Abstract: Given a pair of matrices \((A, B)\) we study the stability of their invariant subspaces from a geometric point of view. The main tool is the manifold of quadruples \(((A, B), F, S)\) where \(S\) is an \((A, B)\)-invariant subspace and \(F\) is such that \((A + BF)S \subseteq S\). From the geometry of this manifold we derive sufficient computable conditions of stability.

Keywords: \((A, B)\)-invariant subspace, stability, manifold, submersion.
So, we identify \( \mathcal{M} \) with the set
\[
\{(A, B, P, F) : (A + BF)P = P(A + BF)P \}.
\]
We also denote this set by \( \mathcal{M} \). If we introduce the map
\[
\varphi : M_n \times M_{n,m} \times \text{Gr}_k(\mathbb{R}^n) \times M_{m,n} \rightarrow M_n
\]
defined by
\[
\varphi(A, B, P, F) = (A + BF)P - P(A + BF)P
\]
it is clear that \( \mathcal{M} = \varphi^{-1}(0) \).

In order to show that \( \mathcal{M} \) is a differentiable manifold we have to study the range of \( d\varphi_\xi \) where \( \xi = (A, B, P, F) \in \mathcal{M} \).

**Lemma 2.1.** With the above notation, \( d\varphi_\xi \) is the linear map from \( M_n \times M_{n,m} \times T_P \text{Gr}_k(\mathbb{R}^n) \times M_{m,n} \) to \( M_n \) defined by
\[
d\varphi_\xi(\hat{A}, \hat{B}, \hat{P}, \hat{F}) = (I - P)(\hat{A}P + BFP + B\hat{F}P + BF\hat{P}) - \hat{P}(A + BF)P
\]
where \( \hat{P} = [P, \Omega], \Omega = -\Omega', [P, \Omega] = PO - OP \).

**Lemma 2.2.** For every \( \xi \in \mathcal{M} \), we have
\[
\text{rank } d\varphi_\xi = k(n - k).
\]

**Proof.** Let \( \delta = \dim \text{Im } d\varphi_\xi \). Then \( L' \in \text{Im } d\varphi_\xi \) if and only if
\[
\text{tr } L(I - P)(\hat{A}P + BFP + B\hat{F}P + BF\hat{P}) - \hat{P}(A + BF)P = 0 \tag{3}
\]
for every \( \hat{A} \in M_n, \hat{B} \in M_{n,m}, \hat{F} \in M_{m,n}, \hat{P} \in T_P \text{Gr}_k(\mathbb{R}^n) \) (\text{tr} stands for trace).

It can be checked that equation (3) is equivalent to
\[
\text{tr } PL(I - P)\hat{A} = 0 \text{ for every } \hat{A} \in M_n
\]
\[
\text{tr } PL(I - P)\hat{B}F = 0 \text{ for every } \hat{B} \in M_{n,m}
\]
\[
\text{tr } L(I - P)\hat{A}P + L(I - P)BFP - L\hat{P}AP - L\hat{P}BF\hat{P} = 0 \text{ for every } \hat{P} \in T_P \text{Gr}_k(\mathbb{R}^n)
\]
\[
\text{tr } PL(I - P)BF = 0 \text{ for every } \hat{F} \in M_{m,n}
\]
which in its turn is equivalent to \( PL(I - P) = 0 \) and
\[
\text{tr } \Omega(L(I - P)(AP + BFP) - APLP + PAPL + BFPLP + PBFPPL) = 0 \text{ for every } \Omega = -\Omega'.
\]
But, \( (A + BF)P = P(A + BF)P \) so that
\[
L(I - P)(AP + BFP) = 0
\]
and
\[
PAPL + PBFPL = APL + BFPL.
\]
Finally, taking into account that \( PL = PLP \) we conclude that
\[
L(I - P)(AP + BFP) - APLP + PAPL - BFPLP + PBFPPL = 0.
\]

Hence,
\[
L' \in \text{Im } d\varphi_\xi \text{ if and only if } PL(I - P) = 0.
\]
In a suitable bases, we have \( P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). Then, from the last equality we obtain
\[
\dim \text{Im } d\varphi_\xi = k(n - k)
\]
and the lemma follows. \( \blacksquare \)

Since \( n^2 + nm + k(n - k) + mn - k(n - k) = n^2 + 2nm \), we have proved the following basic result.

**Theorem 3.3.** With the above notation \( \mathcal{M} \) is a submanifold of \( M_n \times M_{n,m} \times \text{Gr}_k(\mathbb{R}^n) \times M_{m,n} \) of dimension \( n^2 + 2nm \).

Notice that from lemma 2.1 we have that the tangent space to \( \mathcal{M} \) in \( \xi \) is given by
\[
T_\xi \mathcal{M} = \{(\hat{A}, \hat{B}, \hat{P}, \hat{F}); (I - P)(\hat{A}P + BFP + B\hat{F}P + BF\hat{P}) - \hat{P}(A + BF)P = 0 \}.
\]

### 3. Stability

Given an \((A, B)\)-invariant subspace \( S \), it is said that \( S \) is \((A, B)\) stable (or simply stable) if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \((A', B') \in M_n \times M_{m,n} \) verifies \( \|(A', B') - (A, B)\| < \delta \), there exists an \((A', B')\)-invariant subspace \( P' \) such that \( \|P' - P\| < \varepsilon \).

For our purposes it is more convenient to set this definition into a geometric frame. Consider the following diagram:

\[
\begin{array}{ccc}
M_n \times M_{n,m} \times \text{Gr}_k(\mathbb{R}^n) \times M_{m,n} & \xrightarrow{\pi_1} & \mathcal{M} \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
M_n \times M_{n,m} \times M_{m,n} & \xrightarrow{\hat{\pi}_1} & \text{Gr}_k(\mathbb{R}^n) \\
\downarrow \hat{\pi}_2 & & \\
& & \text{Gr}_k(\mathbb{R}^n)
\end{array}
\]

where \( \pi_1 \) and \( \pi_2 \) are the natural projections and \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) the corresponding restrictions.

**Lemma 3.1.** With the above notation, for every \( \xi \in \mathcal{M} \), \( \text{rank } d\hat{\pi}_{2,\xi} = k(n - k) \).

**Corollary 3.2.** The map \( \hat{\pi}_2 : \mathcal{M} \rightarrow \text{Gr}_k(\mathbb{R}^n) \) is a submersion.

We are ready to state the main result of this note.

**Theorem 3.3.** Let \( \xi = (A, B, P, F) \in \mathcal{M} \). Then if \( d\hat{\pi}_{1,\xi} \) is an isomorphism, \( P = (A, B) \) stable.
Proof. Given $\varepsilon > 0$, denote $U_\varepsilon(P)$ the set

$$\mathcal{U}_\varepsilon(P) = \{P' \in \text{Gr}_k(\mathbb{R}^n); \|P' - P\| < \varepsilon\}.$$ 

Since $d\hat{\pi}_{1,\xi}$ is an isomorphism, $\pi_1$ is a local diffeomorphism. Then, taking into account that $\hat{\pi}_2$ is a submersion (Corollary 3.2), there exists open neighbourhoods of $(A, B, P, F)$ in $\mathcal{M}$, $\mathcal{U}_0(A, B, P, F)$, and of $(A, B, F)$ in $M_n \times M_{n,m} \times M_{m,n}$, $\mathcal{U}_0(A, B, F)$ such that for every $(A', B', F') \in \mathcal{U}_0(A, B, F)$ there exists $P' \in \text{Gr}_k(\mathbb{R}^n)$ such that $P' \in \mathcal{U}_\varepsilon(P)$. Since the natural projection $\pi$ from $M_n \times M_{n,m} \times M_{m,n}$ to $M_n \times M_{m,n}$ is open the theorem follows.

Notice that since $d\hat{\pi}_{1,\xi}$ is the linear map

$$d\hat{\pi}_{1,\xi} : T_P \mathcal{M} \longrightarrow M_n \times M_{n,m} \times M_{m,n}$$

defined by $d\hat{\pi}_{1,\xi}(\hat{A}, \hat{B}, \hat{P}, \hat{F}) = (\hat{A}, \hat{B}, \hat{F})$, $d\pi_{1,\xi}$ is an isomorphism if and only if

$$\left\{\tilde{P} \in T_P \text{Gr}_k(\mathbb{R}^n); (I - P)(AP + BF \tilde{P}) - \tilde{P}(A + BF)\tilde{P} = 0\right\} = \{0\}.$$ 

4. REFERENCES

