On monophonic sets in graphs *

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Abstract

In this paper we study monophonic sets in a connected graph \( G \). First, we present a realization theorem proving that there is no general relationship between monophonic and geodetic hull sets. Second, we study the contour of a graph, introduced by Cáceres and alt. [2] as a generalization of the set of extreme vertices where the authors proved that the contour of a graph is a \( g \)-hull set; in this work we show that the contour must also be monophonic. Finally, we focus our attention on the so-called edge Steiner sets. We prove that every edge Steiner set \( W \) in \( G \) is edge monophonic, i.e., every edge of \( G \) lies on some monophonic path joining two vertices of \( W \).

Keywords: Convexity, contour, extreme vertex, geodetic set, hull set, monophonic set, Steiner set.

1 Introduction

A convexity on a finite set \( V \) is a family \( \mathcal{C} \) of subsets of \( V \), convex sets, which is closed under intersection and which contains both \( V \) and the empty set. The pair \((V, \mathcal{C})\) is called a convexity space. A finite graph-convexity space is a pair \((G, \mathcal{C})\), formed by a finite connected graph \( G = (V,E) \) and a convexity \( \mathcal{C} \) on \( V \) such that \((V, \mathcal{C})\) is a convexity space satisfying that every member of \( \mathcal{C} \) induces a connected subgraph of \( G \) [4, 5]. Thus, classical convexity can be extended to graphs in a natural way. We know that a set \( X \) of \( \mathbb{R}^n \) is convex if every segment joining two points of \( X \) is entirely contained in it. Similarly, a vertex set \( W \) of a finite connected graph \( G \) is said to be a convex set of \( G \) if it contains all the vertices lying in a certain kind of path connecting vertices of \( W \).

In this paper we deal with two types of graph convexities, which are the most natural path convexities in a graph and which are defined by a system \( \mathcal{P} \) of paths in \( G \): the geodetic convexity (also called metric convexity) [5, 6, 7, 11] which arises when we consider shortest paths, and the monophonic convexity (also called minimal path convexity) [4, 5] when we consider chordless paths.

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In what follows, $G = (V, E)$ denotes a finite connected graph with no loops or multiple edges. The distance $d(u, v)$ between two vertices $u$ and $v$ is the length of a shortest $u - v$ path in $G$. A chord of a path $u_0u_1 \ldots u_h$ is an edge $u_iu_j$, with $j \geq i + 2$. A $u - v$ path $\rho$ is called monophonic if it is a chordless path, and geodesic if it is a shortest $u - v$ path, that is, if $|E(\rho)| = d(u, v)$.

The geodesic closed interval $I[u, v]$ is the set of vertices of all $u - v$ geodesics. Similarly, the monophonic closed interval $J[u, v]$ is the set of vertices of all monophonic $u - v$ paths. For $W \subseteq V$, the geodesic closure $I(W)$ of $W$ is defined as the union of all geodesic closed intervals $I[u, v]$ over all pairs $u, v \in W$. The monophonic closure $J(W)$ is the set formed by the union of all monophonic closed intervals $J[u, v]$.

A vertex set $W \subseteq V$ is called geodesically convex (or simply g-convex) if $I(W) = W$, while it is said to be geodetic if $I(W) = V$. Likewise, $W$ is called monophonically convex (or simply m-convex) if $J(W) = W$, and is called monophonic if $J(W) = V$. The smallest g-convex set containing $W$ is denoted $[W]_g$ and is called the $g$-convex hull of $W$. Similarly, the $m$-convex hull $[W]_m$ of $W$ is defined as the minimum $m$-convex set containing $W$. Observe that $J(W) \subseteq [W]_m$, $I(W) \subseteq [W]_g$ and $[W]_g \subseteq [S]_m$. A g-hull (m-hull) set of $G$ is a vertex set $W$ satisfying $[W]_g = V$ ($[W]_m = V$).

For a nonempty set $W \subseteq V$, a connected subgraph of $G$ with the minimum number of edges that contains all of $W$ must be clearly a tree. Such a tree is called a Steiner $W$-tree. The Steiner interval $S(W)$ of $W$ consists of all vertices that lie on some Steiner $W$-tree. If $S(W) = V$, then $W$ is called a Steiner set for $G$ [3].

The monophonic (m-hull, geodetic, g-hull, Steiner, respectively) number of $G$, denoted by $mn(G)$ ($mhn(G)$, $gn(G)$, $ghn(G)$, $st(G)$, respectively) is the minimum cardinality of a monophonic (m-hull, geodetic, g-hull, Steiner, respectively) set in $G$. Clearly, $ghn(G) \leq gn(G)$, since every geodetic set is a g-hull set. In [7], the authors showed that, apart from the previous one, no other general relationship among the parameters $ghn(G)$, $gn(G)$ and $st(G)$ exists. In Section 2, we approach the same problem by replacing the parameter $st(G)$ by both $mhn(G)$ and $mn(G)$.

In Section 3, we examine a number of monophonic convexity issues involving three types of vertices: contour, peripheral and extreme vertices. We prove that the contour of a graph is monophonic. It is interesting to notice that these results are closely related to the graph reconstruction problem, in the sense that we want to obtain all the vertices of a graph by considering a certain kind of paths joining vertices of a fixed set $W$.

In [3], it was shown that every Steiner set in $G$ is also geodetic. Unfortunately, this particular result turned out to be wrong and was disproved by Pelayo [10]. In [7], the authors proved that every Steiner set is monophonic. As a consequence, they immediately derived that, in the class of distance-hereditary graphs (i.e., those graphs for which every monophonic path is a geodesic [8]), every Steiner set is geodetic. They approached the problem of determining for which classes of chordal graphs (i.e., without induced cycles of length greater than 3) every Steiner set is geodetic, proving this statement to be true both for Ptolemaic graphs (i.e, distance-hereditary chordal graphs [5]) and interval graphs (i.e., chordal graphs without induced asteroid triples [9]). In Section 4, we focus our attention on the edges of geodesic and monophonic paths, approaching the same problems and obtaining similar results.
2 Monophonic and geodetic parameters

Let us review the main definitions involved in this section. A vertex set $W \subseteq V$ is a $g$-hull set if its $g$-convex hull $[W]_g$ covers all the graph, i.e., if $[W]_g = V$. Moreover, $W$ is called geodetic if $J[W] = V$. The $g$-hull number $ghn(G)$ of $G$ is defined as the minimum cardinality of a hull set. The geodetic number $gn(G)$ of $G$ is the minimum cardinality of a geodetic set [6]. Certainly, $ghn(G) \leq gn(G)$.

Although it has been shown that determining the geodetic number and the hull number of a graph is a $NP$-hard problem [6], it is rather simple to obtain these two parameters for a wide range of classes of graphs as paths, cycles, trees, (bipartite) complete graphs, wheels and hypercubes (Table 1).

A vertex set $W \subseteq V$ is a $m$-hull set if $[W]_m = V$. Moreover, $W$ is called monophonic if $J[W] = V$. The $m$-hull number $mhn(G)$ of $G$ is the minimum cardinality of an $m$-hull set. The monophonic number $mn(G)$ of $G$ is the minimum cardinality of a monophonic set. Certainly, $mhn(G) \leq mn(G) \leq gn(G)$ and $mhn(G) \leq ghn(G)$, since every monophonic set is an $m$-hull set, every geodetic set is monophonic, and every $g$-hull set is an $m$-hull set. Nevertheless, it is not true that every $g$-hull set be monophonic. For example, if we consider the complete bipartite graph $K_{3,3}$, with $V_1 = \{a, b, c\}$ and $V_2 = \{e, f, g\}$, it is easy to see that the set $W = \{a, b\}$ satisfies $[W]_g = V$ and $J[W] = V \setminus \{c\}$.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$P_n$</th>
<th>$C_{2l}$</th>
<th>$C_{2l+1}$</th>
<th>$T_n$</th>
<th>$K_n$</th>
<th>$K_{p,q}$ ($2 \leq p \leq q$)</th>
<th>$W_{1,p}$ ($p \geq 4$)</th>
<th>$Q_n$</th>
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<tbody>
<tr>
<td>$mhn(G)$</td>
<td>2</td>
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<td>$\geq$ leaves</td>
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<td>$mn(G)$</td>
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<td>$ghn(G)$</td>
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<td>$\left\lceil \frac{p}{2} \right\rceil$</td>
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<tr>
<td>$gn(G)$</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>$\geq$ leaves</td>
<td>$n$</td>
<td>$\min{4, p}$</td>
<td>$\left\lceil \frac{p}{2} \right\rceil$</td>
<td>2</td>
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Table 1: $m$-hull, monophonic, $g$-hull and geodetic number of some classes of graphs.

At this point, what remains to be done is to ask the following question: *Is there any other general relationship among the parameters $mhn(G)$, $mn(G)$, $ghn(G)$ and $gn(G)$, apart from the previous known inequalities?* The following realization theorem shows that, unless we restrict ourselves to a specific class of graphs, the answer is negative.

We use the following notation. By $N(v)$ we denote the neighborhood of a vertex $v$. A vertex $v$ is said to be *simplicial* in $G$ if the subgraph induced by its neighborhood $N(v)$ is a clique. The *extreme set of $G$*, denoted $Ext(G)$, is the set of all its simplicial vertices.

**Theorem 2.1.** For any integers $a, b, c, d$ such that $2 \leq a \leq b \leq c \leq d$, there exists a connected graph $G = (V, A)$ such that:

1. $a = mhn(G)$, $b = mn(G)$, $c = ghn(G)$ and $d = gn(G)$,

2. $a = mhn(G)$, $b = ghn(G)$, $c = mn(G)$ and $d = gn(G)$.

**Proof.** Let $G = (V, E)$ be the connected graph illustrated in Figure 1. We consider the following subsets of vertices: $W_1 = Ext(G) = \{a_1, \ldots, a_r, e, f\}$, $W_2 = W_1 \cup \{b_1, \ldots, b_s\}$, $W_3 = W_1 \cup \{c_1, \ldots, c_t\}$, and $W_4 = W_2 \cup \{c_1, \ldots, c_t\} \cup \{d_1, \ldots, d_u\}$. We will prove that $W_1$
Figure 1: The connected graph $G = (V, E)$.

is a minimum monophonic hull set, $W_2$ is a minimum monophonic set, $W_3$ is a minimum hull set, and $W_4$ is a minimum geodetic set.

i) $W_1$ is a minimum monophonic hull set. It is easy to prove that for any vertex $v \in V \setminus \{b_1, \ldots, b_s\}$, $v$ lies in one $a_i - e$ monomorphic path. Then $J[W_1] = V \setminus \{b_1, \ldots, b_s\}$. Given $b_i$, we consider $u_i, v_i \in N(b_i)$ such that $d(v_i, u_i) = 2$. Then $u_i - b_i - v_i$ is a monomorphic path and $u_i, v_i \in J[W_1]$, then $b_i \in J^2[W_1]$. As a conclusion $J^2[W_1] = V$ and $W_1$ is a monomorphic hull set.

ii) $W_2$ is a minimum monophonic set. By i) is obvious that $W_2$ is a monophonic set. In order to prove that $W_2$ is a minimum monophonic set, it is enough to remark the following fact. If $W$ is a monophonic set of $G$, then for $i = 1, \ldots, s$ it holds that either $b_i \in W$ or $u_i, v_i \in W$ because, in any other case, all path containing $b_i$ also contains a chord.

iii) $W_3$ is a minimum hull set. We know that $W_1 = Est(G) \in W$ for all $W$ hull set, and we observe that:

$$I[W_1] = V \setminus \{b_1, \ldots, b_s, c_1, \ldots, c_t, d_1, \ldots, d_u\}$$

$$I^2[W_1] = V \setminus \{c_1, \ldots, c_t, N(c_1), \ldots, N(c_t)\} = [Est(G)].$$

If $W$ is a hull set then $c_i \in W$ or some vertex of $N(c_i)$ is in $W$, for $i = 1, \ldots, t$. Hence $W_3$ is a minimum hull set.

iv) $W_4$ is a minimum geodetic set. It is easy to prove that for any vertex $v \in V$, $v$ lies in one $a_i - e$ geodetic path between two vertices of $W_4$. By the other hand, if we consider $W$ as a geodetic set, then: (a) either $b_i \in W$ or $u_i, v_i \in W$, $i = 1, \ldots, s$; (b) either $c_j \in W$ or $z_j \in W$ for some $z_j \in N(c_j)$, $j = 1, \ldots, t$; (c) either $d_h \in W$ or $N(d_h) \subset W$, $h = 1, \ldots, u$.

Hence $W_4$ is a minimum geodetic set.

Finally, it holds that: $mhn(G) = r + 2$, $mn(G) = r + s + 2$, $ghn(G) = r + t + 2$, $gn(G) = r + s + t + u + 2$. If $s \leq t$ then we obtain the the statement 1 of the theorem, and if $t \leq s$ then we obtain the statement 2 of the theorem. $\square$
Contour, peripheral and extreme vertices

The eccentricity of a vertex $u \in V$ is defined as $ecc_G(u) = ecc(u) = \max\{d(u, v) | v \in V\}$. Hence, the diameter $D$ of $G$ can be defined as the maximum eccentricity of the vertices in $G$. The periphery of $G$, denoted $Per(G)$, is the set of vertices that have maximum eccentricity, i.e., the set of the so-called peripheral vertices. A vertex $v$ is said to be simplicial in $G$ if the subgraph induced by its neighborhood $N(v)$ is a clique. The extreme set of $G$, denoted $Ext(G)$, is the set of all its simplicial vertices. With the aim of generalizing these two definitions, the so-called contour of $G$ was introduced in [2] as follows. Given a set $W \subseteq V$, a vertex $v \in W$ is said to be a contour vertex of $W$ if $ecc(v) \geq ecc(u)$, for all $u \in N(v) \cap W$. The contour $Ct(G)$ of $G$ is the set formed by all the contour vertices of $V$. Notice that $Per(G) \cup Ext(G) \subseteq Ct(G)$.

**Remark.** Figure 2 illustrates examples of graphs showing that there is no general relationship between peripheral and extreme vertices. The vertices inside a square (circle) are peripheral (extremal) vertices. More concretely, in Figure 2a: $Per(G) \not\subseteq Ext(G)$; in Figure 2b: $Ext(G) \not\subseteq Per(G)$; in Figure 2c: $Per(G) \cap Ext(G) = \emptyset$; and in Figure 2d: $Per(G) \cap Ext(G) \not= \emptyset$, $Per(G) \not\subseteq Ext(G)$ and $Ext(G) \not\subseteq Per(G)$. The extreme set in Figure 2b has only one vertex and it is neither a $g$-hull set nor $m$-hull set. The same holds for the periphery in Figure 2a. It is also clear that $Per(G) \cup Ext(G)$ is not necessarily a $m$-hull set, for example the graph in Figure 2b.

![Figure 2: No relationship between peripheral and extreme vertices.](image)

Let $W \subseteq V$ be a $m$-convex ($g$-convex) set and let $F = \langle W \rangle_G$ be the subgraph of $G$ induced by $W$. A vertex $v \in W$ is called a $m$-extreme vertex ($g$-extreme vertex) of $W$ if $W - v$ is a $m$-convex ($g$-convex) set. It is clear that a vertex $v$ of a $m$-convex ($g$-convex) set $W$ is a $m$-extreme ($g$-extreme) vertex of $W$ if and only if $v$ is simplicial in $F$ [5].

A convexity space $(V, C)$ is a convex geometry if it satisfies the so-called Minkowsky-Krein-Milman property: Every convex set is the convex hull of its extreme vertices. Notice that this condition allows us to rebuild every convex set from its extreme vertices, by using the convex hull operator. Farber and Jamison [5] proved that the monophonic (geodesic) convexity of a graph $G$ is a convex geometry if and only if $G$ is chordal (Ptolemaic). Cáceres and alt. [2] obtained a similar property to the previous one, valid for every graph, by considering, instead of the extremes vertices, the so-called contour vertices.
Theorem 3.1. [2] Let $G = (V,E)$ a connected graph and $W \subseteq V$ a $g$-convex set. Then, $W$ is the $g$-convex hull of its contour vertices.

As was pointed in [2], the contour set of a graph needs not to be geodetic. In Figure 3 we illustrate two graphs such that the contour set is $\{u,v,w\}$ and $I[\{u,v,w\}] = V \setminus \{z\}$.

Figure 3: Two graphs where the contour set is not geodetic.

Nevertheless, this assertion is true in the following case.

Proposition 3.1. If $Ct(G) = Per(G)$, then $Ct(G)$ is a geodetic set.

Proof. Let $x$ be a vertex of $V(G) \setminus Ct(G)$. As we have seen in the proof of Theorem 3.2, there exists a shortest $x_0 - x_s$ path, $x_0, x_1, \ldots, x_r$, such that $x = x_0$, $x_i \notin Ct(G)$, for $i \in \{0,\ldots,r-1\}$, $x_r \in Ct(G)$, and $ecc(x_i) = ecc(x_{i-1}) + 1 = l + i$ for $i \in \{1,\ldots,r\}$, where $l = ecc(x)$. But $x_r \in Ct(G) = Per(G)$ implies that $ecc(x_r) = D$ and $D = l + r$. Thus, there exists a vertex $z$ such that $d(z, x_r) = D$, and, therefore, $z \in Per(G)$. The distance satisfies $D = d(z, x_r) \leq d(z, x) + d(x, x_r) \leq ecc(x) + r = l + r = D$, that is $d(z, x_r) = d(z, x) + d(x, x_r)$. Hence $x$ is a shortest path between the vertices $z, x_r \in Per(G) = Ct(G)$.

As a consequence of this result, we have the following corollary.

Corollary 3.1. If $Ct(G)$ has exactly two vertices, $Ct(G)$ is a geodetic set.

Now, we approach the same issues by considering the monophonic convexity.

Theorem 3.2. The contour of any connected graph $G$ is a monophonic set.

Proof. Consider a vertex $x$ of $G$. Since the eccentricities of two adjacent vertices differ by at most one unit, if $x$ is not a contour vertex, there exists a vertex $y \in V$, adjacent to $x$, such that its eccentricity satisfies $ecc(y) = ecc(x) + 1$. This fact implies the existence of a path $\rho(x), x_0, x_1, x_2, \ldots, x_r$, such that $x = x_0$, $x_i \notin Ct(G)$, for $i \in \{0,\ldots,r-1\}$, $x_r \in Ct(G)$, and $ecc(x_i) = ecc(x_{i-1}) + 1 = l + i$ for $i \in \{1,\ldots,r\}$, where $l = ecc(x)$. Moreover, $\rho(x)$ is a shortest $x - x_r$ path, otherwise, the eccentricity of $x_r$ would be less than $l + r$.

Let us now consider the vertices at maximum distance $l$ from $x$. Suppose that all of them at distance less than $l + r$ from $x$. The vertices at distance less than $l$ from $x$ are at distance less than $l + r$ from $x_r$. Hence, the eccentricity of $x_r$ would be less than $l + r$. This implies the existence of a vertex $z$ at distance exactly $l$ from $x$ and $l + r$ from $x_r$, and $x$ lies in a shortest path $\psi = z, x, x_1, \ldots, x_r$ between $z$ and $x_r$ (Figure 4). Since every shortest path is a chordless path, whenever $z$ is a contour vertex, $x$ lies in a monophonic path with end vertices at the contour of the graph.
with a vertex of $V$ account the eccentricities of the vertices, if there is an edge joining a vertex of $V_l = 1$ (Figure 5).

Let $\delta$ be the $z - x$ sub-path of $\psi$. The vertex $z$ satisfies $ecc(z) \geq l + r$, the vertices of $V(\rho(z)) \setminus z$ have eccentricity at least $l + r + 1$ and the vertices of $V(\rho(x))$ have eccentricity at most $l + r$. Therefore, the sets $V(\rho(z))$ and $V(\rho(x))$ are disjoint. Moreover, taking into account the eccentricities of the vertices, if there is an edge joining a vertex of $V(\rho(z)) \setminus z$ with a vertex of $V(\rho(x))$, it must be $z_1 x_r$. In this case, $d(z, x_r) = 2 = l + r$, implying that $l = r = 1$ (Figure 5).

Hence, the eccentricity of $x$ is 1, the diameter of the graph is 2 and $z$ is a contour vertex, which is a contradiction.

Notice that the sets of vertices $V(\rho(z)) \setminus z$ and $V(\delta)$ are not necessarily disjoint. Consider a $z_s - x$ path $P$ contained in the walk $\rho(z) \cup \delta = z_s, \ldots, z_1, z, \ldots, x$. If it has a chord $e = ab$, we can replace the $a - b$ sub-path of $P$ with $e$ obtaining a $z_s - x$ path $P'$. Since $V(P') \subseteq V(P)$, the path $P'$ has strictly less chords than $P$. We proceed in an analogous way with $P'$, until we obtain a chordless $z_s - x$ path, $P^*.$ Notice that $\psi = z, \ldots, x, \ldots, x_r$ was a shortest path, and consequently there are no edges joining vertices of $V(\delta) \cup V(\rho(x))$. Therefore, $P^* \cup \rho(x)$ is a monophonic $z_s - x_r$ path through $x$ with $z_s, x_r$ contour vertices of $G$.

\textbf{Corollary 3.2.} Let $G$ be a connected graph and let $W \subseteq V$ be an $m$-convex set. Then, every vertex of $W$ lies on a monophonic path joining contour vertices of $W$. 

7
Proof. Let $F = \langle W \rangle_G$ be the subgraph of $G$ induced by $W$, and let $\Omega$ be the set of contour vertices of $W$. Certainly, $\Omega = \text{Cl}(F)$. Hence, the previous statement is equivalent to saying that the contour of $F$ is a monophonic set.

**Corollary 3.3.** Let $G$ be a connected graph and let $W \subseteq V$ be a $m$-convex set. Then, $W$ is the $m$-convex hull of its contour vertices.

Other consequence from Theorem 3.2 is the following corollary, which was directly proved in [2].

**Corollary 3.4.** The contour of a distance-hereditary graph is a geodetic set.

4 The edge Steiner problem

In this section, we focus our attention on the edges that lie in paths joining two vertices of $G = (V, E)$. We define the edge intervals of a graph as follows. The edge geodetic closed interval $I_e[u, v]$ is the set of edges of all $u - v$ geodesics. Similarly, the edge monophonic closed interval $J_e[u, v]$ is the set of vertices of all monophonic $u - v$ paths. For $W \subseteq V$, the edge geodetic closure $I_e[W]$ of $W$ is the union of all edge closed intervals $I_e[u, v]$ over all pairs $u, v \in W$. The edge monophonic closure, $J_e[W]$, is defined as the union of all edge closed monophonic intervals over all pairs $u, v \in W$. In other words, we have

$$I_e[W] = \bigcup_{u, v \in W} I_e[u, v], \quad J_e[W] = \bigcup_{u, v \in W} J_e[u, v].$$

A set $W$ of vertices for which $J_e[W] = E$ is called an edge monophonic set. Similarly, $W$ is said to be an edge geodetic set if $I_e[W] = E$ [1]. A set $W \subseteq V$ is an edge Steiner set if the edges lying in some Steiner $W$-tree cover $E$. Notice that: (1) every edge Steiner set is a Steiner set, (2) every edge geodetic set is a geodetic set, (3) every edge monophonic set is a monophonic set, and (4) every edge geodetic set is an edge monophonic set. It is easy to find examples where the converses of these statements are not true. We have obtained the following results.

**Theorem 4.1.** Every edge Steiner set of a connected graph is an edge monophonic set.

**Proof.** Let $e = uw$ be an edge of $G = (V, E)$ and $W \subseteq V$. If $W$ is an edge Steiner set, there exists a Steiner tree $T_W$ of $G$ that contains $e$. Suppose now that every path joining vertices of $W$ that contains $e$ is not a chordless path. It implies that at least $u$ or $v$ is not in $W$. Assume that $v \notin W$.

Consider the tree $T_W(v)$ obtained by travelling in $T_W$ from $v$ in all directions until reaching vertices of $W$. That is, $T_W(v)$ is a subtree of $T_W$ such that its vertices are in $W$ if, and only if, their degree is 1 (Figure 6a). Take a $v - w$ path of $T_W(v)$ not containing $e$, $P = v, \ldots, x, \ldots, y', y, \ldots, w$, where $w \in W$. If $P$ has a chord in $G$ joining $x$ and $y$, let $T'_W(v)$ be the tree obtained by changing the edge $y'y$ by the edge $xy$ (Figure 6b). The vertex $y'$ is not in $W$. Moreover, its degree in $T'_W(v)$ is greater than 1, otherwise, we can delete $y'$ and construct a Steiner $W$-tree of order less than the order of $T_W$ (by adding to $T_W(v) - y'$ the vertices and edges of $T_W$ that are not in $T_W(v)$).

If a $v - w$ $e$-free path of the tree $T_W(v)$, where $w \in W$, has a chord, it was already a chord in the $v - w$ $e$-free path of $T_W(v)$. Therefore, the total number of chords of all
Figure 6: a) In $T_{W(v)}$, $xy$ is a chord of a $v$–$w$ path, b) In $T'_{W(v)}$, $yy'$ is not a chord of a $v$–$w$ path, for any $w \in W$.

$v$–$w$ e-free paths of $T'_{W(v)}$, where $w \in W$, is strictly less than that of $T_{W(v)}$. We apply this process repeatedly to $T'_{W(v)}$, until obtaining a tree with no chords in the $v$–$w$ e-free paths. If $u \notin W$, we consider the $u$–$w$ e-free paths, where $w \in W$, and proceed in the same way.

The final result is a tree, $T^*$, such that: (1) $V(T^*) = V(T_{W(v)})$; (2) a vertex has degree 1 if and only if it is in $W$; (3) there is no edge of $G$ joining vertices of an e-free path from $u$ or $v$ to $w \in W \cap V(T^*)$.

Let $A_1, \ldots, A_r$ and $B_1, \ldots, B_s$ be the connected components obtained by deleting the vertices $u$ and $v$ of $T^*$, and such that $u$ is adjacent to a vertex of $A_i$ and $v$ is adjacent to and a vertex of $B_j$ in $T^*$ (Figure m8).

Figure 7: In $T^*$, the $u$–$w$ paths, $w \in W \cap V(A_i)$, and the $v$–$w$ paths, $w \in W \cap V(B_j)$, are chordless.

Notice that $r = \deg_{T^*}(u) - 1 \geq 0$ and $s = \deg_{T^*}(v) - 1 \geq 1$. We define $X_1, \ldots, X_h$ as the sets of vertices of the connected components of the subgraph induced by the vertices of $A_1, \ldots, A_r$ in $G$ and $Y_1, \ldots, Y_k$ as the sets of vertices of the connected components of the subgraph induced by the vertices of $B_1, \ldots, B_s$ in $G$. Observe that, the vertices of connected component $A_i$ (resp. $B_j$) are all in the same $X_l$ (resp. $Y_m$). By construction, there is no edge of $G$ joining a vertex of $X_i$ with a vertex of $X_j$, $i \neq j$. The same yields for the vertices of the sets $Y_1, \ldots, Y_k$ (Figure m9).

Observe that in each $X_i$ and in each $Y_j$ there is at least a vertex of $W$. Recall that we have assumed that every path joining vertices of $W$ through $e$ has at least a chord in $G$.

Thus, if $u \in W$, there must be a chord of $G$ joining $u$ with a vertex of $Y_j$, for every $j$. 

9
Consequently, any Steiner W

In this case, it is possible to connect all the vertices of W to construct a Steiner W-tree of order less than |V(T_W)|, which contradicts the fact that T_W was a Steiner W-tree (Figure 9).

If u \notin W and there is a chord of G from each Y_j to a vertex of X = X_1 \cup X_2 \cup X_k \cup \{u\}, the graph induced by V(T_W) \setminus v in G is connected. As in the preceding case, it is possible to construct a Steiner W-tree of order less than |V(T_W)|, which is a contradiction.

Finally, if u \notin W and there is no chord of G from a vertex of X to a vertex of one of the sets Y_j, there must be a chord from every X_i to v. The graph induced by V(T_W) \setminus u in G is connected and proceed analogously.

\[ \square \]

Corollary 4.1. In the class of connected distance-hereditary graph, every edge Steiner set is an edge geodetic set.

Analogously to the vertex case, this last result also holds for interval graphs.

Theorem 4.2. In the class of connected interval graphs, every edge Steiner set is an edge geodetic set.

Proof. Let G = (V, E) be a connected interval graph, W an edge Steiner set of G, and e \in E. The edge e lies at least in a Steiner W-tree, T_W. For any vertex u \in V, let I(u) = [a(I(u)), b(I(u))] be the corresponding interval of \( \mathbb{R} \). Consider the vertices x, y \in W such that a(I(x)) = a^* = \min\{a(I(w))|w \in W\} and b(I(y)) = b^* = \max\{b(I(w))|w \in W\}. If x = y, I(w) \subset I(x) for any w \in W, that is, x is adjacent to all the vertices of W \setminus \{x\}. In this case, it is possible to connect all the vertices of W forming a star with center x. Consequently, any Steiner W-tree has W as set of vertices. Therefore, the edge e lies in a shortest path between vertices of W.

We need to prove now the following lemma.

Lemma 4.1. (i) If P is an x–y walk in G, any vertex of W is adjacent to at least a vertex of V(P). (ii) If T_W is a Steiner W-tree, for any u \in V(T_W) \setminus W, u lies in the unique x–y path of T_W. Moreover, there exists a Steiner W-tree, T_W\*, formed by the unique x–y path of T_W and vertices of W adjacent to vertices of that path.

Proof. (i) We have \( \bigcup_{w \in W} I(w) \subset [a^*, b^*] \subset \bigcup_{z \in V(P)} I(z) \), that is, each w \in W is adjacent to a vertex of V(P) in G. (ii) Let T_W be a Steiner W-tree, and P be the unique x–y path
Figure 9: A step of the construction of a $W$-tree of order less than $|V(T_W)|$.

in $T_W$. We know that every vertex $w \in W$ is adjacent to a vertex $p_w$ of $V(P)$. Consider the tree $T^*_W$ with set of vertices $V(P) \cup (W \setminus V(P))$ and set of edges $E(P) \cup \{wp_w, w \in W \setminus V(P)\}$. If at least a vertex $u \in V(T_W) \setminus W$ is not in $V(P)$, we obtain a $W$-tree of order less than the order of $T_W$ containing $W$, which contradicts the fact that $T_W$ was a Steiner $W$-tree.

Suppose now that $x \neq y$. Let $P$ be the unique $x - y$ path in $T_W$, and $T^*_W$, the Steiner $W$-tree obtained as in the preceding Lemma. By Lemma 4.1, there are the following possibilities for the edge $e$: (1) $e = ab \in E(P)$, that is $a, b \in V(P)$; (2) $e = ab \notin E(P)$, $a, b \in W$; (3) $e = ab \notin E(P)$, $a \in V(P) \setminus W, b \in W$.

In the first case, we travel from $a$ and $b$ along $P$ in both directions until arriving to vertices $w', w''$ of $W$. Observe that the internal vertices of the $w'-w''$ sub-path of $P$ are not in $W$. Therefore, if $w'-w''$ is not a shortest path, we change it with a shortest $w'-w''$ path, obtaining an $x - y$ walk, $P^*$. By the preceding Lemma, every vertex of $W$ is in $V(P^*)$ or is adjacent to a vertex of $V(P^*)$, implying the existence of a $W$-tree of order less than $|V(T^*_W)|$, which is a contradiction. Hence, $e$ lies in a shortest path with ends in $W$.

In the second case, $e = ab$ lies in the unique $a - b$ shortest path, where $a, b \in W$.

In the third case, let us travel from $a$ along $P$ until arriving to a vertex $w'$ of $W$. The
internal vertices of the path $b - a - w'$ are not in $W$. If it is a shortest path, the proof is over. Otherwise, we can change it by a shortest $b - w'$ path of length $h < 1 + k$, where $k$ is the length of the sub-path $a - w'$ of $P$. By Lemma 4.1, if $h < k$ we can construct a $W$-tree of order less than $|V(T^*_W)|$, which is a contradiction. In a similar way, if $h = k$, we can construct a Steiner $W$-tree, $T'_W$, such that $e = ab$ lies in its shortest $x - y$ path, and finally proceed as in the first case, obtaining a shortest path between vertices of $W$ that contains the edge $e$.

Figure 10:

In the preceding section we have seen that the contour of a graph is a monophonic set. Nevertheless, it is quite easy to find graphs whose contour is not an edge monophonic set (Figure 2d).

It remains an open question the problem of characterizing those classes of chordal graphs for which every edge Steiner set is edge geodetic. We know this statement to be true for interval and ptolemaic graphs, and false for split graphs (i.e., those chordal graphs whose complementary is also chordal); for example, the set $W = \{u, v, w, y\}$ of the split graph illustrated in Figure 12 is an edge Steiner set, but not an edge geodetic set.

Figure 11: Split graph.

Figure 12: Split graph.
References


