Distance-regular graphs where the distance-d graph has fewer distinct eigenvalues

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Abstract

Let the Kneser graph K of a distance-regular graph Γ be the graph on the same vertex set as Γ, where two vertices are adjacent when they have maximal distance in Γ. We study the situation where the Bose-Mesner algebra of Γ is not generated by the adjacency matrix of K. In particular, we obtain strong results in the so-called 'half antipodal' case.

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Let Γ be a distance-regular graph of diameter d on n vertices. Let Γ_i be the graph with the same vertex set as Γ where two vertices are adjacent when they have distance i in Γ. Let A be the adjacency matrix of Γ, and A_i that of Γ_i. We are interested in the situation where A_d has fewer distinct eigenvalues than A. In this situation the matrix A_d generates a proper subalgebra of the Bose-Mesner algebra of Γ, a situation reminiscent of imprimitivity. We survey the known examples, derive parameter conditions, and obtain strong results in what we called the 'half antipodal' case. Unexplained notation is as in [BCN].

The vertex set X of Γ carries an association scheme with d classes, where the i-th relation is that of having graph distance i $(0 \le i \le d)$. All elements of the Bose-Mesner algebra $\mathcal{A}$ of this scheme are polynomials of degree at most d in the matrix A. In particular, A_i is a polynomial in A of degree i $(0 \le i \le d)$. Let $\mathcal{A}$ have minimal idempotents E_i $(0 \le i \le d)$. The column spaces of the E_i are common eigenspaces of all matrices in $\mathcal{A}$. Let P_{ij} be the corresponding eigenvalue of A_j, so that $A_j E_i = P_{ij} E_i$ $(0 \le i, j \le d)$. Now A has eigenvalues $\theta_i = P_{i1}$ with multiplicities $m_i = \operatorname{rk} E_i = \operatorname{tr} E_i$ $(0 \le i \le d)$. Index the eigenvalues such that $\theta_0 > \theta_1 > \cdots > \theta_d$.

Standard facts about Sturm sequences give information on the sign pattern of the matrix P.

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Proposition 1 *Let Γ be distance-regular, and P its eigenvalue matrix. Then row i and column i $(0 \le i \le d)$ of P both have i sign changes. In particular, row d and column d consist of nonzero numbers that alternate in sign.* $\square$

If $M \in \mathcal{A}$ and $0 \le i \le d$, then $M \prod_{j \ne i}(A - \theta_j I) = c(M,i)E_i$ for some constant $c(M,i)$. We apply this observation to $M = A_d$.

Proposition 2 *Let Γ have intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$. Then for each i $(0 \le i \le d)$ we have*

$$m_i A_d \prod_{j \ne i}(A - \theta_j I) = nb_0 b_1 \cdots b_{d-1} E_i. \tag{1}$$

Proof. Both sides differ by a constant factor. Take traces on both sides. Since $\operatorname{tr} A_d A^h = 0$ for $h < d$ it follows that $\operatorname{tr} A_d \prod_{j \ne i}(A - \theta_j I) = \operatorname{tr} A_d A^d = c_1 c_2 \cdots c_d n k_d = nb_0 b_1 \cdots b_{d-1}$. Now the result follows from $\operatorname{tr} E_i = m_i$. $\square$

This can be said in an equivalent numerical way.

Corollary 3 *We have* $m_i P_{id} \prod_{j \ne i}(\theta_i - \theta_j) = nb_0 b_1 \cdots b_{d-1}$ *for each i $(0 \le i \le d)$.*

Proof. Multiply (1) by E_i. $\square$

We find a criterion for A_d to have two equal eigenvalues P_{gd} and P_{hd}.

Proposition 4 *For $g \ne h$, $P_{gd} = P_{hd}$ if and only if $\sum_i m_i \prod_{j \ne g,h}(\theta_i - \theta_j) = 0$.*

Proof. $P_{gd} = P_{hd}$ if and only if $m_g \prod_{j \ne g}(\theta_g - \theta_j) = m_h \prod_{j \ne h}(\theta_h - \theta_j)$. $\square$

For example, the Biggs-Smith graph has diameter $d = 7$ and spectrum 3^1, θ_1^9, 2^{18}, θ_3^{16}, 0^{17}, θ_5^{16}, θ_6^9, θ_7^{16}, where θ_i, $i = 1, 6$, satisfy $f(\theta) = \theta^2 - \theta - 4 = 0$ and θ_i, $i = 3, 5, 7$, satisfy $g(\theta) = \theta^3 + 3\theta^2 - 3 = 0$. Now $P_{27} = P_{47}$ since

$$\sum_i m_i \prod_{j \ne 2,4}(\theta_i - \theta_j) = \sum_{i=2,4} m_i(\theta_i - 3) f(\theta_i) g(\theta_i) = 0.$$

One can generalize Proposition 4, and see:

Proposition 5 *Let $H \subseteq \{0, \ldots, d\}$. Then all P_{hd} for $h \in H$ take the same value if and only if $\sum_i m_i \theta_i^e \prod_{j \notin H}(\theta_i - \theta_j) = 0$ for $0 \le e \le |H| - 2$.*

Proof. Induction on $|H|$. We just did the case $|H| = 2$. Let $|H| > 2$ and let $h, h' \in H$. We do the 'only if' part. By induction $\sum_i m_i \theta_i^e \prod_{j \notin H \setminus \{x\}}(\theta_i - \theta_j) = 0$ holds for $0 \le e \le |H| - 3$ and $x = h, h'$. Subtract these two formulas and divide by $\theta_h - \theta_{h'}$ to get $\sum_i m_i \theta_i^e \prod_{j \notin H}(\theta_i - \theta_j) = 0$ for $0 \le e \le |H| - 3$. Then add the first formula for $x = h$ and θ_h times the last formula, to get the same conclusion for $1 \le e \le |H| - 2$. The converse is clear. $\square$

Since the P_{id} alternate in sign, the largest sets H that can occur here are $\{0, 2, \ldots, d\}$ and $\{1, 3, \ldots, d-1\}$ for $d = 2e$ and $\{0, 2, \ldots, d-1\}$ and $\{1, 3, \ldots, d\}$ for $d = 2e + 1$. We investigate such sets below (see 'the half-antipodal case').

For small d one can use identities like $\sum_i m_i = n$, $\sum_i m_i \theta_i = 0$, $\sum_i m_i \theta_i^2 = nk$, $\sum_i m_i \theta_i^3 = nk\lambda$ (where $k = b_0$, and $\lambda = k - 1 - b_1$) to simplify the condition of Proposition 4. Let us do some examples. Note that $\theta_0 = k$.

The case $d = 3$

For $d = 3$ we find that $P_{13} = P_{33}$ if and only if $\sum_i m_i(\theta_i - \theta_0)(\theta_i - \theta_2) = 0$, i.e., if and only if $nk + n\theta_0\theta_2 = 0$, i.e., if and only if $\theta_2 = -1$ (cf. [BCN], 4.2.17).

The case $d = 4$

For $d = 4$ we find that $P_{14} = P_{34}$ if and only if $\sum_i m_i(\theta_i - \theta_0)(\theta_i - \theta_2)(\theta_i - \theta_4) = 0$, i.e., if and only if $nk\lambda - nk(\theta_0 + \theta_2 + \theta_4) - n\theta_0\theta_2\theta_4 = 0$. This happens if and only if $(\theta_2+1)(\theta_4+1) = -b_1$. Of course $P_{24} = P_{44}$ will follow from $(\theta_1+1)(\theta_3+1) = -b_1$.

A generalized octagon GO(s,t) has eigenvalues

$$(\theta_i)_i = (s(t+1),\ s-1+\sqrt{2st},\ s-1,\ s-1-\sqrt{2st},\ \ -t-1)$$

and $b_1 = st$. Since $(\theta_2 + 1)(\theta_4 + 1) = -b_1$ it follows that $P_{14} = P_{34}$, so that Γ_4 does not have more than 4 distinct eigenvalues.

A dual polar graph ${}^2D_5(q)$ has eigenvalues

$$(\theta_i)_i = (q^5+q^4+q^3+q^2,\ q^4+q^3+q^2-1,\ q^3+q^2-q-1,\ -q-1,\ -q^3-q^2-q-1)$$

and $b_1 = q^3(q^2 + q + 1)$. Since $(\theta_1 + 1)(\theta_3 + 1) = -b_1$ it follows that $P_{24} = P_{44}$.

Some further examples:

name	n	intersection array	spectrum	equality
Coxeter graph	28	$\{3,2,2,1;\ 1,1,1,2\}$	$3^1\ 2^8\ a^6\ (-1)^7\ b^6$ $a,b = -1 \pm \sqrt{2}$	$P_{14} = P_{34}$
Odd graph O_5	126	$\{5,4,4,3;\ 1,1,2,2\}$	$5^1\ 3^{27}\ 1^{42}\ (-2)^{48}\ (-4)^8$	$P_{24} = P_{44}$
M_{22} graph	330	$\{7,6,4,4;\ 1,1,1,6\}$	$7^1\ 4^{55}\ 1^{154}\ (-3)^{99}\ (-4)^{21}$	$P_{14} = P_{34}$
Unital graph	280	$\{9,8,6,3;\ 1,1,3,8\}$	$9^1\ 4^{64}\ 1^{105}\ (-3)^{90}\ (-5)^{20}$	$P_{14} = P_{34}$

The case $d = 4$ with strongly regular Γ_4

One may wonder whether it is possible that Γ_4 is strongly regular. This would require $p^1_{44} = p^2_{44} = p^3_{44}$. Or, equivalently, that Γ_4 has only two eigenvalues with eigenvector other than the all-1 vector. Since the values P_{i4} alternate in sign, this would mean $P_{14} = P_{34}$ and $P_{24} = P_{44}$.

Proposition 6 *Let Γ be a distance-regular graph of diameter 4. The following assertions are equivalent.*
(i) Γ_4 is strongly regular.
(ii) $b_3 = a_4 + 1$ and $b_1 = b_3c_3$.
(iii) $(\theta_1 + 1)(\theta_3 + 1) = (\theta_2 + 1)(\theta_4 + 1) = -b_1$.

Proof. (i)-(ii) A boring computation (using [BCN], 4.1.7) shows that $p^1_{44} = p^2_{44}$ is equivalent to $b_3 = a_4 + 1$, and that if this holds $p^1_{44} = p^3_{44}$ is equivalent to $b_1 = b_3c_3$.

(i)-(iii) Γ_4 will be strongly regular if and only if $P_{14} = P_{34}$ and $P_{24} = P_{44}$. We saw that this is equivalent to $(\theta_2+1)(\theta_4+1) = -b_1$ and $(\theta_1+1)(\theta_3+1) = -b_1$. □

The fact that (i) implies the first equality in (iii) was proved in [F01] as a consequence of another characterization of (i) in terms of the spectrum only. More generally, a quasi-spectral characterization of those connected regular graphs (with $d+1$ distinct eigenvalues) which are distance-regular, and with the distance-d graph being strongly regular, is given in [F00, Th. 2.2].

No nonantipodal examples are known, but the infeasible array $\{12, 8, 6, 4;\ 1, 1, 2, 9\}$ with spectrum $12^1\ 7^{56}\ 3^{140}\ (-2)^{160}\ (-3)^{168}$ (cf. [BCN], p. 410) would have been an example (and there are several open candidate arrays, such as $\{21, 20, 14, 10;\ 1, 1, 2, 12\}$, $\{24, 20, 20, 10;\ 1, 1, 2, 15\}$, and $\{66, 65, 63, 13;\ 1, 1, 5, 54\}$).

If Γ is antipodal, then Γ_4 is a union of cliques (and hence strongly regular). This holds precisely when $\theta_1+\theta_3 = \lambda$ and $\theta_1\theta_3 = -k$ and $(\theta_2+1)(\theta_4+1) = -b_1$. (Indeed, θ_1, θ_3 are the two roots of $\theta^2 - \lambda\theta - k = 0$ by [BCN], 4.2.5.).
There are many examples, e.g.

name	n	intersection array	spectrum
Wells graph	32	$\{5, 4, 1, 1;\ 1, 1, 4, 5\}$	$5^1\ \sqrt{5}^8\ 1^{10}\ (-\sqrt{5})^8\ (-3)^5$
3.Sym(6).2 graph	45	$\{6, 4, 2, 1;\ 1, 1, 4, 6\}$	$6^1\ 3^{12}\ 1^9\ (-2)^{18}\ (-3)^5$
Locally Petersen	63	$\{10, 6, 4, 1;\ 1, 2, 6, 10\}$	$10^1\ 5^{12}\ 1^{14}\ (-2)^{30}\ (-4)^6$

If Γ is bipartite, then Γ_4 is disconnected, so if it is strongly regular, it is a union of cliques and Γ is antipodal. In this case its spectrum is

$$\{k^1,\ \sqrt{k}^{\,n/2-k},\ 0^{2k-2},\ (-\sqrt{k})^{n/2-k},\ (-k)^1\}.$$

Such graphs are precisely the incidence graphs of symmetric (m, μ)-nets, where $m = k/\mu$ ([BCN], p. 425).

The case $d = 5$

As before, and also using $\sum_i m_i\theta_i^4 = nk(k + \lambda^2 + b_1\mu)$ (where $\mu = c_2$) we find for $\{f, g, h, i, j\} = \{1, 2, 3, 4, 5\}$ that $P_{fd} = P_{gd}$ if and only if

$$(\theta_h + 1)(\theta_i + 1)(\theta_j + 1) + b_1(\theta_h + \theta_i + \theta_j) = b_1(\lambda - \mu - 1).$$

In case $\theta_i = -1$, this says that $\theta_h + \theta_j = \lambda - \mu$.

For example, the Odd graph O_6 has $\lambda = 0$, $\mu = 1$, and eigenvalues 6, 4, 2, -1, -3, -5. It follows that $P_{15} = P_{55}$ and $P_{25} = P_{45}$.

Similarly, the folded 11-cube has $\lambda = 0$, $\mu = 2$, and eigenvalues 11, 7, 3, -1, -5, -9. It follows that $P_{15} = P_{55}$ and $P_{25} = P_{45}$.

An example without eigenvalue -1 is provided by the folded Johnson graph $\bar{J}(20, 10)$. It has $\lambda = 18$, $\mu = 4$ and eigenvalues 100, 62, 32, 10, -4, -10. We see that $P_{35} = P_{55}$.

Combining two of the above conditions, we see that $P_{15} = P_{35} = P_{55}$ if and only if $(\theta_2 + 1)(\theta_4 + 1) = -b_1$ and $\theta_2 + \theta_4 = \lambda - \mu$ (and hence $\theta_2\theta_4 = \mu - k$).
Now $b_3 + b_4 + c_4 + c_5 = 2k + \mu - \lambda$ and $b_3b_4 + b_3c_5 + c_4c_5 = kb_1 + k\mu + \mu$.

Generalized 12-gons

A generalized 12-gon of order $(q, 1)$ (the line graph of the bipartite point-line incidence graph of a generalized hexagon of order (q, q)) has diameter 6, and its P matrix is given by

$$P = \begin{pmatrix} 1 & 2q & 2q^2 & 2q^3 & 2q^4 & 2q^5 & q^6 \\ 1 & q-1+a & q+(q-1)a & 2q(q-1) & -q^2+q(q-1)a & q^2(q-1)-q^2a & -q^3 \\ 1 & q-1+b & -q+(q-1)b & -2qb & -q^2-q(q-1)b & -q^2(q-1)+q^2b & q^3 \\ 1 & q-1 & -2q & -q(q-1) & 2q^2 & q^2(q-1) & -q^3 \\ 1 & q-1-b & -q-(q-1)b & 2qb & -q^2+q(q-1)b & -q^2(q-1)-q^2b & q^3 \\ 1 & q-1-a & q-(q-1)a & 2q(q-1) & -q^2-q(q-1)a & q^2(q-1)+q^2a & -q^3 \\ 1 & -2 & 2 & -2 & 2 & -2 & 1 \end{pmatrix}$$

where $a = \sqrt{3q}$ and $b = \sqrt{q}$. We see that $P_{16} = P_{36} = P_{56}$ and $P_{26} = P_{46}$.

Its dual is a generalized 12-gon of order $(1, q)$, and is bipartite. The P matrix is given by

$$P = \begin{pmatrix} 1 & q+1 & q(q+1) & q^2(q+1) & q^3(q+1) & q^4(q+1) & q^5 \\ 1 & a & 2q-1 & (q-1)a & q(q-2) & -qa & -q^2 \\ 1 & b & -1 & -(q+1)b & -q^2 & qb & q^2 \\ 1 & 0 & -q-1 & 0 & q(q+1) & 0 & -q^2 \\ 1 & -b & -1 & (q+1)b & -q^2 & -qb & q^2 \\ 1 & -a & 2q-1 & -(q-1)a & q(q-2) & qa & -q^2 \\ 1 & -q-1 & q(q+1) & -q^2(q+1) & q^3(q+1) & -q^4(q+1) & q^5 \end{pmatrix}$$

where $a = \sqrt{3q}$ and $b = \sqrt{q}$. We see that $P_{16} = P_{36} = P_{56}$ and $P_{26} = P_{46}$. As expected (cf. [B]), the squares of all P_{id} are powers of q.

Dual polar graphs

According to [B], dual polar graphs of diameter d satisfy

$$P_{id} = (-1)^i q^{d(d-1)/2+de-i(d+e-i)}$$

where e has the same meaning as in [BCN], 9.4.1 (that is, e is 1, 1, 0, 2, $\frac{1}{2}$, $\frac{3}{2}$ in the case of $B_d(q)$, $C_d(q)$, $D_d(q)$, ${}^2D_{d+1}(q)$, ${}^2A_{2d-1}(\sqrt{q})$, ${}^2A_{2d}(\sqrt{q})$, respectively). It follows that $P_{hd} = P_{id}$ when $d+e$ is an even integer and $h+i = d+e$.

For the dual polar graphs $B_d(q)$ and $C_d(q)$ we have $e = 1$, and the condition becomes $h+i = d+1$ where d is odd. Below we will see this in a different way.

For the dual polar graph $D_d(q)$ we have $e = 0$, and the condition becomes $h+i = d$ where d is even. Not surprising, since this graph is bipartite.

For the dual polar graph ${}^2D_{d+1}(q)$ we have $e = 2$, and the condition becomes $h+i = d+2$ where d is even. (We saw the case $d = 4$ above.)

Finally, $h+i = d+e$ is impossible when e is not integral.

Distance-regular distance 1-or-2 graph

The distance 1-or-2 graph $\Delta = \Gamma_1 \cup \Gamma_2$ of Γ (with adjacency matrix $A_1 + A_2$) is distance-regular if and only if $b_{i-1} + b_i + c_i + c_{i+1} = 2k + \mu - \lambda$ for $1 \le i \le d-1$, cf. [BCN], 4.2.18.

Proposition 7 *Suppose that $\Gamma_1 \cup \Gamma_2$ is distance-regular. Then for $1 \le i \le d$ we have $P_{d+1-i,d} = P_{id}$ if d is odd, and $(\theta_{d+1-i} + 1)P_{i,d} = (\theta_i + 1)P_{d+1-i,d}$ if d is even. If $i \ne (d+1)/2$ then $\theta_{d+1-i} = \lambda - \mu - \theta_i$. If d is odd, then $\theta_{(d+1)/2} = -1$.*

Proof. For each eigenvalue θ of Γ, there is an eigenvalue $(\theta^2+(\mu-\lambda)\theta-k)/\mu$ of Δ. If d is odd, then Δ has diameter $(d+1)/2$, and Γ has an eigenvalue -1, and for each eigenvalue $\theta \neq k, -1$ of Γ also $\lambda-\mu-\theta$ is an eigenvalue. Now $\Delta_{(d+1)/2} = \Gamma_d$, and $P_{i'd} = P_{id}$ if i, i' belong to the same eigenspace of Δ. Since the numbers P_{id} alternate, the eigenvalue -1 must be the middle one (not considering θ_0), and we see that $P_{d+1-i,d} = P_{id}$ for $1 \leq i \leq d$.

If d is even, then Δ has diameter $d/2$, and for each eigenvalue $\theta \neq k$ of Γ also $\lambda-\mu-\theta$ is an eigenvalue. Now $\Delta_{d/2} = \Gamma_{d-1} \cup \Gamma_d$. Since $AA_d = b_{d-1}A_{d-1} + a_d A_d$ and our parameter conditions imply $b_{d-1} + c_d = b_1 + \mu$ (so that $b_{d-1}(P_{i,d-1}+P_{i,d}) = (\theta_i - a_d + b_{d-1})P_{i,d} = (\theta_i+\mu-\lambda-1)P_{i,d}$), the equalities $P_{i,d-1}+P_{i,d} = P_{d+1-i,d-1}+P_{d+1-i,d}$ $(1 \leq i \leq d)$ and $\theta_i + \theta_{d+1-i} = \lambda - \mu$ imply $(\theta_{d+1-i}+1)P_{i,d} = (\theta_i+1)P_{d+1-i,d}$. □

The Odd graph O_{d+1} on $\binom{2d+1}{d}$ vertices has diameter d and eigenvalues $\theta_i = d+1-2i$ for $i < (d+1)/2$, and $\theta_i = d-2i$ for $i \geq (d+1)/2$. Since its distance 1-or-2 graph is distance-regular, we have $P_{d+1-i,d} = P_{id}$ for odd d and $1 \leq i \leq d$.

The folded $(2d+1)$-cube on 2^{2d} vertices has diameter d and eigenvalues $\theta_i = 2d+1-4i$ with multiplicities $m_i = \binom{2d+1}{2i}$ $(0 \leq i \leq d)$. Since its distance 1-or-2 graph is distance-regular, it satisfies $P_{d+1-i,d} = (-1)^{d+1}P_{id}$ for $1 \leq i \leq d$. (Note that $\theta_{d+1-i}+1 = -(\theta_i+1)$ since $\mu-\lambda = 2$.)

The dual polar graphs $B_d(q)$ and $C_d(q)$ have diameter d and eigenvalues $\theta_i = (q^{d-i+1}-q^i)/(q-1)-1$. Since their distance 1-or-2 graphs are distance-regular, they satisfy $P_{d+1-i,d} = (-1)^{d+1}P_{id}$ for $1 \leq i \leq d$.

The fact that -1 must be the middle eigenvalue for odd d, implies that $\theta_{(d-1)/2} > \lambda-\mu+1$, so that there is no eigenvalue ξ with $-1 < \xi < \lambda-\mu+1$.

The bipartite case

If Γ is bipartite, then $\theta_{d-i} = -\theta_i$, and $P_{d-i,j} = (-1)^j P_{i,j}$ $(0 \leq i,j \leq d)$. In particular, if d is even, then $P_{d-i,d} = P_{id}$ and Γ_d is disconnected.

The antipodal case

The graph Γ is antipodal when having distance d is an equivalence relation, i.e., when Γ_d is a union of cliques. The graph is called an antipodal r-cover, when these cliques are r-cliques. Now $r = k_d+1$, and P_{id} alternates between k_d and -1.

For example, the ternary Golay code graph (of diameter 5) with intersection array $\{22,20,18,2,1;1,2,9,20,22\}$ has spectrum $22^1\ 7^{132}\ 4^{132}\ (-2)^{330}\ (-5)^{110}\ (-11)^{24}$ and satisfies $P_{05} = P_{25} = P_{45} = 2$, $P_{15} = P_{35} = P_{55} = -1$.

For an antipodal distance-regular graph Γ, the folded graph has eigenvalues $\theta_0, \theta_2, \ldots, \theta_{2e}$ where $e = [d/2]$. In Theorem 9 below we show for odd d that this already follows from $P_{1d} = P_{3d} = \cdots = P_{dd}$.

Proposition 8 *If $P_{0d} = P_{id}$ then i is even. Let $i > 0$ be even. Then $P_{0d} = P_{id}$ if and only Γ is antipodal, or $i = d$ and Γ is bipartite.*

Proof. Since the P_{id} alternate in sign, $P_{0d} = P_{id}$ implies that i is even. If Γ is bipartite, then $P_{dd} = (-1)^d P_{0d}$. If Γ is antipodal, then $P_{id} = P_{0d}$ for all even i.

That shows the 'if' part. Conversely, if $P_{0d} = P_{id}$, then the valency of Γ_d is an eigenvalue of multiplicity larger than 1, so that Γ_d is disconnected, and hence Γ is imprimitive and therefore antipodal or bipartite. If Γ is bipartite but not antipodal, then its halved graphs are primitive and $|P_{id}| < P_{0d}$ for $0 < i < d$ (cf. [BCN], pp. 140–141). □

The half-antipodal case

Given an array $\{b_0, \ldots, b_{d-1};\, c_1, \ldots, c_d\}$ of positive real numbers, define the polynomials $p_i(x)$ for $-1 \le i \le d+1$ by $p_{-1}(x) = 0$, $p_0(x) = 1$, $(x - a_i)p_i(x) = b_{i-1}p_{i-1}(x) + c_{i+1}p_{i+1}(x)$ $(0 \le i \le d)$, where $a_i = b_0 - b_i - c_i$ and c_{d+1} is some arbitrary positive number. The eigenvalues of the array are by definition the zeros of $p_{d+1}(x)$, and do not depend on the choice of c_{d+1}. Each $p_i(x)$ has degree i, and, by the theory of Sturm sequences, each $p_i(x)$ has i distinct real zeros, where the zeros of $p_{i+1}(x)$ interlace those of $p_i(x)$. If $\{b_0, \ldots, b_{d-1};\, c_1, \ldots, c_d\}$ is the intersection array of a distance-regular graph Γ, then the eigenvalues of the array are the eigenvalues of (the adjacency matrix of) Γ.

Let L be the tridiagonal matrix

$$L = \begin{pmatrix} a_0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} \\ & & & & c_d & a_d \end{pmatrix}.$$

The eigenvalues of the array $\{b_0, \ldots, b_{d-1};\, c_1, \ldots, c_d\}$ are the eigenvalues of the matrix L.

Theorem 9 *Let Γ be a distance-regular graph with odd diameter $d = 2e + 1$ and intersection array $\{b_0, \ldots, b_{d-1};\, c_1, \ldots, c_d\}$. Then $P_{1d} = P_{3d} = \cdots = P_{dd}$ if and only if the θ_j with $j = 0, 2, 4, \ldots, 2e$ are the eigenvalues of the array $\{b_0, \ldots, b_{e-1};\, c_1, \ldots, c_e\}$.*

Proof. Let $H = \{1, 3, \ldots, d\}$, so that $|H| = e + 1$. By Proposition 5, $P_{1d} = P_{3d} = \cdots = P_{dd}$ if and only if $\sum_i m_i \theta_i^s \prod_{j \notin H}(\theta_i - \theta_j) = 0$ for $0 \le s \le e - 1$. Let $E = \{0, 2, \ldots, 2e\}$, so that $|E| = e + 1$. Then this condition is equivalent to

$$\operatorname{tr} A^s \prod_{j \in E}(A - \theta_j I) = 0 \qquad (0 \le s \le e - 1).$$

This says that the expansion of $\prod_{j \in E}(A - \theta_j I)$ in terms of the A_i does not contain A_s for $0 \le s \le e-1$, hence is equivalent to $\prod_{j \in E}(A - \theta_j I) = aA_e + bA_{e+1}$ for certain constants a, b. Since $0 \in E$, we find that $ak_e + bk_{e+1} = 0$, and the condition is equivalent to $(A_e/k_e - A_{e+1}/k_{e+1})E_j = 0$ for all $j \in E$.

An eigenvalue θ of Γ defines a right eigenvector u (known as the 'standard sequence') by $Lu = \theta u$. It follows that θ will be an eigenvalue of the array $\{b_0, \ldots, b_{e-1};\, c_1, \ldots, c_e\}$ precisely when $u_e = u_{e+1}$. Up to scaling, the u_i belonging to θ_j are the Q_{ij} (that is, the columns of Q are eigenvectors of L). So, θ_j is an eigenvalue of $\{b_0, \ldots, b_{e-1};\, c_1, \ldots, c_e\}$ for all $j \in E$ precisely when $Q_{ej} = Q_{e+1,j}$ for all $j \in E$. Since $k_i Q_{ij} = m_j P_{ji}$, this holds if and only if

$P_{je}/k_e = P_{j,e+1}/k_{e+1}$, i.e., if and only if $(A_e/k_e - A_{e+1}/k_{e+1})E_j = 0$ for all $j \in E$. $\square$

For example, if $d = 3$ one has $P_{13} = P_{33}$ if and only if θ_0, θ_2 are the eigenvalues k, -1 of the array $\{k; 1\}$. And if $d = 5$ one has $P_{15} = P_{35} = P_{55}$ if and only if $\theta_0, \theta_2, \theta_4$ are the eigenvalues of the array $\{k, b_1; 1, c_2\}$.

The case of even d is slightly more complicated.

Theorem 10 *Let Γ be a distance-regular graph with even diameter $d = 2e$ and intersection array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$. Then $P_{1d} = P_{3d} = \cdots = P_{d-1,d}$ if and only if the θ_j with $j = 0, 2, 4, \ldots, 2e$ are the eigenvalues of the array $\{b_0, \ldots, b_{e-1}; c_1, \ldots, c_{e-1}, c_e + zb_e\}$ for some real number z with $0 < z \leq 1$, uniquely determined by $\sum_{i=0}^{e} \theta_{2i} = \sum_{i=0}^{e} a_i + (1 - z)b_e$. If Γ is antipodal or bipartite, then $z = 1$.*

Proof. Let $E = \{0, 2, \ldots, d\}$. As before we see that $P_{1d} = P_{3d} = \cdots = P_{d-1,d}$ is equivalent to the condition that $\prod_{j \in E}(A - \theta_j I) = aA_{e-1} + bA_e + cA_{e+1}$ for certain constants a, b, c. Comparing coefficients of A^{e+1} we see that $c > 0$. With $j = 0$ we see that $ak_{e-1} + bk_e + ck_{e+1} = 0$.

Take
$$z = -\frac{ak_{e-1}}{ck_{e+1}} = 1 + \frac{bk_e}{ck_{e+1}}.$$
Then $aP_{j,e-1} + bP_{je} + cP_{j,e+1} = 0$ for $j \in E$ gives
$$z\left(\frac{P_{j,e-1}}{k_{e-1}} - \frac{P_{je}}{k_e}\right) = \frac{P_{j,e+1}}{k_{e+1}} - \frac{P_{je}}{k_e}.$$

On the other hand, if $\theta = \theta_j$ for some $j \in E$, then θ is an eigenvalue of the array $\{b_0, \ldots, b_{e-1}; c_1, \ldots, c_{e-1}, c_e+zb_e\}$ precisely when $c_e u_{e-1} + (k-b_e-c_e)u_e + b_e u_{e+1} = (c_e+zb_e)u_{e-1} + (k-c_e-zb_e)u_e$, i.e., when $z(u_{e-1} - u_e) = u_{e+1} - u_e$. Since (up to a constant factor) $u_i = P_{ji}/k_i$, this is equivalent to the condition above.

Noting that $d \in E$, we can apply the above to $\theta = \theta_d$. Since the bottom row of P has d sign changes, it follows that the sequence u_i has d sign changes. In particular, the u_i are nonzero. Now $u_{e-1} - u_e$ and $u_{e+1} - u_e$ have the same sign, and it follows that $z > 0$.

If Γ is an antipodal r-cover of diameter $d = 2e$, then $c_e + zb_e = rc_e$ and $z = 1$ (and $P_{je} = 0$ for all odd j).

If Γ is bipartite, then $\theta_i + \theta_{d-i} = 0$ for all i, so $\sum_{j \in E} \theta_j = 0$, so our tridiagonal matrix (the analog of L) has trace $0 = a_1 + \cdots + a_e + (1 - z)b_e = (1 - z)b_e$, so that $z = 1$.

It remains to show that $z \leq 1$. We use $z(u_{e-1} - u_e) = u_{e+1} - u_e$ to conclude that θ is an eigenvalue of both
$$\begin{pmatrix} 0 & k & & & \\ c_1 & a_1 & b_1 & & \\ & . & . & . & \\ & & c_{e-1} & a_{e-1} & b_{e-1} \\ & & & p & k-p \end{pmatrix} \text{ and } \begin{pmatrix} k-q & q & & & \\ c_{e+1} & a_{e+1} & b_{e+1} & & \\ & . & . & . & \\ & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & c_d & a_d \end{pmatrix},$$
where $p = c_e + zb_e$ and $q = b_e + z^{-1}c_e$. Since this holds for each $\theta = \theta_j$ for $j \in E$, this accounts for all eigenvalues of these two matrices, and $\sum_{j \in E} \theta_j =$

$a_1 + \cdots + a_e + (1-z)b_e = a_e + \cdots + a_d + (1-z^{-1})c_e$. Since $p^{e+1}_{ed} \geq 0$ it follows that $a_1 + \cdots + a_{e-1} \leq a_{e+1} + \cdots + a_d$ (cf. [BCN], 4.1.7), and therefore $(1-z)b_e \geq (1-z^{-1})c_e$. It follows that $z \leq 1$. □

Nonantipodal, nonbipartite examples:

name	array	half array	z
Coxeter	$\{3,2,2,1;\ 1,1,1,2\}$	$\{3,2;\ 1,2\}$	$1/2$
M_{22}	$\{7,6,4,4;\ 1,1,1,6\}$	$\{7,6;\ 1,3\}$	$1/2$
$P\Gamma L(3,4).2$	$\{9,8,6,3;\ 1,1,3,8\}$	$\{9,8;\ 1,4\}$	$1/2$
gen. 8-gon	$\{s(t+1), st, st, st;\ 1,1,1,t+1\}$	$\{s(t+1), st;\ 1, t+1\}$	$1/s$
gen. 12-gon	$\{2q, q, q, q, q, q;\ 1,1,1,1,1,2\}$	$\{2q, q, q;\ 1,1,2\}$	$1/q$

Concerning the value of z, note that both zb_e and $z^{-1}c_e$ are algebraic integers.

If $d = 4$, the case $z = 1$ can be classified.

Proposition 11 *Let Γ be a distance-regular graph with even diameter $d = 2e$ such that θ_j with $j = 0, 2, 4, \ldots, 2e$ are the eigenvalues of the array $\{b_0, \ldots, b_{e-1};$ $c_1, \ldots, c_{e-1}, b_e + c_e\}$. Then Γ satisfies $p^d_{e,e+1} = 0$. If moreover $d \leq 4$, then Γ is antipodal or bipartite.*

Proof. We have equality in the inequality $a_1 + \cdots + a_{e-1} \leq a_{e+1} + \cdots + a_d$, so that $p^{e+1}_{ed} = 0$ by [BCN], 4.1.7. The case $d = 2$ is trivial. Suppose $d = 4$. Then $p^3_{24} = 0$. Let $d(x,z) = 4$, and consider neighbors y, w of z, where $d(x,y) = 3$ and $d(x,w) = 4$. Then $d(y,w) \neq 2$ since there are no 2-3-4 triangles, so $d(y,w) = 1$. If $a_4 \neq 0$ then there exist such vertices w, and we find that the neighborhood $\Gamma(z)$ of z in Γ is not coconnected (its complement is not connected), contradicting [BCN], 1.1.7. Hence $a_4 = 0$, and $a_3 = a_1$. If $b_3 > 1$, then let $d(x,y) = 3$, $y \sim z, z'$ with $d(x,z) = d(x,z') = 4$. Let z'' be a neighbor of z' with $d(z,z'') = 2$. Then $d(x,z'') = 3$ since $a_4 = 0$, and we see a 2-3-4 triangle, contradiction. So if $b_3 > 1$ then $a_2 = 0$, and $a_1 = 0$ since $a_1 \leq 2a_2$, and the graph is bipartite. If $b_3 = 1$, then the graph is antipodal. □

Variations

One can vary the above theme. First, the following general result builds on ideas previously used.

Theorem 12 *Let Γ be a distance-regular graph with diameter d, and let $H \subseteq \{0, \ldots, d\}$. Then all P_{id} for $i \in H$ take the same value if and only if the eigenvalues θ_j with $j \notin H$ are the zeros of the polynomial $\sum_{i=|H|-1}^{r} \frac{\alpha_i}{k_i} p_i(x)$, where $r = d + 1 - |H|$, $\alpha_r = 1$, and*

$$\alpha_i = \frac{\operatorname{tr}(A_i \prod_{j \notin H}(A - \theta_j I))}{\operatorname{tr}(A_r A^r)} \qquad (|H| - 1 \leq i \leq d - |H|). \tag{2}$$

Proof. By Proposition 5, the hypothesis holds if and only if we have the equalities $\sum_i m_i \theta_i^s \prod_{j \notin H}(\theta_i - \theta_j) = 0$ for $0 \leq s \leq |H| - 2$. That is,

$$\operatorname{tr} A^s \prod_{j \notin H}(A - \theta_j I) = 0 \qquad (0 \leq s \leq |H| - 2).$$

It follows that $\prod_{j\notin H}(A-\theta_j I)$, when written on the basis $\{A_i \mid 0 \le i \le d\}$, does not contain A_s for $0 \le s \le |H|-2$. Moreover, the number of factors is $r = d+1-|H|$, so that A_s does not occur either when $s > r$. Therefore, $\prod_{j\notin H}(A-\theta_j I)$ is a linear combination of the A_s with $|H|-1 \le s \le r$ and, hence, for some constants α_i,

$$\frac{n}{\operatorname{tr}(A_r A^r)} \prod_{j\notin H}(A-\theta_j I) = \sum_{s=|H|-1}^{r} \frac{\alpha_s}{k_s} A_s.$$

Now comparing coefficients of A^r, we see that $\alpha_r = 1$ (notice that $\operatorname{tr}(A_r A^r) = nc_1 \cdots c_r k_r$). To obtain the value of α_i for $|H|-1 \le i \le d-|H|$, multiply both terms of the above equation by A_i and take traces. □

In the above we applied this twice, namely for $d = 2e+1$, $H = \{1,3,5,\ldots,d\}$, and for $d = 2e$, $H = \{1,3,5,\ldots,d-1\}$. In the latter case, (2) yields the following expression for $z = -\alpha_{e-1}$.

$$b_0 b_1 \cdots b_e nz = c_1 c_2 \cdots c_{e+1} k_{e+1} nz = -\operatorname{tr}(A_{e-1} \prod_{j\notin H}(A-\theta_j I)).$$

Let us now take for H the set of even indices, with or without 0.

$H = \{0, 2, 4, \ldots, d\}$

Let $d = 2e$ be even and suppose that P_{id} takes the same value (k_d) for all $i \in H = \{0,2,4,\ldots,d\}$. Then $|H| = e+1$, and $\prod_{j\notin H}(A-\theta_j I)$ is a multiple of A_e. By Proposition 8 this happens if and only if Γ is antipodal with even diameter.

$H = \{0, 2, 4, \ldots, d-1\}$

Let $d = 2e+1$ be odd and suppose that P_{id} takes the same value (k_d) for all $i \in H = \{0,2,4,\ldots,2e\}$. Then $|H| = e+1$, and $\prod_{j\notin H}(A-\theta_j I)$ is a linear combination of A_s for $e \le s \le e+1$. By Proposition 8 this happens if and only if Γ is antipodal with odd diameter.

$H = \{2, 4, \ldots, d\}$

Let $d = 2e$ be even and suppose that P_{id} takes the same value for all $i \in H = \{2,4,\ldots,d\}$. Then $|H| = e$, and $\prod_{j\notin H}(A-\theta_j I)$ is a linear combination of A_s for $e-1 \le s \le e+1$. As before we conclude that the θ_j with $j \notin H$ are the eigenvalues of the array $\{b_0,\ldots,b_{e-1};\ c_1,\ldots,c_{e-1},c_e+zb_e\}$ for some real $z \le 1$. This time $d \in H$, and there is no conclusion about the sign of z.

For example, the Odd graph O_5 with intersection array $\{5,4,4,3;\ 1,1,2,2\}$ has eigenvalues 5, 3, 1, -2, -4 and $P_{24} = P_{44}$. The eigenvalues 5, 3, -2 are those of the array $\{5,4;\ 1,-1\}$.

No primitive examples with $d > 4$ are known.

$H = \{2, 4, \ldots, d-1\}$

Let $d = 2e+1$ be odd and suppose that P_{id} takes the same value for all $i \in H = \{2,4,\ldots,2e\}$. Then $|H| = e$, and $\prod_{j\notin H}(A-\theta_j I)$ is a linear combination of A_s for $e-1 \le s \le e+2$.

For example, the Odd graph O_6 with intersection array $\{6,5,5,4,4;\ 1,1,2,2,3\}$ has eigenvalues 6, 4, 2, -1, -3, -5 and $P_{24} = P_{44}$.

No primitive examples with $d > 5$ are known.

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