Distance-regular graphs where the distance-$d$
graph has fewer distinct eigenvalues

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August 5, 2014

Abstract

Let the Kneser graph $K$ of a distance-regular graph $\Gamma$ be the graph on
the same vertex set as $\Gamma$, where two vertices are adjacent when they have
maximal distance in $\Gamma$. We study the situation where the Bose-Mesner
algebra of $\Gamma$ is not generated by the adjacency matrix of $K$.

AMS Classification: 05E30, 05C50.

Keywords: Distance-regular graph; Kneser graph; Bose-Mesner algebra; half-
antipodality.

Let $\Gamma$ be a distance-regular graph of diameter $d$ on $n$ vertices. Let $\Gamma_i$ be
the graph with the same vertex set as $\Gamma$ where two vertices are adjacent when they
have distance $i$ in $\Gamma$. Let $A$ be the adjacency matrix of $\Gamma$, and $A_i$ that of $\Gamma_i$.
We are interested in the situation where $A_d$ has fewer distinct eigenvalues than $A$.
In this situation the matrix $A_d$ generates a proper subalgebra of the Bose-Mesner
algebra of $\Gamma$, a situation reminiscent of imprimitivity. We survey the known
examples, derive parameter conditions, and obtain strong results in what we
called the ‘half antipodal’ case. Unexplained notation is as in [BCN].

The vertex set $X$ of $\Gamma$ carries an association scheme with $d$ classes, where
the $i$-th relation is that of having graph distance $i$ ($0 \leq i \leq d$). All elements of
the Bose-Mesner algebra $A$ of this scheme are polynomials of degree at most $d$
in the matrix $A$. In particular, $A_i$ is a polynomial in $A$ of degree $i$ ($0 \leq i \leq d$).
Let $A$ have minimal idempotents $E_i$ ($0 \leq i \leq d$). The column spaces of the
$E_i$ are common eigenspaces of all matrices in $A$. Let $P_{ij}$ be the corresponding
eigenvalue of $A_i$, so that $A_i E_i = P_{ij} E_i$ ($0 \leq i, j \leq d$). Now $A$ has eigenvalues
$\theta_i = P_{i1}$ with multiplicities $m_i = \text{rk} E_i = \text{tr} E_i$ ($0 \leq i \leq d$). Index the
eigenvalues such that $\theta_0 > \theta_1 > \cdots > \theta_d$.

Standard facts about Sturm sequences give information on the sign pattern
of the matrix $P$.

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1Research supported by the Ministerio de Ciencia e Innovación, Spain, and the
European Regional Development Fund under project MTM2011-28800-C02-01, and the
Catalan Research Council under project 2009SGR1387.
Proposition 1 Let $\Gamma$ be distance-regular, and $P$ its eigenvalue matrix. Then row $i$ and column $i$ ($0 \leq i \leq d$) of $P$ both have $i$ sign changes. In particular, row $d$ and column $d$ consist of nonzero numbers that alternate in sign. 

If $M \in \mathcal{A}$ and $0 \leq i \leq d$, then $M \prod_{j \neq i}(A - \theta_j I) = c(M, i)E_i$ for some constant $c(M, i)$. We apply this observation to $M = A_d$.

Proposition 2 Let $\Gamma$ have intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$. Then for each $i$ ($0 \leq i \leq d$) we have

$$m_i A_d \prod_{j \neq i}(A - \theta_j I) = nb_i b_1 \cdots b_{d-1} E_i. \tag{1}$$

**Proof.** Both sides differ by a constant factor. Take traces on both sides. Since $\text{tr} A_d A^h = 0$ for $h < d$ it follows that $\text{tr} A_d \prod_{j \neq i}(A - \theta_j I) = \text{tr} A_d A^d = c_1 c_2 \cdots c_d n k_d = nb_i b_1 \cdots b_{d-1}$. Now the result follows from $\text{tr} E_i = m_i$. 

This can be said in an equivalent numerical way.

Corollary 3 We have $m_i P_d \prod_{j \neq i}(\theta_i - \theta_j) = nb_i b_1 \cdots b_{d-1}$ for each $i$ ($0 \leq i \leq d$).

**Proof.** Multiply (1) by $E_i$. 

We find a criterion for $A_d$ to have two equal eigenvalues $P_{gd}$ and $P_{hd}$.

Proposition 4 For $g \neq h$, $P_{gd} = P_{hd}$ if and only if $\sum_i m_i \prod_{j \neq h, g}(\theta_i - \theta_j) = 0$.

**Proof.** $P_{gd} = P_{hd}$ if and only if $m_g \prod_{j \neq g}(\theta_g - \theta_j) = m_h \prod_{j \neq h}(\theta_h - \theta_j)$. 

For example, the Biggs-Smith graph has diameter $d = 7$ and spectrum $3^1, 2^{18}, 1^{17}, 0^7, -\theta_7^3$, where $\theta_i, i = 1, 6$, satisfy $f(\theta) = \theta^2 + \theta - 4 = 0$ and $\theta_i, i = 3, 5, 7$, satisfy $g(\theta) = \theta^3 + 3\theta^2 - 3 = 0$. Now $P_{27} = P_{27}$ since

$$\sum_{i} m_i \prod_{j \neq 2, 4} (\theta_i - \theta_j) = \sum_{i=2, 4} m_i (\theta_i - 3) f(\theta_i) g(\theta_i) = 0.$$

One can generalize Proposition 4, and see:

Proposition 5 Let $H \subseteq \{0, \ldots, d\}$. Then all $P_{hd}$ for $h \in H$ take the same value if and only if $\sum_i m_i \theta_i^e \prod_{x \in H \setminus \{x\}} (\theta_i - \theta_j) = 0$ for $0 \leq e \leq |H| - 2$.

**Proof.** Induction on $|H|$. We just did the case $|H| = 2$. Let $|H| > 2$ and let $h, h' \in H$. We do the ‘only if’ part. By induction $\sum_i m_i \theta_i^e \prod_{x \in H \setminus \{x\}} (\theta_i - \theta_j) = 0$ holds for $0 \leq e \leq |H| - 3$ and $x = h, h'$. Subtract these two formulas and divide by $\theta_h - \theta_h'$ to get

$$\sum_i m_i \theta_i^e \prod_{x \in H \setminus \{x\}} (\theta_i - \theta_j) = 0$$

for $0 \leq e \leq |H| - 3$. Subtract the last formula for $x = h$ and $\theta_h$ times the last formula, to get the same conclusion for $1 \leq e \leq |H| - 2$. The converse is clear.

Since the $P_{gd}$ alternate in sign, the largest sets $H$ that can occur here are $\{0, 2, \ldots, d\}$ and $\{1, 3, \ldots, d-1\}$ for $d = 2e$ and $\{0, 2, \ldots, d-1\}$ and $\{1, 3, \ldots, d\}$ for $d = 2e + 1$. We investigate such sets below (see ‘the half-antipodal case’).

For small $d$ one can use identities like $\sum_i m_i = n$, $\sum_i m_i \theta_i = 0$, $\sum_i m_i \theta_i^2 = nk$, $\sum_i m_i \theta_i^3 = nk\lambda$ (where $k = b_0$, and $\lambda = k - 1 - b_1$) to simplify the condition of Proposition 4. Let us do some examples. Note that $\theta_0 = k$.

2
The case $d = 3$

For $d = 3$ we find that $P_{13} = P_{33}$ if and only if $\sum_{i} m_i (\theta_i - \theta_0) (\theta_i - \theta_2) = 0$, i.e., if and only if $nk + nb_0 \theta_2 = 0$, i.e., if and only if $\theta_2 = -1$ (cf. [BCN], 4.2.17).

The case $d = 4$

For $d = 4$ we find that $P_{14} = P_{34}$ if and only if $\sum_{i} m_i (\theta_i - \theta_0) (\theta_i - \theta_2) (\theta_i - \theta_4) = 0$, i.e., if and only if $nk \lambda - nk (\theta_0 + \theta_2 + \theta_4) - nb_0 \theta_2 \theta_4 = 0$. This happens if and only if $(\theta_2 + 1)(\theta_4 + 1) = - b_1$. Of course $P_{24} = P_{44}$ will follow from $(\theta_4 + 1)(\theta_3 + 1) = - b_1$.

A generalized octagon $GO(s, t)$ has eigenvalues

\[(\theta_i)_1 = (s(t + 1), \ s - 1 + \sqrt{2}st, \ s - 1, \ s - 1 - \sqrt{2}st, \ -t - 1)\]

and $b_1 = st$. Since $(\theta_2 + 1)(\theta_4 + 1) = - b_1$ it follows that $P_{14} = P_{34}$, so that $\Gamma_4$ does not have more than 4 distinct eigenvalues.

A dual polar graph $2D_5(q)$ has eigenvalues

\[(\theta_i)_1 = (q^3 + q^3 + q^3 + q^7, \ q^4 + q^4 + q^2 - 1, \ q^3 + q^2 - q - 1, \ -q - 1, \ -q^3 - q^2 - q - 1)\]

and $b_1 = q^4(q^2 + q + 1)$. Since $(\theta_1 + 1)(\theta_3 + 1) = - b_1$ it follows that $P_{24} = P_{44}$.

Some further examples:

<table>
<thead>
<tr>
<th>name</th>
<th>$n$</th>
<th>intersection array</th>
<th>spectrum</th>
<th>equality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coxeter graph</td>
<td>28</td>
<td>{3, 2, 2, 1; 1,1,1,2}</td>
<td>$3^4 \ 2^5 \ a^6 \ (-1)^7 \ b^9$</td>
<td>$P_{14} = P_{34}$</td>
</tr>
<tr>
<td>Odd graph $O_5$</td>
<td>126</td>
<td>{5, 4, 4, 3; 1, 1, 2, 2}</td>
<td>$5^1 \ 3^{27} \ 1^{42} \ (-2)^{48} \ (-4)^8$</td>
<td>$P_{24} = P_{44}$</td>
</tr>
<tr>
<td>$M_{22}$ graph</td>
<td>330</td>
<td>{7, 6, 4, 4; 1, 1, 1, 6}</td>
<td>$7^1 \ 4^{55} \ 1^{154} \ (-3)^{99} \ (-4)^{21}$</td>
<td>$P_{14} = P_{34}$</td>
</tr>
<tr>
<td>Unital graph</td>
<td>280</td>
<td>{9, 8, 6, 3; 1, 1, 3, 8}</td>
<td>$9^1 \ 4^{64} \ 1^{105} \ (-3)^{90} \ (-5)^{20}$</td>
<td>$P_{14} = P_{34}$</td>
</tr>
</tbody>
</table>

The case $d = 4$ with strongly regular $\Gamma_4$

One may wonder whether it is possible that $\Gamma_4$ is strongly regular. This would require $p_{14}^4 = p_{34}^4 = p_{44}^4$. Or, equivalently, that $\Gamma_4$ has only two eigenvalues with eigenvector other than the all-1 vector. Since the values $P_{14}$ alternate in sign, this would mean $P_{14} = P_{34}$ and $P_{24} = P_{44}$.

Proposition 6 Let $\Gamma$ be a distance-regular graph of diameter 4. The following assertions are equivalent.

(i) $\Gamma_4$ is strongly regular.

(ii) $b_3 = a_4 + 1$ and $b_1 = b_3 c_3$.

(iii) $(\theta_1 + 1)(\theta_4 + 1) = (\theta_2 + 1)(\theta_3 + 1) = - b_1$.

Proof. (i)-(ii) A boring computation (using [BCN], 4.1.7) shows that $p_{14}^4 = p_{34}^4$ is equivalent to $b_3 = a_4 + 1$, and that if this holds $p_{14}^3 = p_{34}^3$ is equivalent to $b_1 = b_3 c_3$.

(i)-(iii) $\Gamma_4$ will be strongly regular if and only if $P_{14} = P_{34}$ and $P_{24} = P_{44}$.

We saw that this is equivalent to $(\theta_2 + 1)(\theta_4 + 1) = - b_1$ and $(\theta_1 + 1)(\theta_3 + 1) = - b_1$.

$\square$
The fact that (i) implies the first equality in (iii) was proved in [F01] as a consequence of another characterization of (i) in terms of the spectrum only. More generally, a quasi-spectral characterization of those connected regular graphs (with $d + 1$ distinct eigenvalues) which are distance-regular, and with the distance-$d$ graph being strongly regular, is given in [F00, Th. 2.2].

No nonantipodal examples are known, but the infeasible array $\{12, 8, 6, 4; 1, 1, 2, 9\}$ with spectrum $21^{1} 7^{56} 3^{140} (-2)^{160} (-3)^{168}$ (cf. [BCN], p. 410) would have been an example (and there are several open candidate arrays, such as $\{21, 20, 14, 10; 1, 1, 2, 12\}, \{24, 20, 20, 10; 1, 1, 2, 15\}$, and $\{66, 65, 63, 13; 1, 1, 5, 54\}$).

If $\Gamma$ is antipodal, then $\Gamma_4$ is a union of cliques (and hence strongly regular). This holds precisely when $\theta_1 + \theta_3$ and $\theta_1 \theta_3 = -k$ and $(\theta_2 + 1)(\theta_4 + 1) = -b_1$. (Indeed, $\theta_1, \theta_3$ are the two roots of $\theta^2 - \lambda \theta - k = 0$ by [BCN], 4.2.5.)

There are many examples, e.g.

<table>
<thead>
<tr>
<th>name</th>
<th>$n$</th>
<th>intersection array</th>
<th>spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wells graph</td>
<td>32</td>
<td>${5, 4, 1, 1; 1, 1, 4, 5}$</td>
<td>$5^4 \sqrt{5}^8 1^{10} (-\sqrt{5})^8 (-3)^6$</td>
</tr>
<tr>
<td>3.Sym(6).2 graph</td>
<td>45</td>
<td>${6, 4, 2, 1; 1, 1, 4, 6}$</td>
<td>$6^4 3^{12} 1^9 (-2)^{18} (-3)^5$</td>
</tr>
<tr>
<td>Locally Petersen</td>
<td>63</td>
<td>${10, 6, 4, 1; 1, 2, 6, 10}$</td>
<td>$10^1 5^{12} 1^{14} (-2)^{30} (-4)^6$</td>
</tr>
</tbody>
</table>

If $\Gamma$ is bipartite, then $\Gamma_4$ is disconnected, so if it is strongly regular, it is a union of cliques and $\Gamma$ is antipodal. In this case its spectrum is

$$\{k^1, \sqrt{k}^{n/2-k}, 0^{2k-2}, (-\sqrt{k})^{n/2-k}, (-k)^1\}.$$  

Such graphs are precisely the incidence graphs of symmetric $(m, \mu)$-nets, where $m = k/\mu$ ([BCN], p. 425).

The case $d = 5$

As before, and also using $\sum_i m_i \theta_i^4 = nk(k + \lambda^2 + b_1\mu)$ (where $\mu = c_2$) we find for $\{f, g, h, i, j\} = \{1, 2, 3, 4, 5\}$ that $P_{fd} = P_{gd}$ if and only if

$$(\theta_h + 1)(\theta_i + 1)(\theta_j + 1) + b_1(\theta_h + \theta_i + \theta_j) = b_1(\lambda - \mu - 1).$$

In case $\theta_i = -1$, this says that $\theta_h + \theta_j = \lambda - \mu$.

For example, the Odd graph $O_6$ has $\lambda = 0$, $\mu = 1$, and eigenvalues 6, 4, 2, $-1, -3, -5$. It follows that $P_{15} = P_{25} = P_{35}$ and $P_{45} = P_{55}$.

Similarly, the folded 11-cube has $\lambda = 0$, $\mu = 2$, and eigenvalues 11, 7, 3, $-1, -5, -9$. It follows that $P_{15} = P_{25} = P_{35}$ and $P_{45} = P_{55}$.

An example without eigenvalue $-1$ is provided by the folded Johnson graph $J(20, 10)$. It has $\lambda = 18$, $\mu = 4$ and eigenvalues 100, 62, 32, 10, $-4, -10$. We see that $P_{35} = P_{55}$.

Combining two of the above conditions, we see that $P_{15} = P_{25} = P_{55}$ if and only if $(\theta_2 + 1)(\theta_4 + 1) = -b_1$ and $\theta_2 + \theta_4 = \lambda - \mu$ (and hence $\theta_2\theta_4 = \mu - k$).

Now $b_3 + b_4 + c_4 = 2k + \mu - \lambda$ and $b_3b_4 + b_3c_5 + c_4c_5 = kb_1 + k\mu + \mu$.

**Generalized 12-gons**

A generalized 12-gon of order $(q, 1)$ (the line graph of the bipartite point-line incidence graph of a generalized hexagon of order $(q, q)$) has diameter 6, and its $P$ matrix is given by
For the dual polar graphs $B_P^h$

Suppose that cf. [BCN], 4.2.18.

The distance 1-or-2 graph $\Delta = \Gamma$

Distance-regular distance 1-or-2 graph

And according to [B], dual polar graphs of diameter $d$

Dual polar graphs

According to [B], dual polar graphs of diameter $d$ satisfy

$$P_{id} = (-1)^i q^{d(d-1)/2+de-i(d+e-i)}$$

where $e$ has the same meaning as in [BCN], 9.4.1 (that is, $e$ is 1, 1, 0, 2, 1 $\frac{1}{2}$, $\frac{3}{2}$ in the case of $B_d(q), C_d(q), D_d(q), 2D_{d+1}(q), 2A_{2d-1}(\sqrt{q}), 2A_{2d}(\sqrt{q})$, respectively).

It follows that $P_{hd} = P_{ud}$ when $d+e$ is an even integer and $h+i = d+e$.

For the dual polar graphs $B_d(q)$ and $C_d(q)$ we have $e = 1$, and the condition becomes $h+i = d+1$ where $d$ is odd. Below we will see this in a different way.

For the dual polar graph $D_d(q)$ we have $e = 0$, and the condition becomes $h+i = d$ where $d$ is even. Not surprising, since this graph is bipartite.

For the dual polar graph $2D_{d+1}(q)$ we have $e = 2$, and the condition becomes $h+i = d+2$ where $d$ is even. (We saw the case $d = 4$ above.)

Finally, $h+i = d+e$ is impossible when $e$ is not integral.

Distance-regular distance 1-or-2 graph

The distance 1-or-2 graph $\Delta = \Gamma_1 \cup \Gamma_2$ of $\Gamma$ (with adjacency matrix $A_1 + A_2$) is distance-regular if and only if $b_{i-1} + b_i + c_i + c_{i+1} = 2k + \mu - \lambda$ for $1 \leq i \leq d-1$, cf. [BCN], 4.2.18.

Proposition 7 Suppose that $\Gamma_1 \cup \Gamma_2$ is distance-regular. Then for $1 \leq i \leq d$ we have $P_{d+1-i,d} = P_{id}$ if $d$ is odd, and $(\theta_{d+1-i} + 1)P_{id} = (\theta_i + 1)P_{d+1-i,d}$ if $d$ is even. If $i \neq (d+1)/2$ then $\theta_{d+1-i} = \lambda - \mu - \theta_i$. If $d$ is odd, then $\theta_{(d+1)/2} = -1$. 

\begin{align*}
  P &= \\
  &= \begin{pmatrix}
    1 & 2q & 2q^2 & 2q^3 & 2q^4 & 2q^5 & q^6 \\
    1 & q-1+a & q+(q-1)a & 2q(q-1) & -q^2+q(q-1)a & q^3(q-1)-q^2a & -q^3 \\
    1 & q-1+b & -q+(q-1)b & -2qb & -q^2+q(q-1)b & -q^3(q-1)+q^2b & q^4 \\
    1 & q-1 & 1 & -2q & 2q & q^2(q-1) & -q^3 \\
    1 & q-1 & -q-(q-1)b & 2qb & -q^2+q(q-1)b & -q^3(q-1)-q^2b & q^4 \\
    1 & q-1-a & q-(q-1)a & 2q(q-1) & -q^2-q(q-1)a & q^3(q-1)+q^2a & -q^4 \\
    1 & -2 & 2 & -2 & 2 & -2 & 1
  \end{pmatrix}
\end{align*}

where $a = \sqrt{3q}$ and $b = \sqrt{q}$. We see that $P_{16} = P_{36} = P_{56}$ and $P_{26} = P_{16}$.

Its dual is a generalized 12-gon of order $(1, q)$, and is bipartite. The $P$ matrix is given by

\begin{align*}
  P &= \\
  &= \begin{pmatrix}
    1 & q+1 & q(q+1) & q^2(q+1) & q^3(q+1) & q^4(q+1) & q^5 \\
    1 & a & 2q-1 & (q-1)a & q(q-2) & -qa & -q^2 \\
    1 & b & 1 & -q-1 & -q(q-1)b & -q^2 & qb \\
    1 & 0 & 0 & -q+1 & 0 & q(q+1) & 0 & -q^2 \\
    1 & -b & 0 & -(q+1)b & -q^2 & -qb & q^2 \\
    1 & -a & 2q-1 & -(q-1)a & q(q-2) & qa & -q^2 \\
    1 & -q-1 & q(q+1) & -q^2(q+1) & q^2(q+1) & -q^4(q+1) & q^5 \\
  \end{pmatrix}
\end{align*}

where $a = \sqrt{3q}$ and $b = \sqrt{q}$. We see that $P_{16} = P_{36} = P_{56}$ and $P_{26} = P_{16}$.

As expected (cf. [B]), the squares of all $P_{id}$ are powers of $q$. 

Dual polar graphs
Proof. For each eigenvalue \( \theta \) of \( \Gamma \), there is an eigenvalue \((\theta^2 + (\mu - \lambda)\theta - k)/\mu\) of \( \Delta \). If \( d \) is odd, then \( \Delta \) has diameter \((d + 1)/2\), and \( \Gamma \) has an eigenvalue \(-1\), and for each eigenvalue \( \theta \neq -k, -1 \) of \( \Gamma \) also \( \lambda - \mu - \theta \) is an eigenvalue. Now \( \Delta_{d+1}/2 = \Gamma_d \), and \( P_{i\ell} = P_{i\ell} \) if \( i, i' \) belong to the same eigenspace of \( \Delta \). Since the numbers \( P_{i\ell} \) alternate, the eigenvalue \(-1\) must be the middle one (not considering \( \theta_0 \)), and we see that \( P_{d+1-i,\ell} = P_{d\ell} \) for \( 1 \leq i \leq d \).

If \( d \) is even, then \( \Delta \) has diameter \( d/2 \), and for each eigenvalue \( \theta \neq k \) of \( \Gamma \) also \( \lambda - \mu - \theta \) is an eigenvalue. Now \( \Delta_{d/2} = \Gamma_{d-1} \cup \Gamma_d \). Since \( AA_d = b_{d-1}A_{d-1} + a_dA_d \) and our parameter conditions imply \( b_{d-1} + c_d = b_1 + \mu \) (so that \( b_{d-1}(P_{i,d-1} + P_{i\ell}) = (\theta_1 - a_d + b_{d-1})P_{i\ell} = (\theta_1 + \mu - \lambda - 1)P_{i\ell} \)), the equalities \( P_{i,d-1} + P_{i\ell} = P_{d+1-i,d-1} + P_{d+1-i,\ell} \) \((1 \leq i \leq d)\) and \( \theta_i + \theta_{d+1-i} = \lambda - \mu \) imply \((\theta_{d+1-i} + 1)P_{i\ell} = (\theta_1 + 1)P_{d+1-i,\ell} \).

The Odd graph \( O_{d+1} \) on \((2d+1)/d\) vertices has diameter \( d \) and eigenvalues \( \theta_i = d + 1 - 2i \) for \( i < (d + 1)/2 \), and \( \theta_i = d - 2i \) for \( i \geq (d + 1)/2 \). Since its distance 1-or-2 graph is distance-regular, we have \( P_{d+1-i,\ell} = P_{d\ell} \) for odd \( d \) and \( 1 \leq i \leq d \).

The folded \((2d + 1)\)-cube on \( 2^{2d} \) vertices has diameter \( d \) and eigenvalues \( \theta_i = 2d + 1 - 4i \) with multiplicities \( m_i = (2d+1)/2 \) \((0 \leq i \leq d)\). Since its distance 1-or-2 graph is distance-regular, it satisfies \( P_{d+1-i,\ell} = (-1)^{d+1}P_{d\ell} \) for \( 1 \leq i \leq d \). (Note that \( \theta_{d+1-i} + 1 = -(\theta_i + 1) \) since \( \mu - \lambda = 2 \).)

The dual polar graphs \( B_d(q) \) and \( C_d(q) \) have diameter \( d \) and eigenvalues \( \theta_i = (q^{d-i+1} - q^{i})/(q - 1) - 1 \). Since their distance 1-or-2 graphs are distance-regular, they satisfy \( P_{d+1-i,\ell} = (-1)^{d+1}P_{d\ell} \) for \( 1 \leq i \leq d \).

The fact that \(-1\) must be the middle eigenvalue for odd \( d \), implies that \( \theta_{(d+1)/2} > \lambda - \mu + 1 \), so that there is no eigenvalue \( \xi \) with \(-1 < \xi < \lambda - \mu + 1 \).

The bipartite case

If \( \Gamma \) is bipartite, then \( \theta_{d-i} = -\theta_i \), and \( P_{d-i,j} = (-1)^iP_{i,j} \) \((0 \leq i, j \leq d)\). In particular, if \( d \) is even, then \( P_{d-i,\ell} = P_{d\ell} \) and \( \Gamma_d \) is disconnected.

The antipodal case

The graph \( \Gamma \) is antipodal when having distance \( d \) is an equivalence relation, i.e., when \( \Gamma_d \) is a union of cliques. The graph is called an antipodal \( r \)-cover, when these cliques are \( r \)-cliques. Now \( r = k_d + 1 \), and \( P_{d\ell} \) alternates between \( k_d \) and \(-1\).

For example, the ternary Golay code graph (of diameter 5) with intersection array \( \{22, 20, 18, 2, 1; 1, 2, 9, 20, 22\} \) has spectrum \( 22^1 7^{132} 4^{122} (-2)^{330} (-5)^{110} \) \((-1)\)^d and satisfies \( P_{d5} = P_{25} = P_{15} = 2 \), \( P_{15} = P_{35} = P_{55} = -1 \).

For an antipodal distance-regular graph \( \Gamma \), the folded graph has eigenvalues \( \theta_0, \theta_2, \ldots, \theta_{2e} \), where \( e = [d/2] \). In Theorem 9 below we show for odd \( d \) that this already follows from \( P_{d\ell} = P_{3\ell} = \cdots = P_{d\ell} \).

Proposition 8 If \( P_{0d} = P_{d\ell} \) then \( i \) is even. Let \( i > 0 \) be even. Then \( P_{0d} = P_{d\ell} \) if and only \( \Gamma \) is antipodal, or \( i = d \) and \( \Gamma \) is bipartite.

Proof. Since the \( P_{d\ell} \) alternate in sign, \( P_{0d} = P_{d\ell} \) implies that \( i \) is even. If \( \Gamma \) is bipartite, then \( P_{d\ell} = (-1)^dP_{0d} \). If \( \Gamma \) is antipodal, then \( P_{d\ell} = P_{0d} \) for all even \( i \).
That shows the ‘if’ part. Conversely, if $P_{0d} = P_{dd}$, then the valency of $\Gamma_d$ is an eigenvalue of multiplicity larger than 1, so that $\Gamma_d$ is disconnected, and hence $\Gamma$ is imprimitive and therefore antipodal or bipartite. If $\Gamma$ is bipartite but not antipodal, then its halved graphs are primitive and $|P_{ad}| < P_{0d}$ for $0 < i < d$ (cf. [BCN], pp. 140–141).

The half-antipodal case

Given an array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ of positive real numbers, define the polynomials $p_i(x)$ for $-1 \leq i \leq d+1$ by $p_{-1}(x) = 0, p_0(x) = 1, (x - a_i)p_i(x) = b_{i-1}p_{i-1}(x) + c_{i+1}p_{i+1}(x)$ $(0 \leq i \leq d)$, where $a_i = b_0 - b_1 - c_i$ and $c_{d+1}$ is some arbitrary positive number. The eigenvalues of the array are by definition the zeros of $p_{d+1}(x)$, and do not depend on the choice of $c_{d+1}$. Each $p_i(x)$ has degree $i$, and, by the theory of Sturm sequences, each $p_i(x)$ has $i$ distinct real zeros, where the zeros of $p_{i+1}(x)$ interlace those of $p_i(x)$. If $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ is the intersection array of a distance-regular graph $\Gamma$, then the eigenvalues of the array are the eigenvalues of (the adjacency matrix of) $\Gamma$.

Let $L$ be the tridiagonal matrix

\[ L = \begin{pmatrix}
  a_0 & b_0 &  &  \\
  c_1 & a_1 & b_1 &  \\
  & c_2 & \ddots & \ddots \\
  & & \ddots & b_{d-1} \\
  & & & c_d & a_d
\end{pmatrix}. \]

The eigenvalues of the array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ are the eigenvalues of the matrix $L$.

**Theorem 9** Let $\Gamma$ be a distance-regular graph with odd diameter $d = 2e + 1$ and intersection array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$. Then $P_{1d} = P_{dd} = \cdots = P_{dd}$ if and only if the $\theta_j$ with $j = 0, 2, 4, \ldots, 2e$ are the eigenvalues of the array $\{b_0, \ldots, b_{e-1}; c_1, \ldots, c_e\}$.

**Proof.** Let $H = \{1, 3, \ldots, d\}$, so that $|H| = e + 1$. By Proposition 5, $P_{1d} = P_{dd} = \cdots = P_{dd}$ if and only if $\sum \sum m_i \theta_i^s \prod_{j \not\in H}(\theta_i - \theta_j) = 0$ for $0 \leq s \leq e - 1$. Let $E = \{0, 2, \ldots, 2e\}$, so that $|E| = e + 1$. Then this condition is equivalent to

\[ \text{tr} A^s \prod_{j \in E}(A - \theta_j I) = 0 \quad (0 \leq s \leq e - 1). \]

This says that the expansion of $\prod_{j \in E}(A - \theta_j I)$ in terms of the $A_j$ does not contain $A_s$ for $0 \leq s \leq e - 1$, hence is equivalent to $\prod_{j \in E}(A - \theta_j I) = aA_{e} + bA_{e+1}$ for certain constants $a, b$. Since $0 \in E$, we find that $ak_e + bk_{e+1} = 0$, and the condition is equivalent to $(A_e/k_e - A_{e+1}/k_{e+1})E_j = 0$ for all $j \in E$.

An eigenvalue $\theta$ of $\Gamma$ defines a right eigenvector $u$ (known as the ‘standard sequence’) by $Lu = \theta u$. It follows that $\theta$ will be an eigenvalue of the array $\{b_0, \ldots, b_{e-1}; c_1, \ldots, c_e\}$ precisely when $u_e = u_{e+1}$. Up to scaling, the $u_i$ belonging to $\theta_j$ are the $Q_{ij}$ (that is, the columns of $Q$ are eigenvectors of $L$). So, $\theta_j$ is an eigenvalue of $\{b_0, \ldots, b_{e-1}; c_1, \ldots, c_e\}$ for all $j \in E$ precisely when $Q_{e_j} = Q_{e+1,j}$ for all $j \in E$. Since $k_i Q_{ij} = m_j P_{ji}$, this holds if and only if
$P_{je}/k_e = P_{j,e+1}/k_{e+1}$, i.e., if and only if $(A_e/k_e - A_{e+1}/k_{e+1})E_j = 0$ for all $j \in E$.

For example, if $d = 3$ one has $P_{13} = P_{33}$ if and only if $\theta_0, \theta_2$ are the eigenvalues $k, -1$ of the array $\{k; 1\}$. And if $d = 5$ one has $P_{15} = P_{55}$ if and only if $\theta_0, \theta_2, \theta_4$ are the eigenvalues of the array $\{k, b_1; 1, c_2\}$. The case of even $d$ is slightly more complicated.

**Theorem 10** Let $\Gamma$ be a distance-regular graph with even diameter $d = 2e$ and intersection array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$. Then $P_{td} = P_{td} = \cdots = P_{d-1,t_d}$ if and only if the $\theta_j$ with $j = 0, 2, 4, \ldots, 2e$ are the eigenvalues of the array $\{b_0, \ldots, b_{e-1}; c_1, \ldots, c_{e-1}, c_e + zb_e\}$ for some real number $z$ with $0 < z \leq 1$, uniquely determined by $\sum_{i=0}^e \theta_{2i} = \sum_{i=0}^e a_i + (1 - z)b_c$. If $\Gamma$ is antipodal or bipartite, then $z = 1$.

**Proof.** Let $E = \{0, 2, \ldots, d\}$. As before we see that $P_{td} = P_{td} = \cdots = P_{d-1,t_d}$ is equivalent to the condition that $\prod_{j \in E}(A - \theta_jI) = aA_e + bA_c + cA_e$ for certain constants $a, b, c$. Comparing coefficients of $A^{e+1}$ we see that $c > 0$.

With $j = 0$ we see that $a_{e-1} + bk_e + c_{e+1} = 0$.

Take $z = -\frac{ak_{e-1}}{ck_{e+1}} = 1 + \frac{bk_e}{ck_{e+1}}$.

Then $aP_{j,e-1} + bP_{j,c} + cP_{j,e+1} = 0$ for $j \in E$ gives

\[ z \left( \frac{P_{j,e-1}}{k_{e-1}} - \frac{P_{j,e}}{k_e} \right) = \frac{P_{j,e+1}}{k_{e+1}} - \frac{P_{j,c}}{k_c}. \]

On the other hand, if $\theta = \theta_j$ for some $j \in E$, then $\theta$ is an eigenvalue of the array $\{b_0, \ldots, b_{e-1}; c_1, \ldots, c_{e-1}, c_e + zb_e\}$ precisely when $c_eu_{e-1} + (k - b_e - c_e)u_e + b_{e}u_{e+1} = (c_e + zb_e)u_{e-1} + (k - c_e - zb_e)u_e$, i.e., when $z(u_{e-1} - u_e) = u_{e+1} - u_e$.

Since (up to a constant factor) $u_i = P_{ji}/k_i$, this is equivalent to the condition above.

Noting that $d \in E$, we can apply the above to $\theta = \theta_d$. Since the bottom row of $P$ has $d$ sign changes, it follows that the sequence $u_i$ has $d$ sign changes. In particular, the $u_i$ are nonzero. Now $u_{e-1} - u_e$ and $u_{e+1} - u_e$ have the same sign, and it follows that $z > 0$.

If $\Gamma$ is an antipodal $r$-cover of diameter $d = 2e$, then $c_e + zb_e = rc_e$ and $z = 1$ (and $P_{je} = 0$ for all odd $j$).

If $\Gamma$ is bipartite, then $\theta_j + \theta_{d-j} = 0$ for all $i$, so $\sum_{j \in E} \theta_j = 0$, so our tridiagonal matrix (the analog of $L$) has trace $0 = a_1 + \cdots + a_e + (1 - z)b_e = (1 - z)b_e$, so that $z = 1$.

It remains to show that $z \leq 1$. We use $z(u_{e-1} - u_e) = u_{e+1} - u_e$ to conclude that $\theta$ is an eigenvalue of both

\[
\begin{pmatrix}
0 & k & c_1 & a_1 & b_1 & c_{e-1} & a_{e-1} & b_{e-1} & p & k - p \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
k - q & q & c_{e+1} & a_{e+1} & b_{e+1} & c_{d-1} & a_{d-1} & b_{d-1} & p & c_d & a_d \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
\end{pmatrix}
\]

where $p = c_e + zb_e$ and $q = b_e + z^{-1}c_e$. Since this holds for each $\theta = \theta_j$ for $j \in E$, this accounts for all eigenvalues of these two matrices, and $\sum_{j \in E} \theta_j =$
\(a_1 + \cdots + a_e + (1 - z)b_e = a_e + \cdots + a_d + (1 - z^{-1})c_e.\) Since \(p_{cd}^{e+1} \geq 0\) it follows that \(a_1 + \cdots + a_{e-1} \leq a_{e+1} + \cdots + a_d\) (cf. [BCN], 4.1.7), and therefore \((1 - z)b_e \geq (1 - z^{-1})c_e.\) It follows that \(z \leq 1.\) \(\square\)

Nonantipodal, nonbipartite examples:

<table>
<thead>
<tr>
<th>name</th>
<th>array</th>
<th>half array</th>
<th>(z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coxeter</td>
<td>{3, 2, 2, 1; 1, 1, 1, 2}</td>
<td>{3; 2; 1}</td>
<td>1/2</td>
</tr>
<tr>
<td>(M_{22})</td>
<td>{7, 6, 4, 4; 1, 1, 1, 6}</td>
<td>{7; 6; 1}</td>
<td>1/2</td>
</tr>
<tr>
<td>(PTL(3, 4, 2))</td>
<td>{9, 8, 6, 3; 1, 1, 3, 8}</td>
<td>{9; 8; 1}</td>
<td>1/2</td>
</tr>
<tr>
<td>gen. 8-gon</td>
<td>{s(t + 1), st, st, st; 1, 1, t + 1}</td>
<td>{s(t + 1), st; 1, t + 1}</td>
<td>1/s</td>
</tr>
<tr>
<td>gen. 12-gon</td>
<td>{2q, q, q, q, q; 1, 1, 1, 1, 2}</td>
<td>{2q, q, q; 1, 1, 2}</td>
<td>1/q</td>
</tr>
</tbody>
</table>

Concerning the value of \(z\), note that both \(zb_e\) and \(z^{-1}c_e\) are algebraic integers.

If \(d = 4\), the case \(z = 1\) can be classified.

**Proposition 11** Let \(\Gamma\) be a distance-regular graph with even diameter \(d = 2e\) such that \(\theta_j\) with \(j = 0, 2, 4, \ldots, 2e\) are the eigenvalues of the array \(\{b_0, \ldots, b_{e-1}; c_1, \ldots, c_{e-1}, b_e + c_e\}\). Then \(\Gamma\) satisfies \(p_{e,e+1} = 0.\) If moreover \(d \leq 4\), then \(\Gamma\) is antipodal or bipartite.

**Proof.** We have equality in the inequality \(a_1 + \cdots + a_{e-1} \leq a_{e+1} + \cdots + a_d\), so that \(p_{cd}^{e+1} = 0\) by [BCN], 4.1.7. The case \(d = 2\) is trivial. Suppose \(d = 4\). Then \(p_{d4}^d = 0.\) Let \(d(x, z) = 4\), and consider neighbors \(y, w\) of \(z\), where \(d(x, y) = 3\) and \(d(x, w) = 4\). Then \(d(y, w) \neq 2\) since there are no 2-3-4 triangles, so \(d(y, w) = 1\). If \(a_4 \neq 0\) then there exist such vertices \(w\), and we find that the neighborhood \(\Gamma(z)\) of \(z\) in \(\Gamma\) is not cocomplex (its complement is not connected), contradicting [BCN], 1.1.7. Hence \(a_4 = 0\), and \(a_3 = a_1\). If \(b_3 > 1\), then let \(d(x, y) = 3, y \sim z, z'\) with \(d(x, z) = d(x, z') = 4\). Let \(z''\) be a neighbor of \(z'\) with \(d(z, z'') = 2\). Then \(d(x, z'') = 3\) since \(a_4 = 0\), and we see a 2-3-4 triangle, contradiction. So if \(b_3 > 1\) then \(a_2 = 0\), and \(a_1 = 0\) since \(a_1 \leq 2a_2\), and the graph is bipartite. If \(b_3 = 1\), then the graph is antipodal. \(\square\)

**Variations**

One can vary the above theme. First, the following general result builds on ideas previously used.

**Theorem 12** Let \(\Gamma\) be a distance-regular graph with diameter \(d\), and let \(H \subseteq \{0, \ldots, d\}\). Then all \(P_{id}\) for \(i \in H\) take the same value if and only if the eigenvalues \(\theta_j\) with \(j \notin H\) are the zeros of the polynomial \(\sum_{i=|H|-1}^{r} \frac{\alpha_i}{\alpha_{i+1}} x^i tr(A^i)\), where \(r = d + 1 - |H|, \alpha_0 = 1,\) and

\[
\alpha_i = \frac{tr(A_i \prod_{j \in H} (A - \theta_j I))}{tr(A_r A^r)} \quad (|H| - 1 \leq i \leq d - |H|). \tag{2}
\]

**Proof.** By Proposition 5, the hypothesis holds if and only if we have the equalities \(\sum_i m_i \theta_i^r \prod_{j \notin H} (\theta_i - \theta_j) = 0\) for \(0 \leq s \leq |H| - 2\). That is,

\[
\tr A^r \prod_{j \notin H} (A - \theta_j I) = 0 \quad (0 \leq s \leq |H| - 2).
\]

9
It follows that $\prod_{j \in H} (A - \theta_j I)$, when written on the basis $\{A_i \mid 0 \leq i \leq d\}$, does not contain $A_s$ for $0 \leq s \leq |H| - 2$. Moreover, the number of factors is $r = d + 1 - |H|$, so that $A_s$ does not occur either when $s > r$. Therefore, $\prod_{j \in H} (A - \theta_j I)$ is a linear combination of the $A_s$ with $|H| - 1 \leq s \leq r$ and, hence, for some constants $\alpha_i$,

$$\frac{n}{\text{tr}(A_r A^r)} \prod_{j \in H} (A - \theta_j I) = \sum_{s = |H| - 1}^{r} \frac{\alpha_s}{k_s} A_s.$$ 

Now comparing coefficients of $A^r$, we see that $\alpha_r = 1$ (notice that $\text{tr}(A_r A^r) = nc_1 \cdots c_e k_e$). To obtain the value of $\alpha_i$ for $|H| - 1 \leq i \leq d - |H|$, multiply both terms of the above equation by $A_i$ and take traces.

In the above we applied this twice, namely for $d = 2e + 1$, $H = \{1, 3, 5, \ldots, d\}$, and for $d = 2e$, $H = \{1, 3, 5, \ldots, d-1\}$. In the latter case, (2) yields the following expression for $z = -\alpha_{e-1}$.

$$b_0 b_1 \cdots b_e n_z = c_1 c_2 \cdots c_{e+1} k_{e+1} n_z = -\text{tr}(A_{e-1} \prod_{j \in H} (A - \theta_j I)).$$

Let us now take for $H$ the set of even indices, with or without 0.

$H = \{0, 2, 4, \ldots, d\}$

Let $d = 2e$ be even and suppose that $P_{sd}$ takes the same value $(k_d)$ for all $i \in H = \{0, 2, 4, \ldots, d\}$. Then $|H| = e + 1$, and $\prod_{j \in H} (A - \theta_j I)$ is a multiple of $A_e$. By Proposition 8 this happens if and only if $\Gamma$ is antipodal with even diameter.

$H = \{0, 2, 4, \ldots, d-1\}$

Let $d = 2e + 1$ be odd and suppose that $P_{sd}$ takes the same value $(k_d)$ for all $i \in H = \{0, 2, 4, \ldots, 2e\}$. Then $|H| = e$, and $\prod_{j \in H} (A - \theta_j I)$ is a linear combination of $A_s$ for $e - 2 \leq s \leq e + 1$. As before we conclude that the $\theta_j$ with $j \notin H$ are the eigenvalues of the array $\{b_0, \ldots, b_{2e-1}; c_1, \ldots, c_{e-1}, c_{e+1} \pm z b_e\}$ for some real $z \leq 1$. This time $d \in H$, and there is no conclusion about the sign of $z$.

For example, the Odd graph $O_3$ with intersection array $\{5, 4, 4, 3; 1, 1, 2, 2\}$ has eigenvalues 5, 3, 1, $-2$, $-4$ and $P_{24} = P_{44}$. The eigenvalues 5, 3, $-2$ are those of the array $\{5, 4, 1, -1\}$.

No primitive examples with $d > 4$ are known.

$H = \{2, 4, \ldots, d-1\}$

Let $d = 2e + 1$ be odd and suppose that $P_{sd}$ takes the same value for all $i \in H = \{2, 4, \ldots, 2e\}$. Then $|H| = e$, and $\prod_{j \in H} (A - \theta_j I)$ is a linear combination of $A_s$ for $e - 1 \leq s \leq e + 2$. 

10
For example, the Odd graph $O_6$ with intersection array \{6, 5, 5, 4, 4; 1, 1, 2, 2, 3\} has eigenvalues 6, 4, 2, −1, −3, −5 and $P_{24} = P_{44}$.
No primitive examples with $d > 5$ are known.

Acknowledgments
Part of this note was written while the second author was visiting the Department of Combinatorics and Optimization (C&O), in the University of Waterloo (Ontario, Canada). He sincerely acknowledges to the Department of C&O the hospitality and facilities received. Also, special thanks are due to Chris Godsil for useful discussions on the case of diameter four.

References

