A&A 567, A46 (2014)
DOI: 10.1051/0004-6361/201423813

Astronomy & Astrophysics

Conditions of consistency for multicomponent stellar systems

II. Is a point-axial symmetric model suitable for the Galaxy?

Rafael Cubarsi

Dept. Matemàtica Aplicada IV, Universitat Politècnica de Catalunya, 08034 Barcelona, Catalonia, Spain
e-mail: rcubarsi@ma4.upc.edu

Received 14 March 2014 / Accepted 14 April 2014

ABSTRACT

Under a common potential, a finite mixture of ellipsoidal velocity distributions satisfying the Boltzmann collisionless equation provides a set of integrability conditions that may constrain the population kinematics. They are referred to as conditions of consistency and were discussed in a previous paper on mixtures of axisymmetric populations. As a corollary, these conditions are now extended to point-axial symmetry, that is, point symmetry around the rotation axis or bisymmetry, by determining which potentials are connected with a more flexible superposition of stellar populations. Under point-axial symmetry, the potential is still axisymmetric, but the velocity and mass distributions are not necessarily. A point-axial stellar system is, in a natural way, consistent with a flat velocity distribution of a disc population. Therefore, no additional integrability conditions are required to solve the Boltzmann collisionless equation for such a population. For other populations, if the potential is additively separable in cylindrical coordinates, the populations are not kinematically constrained, although under point-axial symmetry, the potential is reduced to the harmonic function, which, for the Galaxy, is proven to be non-realistic. In contrast, a non-separable potential provides additional conditions of consistency. When mean velocities for the populations are unconstrained, the potential becomes quasi-stationary, being a particular case of the axisymmetric model. Then, the radial and vertical mean velocities of the populations can differ and produce an apparent vertex deviation of the whole velocity distribution. However, single population velocity ellipsoids still have no vertex deviation in the Galactic plane and no tilt in their intersection with a meridional Galactic plane. If the thick disc and halo ellipsoids actually have non-vanishing tilt, as the surveys of the solar neighbourhood that include RAdial Velocity Experiment (RAVE) data seem to show, the point-axial model is unable to fit the local velocity distribution. Conversely, the axisymmetric model is capable of making a better approach. If, in the end, more accurate data confirm a negligible tilt of the populations, then the point-axisymmetric model will be able to describe non-axisymmetric mass and velocity distributions, although in the Galactic plane the velocity distribution will still be axisymmetric.

Key words. galaxies: kinematics and dynamics – solar neighborhood – galaxies: statistics

1. Introduction

The purpose of the present work is to complete some aspects of the analysis of conditions of consistency for mixtures of axisymmetric stellar systems (Cubarsi 2014, hereafter Paper I) by studying the more general point-axial symmetry (or bisymmetry) case, i.e., rotational symmetry of 180° for the potential and the phase space density functions.

To simplify the solution of the Boltzmann collisionless equation (BCE) it is necessary to introduce some symmetries for the mass and the velocity distributions, such as the assumptions of axisymmetry, steady state, or Galactic plane of symmetry. These hypotheses provide serious limitations for describing in a realistic way the kinematic observables of the Galaxy unless a mixture model is assumed. The conditions of consistency are integrability conditions allowing for a mixture of independent populations of generalised Schwarzschild type to share the same potential function. Since the potential may depend on the population parameters involved in the velocity distribution, the less the potential depends on them, the less kinematically constrained the populations will be.

Several kinematic analyses using the newest radial velocity data from the RAdial Velocity Experiment (RAVE) survey (Siebert et al. 2011; Zwitter et al. 2008; Steinmetz et al. 2006) confirmed that the thin disc has non-vanishing vertex deviation, the thick disc has a radial mean motion differing from that of the thin disc, and the halo velocity ellipsoid is likely to be tilted (e.g., Pasetto et al. 2012a,b; Moni Bidin et al. 2012; Casetti-Dinescu et al. 2011; Carollo et al. 2010; Smith et al. 2009a,b).

It was suggested (Pasetto et al. 2012b; Steinmetz 2012) that the axisymmetry assumption should be relaxed towards a model with point-axial symmetry to account for these features. However, in Paper I we proved that an axisymmetric mixture model is able to describe the actual velocity distribution in the solar neighbourhood provided that the potential is quasi-stationary[1] and the phase space density function is time-dependent. This family of potentials is consistent with populations having different mean velocities producing a non-null vertex deviation of the disc distribution. In addition, if the potential is separable in cylindrical coordinates, the velocity ellipsoids may have an arbitrary tilt.

Unlike in the axisymmetric model, in steady state point-axial systems, non-null radial and vertical differential motions are also possible (Sanz-Subirana 1987; Juan-Zornoza 1995). Point-axial symmetry is indeed not a relaxation of the axial symmetry, but a more informative symmetry which may account for ellipsoidal,

[1] In cylindrical coordinates (ϖ, θ, z) the stationary potential consistent with a quadratic velocity distribution is $U = \frac{1}{2}A(\varpi^2 + z^2) + V(z/\varpi)/(\varpi^2 + z^2)$, with A constant and V an arbitrary function that does not depend on time. When A is a time dependent function, the potential was called quasi-stationary in Paper I.

spiral, or bar structures, and includes axial symmetry as a degenerate case. In particular, along with a quadratic velocity distribution, a point-axial symmetry model provides triaxial mass distributions and velocity ellipsoids with non-vanishing vertex deviation. However, a quadratic point-axial velocity distribution is still symmetric in the peculiar velocities, so that it has null odd-order central moments. Therefore, either in axial or in point-axial symmetric systems, a mixture model is compulsory to fit the full set of local velocity moments. Nevertheless, if each population of the mixture had a velocity ellipsoid with an arbitrary orientation, as, in principle, in the point-axial model, a lower number of populations would likely be required to fit the overall velocity distribution.

Hereafter, this analysis is organised as follows. In the next two sections we review the solution of the BCE for a single point-axial population. In a first step, Chandrasekhar's system of equations provides the kinematic parameters involved in the velocity distribution function whilst, in a second step, they provide an axisymmetric potential. In the fourth section we find the general solution for the potential, in both the separable and non-separable cases. In the fifth section we study the conditions of consistency for point-axisymmetric mixtures. In the last section we discuss the results in contrast to the axisymmetric case.

2. Point-axial system

For fixed position and time $(\boldsymbol{r}, t)$, a single stellar population is usually described through a Gaussian velocity distribution function, which is a particular case of a generalised quadratic velocity distribution function in terms of the peculiar velocities (u_1, u_2, u_3), that is, $f(Q + \sigma)$ with $Q = \sum_{i,j} A_{ij}(\boldsymbol{r}, t)\, u_i u_j$, where A_{ij} are the elements of a symmetric, positive definite second-rank tensor. Under point-axial symmetry these functions satisfy, in cylindrical coordinates, $A_{ij}(\varpi, \theta, z, t) = A_{ij}(\varpi, \theta + \pi, z, t)$, likewise the function σ and the components of the mean velocity $\boldsymbol{v}$.

For the above generalised Schwarzschild velocity distribution, the BCE yields the Chandrasekhar equations (Chandrasekhar 1960), which are equivalent to the moment equations (Cubarsi 2007, 2010). Their solution provides the tensor $\mathbf{A}$, the function σ, the mean velocity $\boldsymbol{v}$, and the potential U. The two first Chandrasekhar equations are Eqs. (1) and (2) in Paper I, that may be written, with the notation used in Paper I and using the variable $\boldsymbol{\Delta} = \mathbf{A} \cdot \boldsymbol{v}$, as

$$3\, \nabla \star \mathbf{A} = (\mathbf{0})^3, \tag{1}$$

$$\frac{\partial \mathbf{A}}{\partial t} = 2\, \nabla \star \boldsymbol{\Delta}, \tag{2}$$

which yield the elements of the second-rank tensor $\mathbf{A}$ and the vector $\boldsymbol{\Delta}$. In the point-axial model these elements have the functional form (Juan-Zornoza et al. 1990; Juan-Zornoza & Sanz-Subirana 1991)

$$\begin{aligned} &A_{\varpi\varpi} = K_1 + K_4 z^2; \quad A_{\varpi\theta} = \tfrac{1}{2}(K_1' + K_4' z^2);\\ &A_{\varpi z} = -K_4 \varpi z; \quad A_{\theta\theta} = K_1^* + k_2 \varpi^2 + K_4^* z^2;\\ &A_{\theta z} = -\tfrac{1}{2} K_4' \varpi z; \quad A_{zz} = k_3 + K_4 \varpi^2; \end{aligned} \tag{3}$$

and

$$\Delta_\varpi = \tfrac{1}{2} \dot{K}_1 \varpi; \;\; \Delta_\theta = \tfrac{1}{4}(\dot{K}_1' - 4\beta)\varpi; \;\; \Delta_z = \tfrac{1}{2} \dot{k}_3 z; \tag{4}$$

with

$$\begin{aligned} &K_1 = k_1 + q \sin(2\theta + \varphi_1); \quad K_1^* = k_1 - q \sin(2\theta + \varphi_1);\\ &K_4 = k_4 + n \sin(2\theta + \varphi_2); \quad K_4^* = k_4 - n \sin(2\theta + \varphi_2); \end{aligned} \tag{5}$$

where k_1, k_3, q, φ_1 are time dependent functions and $k_2, k_4, n, \varphi_2, \beta$ constants. The condition that $\mathbf{A}$ is positive-definite implies that k_1, k_3 are positive functions and k_2, k_4 non-negative constants, with the requirements $k_1 > q \geq 0$, $k_4 > n \geq 0$. If K_4 is null, the velocity distribution is independent from z and, in addition, if k_2 is null, then it is independent from ϖ too, which makes no sense in a three-dimensional and finite Galaxy. Thus, in general, these constants are assumed to be positive, with the exception of the limiting case $K_4 = 0$ of a two-dimensional disc distribution[2]. The particular case $k_2 = 0$ will not be considered here, although it is discussed at the end of the Conclusions section. It would correspond to a particular stellar component with constant angular rotation at fixed height z, similarly to the axisymmetric model.

The uppercase letter K is used for a function also depending on θ. The accents mean derivatives with respect to the angle and the dots with respect to the time. As expected, the functional form of $\mathbf{A}$ is similar to the axisymmetric case in Paper I, with the difference that some parameters, those written in capital letters, have an additional term depending on $\cos 2\theta$ and $\sin 2\theta$, responsible for the rotational symmetry of order 2.

As in the axisymmetric case, the model also provides a time dependent parameter k_5 that determines a plane of symmetry for the tensor $\mathbf{A}$ at $z = -k_5/k_4$. Without loss of generality (Camm 1941) this symmetry plane may be assumed as the Galactic plane, being fixed by taking $k_5 = 0$, resulting then in a symmetric velocity distribution about this plane. Therefore, the point-axisymmetric model, with the inherent symmetry plane, also possesses point-to-point central symmetry.

3. Equations for the potential

The remaining Chandrasekhar equations are Eqs. (3) and (4) in Paper I, which provide the potential U and the function σ. They can be written more easily by using the variable $\mathcal{X} = -\boldsymbol{\Delta} \cdot \boldsymbol{v} - \sigma$ as follows:

$$\mathbf{A} \cdot \nabla U + \frac{\partial \boldsymbol{\Delta}}{\partial t} = -\frac{1}{2} \nabla \mathcal{X}, \tag{6}$$

$$\boldsymbol{\Delta} \cdot \nabla U = \frac{1}{2} \frac{\partial \mathcal{X}}{\partial t}\cdot \tag{7}$$

By elimination of $\mathcal{X}$ between Eqs. (6) and (7), with the new variables $\tau = \frac{1}{2}\varpi^2$ and $\zeta = \frac{1}{2} z^2$, which are appropriate to the symmetry plane of the system, six second-order partial differential equations for the potential are obtained. In their vector notation they can be found in Chandrasekhar (1960, Eqs. (3.448) and (3.450), p.100). After substitution of the elements of $\mathbf{A}$ and the components of $\boldsymbol{\Delta}$, Sanz-Subirana (1987) and Juan-Zornoza (1995) proved that continuity conditions on the function $\mathcal{X}$ force the potential to be axisymmetric. A similar result was obtained by Vandervoort (1979) for point-axial systems, which he called galactic bars, although the study was limited to a two-dimensional disc with a steady-state potential.

3.1. The potential is axisymmetric

These equations are explicitly written in the Appendix. Their solution is tedious and long, and, unfortunately, the

[2] In Paper I, the asymptotic case $K_4 \to 0$ was called flat velocity distribution, which, according to Chandrasekhar (1962), applies to the velocity distribution of an ideal disc. Although a disc stellar population can be approximated by this model, the other populations have a velocity distribution that must depend on z. Therefore, in general we must assume that K_4 is non-null.

above-mentioned thesis papers cannot be accessed easily. Since this is one of the key properties of the point-axial model, we shall see a shorter and alternative justification to this crucial fact.

We note that in the Galactic plane $\zeta = 0$, the three Eqs. (28)–(30) in the Appendix are reduced to Eq. (30) by providing the basic dependence of the potential on the radius and the angle variables. Hence, we focus on this equation in its complete form. First, we consider the main case $K_1' \neq 0$, hence $q \neq 0$, since if K_1 does not depend on θ, the velocity distribution has no vertex deviation in the Galactic plane[3], which was one of the most important observables that justified trying a non-cylindrical model.

If K_1' is non-null, we write Eq. (30) as

$$\frac{\partial^2 U}{\partial \theta^2} - 2\frac{4k_2\tau+(K_1-K_1^*)+2(K_4-K_4^*)\zeta}{K_1'+2K_4'\zeta}\frac{\partial U}{\partial \theta} = 4\tau^2\left[\frac{\partial^2 U}{\partial \tau^2} - \frac{2K_4'\zeta}{K_1'+2K_4'\zeta}\frac{\partial^2 U}{\partial \tau \partial \zeta}\right] + 4\tau\frac{2k_2\tau-(K_1-K_1^*)-2(K_4-K_4^*)\zeta}{K_1'+2K_4'\zeta}\frac{\partial^2 U}{\partial \tau \partial \theta} + \frac{8K_4\tau\zeta}{K_1'+2K_4'\zeta}\frac{\partial^2 U}{\partial \zeta \partial \theta}. \tag{8}$$

We define the function $V = \frac{\partial U}{\partial \theta}$ and bear in mind that the continuity and differentiability of the potential, at least up to the second derivative, implies that V is also differentiable. In the Galactic plane, the foregoing equation becomes

$$\frac{\partial V}{\partial \theta} - 4\tau^2\frac{\partial^2 U}{\partial \tau^2} = 2\frac{4k_2\tau + (K_1 - K_1^*)}{K_1'}V + 4\tau\frac{2k_2\tau - (K_1 - K_1^*)}{K_1'}\frac{\partial V}{\partial \tau}. \tag{9}$$

Since K_1 is a π-periodic function of the angle, a simple recall to the mean value theorem provides us with a value $\theta_0 \in [0, \pi)$ for which $K_1'(\theta_0) = 0$. Then, if V and $\frac{\partial V}{\partial \theta} - 4\tau^2\frac{\partial^2 U}{\partial \tau^2}$ are non-null functions, in order to avoid any singularity, the right-hand side member of the above equation must vanish, at least for $\theta = \theta_0$. We see that the potential does not satisfy

$$\frac{\partial^2 U}{\partial \theta^2} - 4\tau^2\frac{\partial^2 U}{\partial \tau^2} = 0.$$

If so, the solution would be that of the wave equation in the new variable $x = \ln\tau$, hence the solution satisfies $U = F_1(x + 2\theta) + F_2(x - 2\theta)$, but the potential is a one-valued function and a periodic function of θ with period[4] 2π; therefore, $U(x + 2\theta) = U(x + 2(\theta + 2k\pi)) = U((x + 4k\pi) + 2\theta)$ is fulfilled for all $k \in \mathbb{Z}$. However, as the Galaxy is of finite extent, such a potential taking the same value at all points $x + 4k\pi$ is unrealistic.

On the other hand, the right-hand side of Eq. (9) is non-null. If so, it would be a linear and homogeneous differential equation in V, with solution $V \propto \tau^{\frac{1}{2}}(K_1 - K_1^* - 2k_2\tau)^{-\frac{3}{2}}$, which is discontinuous at $\tau = \frac{K_1-K_1^*}{2k_2}$. In particular, when $K_1'(\theta_0) = 0$, according to Eq. (5), the singularity takes place at $\tau = \frac{q}{k_2}$. It is worth noticing that for $k_2 = 0$ such a singularity does not exist, so that we might have non-cylindrical potentials in that degenerate case. Therefore, the only admissible, continuous, and differentiable solutions to Eq. (9) are axisymmetric potentials satisfying $V = 0$, otherwise, in the Galactic plane, the potential is not differentiable.

This means that, the axial symmetry is the way that the differential equations for the potential avoid the singularity produced by any root of the function K_1'. It is actually a situation similar to the axisymmetric model, where the equations for the potential, in the quasi-stationary case in Paper I, did avoid the singularity produced by the zero of the time-dependent function $\dot{k}_1$ by providing a solution that does not depend on $\dot{k}_1/k_1$.

The case where K_1' is null, consequently $K_1 - K_1^*$ also vanishes, requires that K_4' be non-null, otherwise the velocity distribution is axisymmetric. Similarly, as in the above case, there is also an angle θ_1 for which $K_4'(\theta_1) = 0$ that would produce a singularity in the solution of Eq. (8), for $\zeta \neq 0$, unless the potential is axisymmetric[5].

4. Potential

Therefore, in a rotating point-axial system, the potential consistent with a quadratic velocity distribution is still axisymmetric. The set of partial differential equations for the potential in the Appendix generalises the ones for the axisymmetric model in Paper I. We write these equations once they are simplified by taking advantage of the potential satisfying $\frac{\partial U}{\partial \theta} = 0$. The first three equations, derived from Eqs. (28)–(30), are

$$2K_4\left[2\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) + \tau\frac{\partial}{\partial \tau}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) + \zeta\frac{\partial}{\partial \zeta}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right)\right] + (K_1 - k_3)\frac{\partial^2 U}{\partial \tau \partial \zeta} = 0, \tag{10}$$

$$2K_4'\left[2\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) + \zeta\frac{\partial}{\partial \zeta}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right)\right] + K_1'\frac{\partial^2 U}{\partial \tau \partial \zeta} = 0, \tag{11}$$

$$2K_4'\zeta\frac{\partial}{\partial \tau}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) + K_1'\frac{\partial^2 U}{\partial \tau^2} = 0. \tag{12}$$

The remaining three equations, obtained from Eqs. (31)–(33), are

$$2K_4\zeta\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) + \dot{K}_1\tau\frac{\partial^2 U}{\partial \tau^2} + K_1\frac{\partial^2 U}{\partial t \partial \tau} + 2\dot{K}_1\frac{\partial U}{\partial \tau} + \frac{1}{2}\dddot{K}_1 + \dot{k}_3\zeta\frac{\partial^2 U}{\partial \tau \partial \zeta} = 0, \tag{13}$$

$$2K_4\tau\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial \zeta} - \frac{\partial U}{\partial \tau}\right) + \dot{k}_3\zeta\frac{\partial^2 U}{\partial \zeta^2} + k_3\frac{\partial^2 U}{\partial t \partial \zeta} + 2\dot{k}_3\frac{\partial U}{\partial \zeta} + \frac{1}{2}\dddot{k}_3 + \dot{K}_1\tau\frac{\partial^2 U}{\partial \zeta \partial \tau} = 0, \tag{14}$$

$$2K_4'\zeta\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) + K_1'\frac{\partial^2 U}{\partial t \partial \tau} + 2\dot{K}_1'\frac{\partial U}{\partial \tau} + \frac{1}{2}\dddot{K}_1' = 0. \tag{15}$$

However, Eqs. (11) and (12) can be simplified further. By taking the θ-derivative in Eq. (10) and subtracting from Eq. (11), we get

$$K_4'\frac{\partial}{\partial \tau}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) = 0. \tag{16}$$

Also, by taking into account Eq. (12), we get

$$K_1'\frac{\partial^2 U}{\partial \tau^2} = 0. \tag{17}$$

Similarly, Eq. (15) can be expressed in a simpler form. By taking the θ-derivative in Eq. (13) and subtracting from Eq. (15) we

---

[3] As pointed out in Paper I, the second-order central velocity moments satisfy $\boldsymbol{\mu} \propto \mathbf{A}^{-1}$. In the Galactic plane, from Eq. (3), we get $\mu_{\varpi\theta} \propto K_1'$.

[4] Although we assume that the velocity distribution is π-periodic, we cannot discard that part of the potential function could still admit a 2π-periodic solution.

[5] In this case, we first prove that, if $\frac{\partial U}{\partial \theta} \neq 0$, the equation $\frac{\partial V}{\partial \theta} - 4\tau^2\left(\frac{\partial^2 U}{\partial \tau^2} - \frac{\partial^2 U}{\partial \tau \partial \zeta}\right) = 0$, with the variables $x = \ln\tau, y = \zeta/\tau$, provides a solution proportional to an exponential function on the argument $x + \theta$, which is not periodic and, hence, unacceptable. We then verify that the remaining terms of Eq. (8) do not vanish, otherwise, its solution, which takes the general form $V = F\left(\frac{(k_2-K_4)\tau-(K_4-K_4^*)\zeta}{(k_2-K_4)\zeta^{k_2/K_4}}\right)\tau^{\frac{1}{2}}\zeta^{-\frac{3}{2}\frac{k_2}{K_4}}$, has discontinuities either at $\zeta = 0$ or at points satisfying $(k_2 - K_4)\tau - (K_4 - K_4^*)\zeta = 0$.

get $\dot{K}'_1 \frac{\partial^2 U}{\partial \tau^2} = 0$, which does not add any new condition to the previous equation.

Therefore, the equations for the potential in the pont-axial model are the set of Eqs. (10)–(14), which are similar to the ones of the axial case (Paper I, Eqs. (7)–(9)), with the additional integrability conditions given by Eqs. (16) and (17). We note that the equations for the potential do not depend on the parameters K_1^* and K_4^*. Under axial symmetry, the conditions depending on the θ-derivatives $K'_1(\theta, t)$ and $K'_4(\theta)$ are identically null. Thus, in a mixture model these equations are similarly planned for each population component, and depend on the respective population parameters $K_1(\theta, t)$, $k_3(t)$, and $K_4(\theta)$.

In the axisymmetric case, when applying the conditions of consistency for a flat velocity distribution, that is, for a potential independent from the population parameter K_4, the potential becomes dramatically simplified. In the point-axial case, we shall see that a similar reasoning and solution are inherent to the point-axial symmetry assumption, since the reasoning can be done in regard to the angle dependence as well as to the population dependence of the parameters. In other words, a point-axial system is consistent with a flat velocity distribution unless it degenerates towards an axisymmetric system.

Thus, being at least one of the population parameters K_1, K_4 functions of the angle θ (otherwise the system is axisymmetric), Eq. (10), once divided by K_4, becomes separated into two parts, one independent from θ and the other depending on θ, which must be null separately[6],

$$2\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) + \tau\frac{\partial}{\partial \tau}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) + \zeta\frac{\partial}{\partial \zeta}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) = 0,$$
$$(K_1 - k_3)\frac{\partial^2 U}{\partial \tau \partial \zeta} = 0. \tag{18}$$

These equations are equivalent to the conditions of a potential independent from K_4 in Paper I, Eqs. (12) and (14). The latter equation leads to the two typical cases of a potential additively separable in cylindrical coordinates, or a non-separable potential.

4.1. Separable potential

The separable potential satisfies

$$\frac{\partial^2 U}{\partial \tau \partial \zeta} = 0.$$

In the point-axial model, at least one of the parameters K_1 or K_4 depends on the angle. In particular, if $K'_4 = 0$, owing to Eq. (11), the potential must be separable, otherwise $K'_1 = 0$ would be held, rendering the axisymmetric model.

For a separable potential, either with K'_4 null or non-null, we are led to the same equations as for the axisymmetric case in Paper I, Eq. (15), with the addition of Eqs. (16) and (17), which add the new condition $\frac{\partial^2 U}{\partial \tau^2} = 0$, yielding a separable potential in their harmonic form

$$U = A(t)\,(\tau + \zeta), \tag{19}$$

where continuity conditions in the Galactic plane have been applied in order to neglect the term proportional to $\frac{1}{\zeta}$. Therefore, the separable potential reduces to the simple case of the harmonic function, and does not depend on the population kinematic parameters except for the unique function $A(t)$ discussed in Paper I.

Hence, under a separable potential, the kinematics of a point-axial symmetric system is totally free from conditions of consistency in regard to a mixture of populations. The population's mean velocities, the semiaxes of the velocity ellipsoids, and their orientations remain unconstrained.

4.2. Non-separable potential

The non-separable potential satisfies

$$\frac{\partial^2 U}{\partial \tau \partial \zeta} \neq 0$$

and

$$k(t) \equiv K_1 = k_3.$$

Then, $K'_1 = 0$. According to Eqs. (16) and (17), the point-axial symmetry assumption requires $K'_4 \neq 0$. Hence, Eqs. (11) and (12) provide the conditions

$$2\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) + \zeta\frac{\partial}{\partial \zeta}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) = 0,$$
$$\frac{\partial}{\partial \tau}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) = 0, \tag{20}$$

which separate Eq. (18) into two identically null equations. Hence, we can consider only one of them. Similarly, the same reasoning of the preceeding section (either in regard to the dependency on the angle or on the population) applied to Eqs. (13)–(15) yields the conditions

$$\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial \zeta}\right) = 0, \tag{21}$$

$$\dot{k}\tau\frac{\partial^2 U}{\partial \tau^2} + k\frac{\partial^2 U}{\partial t \partial \tau} + 2\dot{k}\frac{\partial U}{\partial \tau} + \frac{1}{2}\dddot{k} + \dot{k}\zeta\frac{\partial^2 U}{\partial \tau \partial \zeta} = 0,$$
$$\dot{k}\zeta\frac{\partial^2 U}{\partial \zeta^2} + k\frac{\partial^2 U}{\partial t \partial \zeta} + 2\dot{k}\frac{\partial U}{\partial \zeta} + \frac{1}{2}\dddot{k} + \dot{k}\tau\frac{\partial^2 U}{\partial \zeta \partial \tau} = 0. \tag{22}$$

Thus, we reach the same set of equations as for an axisymmetric model consistent with a flat velocity distribution (Paper I, Eqs. (13) and (15)), by providing the potential $U = A(t)\,(\tau + \zeta) + \frac{1}{k}U_1\left(\frac{\tau+\zeta}{k}\right) + \frac{1}{\tau+\zeta}\,U_2(\zeta/\tau)$, although in the point-axial model we still have to submit it to Eq. (20). Therefore, the resulting potential must adopt the separable form $U = f_1(\tau + \zeta) + f_2(\zeta)$, so that

$$U = A(t)\,(\tau + \zeta) + \frac{1}{k}U_1\left(\frac{\tau + \zeta}{k}\right), \tag{23}$$

where, by continuity conditions in the Galactic plane, an additional term proportional to $\frac{1}{\zeta}$ is neglected.

5. Conditions of consistency

For a separable potential, there are no conditions of consistency, similarly to the axisymmetric model.

For a non-separable potential, all the system dependency on θ is carried through $K_4(\theta)$. Therefore, according to Eq. (3), and bearing in mind that $K'_1 = 0$, in the Galactic plane the tensor

[6] Similarly, by reasoning in regard to a population mixture, the potential is independent from the population parameters only if both parts are null separately.

elements $A_{\varpi\theta}$ and $A_{\theta z}$ are null, as in the axisymmetric model. Hence, the velocity ellipsoid has no vertex deviation in $z = 0$. In addition, according to Paper I, since $K_1 = k_3$ the ellipsoid has no tilt in a meridional Galactic plane (i.e., the intersection of the ellipsoid with a meridional Galactic plane has an axis pointing toward the Galactic centre), and the mean velocities Π_0 and Z_0 are the same as in the axisymmetric case. In the Galactic plane, the only moment depending on θ is μ_{zz}, whilst Θ_0 and the other second moments are also axisymmetric.

In summary, in the Galactic plane the velocity distribution of such a stellar system is basically axisymmetric and does not provide the most important feature we expected a point-axial system should provide, that is, the vertex deviation.

Similarly, as for the axisymmetric case, the potential of Eq. (23) constrains the mean velocity components Π_0 and Z_0 to satisfy $\frac{Z_0}{\Pi_0} = \frac{z}{\varpi}$. For a two population mixture we get $\Pi_0' - \Pi_0'' = 0$ and $Z_0' - Z_0'' = 0$, unless, according to Paper I, the function $k(t)$ is linearly independent among populations and the potential does not depend on $\dot{k}(t)/k(t)$. In that case, an apparent vertex deviation of the mixture distribution is possible. The potential allowing unconstrained population mean velocities must then satisfy the condition

$$\frac{\partial^2}{\partial\tau\partial\zeta}\frac{\partial U}{\partial t} = 0, \tag{24}$$

obtained in Paper I, and the potential takes the quasi-stationary form

$$U = A(t)\,(\tau + \zeta) + \frac{B}{\tau + \zeta}, \tag{25}$$

with $B =$ const, which is a particular, spherical case in Paper I, Eq. (31).

6. Conclusions

The conditions of consistency studied in Paper I proved that a finite mixture of stellar populations was able to describe the main features of the local velocity distribution without having to change the axisymmetry hypothesis. However, in the Galactic plane, single populations had velocity ellipsoids without vertex deviation, so that the apparent vertex deviation of the disc velocity distribution was the result of different radial and rotation mean motions of the populations. Now, as a corollary, we have investigated the same problem under point-axial symmetry, in order to see how it might improve the velocity distribution approximation.

For the point-axial symmetry case, the local kinematic features are similarly derived from a mixture of stellar populations, each one according to a quadratic velocity distribution in the peculiar velocities satisfying the BCE, with a common potential allowing for the populations to be kinematically independent. This means that the populations should differ not only in rotation, but also in radial and vertical mean motions. Under the point-axial hypothesis we should also expect single populations with velocity ellipsoids having non-null vertex deviation and non-vanishing tilt, as well as a point-axial mass distribution.

An important fact is that the potential must be axisymmetric in order to support a quadratic integral of motion for each population, which usually represents a stellar system in statistical equilibrium. That is, we assume that the stellar system has achieved relaxation and satisfies regularity conditions about the definition of the local standard of rest, continuity, and differentiability of its velocity, and that higher-order velocity moments exist. Although dissipative forces related to third and odd-order moments does not appear in the moment equations planned for a single population, they are indirectly connected with the assumption of the mixture model.

The first result we obtain is that the point-axial symmetry is, in a natural way, consistent with the flat velocity distribution of a disc population, by providing potentials not depending on the population parameter K_4, which is responsible for non-isothermal velocity distributions. In axisymmetric systems, only a particular family of potentials is consistent with a flat velocity distribution, while in point-axial systems any potential always is.

We find two possible solutions depending on the separability of the potential:

(a) The point-axial model admits a potential additively separable in cylindrical coordinates that is the harmonic potential. As in the axisymmetric model, for a separable potential there is no need of conditions of consistency in regard to a mixture distribution, since the potential only constrains the population parameters through the function $A(t)$ (Paper I, Eq. (20)). For each population, the radial and vertical mean velocities can be different, and their velocity ellipsoids can have different orientations, including the both vertex deviation and tilt.

(b) For a non-separable potential, the condition given by Eq. (24) provides nearly non-constrained population kinematics, by leading to a spherical and quasi-stationary potential. Then, the radial and the vertical mean velocities can differ among populations, although they are coupled, and they may produce an apparent vertex deviation of the whole velocity distribution. However, single population velocity ellipsoids have no vertex deviation in the Galactic plane and no tilt in their intersection with a meridional Galactic plane, similarly to the axisymmetric case.

In both of these cases, the potential for the point-axial model becomes a particular function of the potential for the axisymmetric model. The non-separable potential loses the dependency on the elevation angle, and the separable potential loses the non-harmonic term that they showed in Paper I (Eqs. (31) and (33), respectively).

We can check the foregoing cases according to the main local kinematic trends analysed in Paper I. Against option (a) is the evidence that there are halo stars near the Sun with no net rotation velocity for which the harmonic potential is not able to support their orbits. On the other hand, option (b) really provides a potential, Eq. (25), with a non-harmonic term, which may be associated with a repulsive force (if $B > 0$) produced by the outer dark matter halo, as discussed in Paper I. This allows stable orbits for stars with no net rotation. However, the potential forces the population velocity ellipsoids to point toward the Galactic centre, although, out of the Galactic plane, the ellipsoids may show some vertex deviation.

Then, according to the point-axial model, how can we explain that the thick disc and the halo ellipsoids have no vanishing tilt, as Casetti-Dinescu (2011), Carollo et al. (2010), Fuchs et al. (2009), Smith et al. (2009a), and Siebert et al. (2008) suggest? By assuming that the harmonic potential is not realistic, under the point-axial model we cannot explain it. Similarly, the point-axial model is unable to explain the trend of the moment $\mu_{\varpi z}$ for the thin disc population described by Pasetto et al. (2012b), which was only possible under a separable potential, as discussed in Paper I.

Conversely, if the thick disc and the halo ellipsoids actually have a non-vanishing tilt, the axisymmetric model is capable of making a reasonable approach to the local features of the local velocity distribution. Therefore, we must conclude that the velocity distribution in the solar neighbourhood reflects a basically axisymmetric Galaxy.

Nevertheless, we should not discard the fact that some of the stellar samples used to describe the thick disc and the halo could have stars that were not sufficiently mixed to produce well defined velocity ellipsoids, or were contaminated by disc stars, as Smith et al. (2009a) suggest. If newer and more accurate analyses yielded non-tilted velocity ellipsoids for the thick disc and the halo, both models would be capable of describing the local velocity distribution from a non-separable potential, which, in all cases, would provide an axisymmetric velocity distribution in the Galactic plane.

However, for the stellar density, point-axial symmetry matters. We may assume a Schwarzschild velocity distribution without loss of generality to discuss the shape of mass distribution. In that case, the stellar density is $N \propto \frac{1}{\sqrt{\det \mathbf{A}}} \mathrm{e}^{-\frac{1}{2}\sigma}$ (Eq. (40) in Appendix A.2 in Paper I) and depends on the angle through $K_4(\theta)$. Leaving aside the simple and unrealistic case of a separable and harmonic potential, for the non-separable potential with $k \equiv K_1 = k_3$, Eq. (25) is a particular case of Eq. (23) with $U_1 = \frac{c_1}{k}(\tau+\zeta) + \frac{c_2}{\tau+\zeta}$. The function σ involved in the stellar density satisfies

$$\frac{1}{2}\sigma = \frac{c_1}{k}(\tau+\zeta) + \frac{c_2 k}{\tau+\zeta} - \frac{\beta^2\tau}{k+2k_2\tau+2K_4\zeta} + \text{const.} \tag{26}$$

For $\zeta = 0$, σ does not depend on θ. However, for $\zeta = 0$, we have

$$\det \mathbf{A} = k(k+k_2\varpi^2)(k+K_4\varpi^2), \tag{27}$$

so that, in the Galactic plane, the stellar density N depends on θ. This dependency of the mass distribution on the angle is balanced out by the velocity distribution, which also depends on θ, while the potential maintains the axisymmetry. This is the only basic feature that the point-axial model adds to the axisymmetric model. While, in the Galactic plane, for the velocity distribution, according to Eqs. (3) and (27), the tensor element A_{zz} and $\det \mathbf{A}$, which depend on θ, lead to moments $\mu_{\varpi\varpi}, \mu_{zz}$ also depending on the angle, although each population component is unable to provide a non-vanishing moment $\mu_{\varpi\theta}$. Thus, similarly to the axisymmetric case, for ellipsoidal velocity distributions under a non-separable potential, the apparent vertex deviation of the velocity distribution is a consequence of the coexistence of two or more populations with different radial and rotation mean velocities.

Then, a point-axisymmetric stellar system would, in principle, be able to show a triaxial or bar-like structure in any of the population's mass distributions. It would be a matter of time that, for the specific population, the rotation curve could transform a bar-like into a spiral-like structure. In that situation, the point-axial model would have the ability to describe a point-axial mass distribution, while the axial model would not. However, we note that the degenerate case of a rigid rotating bar, with $k_2 = 0$, which was the only remaining case that could admit, a priori, a non-cylindrical potential as a solution of Eq. (9), cannot coexist with three-dimensional ellipsoidal velocity distributions under a common differentiable point-axial potential. Hence, any phase mixing process involving a rigid rotating bar with a non-cilyndrical potential must be considered as a state previous to statistical equilibrium, which is not associated with stellar populations having ellipsoidal velocity distributions, even with point-axial symmetry.

Appendix

The first three partial differential equations for the potential, obtained by taking the curl in Eq. (6), are

$$\begin{aligned}&8K_4\tau\left[2\left(\frac{\partial U}{\partial\tau}-\frac{\partial U}{\partial\zeta}\right)+\tau\frac{\partial}{\partial\tau}\left(\frac{\partial U}{\partial\tau}-\frac{\partial U}{\partial\zeta}\right)+\zeta\frac{\partial}{\partial\zeta}\left(\frac{\partial U}{\partial\tau}-\frac{\partial U}{\partial\zeta}\right)\right]\\&+4\tau(K_1-k_3)\frac{\partial^2 U}{\partial\tau\partial\zeta}+2K_4'\frac{\partial}{\partial\theta}\left(\tau\frac{\partial U}{\partial\tau}+U+\zeta\frac{\partial U}{\partial\zeta}\right)+K_1'\frac{\partial^2 U}{\partial\theta\partial\zeta}=0,\end{aligned} \tag{28}$$

where a common factor proportional to $\zeta^{\frac{1}{2}}$ was simplified;

$$\begin{aligned}&4K_4'\tau\left[2\left(\frac{\partial U}{\partial\tau}-\frac{\partial U}{\partial\zeta}\right)+\zeta\frac{\partial}{\partial\zeta}\left(\frac{\partial U}{\partial\tau}-\frac{\partial U}{\partial\zeta}\right)\right]+2K_1'\tau\frac{\partial^2 U}{\partial\tau\partial\zeta}\\&+4K_4'\tau\frac{\partial^2 U}{\partial\theta^2}+2(3K_4^*-K_4)\frac{\partial U}{\partial\theta}+4K_4\tau\frac{\partial^2 U}{\partial\theta\partial\tau}\\&-\left[2(k_3-K_1^*)+4(K_4-k_2)\tau-4K_4^*\zeta\right]\frac{\partial^2 U}{\partial\theta\partial\zeta}=0,\end{aligned} \tag{29}$$

where a common factor proportional to $\zeta^{\frac{1}{2}}$ was simplified; and

$$\begin{aligned}&4\tau^2\left[2K_4'\zeta\frac{\partial}{\partial\tau}\left(\frac{\partial U}{\partial\tau}-\frac{\partial U}{\partial\zeta}\right)+K_1'\frac{\partial^2 U}{\partial\tau^2}\right]-(K_1'+2K_4'\zeta)\frac{\partial^2 U}{\partial\theta^2}\\&+8K_4\tau\zeta\frac{\partial^2 U}{\partial\theta\partial\zeta}+4\tau[2k_2\tau-(K_1-K_1^*)-2(K_4-K_4^*)\zeta]\frac{\partial^2 U}{\partial\theta\partial\tau}\\&+2\left[4k_2\tau+(K_1-K_1^*)+2(K_4-K_4^*)\zeta\right]\frac{\partial U}{\partial\theta}=8\dot\beta\tau.\end{aligned} \tag{30}$$

Those equations which were proportional to $\zeta^{\frac{1}{2}}$ become null at the Galactic plane.

The remaining three equations, which are obtained by taking the gradient in Eq. (7) and the time derivative in Eq. (6), are

$$\begin{aligned}&2K_4\zeta\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial\tau}-\frac{\partial U}{\partial\zeta}\right)\\&+\dot K_1\tau\frac{\partial^2 U}{\partial\tau^2}+K_1\frac{\partial^2 U}{\partial t\partial\tau}+2\dot K_1\frac{\partial U}{\partial\tau}+\frac{1}{2}\dddot K_1+\dot k_3\zeta\frac{\partial^2 U}{\partial\tau\partial\zeta}\\&+\frac{1}{4\tau}\left[(\dot K_1'-4\beta)\tau\frac{\partial^2 U}{\partial\theta\partial\tau}+(K_1'+2K_4'\zeta)\frac{\partial^2 U}{\partial\theta\partial t}+(\dot K_1'+2\dot K_4'\zeta)\frac{\partial U}{\partial\theta}\right]=0;\end{aligned} \tag{31}$$

$$\begin{aligned}&2K_4\tau\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial\zeta}-\frac{\partial U}{\partial\tau}\right)\\&+\dot k_3\zeta\frac{\partial^2 U}{\partial\zeta^2}+k_3\frac{\partial^2 U}{\partial t\partial\zeta}+2\dot k_3\frac{\partial U}{\partial\zeta}+\frac{1}{2}\dddot k_3+\dot K_1\tau\frac{\partial^2 U}{\partial\zeta\partial\tau}\\&+(\dot K_1'-4\beta)\frac{\partial^2 U}{\partial\theta\partial\zeta}-\frac{1}{4}K_4'\frac{\partial^2 U}{\partial\theta\partial t}=0;\end{aligned} \tag{32}$$

in the last one, a common factor $\zeta^{\frac{1}{2}}$ was also simplified; and

$$\begin{aligned}&\tau\left[2K_4'\zeta\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial\tau}-\frac{\partial U}{\partial\zeta}\right)+K_1'\frac{\partial^2 U}{\partial t\partial\tau}+2\dot K_1'\frac{\partial U}{\partial\tau}+\frac{1}{2}\dddot K_1'\right]\\&+\dot K_1\tau\frac{\partial^2 U}{\partial\theta\partial\tau}+\frac{1}{4}(\dot K_1'-4\beta)\frac{\partial^2 U}{\partial\theta^2}+\dot k_3\zeta\frac{\partial^2 U}{\partial\theta\partial\zeta}\\&+(K_1^*+2k_2\tau+2K_4\zeta)\frac{\partial^2 U}{\partial\theta\partial t}+\frac{1}{4}(\dot K_1''+4K_1^*)\frac{\partial U}{\partial\theta}=2\dot\beta\tau.\end{aligned} \tag{33}$$

These equations complete the first set of three equations given by Sanz-Subirana (1987). They may be simplified by assuming that the parameter β is constant, which was actually derived by this author in solving them for the separable potential case, and by Juan-Zornoza (1995) for the general case. This fact is not relevant for our purpose.

Acknowledgements. The author wishes to thank the anonymous referee for his/her useful comments that have contributed to improving this paper.

References

Camm, G. L. 1941, MNRAS, 101, 195
Carollo, D., Beers, T. C., Chiba, M., et al. 2010, ApJ, 712, 692
Casetti-Dinescu, D. I., Girard, T. M., Korchagin, V. I., & van Altena, W. F. 2011, ApJ, 728,7
Chandrasekhar, S. 1960, Principles of Stellar Dynamics (New York: Dover Publications Inc.)
Cubarsi, R. 2007, MNRAS, 207, 380
Cubarsi, R. 2010, A&A, 522, A30
Cubarsi, R. 2014, A&A, 561, A141
Fuchs, B., Dettbarn, C., & Rix, H-W. 2009, AJ, 137, 4149
Juan-Zornoza, J. M. 1995, Ph.D. Thesis (ISBN: 844750767-X), University of Barcelona
Juan-Zornoza, J. M., & Sanz-Subirana, J. 1991, Ap&SS, 185, 95
Juan-Zornoza, J. M., Sanz-Subirana, J., & Cubarsi, R. 1990, Ap&SS, 170, 343
Moni Bidin, C., Carraro, G., & Méndez, R. A. 2012, ApJ, 747, 101
Pasetto, S., Grebel, E. K., Zwitter, T., et al. 2012a, A&A, 547, A70
Pasetto, S., Grebel, E. K., Zwitter, T., et al. 2012b, A&A, 547, A71
Sanz-Subirana, J. 1987, Ph.D. Thesis (ISBN: 8475282326), University of Barcelona
Siebert, A., Bienaymé, O., Binney, J., et al. 2008, MNRAS, 391, 793
Siebert, A., Williams, M. E. K., Siviero, A., et al. 2011, AJ, 141, 187
Smith, M. C., Evans, N. W., An, J. H. 2009a, ApJ, 698, 1110
Smith, M. C., Evans, N. W., Belokurov, V., et al. 2009b, MNRAS, 399, 1223
Steinmetz, M. 2012, Astron. Nachr., 333, 523
Steinmetz, M., Zwitter, T., Siebert, A., et al. 2006, AJ, 132, 1645
Vandervoort, P. O. 1979, ApJ, 232, 91
Zwitter, T., Siebert, A., Munari, U., et al. 2008, AJ, 136, 421