dot{\gamma} = \frac{d}{dt}(X \circ \gamma) = \dot{\eta}.$$

In addition, as $\eta(0) = X(\gamma(0)) \in \mathcal{D}$, it follows that η starts at \mathcal{D} , and hence it is a solution to the constrained dynamics. ■

We can rewrite the above statement as follows: a vector field X is a solution to the general nonholonomic Hamilton–Jacobi problem if $\text{Im}(X)$ is a submanifold of \mathcal{D} and Γ is tangent to this submanifold. Conversely, if N is an n -dimensional submanifold of \mathcal{D} , transverse to the fibers and invariant under Γ , then locally there exists $X \in \mathfrak{X}(Q)$ such that $N = X(Q)$ and it is a local solution to the general Hamilton–Jacobi problem.

Remark 1 As in the unconstrained case (when $\mathcal{D} = TQ$) the above result can be stated in a more general framework, and in fact, it can be applied to any vector field on \mathcal{D} which satisfies the second-order condition.

The SODE Γ being the solution of the Lagrange–d’Alembert equation (1), we can take the pullback of such equation by X , and then obtain an equation that does not involve Γ explicitly.

Theorem 3 *A vector field $X \in \mathfrak{X}(Q)$ is a solution to the general Lagrangian nonholonomic Hamilton–Jacobi problem if, and only if, $X \in \text{Sec}(\mathcal{D})$ and $i_X(X^*\omega_L) - d(X^*E_L) \in \text{Sec}(\mathcal{D}^\circ)$.*

Proof We will use the following preliminary results:

1. If $\lambda \in \widetilde{\mathcal{D}}^\circ$, then $\lambda = \alpha \circ T\tau_Q$ for $\alpha \in \mathcal{D}^\circ$, and we have that $X^*\lambda = \alpha$. In fact,

$$\langle X^*\lambda, v \rangle = \langle \alpha \circ T\tau_Q, TX(v) \rangle = \langle \alpha, T\tau_Q(TX(v)) \rangle = \langle \alpha, v \rangle$$

for every $v \in TQ$. We will write symbolically this equation as $X^*\widetilde{\mathcal{D}}^\circ = \mathcal{D}^\circ$.

2. Given a vector field $X \in \mathfrak{X}(Q)$, let Y be the vector field along X defined by $Y = \Gamma \circ X - TX \circ X$. Consider the one-form α in Q given by

$$\alpha = [X^*(i_\Gamma\omega_L - dE_L)] - [i_X(X^*\omega_L) - d(X^*E_L)].$$

A straightforward calculation (see [8]) leads to

$$\alpha_q(v) = \omega_L(X(q))(Y(q), TX(v)), \quad q \in Q, \quad v \in T_qQ.$$

3. If X is a section of \mathcal{D} , then for every $q \in Q$ there exists $v \in \mathcal{D}$ such that $Y(q) = \xi^V(X(q), v)$, where ξ^V denotes the vertical lift in TQ . Indeed, it is clear that Y take values in the vertical bundle, so that, for every $q \in Q$ there exists $v \in T_q Q$ such that $Y(q) = \xi^V(X(q), v)$. We have just to prove that v is in \mathcal{D} . On the one hand $\Gamma|_{\mathcal{D}}$ is tangent to \mathcal{D} , so that $\Gamma \circ X$ takes values in $T\mathcal{D}$, and on the other hand, $TX \circ X$ also takes values in $T\mathcal{D}$, therefore we get that $Y(q) \in T_{X(q)}\mathcal{D}$. Taking into account that linear constraints for \mathcal{D} are given by the linear functions $\hat{\alpha}$ associated with 1-forms α taking values in \mathcal{D}° , we have that, for every $\alpha \in \text{Sec}(\mathcal{D}^\circ)$,

$$0 = Y(q)\hat{\alpha} = \xi^V(X(q), v)\hat{\alpha} = \langle \alpha_q, v \rangle,$$

and hence $v \in \mathcal{D}$.

Bearing this in mind, the proof of the theorem is as follows:

[\Leftarrow] Let $X \in \text{Sec}(\mathcal{D})$ such that $i_X(X^*\omega_L) - d(X^*E_L) \in \text{Sec}(\mathcal{D}^\circ)$. As $i_\Gamma\omega_L - dE_L \in \text{Sec}(\widetilde{\mathcal{D}}^\circ)$, then $X^*(i_\Gamma\omega_L - dE_L) \in \text{Sec}(\mathcal{D}^\circ)$, and hence

$$\alpha = [X^*(i_\Gamma\omega_L - dE_L)] - [i_X(X^*\omega_L) - d(X^*E_L)] \in \text{Sec}(\mathcal{D}^\circ).$$

The vector field Y along X is vertical, and at every point is the vertical lift of an element in \mathcal{D} : for every $q \in Q$, there exists $v \in \mathcal{D}$ such that $Y(q) = \xi^V(X(q), v)$. Then for every $w \in \mathcal{D}$ we have that

$$0 = \alpha_q(w) = (\omega_L(X(q)))(\xi^V(X(q), v), T_q X(w)) = G_{X(q)}^L(v, w) = G_{X(q)}^{L\mathcal{D}}(v, w).$$

Since this equation holds for every $w \in \mathcal{D}$ and $G^{L\mathcal{D}}$ is regular, we have that $v = 0$, and hence $Y = 0$, which proves the statement.

[\Rightarrow] If $X \in \text{Sec}(\mathcal{D})$ and $\Gamma \circ X = TX \circ X$, then $Y = 0$, and hence $\alpha = 0$. Therefore

$$i_X(X^*\omega_L) - d(X^*E_L) = X^*(i_\Gamma\omega_L - dE_L) \in \text{Sec}(\mathcal{D}^\circ).$$

This completes the proof. ■

3.2 Restricted nonholonomic Lagrangian Hamilton–Jacobi problem

As in the unconstrained case, to solve the generalized Lagrangian nonholonomic Hamilton–Jacobi problem can be a difficult task; thus it is convenient to consider a simplified, and hence less general, problem.

We have seen that $X \in \mathfrak{X}(Q)$ is a solution to the generalized problem if, and only if, the difference $i_X(X^*\omega_L) - d(X^*E_L)$ takes values in \mathcal{D}° . So we can look for solutions satisfying that both terms $i_X(X^*\omega_L)$ and $d(X^*E_L)$ are in \mathcal{D}° . Furthermore the condition $i_X(X^*\omega_L) \in \text{Sec}(\mathcal{D}^\circ)$ can be ensured by imposing that

$$(X^*\omega_L)(\mathcal{D}, \mathcal{D}) = 0,$$

or equivalently $(X^*\omega_L)(\mathcal{D}, \cdot) \in \text{Sec}(\mathcal{D}^\circ)$. We will plainly say that the restriction of $X^*\omega_L$ to \mathcal{D} vanishes.

Another possibility could be to impose that $(X^*\omega_L)(\mathcal{D}, \cdot) = 0$; but this is a less general condition. An additional justification for our choice will be provided in the next section.

In this way, we can state the following restricted Lagrangian nonholonomic Hamilton–Jacobi problem:

Statement 2 (Restricted Lagrangian nonholonomic Hamilton–Jacobi problem) *Given a regular nonholonomic Lagrangian system (L, \mathcal{D}) , find those solutions X to the generalized Lagrangian nonholonomic Hamilton–Jacobi problem such that the restriction of $X^*\omega_L$ to \mathcal{D} vanishes.*

As a consequence, it follows that if X is a solution to the Lagrangian nonholonomic Hamilton–Jacobi problem, then $d(X^*E_L) \in \text{Sec}(\mathcal{D}^\circ)$.

Proposition 1 *A vector field $X \in \mathfrak{X}(Q)$ is a solution to the Lagrangian nonholonomic Hamilton–Jacobi problem if, and only if,*

1. $X \in \text{Sec}(\mathcal{D})$,
2. $(X^*\omega_L)|_{\mathcal{D}} = 0$,
3. $d(X^*E_L)|_{\mathcal{D}} = 0$.

Proof The direct statement is obvious. For the converse, we have that $X \in \text{Sec}(\mathcal{D})$ and both $i_X(X^*\omega_L)$ and $d(X^*E_L)$ take their values in \mathcal{D}° , so that $i_X(X^*\omega_L) - d(X^*E_L) \in \text{Sec}(\mathcal{D}^\circ)$, and by Theorem 3 the statement holds. ■

Remark 2 It is important to point out that every solution to the general (restricted) Lagrangian Hamilton–Jacobi problem for the unconstrained system which takes values on \mathcal{D} is, automatically, a solution to the general (restricted) Lagrangian nonholonomic Hamilton–Jacobi problem. This may be helpful when looking for solutions as we will see in an example later on.

A particular important case is that of bracket-generating distributions (also known as completely nonholonomic distributions). A distribution $\mathcal{D} \subset TQ$ is bracket-generating if the smallest Lie subalgebra $L_{\mathcal{D}} \subset \mathfrak{X}(Q)$ containing $\text{Sec}(\mathcal{D})$ is the full $\mathfrak{X}(Q)$. In other words, we can get a family of vector fields in the distribution \mathcal{D} such that every vector $v \in T_qQ$ can be obtained as a linear combination of the values at q of such vector fields together with repeated brackets. In this case we have the following simplification (see [32]):

Proposition 2 *Assume that $\mathcal{D} \subset TQ$ is a bracket-generating distribution. A vector field $X \in \mathfrak{X}(Q)$ is a solution to the Lagrangian nonholonomic Hamilton–Jacobi problem if, and only if,*

1. $X \in \text{Sec}(\mathcal{D})$,
2. $(X^*\omega_L)|_{\mathcal{D}} = 0$,
3. $X^*E_L = \text{constant}$.

Proof We just have to prove that, for a bracket-generating distribution, the condition $d(X^*E_L) \in \text{Sec}(\mathcal{D}^\circ)$ is equivalent to $X^*E_L = \text{constant}$. The result is true for any function f in Q , our case being $f = X^*E_L$.

Let f be a smooth function on a manifold Q such that $df \in \text{Sec}(\mathcal{D}^\circ)$. We first prove that f is constant on the orbits of the family $\mathcal{F}_{\mathcal{D}}$ of local vector fields taking values in \mathcal{D} . Indeed, given a point q_0 in the orbit, any other point q_1 of the orbit is of the form $q_1 = (\phi_{t_k}^{X_k} \circ \dots \circ \phi_{t_1}^{X_1})(q_0)$ for some vector fields $X_i \in \mathcal{F}_{\mathcal{D}}$ and times $t_i \in \mathbb{R}$. Therefore we can get such point by concatenation of a finite number of curves of the form $C : t \in [0, T] \mapsto \phi_t^X(q)$, with q a point in the orbit and

$X \in \mathcal{F}_D$. Integrating df along a curve C of such type we get on one hand $\int_C df = f(\phi_T^X(q)) - f(q)$ and on the other $\int_C df = \int_0^T \langle df, X \rangle_{\phi_t^X(q)} dt = \int_0^T 0 dt = 0$. Therefore $f(\phi_T^X(q)) = f(q)$ and f is constant along the orbit.

Finally, for a bracket-generating distribution, Chow–Rashevsky theorem (see for instance [2]) ensures that there is only one orbit, the full manifold Q . Therefore, if df takes its values in \mathcal{D}° , then f is a constant function on Q . ■

In the general case, provided that the distribution associated with the Lie algebra $L_{\mathcal{D}}$ is of constant rank, we can restrict our dynamical system to each one of the orbits (which are the integral manifolds of the distribution associated with $L_{\mathcal{D}}$, and hence immersed submanifolds of Q), thus obtaining a Lagrangian system with nonholonomic constraints defined by a bracket-generating distribution. Hence X^*E_L is constant on every orbit of $L_{\mathcal{D}}$.

4 The Hamilton–Jacobi problem in the intrinsic formalism

In the above sections we have been using the standard Lagrangian formalism of nonholonomic constrained problems. Next we develop the theory using the intrinsic Lagrangian formalism (also called the distributional approach). This will allow us to justify the choice made for stating the Lagrangian nonholonomic Hamilton–Jacobi problem. The distributional approach was initiated by Bocharov and Vinogradov [7] and further developed by Śniatycki and coworkers [4, 34]. Similar equations, within the more general framework of Lie algebroids, appear also in [12].

4.1 Intrinsic Lagrangian formalism for nonholonomic systems

In the above standard Lagrangian formalism of nonholonomic constrained problems, the theory is developed on the whole TQ by introducing the constraint forces. But it is clear that only the values in \mathcal{D} are relevant: while the theory depends on the value of the Lagrangian in an open neighbourhood of \mathcal{D} , the final dynamics is defined only on the submanifold \mathcal{D} . Therefore it is interesting to develop the theory intrinsically in \mathcal{D} .

Recall that we defined the rank $2r$ vector subbundle $\mathcal{T}^{\mathcal{D}}\mathcal{D} \rightarrow \mathcal{D}$ of $T\mathcal{D}$ by

$$\mathcal{T}^{\mathcal{D}}\mathcal{D} = \{ V \in T\mathcal{D} \mid T\tau_Q(V) \in \mathcal{D} \},$$

and that item 3 in Theorem 1 expresses the fact that the nonholonomic Lagrangian system is regular if, and only if, $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ is a symplectic subbundle of $T(TQ)|_{\mathcal{D}}$, that is, $\mathcal{T}^{\mathcal{D}}\mathcal{D} \cap (\mathcal{T}^{\mathcal{D}}\mathcal{D})^\perp = \{0\}$. Therefore, the restriction $\omega^{L\mathcal{D}}$ of the symplectic form ω_L to the subbundle $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ is regular, and hence the pair $(\mathcal{T}^{\mathcal{D}}\mathcal{D}, \omega^{L\mathcal{D}})$ is a symplectic vector bundle.

Similarly, we denote by $\varepsilon^{L\mathcal{D}}$ the restriction of dE_L to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$. It follows that there exists a unique $\Gamma \in \text{Sec}(\mathcal{T}^{\mathcal{D}}\mathcal{D})$ such that

$$i_\Gamma \omega^{L\mathcal{D}} = \varepsilon^{L\mathcal{D}}. \tag{2}$$

From the definition of $\omega^{L\mathcal{D}}$ and $\varepsilon^{L\mathcal{D}}$ one obtains that the section Γ here is just the restriction to \mathcal{D} of the dynamical vector field Γ of the last section, and it is a SODE in the sense that $T\tau_Q(\Gamma(v)) = v$, for every $v \in \mathcal{D}$. We will not make any notational distinction between the two views of the dynamical vector field.

The advantage of this formulation of the nonholonomic problem is that we can work entirely in the bundle $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ following similar arguments to those given for the unconstrained case. There

is only one relevant difference: the 2-form $\omega^{L\mathcal{D}}$ is not exact. In fact it even does not make sense to talk about closed forms because $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ is not a tangent bundle, neither a Lie algebroid, except for integrable constraints.

4.2 The general Hamilton–Jacobi problem

In this framework, a solution to the *general nonholonomic Lagrangian Hamilton–Jacobi problem* is a section $\sigma \in \text{Sec}(\mathcal{D})$ of the vector bundle $\tau: \mathcal{D} \rightarrow Q$ such that the natural lift of its integral curves are integral curves of Γ . This statement has sense obviously because our bundles are subbundles of a tangent bundle, and hence its sections are vector fields. It is also clear that this corresponds exactly to the definition in the above section, with a change of notation $X \leftrightarrow \sigma$.

Given a section $\sigma \in \text{Sec}(\mathcal{D})$ we can define the map $\mathcal{T}\sigma: \mathcal{D} \rightarrow \mathcal{T}^{\mathcal{D}}\mathcal{D}$ as the restriction of the tangent map $T\sigma: TQ \rightarrow T\mathcal{D}$. It is well-defined since $T\tau_Q(T\sigma(v)) = v \in \mathcal{D}$, so that $\mathcal{T}\sigma(v) \in \mathcal{T}^{\mathcal{D}}\mathcal{D}$. With this definition, and according to Theorem 2, a section $\sigma \in \text{Sec}(\mathcal{D})$ is a solution to the general nonholonomic Lagrangian Hamilton–Jacobi problem if, and only if,

$$\Gamma \circ \sigma = \mathcal{T}\sigma \circ \sigma. \quad (3)$$

For the following proposition we need a somehow extended notion of the pullback. In particular, for a section θ of the exterior bundle of $(\mathcal{T}^{\mathcal{D}}\mathcal{D})^*$, we are redefining the meaning of $\sigma^*\theta$ as the section of the exterior bundle of \mathcal{D}^* given by

$$(\sigma^*\theta)_q(v_1, \dots, v_p) = \theta_{\sigma(q)}(\mathcal{T}\sigma(v_1), \dots, \mathcal{T}\sigma(v_p)),$$

for $q \in Q$ and $v_1, \dots, v_p \in \mathcal{D}$.

Proposition 3 *A section $\sigma \in \text{Sec}(\mathcal{D})$ is a solution to the general nonholonomic Lagrangian Hamilton–Jacobi problem if, and only if,*

$$i_{\sigma}(\sigma^*\omega^{L\mathcal{D}}) = \sigma^*\varepsilon^{L\mathcal{D}}.$$

Proof From equation (2), $i_{\Gamma}\omega^{L\mathcal{D}} = \varepsilon^{L\mathcal{D}}$, we have that $\sigma^*(i_{\Gamma}\omega^{L\mathcal{D}}) = \sigma^*\varepsilon^{L\mathcal{D}}$. Now, taking into account that $\Gamma \circ \sigma = \mathcal{T}\sigma \circ \sigma$, for every $v_q \in \mathcal{D}$ we obtain

$$\begin{aligned} \sigma^*(i_{\Gamma}\omega^{L\mathcal{D}})(v_q) &= (i_{\Gamma}\omega^{L\mathcal{D}})_{\sigma(q)}(\mathcal{T}_q\sigma(v_q)) = \omega^{L\mathcal{D}}_{\sigma(q)}(\Gamma(\sigma(q)), \mathcal{T}_q\sigma(v_q)) \\ &= \omega^{L\mathcal{D}}_{\sigma(q)}(\mathcal{T}_q\sigma(\sigma(q)), \mathcal{T}_q\sigma(v_q)) = (\sigma^*\omega^{L\mathcal{D}})_q(\sigma(q), v_q) = i_{\sigma}(\sigma^*\omega^{L\mathcal{D}})(v_q), \end{aligned}$$

and the result follows. ■

4.3 The restricted Hamilton–Jacobi problem

A solution of the (*restricted*) *Lagrangian nonholonomic Hamilton–Jacobi problem* is a solution σ of the general Lagrangian nonholonomic Hamilton–Jacobi problem which moreover satisfies the condition

$$\sigma^*\omega^{L\mathcal{D}} = 0.$$

According to the preceding Proposition, it follows that σ must also satisfy $\sigma^*\varepsilon^{L\mathcal{D}} = 0$. Notice that, for $v \in \mathcal{D}$, we have

$$\langle \sigma^*\varepsilon^{L\mathcal{D}}, v \rangle = \langle \sigma^*(dE_L|_{\mathcal{T}^{\mathcal{D}}\mathcal{D}}), v \rangle = \langle dE_L, \mathcal{T}\sigma(v) \rangle = \langle d(\sigma^*E_L), v \rangle,$$

so that $\sigma^*\varepsilon^{L\mathcal{D}} = d(\sigma^*E_L)|_{\mathcal{D}}$.

Summarizing, we have proved the following:

Proposition 4 *A section $\sigma \in \text{Sec}(\mathcal{D})$ is a solution to the Lagrangian nonholonomic Hamilton–Jacobi problem if, and only if, $T\sigma(\mathcal{D}) \subset \mathcal{T}^{\mathcal{D}}\mathcal{D}$ is a Lagrangian subbundle of $(\mathcal{T}^{\mathcal{D}}\mathcal{D}, \omega^{L\mathcal{D}})$ and $d(\sigma^*E_L) \in \text{Sec}(\mathcal{D}^\circ)$.*

Note that, when the distribution is bracket-generating, the last condition means that the energy is constant, as we have seen in Proposition 2 at the end of the preceding section.

5 Coordinate expressions and quasivelocities

In order to find local expressions for the objects we have defined, we can use local coordinates in the base Q and a set of linear coordinates (quasivelocities [6, 10]) on the tangent bundle adapted to the distribution \mathcal{D} . This will greatly simplify many expressions.

Let (x^i) be local coordinates on Q and choose a local basis $\{e_\alpha\}$ of sections of \mathcal{D} . Complete with $\{e_A\}$ to a local basis $\{e_\alpha, e_A\}$ of $\mathfrak{X}(Q)$, and denote the associated linear coordinates by (y^α, y^A) , that is, $y^\alpha = \widehat{e^\alpha}$ and $y^A = \widehat{e^A}$, where $\{e^\alpha, e^A\}$ is the dual basis. So we have coordinates (x^i, y^α, y^A) of TQ . In these coordinates the constraints read $y^A = 0$, so they are adapted to the submanifold $D \subset TQ$, and (x^i, y^α) can be used as coordinates for \mathcal{D} .

In the local coordinate system (x^i) on Q , the elements of the basis $e_\alpha \in \mathfrak{X}(Q)$ are given by

$$e_\alpha = \rho_\alpha^i \frac{\partial}{\partial x^i},$$

for some local functions $\rho_\alpha^i \in C^\infty(Q)$. The bracket of the sections e_α is of the form

$$[e_\alpha, e_\beta] = C_{\alpha\beta}^\gamma e_\gamma + C_{\alpha\beta}^A e_A,$$

with $C_{\alpha\beta}^i \in C^\infty(Q)$ local functions on Q . The constraints are integrable (holonomic) if, and only if, $C_{\alpha\beta}^A = 0$. The vector fields e_A have a similar expression, $e_A = \rho_A^i \frac{\partial}{\partial x^i}$.

Remark 3 In the classical literature, the functions of the type C_{bc}^a are known as Hamel’s transpositional symbols [15], which obviously are nothing but the structure coefficients (in the Cartan’s sense) of the moving frame $\{e_a\}$, see e.g. [13]. Similar expressions arise in the theory of Lie algebroids, where the use of quasivelocities appears naturally [12, 23, 29].

Associated with the above basis and coordinates we can find a local basis of sections of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$, that is, a family of $2r$ vector fields tangent to \mathcal{D} which moreover project point-wise to vectors on \mathcal{D} . The coordinate vector fields $\partial/\partial x^i$ and $\partial/\partial y^\alpha$ are a basis of vector fields tangent to \mathcal{D} . The vector fields $\partial/\partial y^\alpha$ are vertical so that they project (through $T\tau$) to the zero section of \mathcal{D} . However, the vectors $\partial/\partial x^i$ do not (in general) project to admissible velocities. Taking an appropriate linear combination we have that the vector fields $\rho_\alpha^i \partial/\partial x^i$ project to e_α , so that they are sections of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$. Moreover, since they are linearly independent, we have got a basis of sections of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$:

$$\mathcal{X}_\alpha = \rho_\alpha^i \frac{\partial}{\partial x^i} \quad \text{and} \quad \mathcal{V}_\alpha = \frac{\partial}{\partial y^\alpha}.$$

This basis of sections of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ can be completed to a basis of sections of $T\mathcal{D}$ by adding the vector fields

$$\mathcal{X}_A = \rho_A^i \frac{\partial}{\partial x^i},$$

and the brackets of the vector fields of such a basis are given by

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = C_{\alpha\beta}^\gamma \mathcal{X}_\gamma + C_{\alpha\beta}^A \mathcal{X}_A, \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta] = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0.$$

A SODE $\Gamma \in \text{Sec}(\mathcal{T}^{\mathcal{D}}\mathcal{D})$ is a vector field tangent to \mathcal{D} and such that $T\tau(\Gamma(v)) = v$ for every $v \in \mathcal{D}$. It follows that it is of the form

$$\Gamma = y^\alpha \mathcal{X}_\alpha + f^\alpha(x^i, y^\beta) \mathcal{V}_\alpha,$$

for some local functions $f^\alpha \in C^\infty(\mathcal{D})$. The differential equations for its integral curves are

$$\dot{x}^i = \rho_\alpha^i y^\alpha \quad \dot{y}^\alpha = f^\alpha(x^i, y^\beta).$$

The above expressions can be specialized to the frequent case when the constraints are given by expressing some velocities as linear functions of some other velocities,

$$\dot{x}^A - B_\alpha^A(x) \dot{x}^\alpha = 0.$$

The local basis $\{e_\alpha, e_A\}$ can be taken to be

$$e_\alpha = \frac{\partial}{\partial x^\alpha} + B_\alpha^A \frac{\partial}{\partial x^A}, \quad e_A = \frac{\partial}{\partial x^A},$$

and therefore the adequate quasivelocities are

$$y_\alpha = \dot{x}_\alpha, \quad y^A = \dot{x}^A - B_\alpha^A(x) \dot{x}^\alpha.$$

The natural velocities are then given in terms of the quasivelocities by

$$\dot{x}_\alpha = y_\alpha, \quad \dot{x}^A = y^A + B_\alpha^A(x) y^\alpha.$$

The local basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ of sections of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ can be given in terms of the natural coordinates by

$$\mathcal{X}_\alpha = \frac{\partial}{\partial x^\alpha} + B_\alpha^A \frac{\partial}{\partial x^A} + \dot{x}^\beta e_\alpha(B_\beta^A) \frac{\partial}{\partial \dot{x}^A}, \quad \mathcal{V}_\alpha = \frac{\partial}{\partial \dot{x}^\alpha} + B_\alpha^A \frac{\partial}{\partial \dot{x}^A},$$

and we can complete it to a local basis of $\mathfrak{X}(\mathcal{D})$ with

$$\mathcal{X}_A = \frac{\partial}{\partial x^A}.$$

We can further complete to a local basis of $\mathfrak{X}(TQ)$ with

$$\mathcal{V}_A = \frac{\partial}{\partial \dot{x}^A}.$$

The commutators of the elements of this basis of sections of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ are

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = R_{\alpha\beta}^A \left(\frac{\partial}{\partial x^A} + \frac{\partial B_\gamma^B}{\partial x^A} v^\gamma \frac{\partial}{\partial v^B} \right), \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta] = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0,$$

where $R_{\alpha\beta}^A \equiv [e_\alpha, e_\beta]x^A = e_\alpha(B_\beta^A) - e_\beta(B_\alpha^A)$.

Lagrange–D’Alembert equations The constrained Lagrangian system (L, \mathcal{D}) is regular if the restriction of the fibered Hessian G^L to \mathcal{D} is a regular bilinear tensor at every point. This restricted Hessian $G^{L\mathcal{D}}$ has a particularly simple expression in the coordinates (x^i, y^α, y^A) ,

$$G_{(x^i, y^\alpha)}^{L\mathcal{D}}(e_\alpha, e_\beta) = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}(x^i, y^\alpha, 0).$$

The local expression of Lagrange–d’Alembert equations can be easily written in these coordinates without the need of Lagrange multipliers. By contracting the equations $i_\Gamma \omega^{L\mathcal{D}} - \varepsilon^{L\mathcal{D}} = 0$ with the elements of the basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$, and taking into account the constraints $y^A = 0$, these equations read

$$\Gamma \left(\frac{\partial L}{\partial y^\alpha} \right) + \frac{\partial L}{\partial y^\gamma} C_{\alpha\beta}^\gamma y^\beta - \rho_\alpha^i \frac{\partial L}{\partial x^i} = - \frac{\partial L}{\partial y^A} C_{\alpha\beta}^A y^\beta, \quad (4)$$

where $\Gamma = y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha$ is the SODE vector field we are looking for.

Taking into account the second-order condition, the differential equations for the solutions of the dynamics are

$$\begin{aligned} \dot{x}^i &= \rho_\alpha^i y^\alpha, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial y^\alpha} \right) + \frac{\partial L}{\partial y^\gamma} C_{\alpha\beta}^\gamma y^\beta - \rho_\alpha^i \frac{\partial L}{\partial x^i} &= - \frac{\partial L}{\partial y^A} C_{\alpha\beta}^A y^\beta, \\ y^A &= 0. \end{aligned} \quad (5)$$

General Hamilton–Jacobi problem By evaluating the equations (4) on the image of the section σ , *i.e.* at a point of the form $y^\alpha = \sigma^\alpha(x)$, and taking into account the general Hamilton–Jacobi condition, Eq. (3), which can be expressed as a relation between differential operators as $\sigma^* \circ \mathcal{L}_\Gamma = \mathcal{L}_\sigma \circ \sigma^*$, we obtain

$$\mathcal{L}_\sigma \left(\frac{\partial L}{\partial y^\alpha} \circ \sigma \right) + \left(\frac{\partial L}{\partial y^\gamma} \circ \sigma \right) C_{\alpha\beta}^\gamma \sigma^\beta - \rho_\alpha^i \left(\frac{\partial L}{\partial x^i} \circ \sigma \right) = - \left(\frac{\partial L}{\partial y^A} \circ \sigma \right) C_{\alpha\beta}^A \sigma^\beta, \quad (6)$$

which is the local expression of the general Hamilton–Jacobi equation. In order to find the solutions of the dynamics these equations must be supplemented with the differential equations for the integral curves of σ , *i.e.* $\dot{x}^i = \rho_\alpha^i \sigma^\alpha$.

With a simplified notation, the equations to be solved are

$$\begin{aligned} \dot{x}^i &= \rho_\alpha^i \sigma^\alpha, \\ \mathcal{L}_\sigma \left(\frac{\partial L}{\partial y^\alpha} \right) + \frac{\partial L}{\partial y^\gamma} C_{\alpha\beta}^\gamma \sigma^\beta - \rho_\alpha^i \frac{\partial L}{\partial x^i} &= - \frac{\partial L}{\partial y^A} C_{\alpha\beta}^A \sigma^\beta, \end{aligned} \quad (7)$$

where all the partial derivatives of the Lagrangian must be evaluated at points of the form $(x^i, \sigma^\alpha(x))$ before taking further derivatives. We remark the formal similarity of these equations (7) and equations (5), which are obtained formally by the substitution $y^\alpha = \sigma^\alpha(x)$ everywhere.

Restricted Hamilton–Jacobi problem For the expression of the Hamilton–Jacobi equation we need the explicit expression of the form $\omega^{L\mathcal{D}}$. This coordinate expression can be easily found by using the relation

$$\omega^{L\mathcal{D}}(X, Y) = d\theta_L(Y, X) = \mathcal{L}_Y(\theta_L(X)) - \mathcal{L}_X(\theta_L(Y)) + \theta_L([X, Y]), \quad X, Y \in \text{Sec}(\mathcal{T}^{\mathcal{D}}\mathcal{D}),$$

applied to the elements of the basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$. It turns out that

$$\begin{aligned}\omega^{L\mathcal{D}}(\mathcal{X}_\alpha, \mathcal{X}_\beta) &= \rho_\beta^i \frac{\partial^2 L}{\partial y^\alpha \partial x^i} - \rho_\alpha^i \frac{\partial^2 L}{\partial y^\beta \partial x^i} + \frac{\partial L}{\partial y^\gamma} C_{\alpha\beta}^\gamma + \frac{\partial L}{\partial y^A} C_{\alpha\beta}^A, \\ \omega^{L\mathcal{D}}(\mathcal{X}_\alpha, \mathcal{V}_\beta) &= \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}, \\ \omega^{L\mathcal{D}}(\mathcal{V}_\alpha, \mathcal{V}_\beta) &= 0,\end{aligned}$$

where all the partial derivatives of the Lagrangian are taken at points in the constraint subbundle \mathcal{D} , *i.e.* in the submanifold $y^A = 0$. Therefore, the local expression of $\omega^{L\mathcal{D}}$ is

$$\omega^{L\mathcal{D}} = \frac{1}{2} \left(\rho_\beta^i \frac{\partial^2 L}{\partial y^\alpha \partial x^i} - \rho_\alpha^i \frac{\partial^2 L}{\partial y^\beta \partial x^i} + \frac{\partial L}{\partial y^\gamma} C_{\alpha\beta}^\gamma + \frac{\partial L}{\partial y^A} C_{\alpha\beta}^A \right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta + \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \mathcal{X}^\alpha \wedge \mathcal{V}^\beta,$$

where $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$.

To compute pullbacks we can proceed as follows. If $\sigma: Q \rightarrow D$ is a section of \mathcal{D} then the map $\mathcal{T}\sigma$ is determined by

$$\mathcal{T}\sigma(e_\alpha) = \mathcal{X}_\alpha + \rho_\alpha^i \frac{\partial \sigma^\beta}{\partial x^i} \mathcal{V}_\beta,$$

and therefore the pullback of the dual basis $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ is

$$\sigma^* \mathcal{X}^\alpha = e^\alpha \quad \text{and} \quad \sigma^* \mathcal{V}^\alpha = \rho_\beta^i \frac{\partial \sigma^\alpha}{\partial x^i} e^\beta.$$

From this it is straightforward to calculate the local expression of $\sigma^* \omega^{L\mathcal{D}}$:

$$\begin{aligned}\sigma^* \omega^{L\mathcal{D}} &= \\ \frac{1}{2} \left[\rho_\beta^i \left(\frac{\partial^2 L}{\partial y^\alpha \partial x^i} + \frac{\partial^2 L}{\partial y^\alpha \partial y^\gamma} \frac{\partial \sigma^\gamma}{\partial x^i} \right) - \rho_\alpha^i \left(\frac{\partial^2 L}{\partial y^\beta \partial x^i} + \frac{\partial^2 L}{\partial y^\beta \partial y^\gamma} \frac{\partial \sigma^\gamma}{\partial x^i} \right) + \frac{\partial L}{\partial y^\gamma} C_{\alpha\beta}^\gamma + \frac{\partial L}{\partial y^A} C_{\alpha\beta}^A \right] e^\alpha \wedge e^\beta.\end{aligned}$$

The equation $\sigma^* \omega^{L\mathcal{D}} = 0$ is equivalent to the vanishing of the expression between braces:

$$\rho_\beta^i \left(\frac{\partial^2 L}{\partial y^\alpha \partial x^i} + \frac{\partial^2 L}{\partial y^\alpha \partial y^\gamma} \frac{\partial \sigma^\gamma}{\partial x^i} \right) - \rho_\alpha^i \left(\frac{\partial^2 L}{\partial y^\beta \partial x^i} + \frac{\partial^2 L}{\partial y^\beta \partial y^\gamma} \frac{\partial \sigma^\gamma}{\partial x^i} \right) + \frac{\partial L}{\partial y^\gamma} C_{\alpha\beta}^\gamma + \frac{\partial L}{\partial y^A} C_{\alpha\beta}^A = 0. \quad (8)$$

Alternatively, one can calculate $\sigma^* \omega^{L\mathcal{D}}(e_\alpha, e_\beta) = -d(\sigma^* \theta_L)(e_\alpha, e_\beta)$, from where the vanishing of $\sigma^* \omega^{L\mathcal{D}} = 0$ is equivalent to the equations

$$\mathcal{L}_{e_\alpha} \left(\frac{\partial L}{\partial y^\beta} \circ \sigma \right) - \mathcal{L}_{e_\beta} \left(\frac{\partial L}{\partial y^\alpha} \circ \sigma \right) - \left(\frac{\partial L}{\partial y^\gamma} \circ \sigma \right) C_{\alpha\beta}^\gamma = \left(\frac{\partial L}{\partial y^A} \circ \sigma \right) C_{\alpha\beta}^A.$$

Finally, the pullback by σ of the energy is easily calculated:

$$\sigma^* E_L = \sigma^* \left(y^\alpha \frac{\partial L}{\partial y^\alpha} + y^A \frac{\partial L}{\partial y^A} - L \right) = \sigma^\alpha \left(\frac{\partial L}{\partial y^\alpha} \circ \sigma \right) - L \circ \sigma.$$

If the distribution is bracket-generating, then the equation $d(\sigma^* \varepsilon^{L\mathcal{D}}) \in \text{Sec}(\mathcal{D}^\circ)$ can be substituted by

$$\sigma^\alpha(x) \frac{\partial L}{\partial y^\alpha}(x^i, \sigma^\beta(x)) - L(x^i, \sigma^\beta(x)) = \text{constant}.$$

6 Complete solutions

The essential idea in the standard (unconstrained) Hamilton–Jacobi theory consists in finding a complete family of solutions to the problem (not only one particular solution). In the present context a complete solution can be defined as follows:

Definition 1 *Consider a solution σ_λ to the general (respectively, restricted) Lagrangian nonholonomic Hamilton–Jacobi problem depending on $r = \text{rank}(\mathcal{D})$ additional parameters $\lambda \in \Lambda$ (where $\Lambda \subseteq \mathbb{R}^r$ is some open set) and suppose that the map $\Phi: Q \times \Lambda \rightarrow \mathcal{D}$ given by $\Phi(q, \lambda) = \sigma_\lambda(q)$ is a local diffeomorphism. In this case the family $\{\sigma_\lambda; \lambda \in \Lambda\}$ is said to be a complete solution to the general (respectively, restricted) Lagrangian nonholonomic Hamilton–Jacobi problem.*

In other words, a *complete solution* is a local diffeomorphism $\Phi: Q \times \Lambda \rightarrow \mathcal{D}$ over the identity in Q , such that for every $\lambda \in \Lambda$ the section $\sigma_\lambda \in \text{Sec}(\mathcal{D})$ given by $\sigma_\lambda(q) = \Phi(q, \lambda)$, is a solution to the general (restricted) Hamilton–Jacobi problem.

The interest of this notion is that all the integral curves of Γ can be actually described as integral curves of appropriate vector fields in the complete solution. For every point $v \in \text{Im } \Phi$ we take $q = \tau_Q(v)$ and we find $\lambda \in \mathbb{R}^r$ such that $\Phi(q, \lambda) = v$. The vector field σ_λ is a solution to the generalized Hamilton–Jacobi problem, with $\sigma_\lambda(q) = v$. Taking the integral curve $\gamma(t)$ of σ_λ passing through q , we have that $\dot{\gamma}(t)$ is the solution to the dynamics starting at v .

In what follows, for simplicity, we will assume that $\Phi: Q \times \Lambda \rightarrow \mathcal{D}$ is a global diffeomorphism. We then define the map $F: \mathcal{D} \rightarrow \mathbb{R}^r$ by $F = \text{pr}_2 \circ \Phi^{-1}$, where $\text{pr}_2: Q \times \Lambda \rightarrow \Lambda$ is the projection onto the second factor.

From the very definition, it follows that a complete solution provides the manifold \mathcal{D} with a foliation transverse to the fibers of $\tau: \mathcal{D} \rightarrow Q$, the leaves being the image of the vector fields σ_λ , and that the solution vector field Γ is tangent to the leaves. We now study this foliation with more detail, specially in the case of a complete solution to the restricted problem.

Proposition 5 *The following properties hold.*

1. *For every $\lambda \in \Lambda$ we have $F^{-1}(\lambda) = \text{Im}(\sigma_\lambda)$.*
2. *The map $T\Phi: TQ \times T\Lambda \rightarrow T\mathcal{D}$ restricts to a map $\mathcal{T}\Phi: \mathcal{D} \times T\Lambda \rightarrow \mathcal{T}^{\mathcal{D}}\mathcal{D}$. Moreover $\mathcal{T}\Phi$ is a diffeomorphism (a local diffeomorphism if Φ is a local diffeomorphism).*
3. *The section $\bar{\Gamma} \in \text{Sec}(\mathcal{D} \times T\Lambda \rightarrow Q \times \Lambda)$ defined by $\bar{\Gamma} = (\mathcal{T}\Phi)^{-1} \circ \Gamma \circ \Phi$ has the form $\bar{\Gamma}(q, \lambda) = (\sigma_\lambda(q), 0_\lambda)$.*
4. *The components of the map F are constants of motion.*
5. *If Φ is a complete solution to the restricted problem, then the subbundles $\{(v, 0) \in \mathcal{D} \times T\Lambda\}$ and $\{(0, z) \in \mathcal{D} \times T\Lambda\}$ are Lagrangian subbundles of the symplectic bundle $(\mathcal{D} \times T\Lambda, \Phi^*(\omega^{L\mathcal{D}}))$.*

Proof We will make use of the following fact:

$$\mathcal{T}\Phi(v_q, 0_\lambda) = T\sigma_\lambda(v_q). \tag{9}$$

Indeed, if $\gamma(s)$ is a curve in Q such that $\dot{\gamma}(0) = v_q$, then

$$\begin{aligned} T\Phi(v_q, 0_\lambda)f &= (v_q, 0_\lambda)(f \circ \Phi) = \left. \frac{d}{ds} f(\Phi(\gamma(s), \lambda)) \right|_{s=0} \\ &= \left. \frac{d}{ds} f(\sigma_\lambda(\gamma(s))) \right|_{s=0} = \left. \frac{d}{ds} (\sigma_\lambda^* f)(\gamma(s)) \right|_{s=0} \\ &= v_q(\sigma_\lambda^* f) = T\sigma_\lambda(v_q)f \end{aligned}$$

for every function $f \in C^\infty(\mathcal{D})$, which proves the equality (9).

Now let us proceed with the proof of the proposition:

1. $v \in F^{-1}(\lambda) \iff F(v) = \lambda \iff \Phi^{-1}(v) = (\tau(v), \lambda) \iff v = \Phi(\tau(v), \lambda) \iff v = \sigma_\lambda(\tau(v)) \iff v \in \text{Im}(\sigma_\lambda)$.

2. For every $(v_q, z_\lambda) \in TQ \times T\Lambda$

$$T\tau(T\Phi(v_q, z_\lambda)) = T(\tau \circ \Phi)(v_q, z_\lambda) = T\text{pr}_1(v_q, z_\lambda) = v_q .$$

So, $T\Phi(v_q, z_\lambda)$ belongs to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ if, and only if, $v_q \in \mathcal{D}$.

3. We have just to prove that $T\Phi(\sigma_\lambda(q), 0_\lambda) = \Gamma(\Phi(q, \lambda))$, for every $(q, \lambda) \in Q \times \Lambda$. Using (9), for $v_q = \sigma_\lambda(q)$ we have that $T\Phi(\sigma_\lambda(q), 0_\lambda) = T\sigma_\lambda(\sigma_\lambda(q))$, and taking into account that $T\sigma_\lambda \circ \sigma_\lambda = \Gamma \circ \sigma_\lambda$, we finally get

$$T\Phi(\sigma_\lambda(q), 0_\lambda) = T\sigma_\lambda(\sigma_\lambda(q)) = \Gamma(\sigma_\lambda(q)) = \Gamma(\Phi(q, \lambda)).$$

4. F is constant on $\text{Im} \sigma_\lambda$, and Γ is tangent to $\text{Im} \sigma_\lambda$, so the result follows.

5. First, using (9) we have

$$\begin{aligned} (\Phi^* \omega^{L\mathcal{D}})_{(q,\lambda)}((v_q, 0_\lambda), (w_q, 0_\lambda)) &= \omega_{\Phi(q,\lambda)}^{L\mathcal{D}}(T\Phi(v_q, 0_\lambda), T\Phi(w_q, 0_\lambda)) \\ &= \omega_{\Phi(q,\lambda)}^{L\mathcal{D}}(T\sigma_\lambda(v_q), T\sigma_\lambda(w_q)) \\ &= (\sigma_\lambda^* \omega^{L\mathcal{D}})_q(v_q, w_q) = 0 \end{aligned}$$

Furthermore, notice that $T\Phi(\text{Ver}(\text{pr}_1)) = \text{Ver}(\tau)$, which can be easily proved. Thus

$$(\Phi^* \omega^{L\mathcal{D}})_{(q,\lambda)}((0_q, y_\lambda), (0_q, z_\lambda)) = \omega_{\Phi(q,\lambda)}^{L\mathcal{D}}(T\Phi(0_q, y_\lambda), T\Phi(0_q, z_\lambda)) = 0$$

because $\text{Ver}(\tau)$ is an isotropic (in fact Lagrangian) subbundle of the symplectic bundle $(\mathcal{T}^{\mathcal{D}}\mathcal{D}, \omega^{L\mathcal{D}})$.

This finishes the proof. ■

In the case of a complete solution to the restricted problem, only the terms of the form $\Phi^* \omega^{L\mathcal{D}}((v, 0), (0, z))$ can possibly be nonzero, and they can be expressed in terms of the Hessian,

$$(\Phi^* \omega^{L\mathcal{D}})_{(q,\lambda)}((v_q, 0_\lambda), (0_q, z_\lambda)) = G_{\sigma_\lambda(q)}^{L\mathcal{D}}(v_q, \bar{z}(q, \lambda)),$$

where \bar{z} is defined by $\xi^V(0_q, \bar{z}(q, \lambda)) = T\Phi(0_q, z_\lambda)$. Indeed, by the definition of $G^{L\mathcal{D}}$ we have

$$\begin{aligned} (\Phi^* \omega^{L\mathcal{D}})_{(q,\lambda)}((v_q, 0_\lambda), (0_q, z_\lambda)) &= \omega_{\Phi(q,\lambda)}^{L\mathcal{D}}(T\sigma_\lambda(v_q), T\Phi(0_q, z_\lambda)) \\ &= \omega_{\sigma_\lambda(q)}^{L\mathcal{D}}(T\sigma_\lambda(v_q), \xi^V(0_q, \bar{z}(q, \lambda))) \\ &= G_{\sigma_\lambda(q)}^{L\mathcal{D}}(v, \bar{z}(q, \lambda)). \end{aligned}$$

The nonholonomic bracket For a complete solution the map F is a constant of the motion, that is, if we denote by f_1, \dots, f_r the components of F , then every function f_i is a constant of the motion for Γ . Conversely, a family of functionally independent first integrals $f_1, \dots, f_r \in C^\infty(\mathcal{D})$, satisfying the transversality condition $\det[\langle df_\alpha, e_\beta^V \rangle] \neq 0$ (where $\{e_\alpha\}$ is any local basis for \mathcal{D}), defines a complete integral by means of $\Phi^{-1}(v) = (\tau(v), (f_1(v), \dots, f_r(v)))$.

We now show that this functions are in involution with respect to the nonholonomic bracket. One of the possible constructions of such bracket is as follows. Given a function $g \in C^\infty(\mathcal{D})$, we consider the section $\delta g \in \text{Sec}((\mathcal{T}^{\mathcal{D}}\mathcal{D})^*)$ as the restriction of the differential of g to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$; that is, $\delta g = dg|_{\mathcal{T}^{\mathcal{D}}\mathcal{D}}$. Since the constrained system is regular, we can define the nonholonomic Hamiltonian section $\eta_g \in \text{Sec}(\mathcal{T}^{\mathcal{D}}\mathcal{D})$ by means $i_{\eta_g}\omega^{L\mathcal{D}} = \delta g$. Then we define the nonholonomic bracket of two functions $f, g \in C^\infty(\mathcal{D})$ by means of $\{f, g\}^{\text{nh}} = \omega^{L\mathcal{D}}(\eta_f, \eta_g)$. This bracket is skewsymmetric but it does not satisfy the Jacobi identity, except if the constraints are actually holonomic.

Theorem 4 *If $F = (f_1, f_2, \dots, f_r)$ then $\{f_i, f_j\}^{\text{nh}} = 0$.*

Proof We will show that the sections η_{f_i} are of the form $\eta_{f_i} = \mathcal{T}\Phi(X_{f_i}, 0)$. Indeed, let $Z \in \text{Sec}(\mathcal{D} \times T\Lambda \rightarrow Q \times \Lambda)$ be the section such that $\mathcal{T}\Phi \circ Z_i = \eta_{f_i} \circ \Phi$. For every $v \in \mathcal{D}$, let $q = \tau(v)$ and $\lambda = F(v)$, so that $\Phi(q, \lambda) = v$. For every $w \in \mathcal{D}_q$ we have

$$\begin{aligned} (\Phi^*\omega^{L\mathcal{D}})_v(Z_i(v), (w, 0)) &= \omega_v^{L\mathcal{D}}(\mathcal{T}\Phi(Z_i(v)), \mathcal{T}\Phi(w, 0)) = \omega_v^{L\mathcal{D}}(\eta_{f_i}(v), T\sigma_\lambda(w)) \\ &= T\sigma_\lambda(w) \cdot f_i = w \cdot (f_i \circ \sigma_\lambda) = w \cdot (\lambda_i) = 0 \end{aligned}$$

where we have used that $F \circ \sigma_\lambda = \lambda$ (constant), and hence $f_i \circ \sigma_\lambda = \lambda_i$. Therefore Z_i takes values in the orthogonal with respect to $\Phi^*\omega^{L\mathcal{D}}$ of the subbundle $\{(v, 0) \in \mathcal{D} \times T\Lambda\}$. Since this subbundle is Lagrangian, we have that Z_i takes values on it, *i.e.* it is of the form $Z_i = \mathcal{T}\Phi(W_i, 0)$. But then

$$\{f_i, f_j\}^{\text{nh}} = \omega^{L\mathcal{D}}(\eta_{f_i}, \eta_{f_j}) = \omega^{L\mathcal{D}}(\mathcal{T}\Phi(W_i, 0), \mathcal{T}\Phi(W_j, 0)) = (\Phi^*\omega^{L\mathcal{D}})((W_i, 0), (W_j, 0)) = 0,$$

which finishes the proof. ■

7 Example

7.1 The nonholonomic free particle

Every one-dimensional distribution is integrable, so that the easiest example of a nonholonomic system is obtained in \mathbb{R}^3 by a 2-dimensional distribution. By an adequate change of coordinates, the annihilator of \mathcal{D} is generated by the 1-form $dx_3 - x_2 dx_1$. The following example consists on a free particle under the action of such a constraint, and it is known as the nonholonomic free particle [4, 14, 33].

Consider a particle moving in $Q = \mathbb{R}^3$, with Lagrangian function

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2).$$

We have $\omega_L = dx_1 \wedge d\dot{x}_1 + dx_2 \wedge d\dot{x}_2 + dx_3 \wedge d\dot{x}_3$ and $dE_L = \dot{x}_1 d\dot{x}_1 + \dot{x}_2 d\dot{x}_2 + \dot{x}_3 d\dot{x}_3$, so the unconstrained dynamics is the well-known free dynamics described by the vector field

$$\Gamma_0 = \widehat{\omega}_L^{-1} \circ dE_L = \dot{x}_1 \frac{\partial}{\partial x_1} + \dot{x}_2 \frac{\partial}{\partial x_2} + \dot{x}_3 \frac{\partial}{\partial x_3}.$$

We introduce the nonholonomic constraint

$$\phi = \dot{x}_3 - x_2\dot{x}_1 = 0,$$

so that the constraint submanifold is $\mathcal{D} = \{(x_1, x_2, x_3; \dot{x}_1, \dot{x}_2, \dot{x}_3) \in TQ \mid \dot{x}_3 = x_2\dot{x}_1\}$. Applying D’Alembert’s principle for nonholonomic dynamics we get

$$\Gamma = \left(\dot{x}_1 \frac{\partial}{\partial x_1} + \dot{x}_2 \frac{\partial}{\partial x_2} + x_2\dot{x}_1 \frac{\partial}{\partial x_3} - \frac{x_2\dot{x}_2\dot{x}_1}{x_2^2 + 1} \frac{\partial}{\partial \dot{x}_1} + \frac{x_2\dot{x}_1}{x_2^2 + 1} \frac{\partial}{\partial \dot{x}_3} \right) \Big|_{\mathcal{D}}.$$

As a basis $\{e_\alpha\}$ of sections of \mathcal{D} we can take,

$$e_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}, \quad e_2 = \frac{\partial}{\partial x_2},$$

which we can complete with the vector field

$$e_3 = \frac{\partial}{\partial x_3}.$$

The associated quasivelocities are related to the velocities by

$$\begin{aligned} y_1 &= \dot{x}_1 & \dot{x}_1 &= y_1 \\ y_2 &= \dot{x}_2 & \dot{x}_2 &= y_2 \\ y_3 &= \dot{x}_3 - x_2\dot{x}_1 & \dot{x}_3 &= y_3 + x_2y_1. \end{aligned} \tag{10}$$

The corresponding basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ can be expressed in terms of the natural coordinates on the tangent bundle as

$$\begin{aligned} \mathcal{X}_1 &= \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} & \mathcal{X}_2 &= \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} \\ \mathcal{V}_1 &= \frac{\partial}{\partial \dot{x}_1} + x_2 \frac{\partial}{\partial \dot{x}_3} & \mathcal{V}_2 &= \frac{\partial}{\partial \dot{x}_2}, \end{aligned}$$

and it is completed to a basis of sections of $T\mathcal{D}$ with the vector field

$$\mathcal{X}_3 = \frac{\partial}{\partial x_3}.$$

We have that the symplectic section is given by

$$\omega^{L\mathcal{D}} = x_2y_1 \mathcal{X}^1 \wedge \mathcal{X}^2 + (1 + x_2^2) \mathcal{X}^1 \wedge \mathcal{V}^1 + \mathcal{X}^2 \wedge \mathcal{V}^2,$$

and the 1-form $\varepsilon^{L\mathcal{D}}$ is

$$\varepsilon^{L\mathcal{D}} = x_2y_1^2 \mathcal{X}^2 + (1 + x_2^2)y_1 \mathcal{V}^1 + y_2 \mathcal{V}^2.$$

From here, the dynamical section Γ , such that $i_\Gamma \omega^{L\mathcal{D}} = \varepsilon^{L\mathcal{D}}$, is

$$\Gamma = y_1 \mathcal{X}_2 + y_2 \mathcal{X}_2 - \frac{x_2}{1 + x_2^2} y_1 y_2 \mathcal{V}_1,$$

and the integral curves of Γ are the solutions to

$$\dot{x}_1 = y_1, \quad \dot{x}_2 = y_2, \quad \dot{y}_1 = -\frac{x_2}{1 + x_2^2} y_1 y_2, \quad \dot{y}_2 = 0, \tag{11}$$

together with the constraint $\dot{x}_3 = x_2\dot{x}_1$.

Note that in this example \mathcal{D} is a bracket-generating distribution. Indeed, e_1, e_2 and

$$[e_1, e_2] = \frac{1}{(x_2^2 + 1)^{3/2}} \left(\frac{\partial}{\partial x_3} - x_2 \frac{\partial}{\partial x_1} \right)$$

are linearly independent at each point of \mathbb{R}^3 . It follows that there is only one orbit for this distribution, the full space \mathbb{R}^3 , and hence any pair of points can be joined by concatenation of integral curves of vector fields belonging to the distribution \mathcal{D} .

7.2 The Hamilton–Jacobi problem

Let us state the Hamilton–Jacobi problem for this dynamics. According to the general discussion, we wish to find the vector fields X in Q such that:

1. X takes values in \mathcal{D} , and
2. X and Γ are X -related: $TX \circ X = \Gamma \circ X$.

From the first condition we have that X has the form

$$X = f \left(\frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} \right) + g \frac{\partial}{\partial x_2}, \quad (12)$$

and the second condition leads to

$$\mathcal{L}_X f = -\frac{x_2}{x_2^2 + 1} fg, \quad \mathcal{L}_X g = 0,$$

or, more explicitly,

$$f \frac{\partial f}{\partial x_1} + g \frac{\partial f}{\partial x_2} + x_2 f \frac{\partial f}{\partial x_3} = -fg \frac{x_2}{x_2^2 + 1}, \quad f \frac{\partial g}{\partial x_1} + g \frac{\partial g}{\partial x_2} + x_2 f \frac{\partial g}{\partial x_3} = 0. \quad (13)$$

It is easy to find particular solutions of these equations. For instance, $f = 1, g = 0$ is a solution and also $f = 0, g = 1$ is a solution. From them we can find some integral curves of the dynamics Γ . However, to obtain all the integral curves of the dynamical vector field we need to look for a complete solution.

Remark 4 An easier way to find these equations (together with the equations for the integral curves of X) is from D’Alembert equations (11) by the substitution $y_1 = f$ and $y_2 = g$. We get the equations

$$\dot{x}_1 = f, \quad \dot{x}_2 = g, \quad \dot{f} = -\frac{x_2}{1 + x_2^2} fg, \quad \text{and} \quad \dot{g} = 0.$$

Equations (13) follow from this when expanding the total time derivatives of the functions f and g and using the first two equations.

7.3 A complete solution

In order to get a complete solution, we look for a diffeomorphism $\Phi: Q \times \Lambda \rightarrow \mathcal{D}$, with $\Lambda = \mathbb{R}^2$, such that $\Phi(q, \lambda) = X_\lambda(q)$. In our case, taking (12) into account, this means $\Phi(x_1, x_2, x_3; \lambda_1, \lambda_2) = (x_1, x_2, x_3; f, g, x_2 f)$, where the functions f, g satisfy (13).

As we already know, any solution to the free problem with values in \mathcal{D} is also a solution to the constrained problem. There are some obvious solutions of the free problem that are constant vector fields, one of which, $X = (0, \text{constant}, 0)$, takes values in \mathcal{D} . Thus, as we look for a particular complete solution, we can choose $g = \lambda_1$ (constant) and try to find a corresponding value of f satisfying

$$f \frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial f}{\partial x_2} + x_2 f \frac{\partial f}{\partial x_3} = -f \lambda_1 \frac{x_2}{x_2^2 + 1}$$

If we assume that f depends only on x_2 , $f = f(x_2)$, then the above equation is

$$\frac{df}{dx_2} = -\frac{x_2}{x_2^2 + 1} f,$$

whose solution is $f = \lambda_2/\sqrt{x_2^2 + 1}$. Hence we have a complete solution that can be expressed as

$$X_{\lambda_1, \lambda_2} = \lambda_1 \frac{\partial}{\partial x_2} + \lambda_2 \frac{1}{\sqrt{x_2^2 + 1}} \left(\frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} \right) = \lambda_1 X_1 + \lambda_2 X_2, \quad (14)$$

with

$$X_1 = \frac{\partial}{\partial x_2}, \quad X_2 = \frac{1}{\sqrt{x_2^2 + 1}} \left(\frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} \right).$$

In other words, we have the diffeomorphism given by

$$\Phi(x_1, x_2, x_3; \lambda_1, \lambda_2) = \left(x_1, x_2, x_3; \frac{\lambda_2}{\sqrt{x_2^2 + 1}}, \lambda_1, \frac{x_2 \lambda_2}{\sqrt{x_2^2 + 1}} \right) \equiv (x_1, x_2, x_3; \dot{x}_1, \dot{x}_2, \dot{x}_3).$$

From here we get

$$\Phi^{-1}(x_1, x_2, x_3; \dot{x}_1, \dot{x}_2, \dot{x}_3) = \left(x_1, x_2, x_3; \dot{x}_1 \sqrt{x_2^2 + 1}, \dot{x}_2 \right) \equiv (x_1, x_2, x_3; \lambda_1, \lambda_2)$$

and, therefore, we obtain the following constants of motion

$$f_1 = \dot{x}_1 \sqrt{x_2^2 + 1}, \quad f_2 = \dot{x}_2.$$

Let us remark the linear expression of X_{λ_1, λ_2} , which is related to the fact that the conserved quantities are linear (in the velocities).

A straightforward calculation shows that the solution that we have found is a solution to the restricted Hamilton–Jacobi problem, that is $X_{\lambda_1, \lambda_2}^* \omega^{L\mathcal{D}} = 0$. Alternatively, we can calculate the pullback of ω_L

$$X^*(\omega_L) = \frac{\lambda_2}{(x_2^2 + 1)^{3/2}} (dx_3 - x_2 dx_1) \wedge dx_2,$$

which is in the exterior ideal generated by \mathcal{D}° .

The flow of X_{λ_1, λ_2} can also be easily computed: when $\lambda_1 \neq 0$, its integral curves are

$$\begin{aligned} x_1(t) &= x_1^0 + \frac{\lambda_2}{\lambda_1} (\arg \sinh(x_2^0 + \lambda_1 t) - \arg \sinh(x_2^0)), \\ x_2(t) &= x_2^0 + \lambda_1 t, \\ x_3(t) &= x_3^0 + \frac{\lambda_2}{\lambda_1} \left(\sqrt{1 + (x_2^0 + \lambda_1 t)^2} - \sqrt{1 + (x_2^0)^2} \right); \end{aligned}$$

when $\lambda_1 = 0$, the expression of the flow is

$$\begin{aligned} x_1(t) &= x_1^0 + \frac{\lambda_2}{\sqrt{1 + (x_2^0)^2}} t, \\ x_2(t) &= x_2^0, \\ x_3(t) &= x_3^0 + \frac{\lambda_2 x_2^0}{\sqrt{1 + (x_2^0)^2}} t. \end{aligned}$$

It follows that (the tangent lift of) these curves are the solutions of the nonholonomic problem.

Another complete solution We can obtain another complete solution by choosing $g = \lambda_1$ (constant), as above, but now we try $f = f(x_2, x_3)$. Then the second equations of (13) holds, and the first one reads

$$\lambda_1 \frac{\partial f}{\partial x_2} + x_2 f \frac{\partial f}{\partial x_3} = -f \lambda_1 \frac{x_2}{x_2^2 + 1},$$

which has a solution $f = \frac{x_3 \lambda_1 - \lambda_2}{x_2^2 + 1}$. Hence we obtain another complete solution to the Hamilton–Jacobi problem:

$$X_{\lambda_1, \lambda_2} = \frac{\lambda_1 x_3 - \lambda_2}{x_2^2 + 1} \frac{\partial}{\partial x_1} + \lambda_1 \frac{\partial}{\partial x_2} + x_2 \frac{\lambda_1 x_3 - \lambda_2}{x_2^2 + 1} \frac{\partial}{\partial x_3} = \lambda_1 X_1 + \lambda_2 X_2, \quad (15)$$

with

$$X_1 = \frac{x_3}{x_2^2 + 1} \left(\frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} \right) + \frac{\partial}{\partial x_2}, \quad X_2 = \frac{-1}{x_2^2 + 1} \left(\frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} \right). \quad (16)$$

This solution leads to the following constants of motion:

$$f_1 = x_3 \dot{x}_2 - \dot{x}_1 (x_2^2 + 1), \quad \dot{f}_2 = x_2.$$

In this case the solution that we found is not a solution to the restricted problem, that is $X_{\lambda_1, \lambda_2}^* \omega^{LD} \neq 0$. In fact, we have

$$X^*(\omega_L) = -2x_2 \frac{\lambda_1 x_3 - \lambda_2}{(x_2^2 + 1)^2} dx_1 \wedge dx_2 + \frac{\lambda_1}{x_2^2 + 1} dx_1 \wedge dx_3 + (\lambda_1 x_3 - \lambda_2) \frac{x_2^2 - 1}{(x_2^2 + 1)^2} dx_2 \wedge dx_3.$$

and hence

$$X^* \omega^{LD} = -\frac{x_2}{1 + x_2^2} (\lambda_1 x_3 - \lambda_2) e^1 \wedge e^2.$$

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References

- [1] R. ABRAHAM, J.E. MARSDEN, *Foundations of Mechanics*, 2nd edition, Addison–Wesley, 1978.
- [2] A. AGRACHEV, YU. SACHKOV, *Control theory from the geometric viewpoint*, Springer, 2004.
- [3] V.I. ARNOL'D, *Mathematical methods of classical mechanics*, 2nd edition, Springer, 1989.
- [4] L. BATES, J. ŚNIATYCKI, “Nonholonomic reduction”, *Rep. Math. Phys.* **32** (1993) 99–115.
- [5] A. M. BLOCH, P. S. KRISHNAPRASAD, J. E. MARSDEN, R. M. MURRAY, “Nonholonomic mechanical systems with symmetry”, *Arch. Rational Mech. Anal.* **36** (1996) 21–99.
- [6] A. M. BLOCH, J. E. MARSDEN, D. V. ZENKOV, “Quasivelocities and symmetries in non- holonomic systems”, *Dynamical Systems* **24**(2) (2009) 187–222.
- [7] A.V. BOCHAROV, A.M. VINOGRADOV, “The Hamiltonian form of mechanics with friction, non-holonomic mechanics, invariant mechanics, the theory of refraction and impact”, appendix II in: A.M. VINOGRADOV, B.A. KUPERSHMIDT, “The structures of Hamiltonian mechanics”, *Russ. Math. Surveys* **32** (1977) 177–243.

- [8] J.F. CARIÑENA, X. GRÀCIA, G. MARMO, E. MARTÍNEZ, M.C. MUÑOZ–LECANDA, N. ROMÁN–ROY, “Geometric Hamilton–Jacobi theory”, *Int. J. Geometric Methods Mod. Phys.* **3** (2006) 1417–1458.
- [9] J.F. CARIÑENA, X. GRÀCIA, G. MARMO, E. MARTÍNEZ, M.C. MUÑOZ–LECANDA, N. ROMÁN–ROY, “Hamilton–Jacobi theory and the evolution operator”, pp. 177–186 in *Mathematical physics and field theory. Julio Abad, in memoriam*, edited by M. Asorey, J.V. García Esteve, M.F. Rañada and J. Sesma; Prensas Universitarias de Zaragoza, 2009.
- [10] J.F. CARIÑENA, J. M. NUNES DA COSTA, P. S. SANTOS, “Quasi-coordinates from the point of view of Lie algebroid structures”, *J. Phys. A: Math. Theor.* **40** (2007) 10031–10048, doi:10.1088/1751-8113/40/33/008.
- [11] J. CORTÉS, M. DE LEÓN, “Reduction and reconstruction of the dynamics of nonholonomic systems”, *J. Phys. A: Math. Gen.* **32** (1999) 8615–8645.
- [12] J. CORTÉS, M. DE LEÓN, J.C. MARRERO, E. MARTÍNEZ, “Nonholonomic Lagrangian systems on Lie algebroids”, *Discrete and Continuous Dynamical Systems A* **24 2** (2009) 213–271.
- [13] K. EHLERS, J. KOILLER, R. MONTGOMERY, P.M. RIOS, “Nonholonomic systems via moving frames: Cartan equivalence and Chaplygin Hamiltonization”, pp. 75–120 in *The breadth of symplectic and Poisson geometry*, Progr. in Math., **232**, Birkhäuser, 2005.
- [14] X. GRÀCIA, R. MARTÍN, “Regularity and symmetries of nonholonomic systems”, *J. Phys. A: Math. Gen.* **38** (2005) 1071–1087.
- [15] G. HAMEL, *Theoretische Mechanik: Eine einheitliche Einführung in die gesamte Mechanik*, Grundlehren der Mathematischen Wissenschaften, **57**, Springer–Verlag, 1949.
- [16] A. IBORT, M. DE LEÓN, G. MARMO, D. MARTÍN DE DIEGO, “Nonholonomic constrained systems as implicit differential equations”, *Rend. Sem. Mat. Univ. Pol. Torino* **54** (1996) 295–317.
- [17] D. IGLESIAS-PONTE, M. DE LEÓN, D. MARTÍN DE DIEGO, “Towards a Hamilton–Jacobi theory for nonholonomic mechanical systems”, *J. Phys. A: Math. Theor.* **41** (2008) 015205, 14 pp.
- [18] J.V. JOSÉ, E.J. SALETAN, *Classical dynamics: a contemporary approach*, Cambridge University Press, 1998.
- [19] J. KOILLER, “Reduction of some classical non-holonomic systems with symmetry”, *Arch. Rational Mech. Anal.* **118** (1992) 113–148.
- [20] O. KRUPKOVÁ, J. MUSILOVÁ, “The relativistic particle as a mechanical system with non-holonomic constraints”, *J. Phys. A: Math. Gen.* **34** (2001) 3859–3875.
- [21] M. DE LEÓN, J.C. MARRERO, D. MARTÍN DE DIEGO: “Linear almost Poisson structures and Hamilton–Jacobi equation. Applications to nonholonomic mechanics”. arXiv:0801.1181 (2008).
- [22] M. DE LEÓN, J.C. MARRERO, D. MARTÍN DE DIEGO: “A geometric Hamilton–Jacobi theory for classical field theories”. arXiv:0801.1181 [math-ph] (2008)
- [23] M. DE LEÓN, J.C. MARRERO, E. MARTÍNEZ, “Lagrangian submanifolds and dynamics on Lie algebroids” *J. Phys. A: Math. Gen.* **38** (2005) R241–R308.
- [24] M. DE LEÓN, D. MARTÍN DE DIEGO, “On the geometry of non-holonomic Lagrangian systems”, *J. Math. Phys.* **37** (1996) 3389–3414.
- [25] P. LIBERMANN, C.M. MARLE, *Symplectic geometry and analytical dynamics*, D. Reidel Publishing Company, Dordrecht, 1987.
- [26] C.-M. MARLE “Various approaches to conservative and nonconservative nonholonomic systems”, *Rep. Math. Phys.* **42** (1998) 211–229.
- [27] G. MARMO, G. MORANDI, N. MUKUNDA, “A geometrical approach to the Hamilton–Jacobi form of dynamics and its generalizations”, *Riv. Nuovo Cimento* **13** (1990) 1–74.

- [28] G. MARMO, G. MORANDI, N. MUKUNDA, “The Hamilton-Jacobi theory and the analogy between classical and quantum mechanics”, preprint (2009).
- [29] E. MARTÍNEZ, “Lagrangian mechanics on Lie algebroids”, *Acta Appl. Math.* **67** (2001) 295–320.
- [30] E. MASSA, S. VIGNOLO, D. BRUNO, “Non-holonomic lagrangian and hamiltonian mechanics: an intrinsic approach”, *J. Phys. A: Math. Gen.* **35** (2002) 6713–6742.
- [31] T. MESTDAG, B. LANGEROCK, “A Lie algebroid framework for nonholonomic systems”, *J. Phys. A: Math. Gen* **38** (2005) 1097–1111.
- [32] T. OHSAWA, A.M. BLOCH, “Nonholomic Hamilton–Jacobi equation and integrability” preprint arXiv:0906.3357.
- [33] R. ROSENBERG, *Analytical dynamics*, Plenum, New York, 1977.
- [34] J. ŚNIATYCKI, “Nonholonomic Noether theorem and reduction of symmetries” *Rep. Math. Phys.* **42** (1998) 5–23.
- [35] A. M. VERSHIK, L. D. FADDEEV, “Differential geometry and Lagrangian mechanics with constraints”, *Sov. Phys. Dokl.* **17** (1972) 34–36.