

Linearization of class C^1 for contractions on Banach Spaces

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Abstract

In this work we prove a C^1 -linearization result for contraction diffeomorphisms, near a fixed point, valid in infinite dimensional Banach spaces. As an intermediate step, we prove a specific result of existence of invariant manifolds, which can be interesting by itself and that was needed on the proof of our main theorem. Our results essentially generalize some classical results by P. Hartman in finite dimensions, and a result of X. Mora-J. Sola-Morales in the infinite dimensional case. It is shown that the result can be applied to some abstract systems of semilinear damped wave equations.

Key words: linearization, conjugacy, damped wave equation.

1 Introduction and Main Result.

In finite-dimensional systems, results on linearization have been extensively used for systems of ordinary differential equations and for local diffeomorphisms. The idea of linearization is very old, going back to the work of N.H. Abel (See Hartman [4]). Its main use is either to analyze asymptotic behavior

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of solutions or iterates, in particular situations, to study structural stability problems, for local phase portrait analysis, or even to prove existence of invariant manifolds.

For \mathcal{C}^0 linearization, the result is called Hartman-Grobman Theorem (see, for example [5]) and it is a well-known result, even in general Banach spaces (see the paper by C. Pugh [12]). But some of the geometric properties of this kind of linearization are still a relevant object of current research, as one can see in the recent results by H.M. Rodrigues and J.G. Ruas Filho [14], K. Lu [8] and L. R. Volevich and A. R. Shirikyan [22]. An important characteristic of \mathcal{C}^0 linearization is that it is free from any non-resonance condition: it merely requires hyperbolicity.

The situation is very different for \mathcal{C}^k linearization with $k \geq 1$, since non-resonance conditions do appear naturally, as one can see in Poincaré [11], Aronson-Belitskii-Zhuzhoma [1], Belitskii [2], Sell [16], Sternberg [17,18] and Stowe [19]. Other relevant works in related problems are V. S. Samovol [15] and Fenichel [3]. V. Rayskin [13] and B. Tan [20] obtained results concerning Hölder-Continuous linearization. All of these papers deal only with finite dimensions. Linearization of some partial differential equations by using scattering techniques was obtained by H. P. McKean and J. Shatah [9].

But to our knowledge, the first published abstract result, for maps and flows of general type, of smooth linearization in infinite dimensions is that of the paper by X. Mora and J. Solà-Morales [10], where a partial result for the case $k = 1$ and when the linear part is a contraction, under a quite restrictive non-resonance condition, is obtained and applied to the problem of existence of attracting invariant manifolds for semilinear damped wave equations. A similar result was obtained by B. Tan in his PhD-Thesis (See Theorem 8.2 and Theorem 8.3 [21]). The two previous results correspond to our main result (Theorem1) in the particular case $n = 1$, or in other words, when we have just one block in the linear part.

But it is known that for the $k = 1$ case and finite dimension one does not need non-resonance conditions for the case of contractions (see [4] and [2]), and only very simple ones for the case of the saddle ([2]). This is why the purpose of this work has been to obtain \mathcal{C}^1 linearization results in infinite dimensions for contractions with as less as possible non-resonance conditions.

The ideas of Hartman [4] were important in our approach, but in order to obtain similar results for maps on infinite dimensional spaces, we had to introduce new techniques and to use some very special norms. Nevertheless, the structure of the proof of our main result follows the same steps as those of Hartman [4]. Perhaps the main difference is that we use spectral sets (or blocks) instead of the finite set of eigenvalues of the linear part. We make a

systematic use of The Contraction Principle in appropriate Banach spaces to prove The Linearization Theorem (Theorem 1) and The Invariant Manifold Theorem (Theorem 8). Our results essentially generalize those of Hartman [4] in finite dimensions and the result of X. Mora-J. Solà-Morales [10].

The contents of this paper are the following. The next theorem contains our main result. The proof will be done in Chapter 4. One of its main ingredients is the solution of a functional equation, that will be analyzed in detail in Chapter 3. Chapter 2 is devoted to two preliminary results that can have some interest by themselves: the first is on existence of suitable equivalent norms in Banach and Hilbert spaces according to spectral bounds for families of operators and the second about existence of invariant manifolds under the precise hypotheses that will be needed later. Chapter 5 is devoted to an application of Theorem 1 to systems of abstract damped wave equations.

Theorem 1 (The Linearization Theorem.) *Let \mathbb{X} be a Banach space with the property that there exists function γ such that*

$$\gamma \in C^{1,1}(\mathbb{X}, \mathbb{R}), \quad \text{with } \gamma(x) = 1, \text{ when } |x| \leq 1/2 \text{ and } \gamma(x) = 0, \text{ when } |x| \geq 1. \quad (1)$$

Suppose that $A, A^{-1} \in L(\mathbb{X})$. We assume that there exist real numbers ν_i^-, ν_i^+ , $i = 1, \dots, n$ such that:

$$\left. \begin{aligned} 0 < \nu_n^- < \nu_n^+ < \nu_{n-1}^- < \nu_{n-1}^+ < \dots < \nu_1^- < \nu_1^+ < 1 \\ \nu_1^+ \nu_i^+ < \nu_i^-, \quad i = 1, \dots, n \quad (\text{nonresonance condition}) \\ |\sigma(A)| \subset (\nu_n^-, \nu_n^+) \cup (\nu_{n-1}^-, \nu_{n-1}^+) \cup \dots \cup (\nu_1^-, \nu_1^+). \end{aligned} \right\} \quad (2)$$

Let $\mathcal{X} = \mathcal{X}(x)$ be a $C^{1,1}$ -function in a neighborhood of the origin with values in \mathbb{X} , such that $\mathcal{X} = 0, \partial_x \mathcal{X} = 0$, at $x = 0$.

Then, for the map $T : x \mapsto x^1, x^1 = Ax + \mathcal{X}(x)$, there exists a C^1 -map $R : x \mapsto u, u = x + \phi(x)$, satisfying $\phi = 0, \partial_x \phi = 0$, at $x = 0$, such that $RTR^{-1} : u \mapsto u^1$ has the form $u^1 = Au$ in a sufficiently small neighborhood of the origin.

Remark 2 It is easy to see that if $\dim \mathbb{X} < \infty$ and $|\sigma(A)| \subset (0, 1)$ then there is always a choice of numbers ν_i^-, ν_i^+ , such that (2) holds. In this sense, our result implies the corresponding result of Hartman [4], as it is stated.

But after the statement, it is said in [4] that the same proof proves that the change of variables is not only C^1 , but of class $C^{1,\alpha}$, $0 < \alpha < 1$. In our proof the Hölder condition with exponent α will be proved only at the origin. We believe that this could also be proved using our approach, with technical modifications, but we will not do it in the present paper.

Remark 3 Theorem 3.1 in [10] proved the same result when $\partial_x \mathcal{X}(x) = o(|x|^\eta)$, for some $\eta > 0$, and A is restricted to satisfy $|A^{-1}||A|^{1+\eta} < 1$. The most common case happens when $\partial_x \mathcal{X}(x) = O(|x|)$, and in this case one can take $\eta < 1$, but $\eta \simeq 1$. This most common case is covered by our result with $n = 1$ and $\nu_1^- < |A^{-1}|^{-1} \leq |A| < \nu_1^+$ and $(\nu_1^+)^2 < \nu_1^-$.

Remark 4 By an argument due to Sternberg [18], if $T(t) := e^{Lt} + S(t)$ is a group of $C^{1,1}$ -diffeomorphisms, where $S(t)0 = 0$, $(D_x S(t))|_{x=0} = 0$ and $A := e^L$, $T := T(1)$, and if the above hypotheses are satisfied and we define $\bar{R} := \int_0^1 e^{-Ls} R T(s) ds$, then one has $\bar{R} T(t) \bar{R}^{-1} = e^{Lt}$, $\forall t \in \mathbb{R}$.

Some sufficient conditions for the groups $T(t)$ be of class $C^{1,1}$ are included in a general result by D. Henry in [7].

2 Preliminary Results.

The following is a general result, that we will use in the sequel. The proof was inspired on a related proof presented in M. C. Irwin [6].

Theorem 5 *Let \mathbb{X} be a Banach space with norm $|\cdot|$. We indicate also by $|\cdot|$ the induced norm of bounded operators on $L(\mathbb{X})$. Let us denote by $r(A)$ the spectral radius of A . Let $A_1, A_2, \dots, A_p \in L(\mathbb{X})$, such that $A_i A_j = A_j A_i$, for $i, j = 1, 2, \dots, p$. Then, given $\varepsilon > 0$, there exists a norm, $\|\cdot\|$ on \mathbb{X} , equivalent to $|\cdot|$ such that $r(A_i) \leq \|A_i\| \leq r(A_i) + \varepsilon$, for $i = 1, 2, \dots, p$.*

Proof: We prove the above statement for two operators and the general case follows by induction.

Without loss of generality we can assume that for the norm $|\cdot|$ we have $r(A_1) \leq |A_1| \leq r(A_1) + \varepsilon$, (this also follows from the first part of the next proof).

Let $M := r(A_2) + \varepsilon$. Since $|A_2|^k \rightarrow r(A_2)$, as $k \rightarrow \infty$, there exists m , such that $|A_2^k| < M^k$, for $k \geq m$. We define the following norm:

$$\|x\| := \left[|x|^2 + \left(\frac{|A_2 x|}{M} \right)^2 + \dots + \left(\frac{|A_2^m x|}{M^m} \right)^2 \right]^{1/2}$$

The norm $\|\cdot\|$ is equivalent to $|\cdot|$. Moreover,

$$\|A_2 x\| = \left[|A_2 x|^2 + \left(\frac{|A_2 A_2 x|}{M} \right)^2 + \dots + \left(\frac{|(A_2)^{m+1} x|}{M^m} \right)^2 \right]^{1/2} \leq$$

$$\begin{aligned}
& M \left[\left(\frac{|A_2 x|}{M} \right)^2 + \cdots + \left(\frac{|(A_2)^m x|}{M^m} \right)^2 + \left(\frac{|(A_2)^{m+1} x|}{M^{m+1}} \right)^2 \right]^{1/2} \\
& \leq M \left[|x|^2 + \left(\frac{|A_2 x|}{M} \right)^2 + \cdots + \left(\frac{|A_2^m x|}{M^m} \right)^2 \right]^{1/2} = M \|x\| \\
\|A_1 x\| & := \left[|A_1 x|^2 + \left(\frac{|A_2 A_1 x|}{M} \right)^2 + \cdots + \left(\frac{|A_2^m A_1 x|}{M^m} \right)^2 \right]^{1/2} \leq |A_1| \|x\| \\
& \leq (r(A_1) + \varepsilon) \|x\| \quad \blacksquare
\end{aligned}$$

Remark 6 When \mathbb{X} is a Hilbert space, from the above proof one can obtain the same result for an equivalent hilbertian norm.

Corollary 7 *Let \mathbb{X} be a Banach space with norm $|\cdot|$. Suppose that $A, A^{-1} \in L(\mathbb{X})$ and that $|\sigma(A)| \subset (c, d)$, with $c > 0$. Then there exists an equivalent norm, $\|\cdot\|$ on \mathbb{X} , such that we have $\|A\| < d$ and $\|A^{-1}\| < \frac{1}{c}$.*

Proof: Let a, b be such that $c < a < b < d$ and $|\sigma(A)| \subset (a, b)$. Our assumptions imply that $r(A) < b$ and $r(A^{-1}) < \frac{1}{a}$. From the above lemma it follows that we can obtain an equivalent norm $\|\cdot\|$ on \mathbb{X} , such that $\|A\| \leq b < d$ and $\|A^{-1}\| \leq \frac{1}{a} < \frac{1}{c}$. \blacksquare

The following result will be used as one of the steps in the proof of The Linearization Theorem, but since it has some interest by itself, we state it here separately, together with the proof.

Theorem 8 (The Invariant Manifold Theorem.) *Let \mathbb{X}, \mathbb{Z} be Banach spaces. Suppose $\mathcal{X} \in C^1(\mathbb{X} \times \mathbb{Z}; \mathbb{X})$ be such that $\mathcal{X}(0, 0) = 0$, $\partial_x \mathcal{X}(0, 0) = 0$, $\partial_z \mathcal{X}(0, 0) = 0$.*

Suppose also that $\mathcal{X}(x, z) = 0$ when $|x| + |z| \geq r > 0$ and that

$$\frac{|\partial_x \mathcal{X}(x_1, z_1) - \partial_x \mathcal{X}(x_2, z_2)|}{|x_1 - x_2| + |z_1 - z_2|} < M_1, \quad \frac{|\partial_z \mathcal{X}(x_1, z) - \partial_z \mathcal{X}(x_2, z)|}{|x_1 - x_2|} < M_2,$$

$$\frac{|\partial_z \mathcal{X}(x, z) - \partial_z \mathcal{X}(x, 0)|}{|z|^\alpha} < M_3$$

for all $x, z, x_1 \neq x_2, z_1 \neq z_2$ and some $M_1, M_2, M_3 > 0$ and $0 < \alpha \leq 1$.

Suppose $A \in L(\mathbb{X})$, $C \in L(\mathbb{Z})$ be such that $|A|, |C| < 1$ and $|A||C^{-1}|^{1+\alpha} < 1$.

Then, for all δ with $0 < \delta < \alpha$ there exists a r_0 , depending on $\delta, \alpha, M_1, M_2, M_3, |A|, |C|$, such that, if $r_0 > r > 0$, then the functional equation

$$x(z) = Ax(C^{-1}z) + \mathcal{X}(x(C^{-1}z), C^{-1}z) \quad (3)$$

has a solution $x(\cdot) \in \mathcal{C}^1(\mathbb{Z}, \mathbb{X})$ (bounded, and with bounded derivative) such that $x(0) = 0$ and $|x'(z)| = O(|z|^\delta)$ as $z \rightarrow 0$. Moreover, this solution is unique in the class of bounded functions. The set $\{(x(z), z) \in \mathbb{X} \times \mathbb{Z} : z \in \mathbb{Z}\}$ is an invariant manifold for the map

$$(A + \mathcal{X}, C) : \mathbb{X} \times \mathbb{Z} \rightarrow \mathbb{X} \times \mathbb{Z}.$$

Proof: For $x(\cdot) \in \mathcal{C}^1(\mathbb{Z}, \mathbb{X})$ let us define the auxiliary norm $\|x\|_a := \sup_{z \neq 0} |z|^{-\delta} |x'(z)|$, and consider the space

$$E_a = \{x(\cdot) \in \mathcal{C}^1(\mathbb{Z}, \mathbb{X}) : x(0) = 0, x(z) = 0 \text{ for } |z| \geq r, x'(0) = 0, \|x\|_a < \infty\},$$

that turns out to be a Banach space with the norm $\|\cdot\|_a$. We are going to show that the map T defined by

$$T[x(\cdot)](z) = Ax(C^{-1}z) + \mathcal{X}(x(C^{-1}z), C^{-1}z) \quad (4)$$

is a contraction in a suitable closed (small) ball of E_a , provided that r is small enough.

Observe first, after the definition of $\|\cdot\|_a$, that one has always the bound $|x(z)| \leq \|x\|_a r |z|^\delta$ if $x \in E_a$.

Observe also that

$$(T[x])'(z) = Ax'(C^{-1}z)C^{-1} + \partial_x \mathcal{X}(x(C^{-1}z), C^{-1}z)x'(C^{-1}z)C^{-1} + \partial_z \mathcal{X}(x(C^{-1}z), C^{-1}z)C^{-1} \quad (5)$$

and we can easily see that $T(E_a) \subset E_a$. Also,

$$\|T[x]\|_a \leq |A||C^{-1}|^{1+\delta}\|x\|_a + M_1 r |C^{-1}|^{1+\delta}\|x\|_a + M_2 r |C^{-1}|^{1+\delta}\|x\|_a + M_3 |C^{-1}|^{1+\alpha} r^{\alpha-\delta}.$$

So, if r is small enough, the ball of E_a of radius $s(r)$ around zero is invariant under T , where

$$s(r) = \frac{r^{\alpha-\delta} M_3 |C^{-1}|^{1+\alpha}}{1 - |A||C^{-1}|^{1+\delta} - r M_1 |C^{-1}|^{1+\delta} - r M_2 |C^{-1}|^{1+\delta}}.$$

Observe also that $s(r) \rightarrow 0$ as $r \rightarrow 0$.

Now we see that T is a contraction in this ball of radius $s(r)$, if r is small enough, by bounding separately the three terms in (5):

$$\begin{aligned}
& |z|^{-\delta} |Ax'_1(C^{-1}z)C^{-1} - Ax'_2(C^{-1}z)C^{-1}| \leq |A||C^{-1}|^{1+\delta} \|x_1 - x_2\|_a, \\
& |z|^{-\delta} |\partial_x \mathcal{X}(x_1(C^{-1}z), C^{-1}z)x'_1(C^{-1}z)C^{-1} - \partial_x \mathcal{X}(x_2(C^{-1}z), C^{-1}z)x'_2(C^{-1}z)C^{-1})| \\
& \leq |z|^{-\delta} (|\partial_x \mathcal{X}(x_1(C^{-1}z), C^{-1}z)| \|x'_1(C^{-1}z)C^{-1} - x'_2(C^{-1}z)C^{-1}\| + \\
& \quad + |\partial_x \mathcal{X}(x_1(C^{-1}z), C^{-1}z) - \partial_x \mathcal{X}(x_2(C^{-1}z), C^{-1}z)| \|x'_2(C^{-1}z)C^{-1}\|) \leq \\
& \leq M_1 r \|x_1 - x_2\|_a |C^{-1}|^{1+\delta} + M_1 \|x_1 - x_2\|_a r^{1+\delta} |C^{-1}|^\delta s(r) |C^{-1}|^{1+\delta}, \\
& |z|^{-\delta} |\partial_z \mathcal{X}(x_1(C^{-1}z), C^{-1}z)C^{-1} - \partial_z \mathcal{X}(x_2(C^{-1}z), C^{-1}z)C^{-1})| \leq M_2 \|x_1 - x_2\|_a r |C^{-1}|^{1+\delta}.
\end{aligned}$$

To see the uniqueness we observe that if x_1 and x_2 are two solutions, then $\|x_1 - x_2\|_\infty \leq (|A| + M_1 r) \|x_1 - x_2\|_\infty$.

The last statement of our theorem goes as follows:

$$(A + \mathcal{X}, C)(x(C^{-1}z), C^{-1}z) = (Ax(C^{-1}z) + \mathcal{X}(x(C^{-1}z), C^{-1}z), CC^{-1}z) = (x(z), z) \quad \blacksquare$$

Remark 9 In the above theorem we loose some regularity on the variable z , that is, in our assumptions we started with the Hölder exponent α and obtained the invariant manifold with Hölder exponent $\delta < \alpha$. The same problem appears in the finite dimensional case (see Hartman [4]). In [1] the following example is discussed:

$$F(x, z) = (\lambda^2 x + \frac{1}{2} \lambda^2 z^2, \lambda z).$$

This diffeomorphism is analytic. It is shown that there is no local invariant manifold of class $C^{1,1}$ of the form: $M_2 = \{x = \gamma(z)\}$, $\gamma(0) = 0$, $\gamma'(0) = 0$, but of course there is a local invariant manifold of class $C^{1,\delta}$, $\delta < 1$.

3 The Functional Equation.

Now let \mathbb{U} , \mathbb{V} and \mathbb{W} be Banach spaces. For $\mathbb{U} \times \mathbb{V}$ we consider the norm $|(u, v)| := |u| + |v|$.

Let $0 < \beta < \eta < 1$, $\beta + \eta < 1$ and $\varepsilon > 0$.

Let $\phi : B_\varepsilon \subset \mathbb{U} \times \mathbb{V} \rightarrow \mathbb{W}$ be a C^1 function such that

$$\phi(0, v) \equiv 0 \text{ and } \partial_u \phi = 0, \text{ at } (0, 0). \quad (6)$$

Now we define: $\|\phi\| := \max\{\|\phi\|_1, \|\phi\|_2, \|\phi\|_3\}$, where:

$$\|\phi\|_1 := \sup \frac{|\partial_u \phi(u_1, v_1) - \partial_u \phi(u_2, v_2)|}{|u_1 - u_2| + |v_1 - v_2|}, \quad \|\phi\|_2 := \sup \frac{|\partial_v \phi(u_1, v) - \partial_v \phi(u_2, v)|}{|u_1 - u_2|},$$

$$\|\phi\|_3 := \sup \frac{|\partial_v \phi(u, v) - \partial_v \phi(u, 0)|}{|u|^\eta |v|^\beta},$$

for $(u_1, v_1) \neq (u_2, v_2)$, $u \neq 0$, $v \neq 0$, (u_1, v_1) , (u_2, v_2) , (u_1, v) , (u_2, v) , $(u, 0)$ and (u, v) in B_ε .

Lemma 10 *Let E be the space of the functions ϕ , such that (6) is satisfied and $\|\phi\| < \infty$. Then E is a Banach space.*

Proof: Let (ϕ_n) a Cauchy sequence in E . Given $\gamma > 0$ there exists n_0 , such that if $m, n \geq n_0$ and (u, v) , $(u, 0)$, (u_1, v_1) , (u_2, v_2) , (u_1, v) , $(u_2, v) \in B_\varepsilon \subset \mathbb{U} \times \mathbb{V}$, then:

$$|\partial_u(\phi_n - \phi_m)(u_1, v_1) - \partial_u(\phi_n - \phi_m)(u_2, v_2)| \leq \gamma(|u_1 - u_2| + |v_1 - v_2|) \quad (7)$$

$$|\partial_v(\phi_n - \phi_m)(u_1, v) - \partial_v(\phi_n - \phi_m)(u_2, v)| \leq \gamma(|u_1 - u_2|) \quad (8)$$

$$|\partial_v(\phi_n - \phi_m)(u, v) - \partial_v(\phi_n - \phi_m)(u, 0)| \leq \gamma(|u|^\eta |v|^\beta) \quad (9)$$

If we let $(u_1, v_1) = (u, v)$, $(u_2, v_2) = (0, 0)$ in (7), we obtain that $\partial_u \phi_n(u, v)$ is a Cauchy sequence and so it converges uniformly to a continuous function $F(u, v)$, such that $F(0, 0) = 0$.

Therefore,

$\phi_n(u, v) = \phi_n(u, v) - \phi_n(0, v) = \int_0^1 \frac{d}{dt} \phi_n(tu, v) dt = \int_0^1 \partial_u \phi_n(tu, v) u dt$, and so $\phi_n(u, v) \rightarrow \int_0^1 F(tu, v) u dt$, as $n \rightarrow \infty$. If we define $\phi(u, v) := \int_0^1 F(tu, v) u dt$, we have $\phi(0, v) \equiv 0$. For $h \neq 0$, $(u, v) \in B_\varepsilon$, we obtain:

$$\begin{aligned} \frac{|\phi_n(u+h, v) - \phi_n(u, v) - \partial_u \phi_n(u, v)h|}{|h|} &= \frac{|\int_0^1 [\frac{d}{dt} \phi_n(u+th, v) - \partial_u \phi_n(u, v)h] dt|}{|h|} \\ &= \frac{|\int_0^1 [\partial_u \phi_n(u+th, v) - \partial_u \phi_n(u, v)]h dt|}{|h|} \leq \int_0^1 |\partial_u \phi_n(u+th, v) - \partial_u \phi_n(u, v)| dt. \end{aligned}$$

Now we let $n \rightarrow \infty$ to obtain:

$$\frac{|\phi(u+h, v) - \phi(u, v) - F(u, v)h|}{|h|} \leq \int_0^1 |F(u+th, v) - F(u, v)| dt$$

The above inequality and the continuity of F imply that $\partial_u \phi(u, v) = F(u, v)$.

If we let $u_2 = 0$, $u_1 = u$ in (8) we obtain:

$$|\partial_v \phi_n(u, v) - \partial_v \phi_m(u, v)| \leq \gamma |u|,$$

which implies that $\partial_v \phi_n(u, v)$ is a Cauchy sequence. Therefore there exists a continuous function H , such that $\partial_v \phi_n(u, v) \rightarrow H(u, v)$, as $n \rightarrow \infty$, uniformly in B_ε .

As above one can show that $\partial_v \phi(u, v) = H(u, v)$.

To prove that $\phi \in E$, we let $n = n_0$ and $m \rightarrow \infty$ in (7), (8), (9), to obtain:

$$|\partial_u(\phi_{n_0} - \phi)(u_1, v_1) - \partial_u(\phi_{n_0} - \phi)(u_2, v_2)| \leq \gamma(|u_1 - u_2| + |v_1 - v_2|)$$

$$|\partial_v(\phi_{n_0} - \phi)(u_1, v) - \partial_v(\phi_{n_0} - \phi)(u_2, v)| \leq \gamma(|u_1 - u_2|)$$

$$|\partial_v(\phi_{n_0} - \phi)(u, v) - \partial_v(\phi_{n_0} - \phi)(u, 0)| \leq \gamma(|u|^\eta |v|^\beta)$$

The above inequalities imply that $\phi_{n_0} - \phi \in E$ and so $\phi \in E$.

To prove that $\phi_n \rightarrow \phi$ in E , we let $m \rightarrow \infty$ in (7), (8), (9), to obtain:

$$|\partial_u(\phi_n - \phi)(u_1, v_1) - \partial_u(\phi_n - \phi)(u_2, v_2)| \leq \gamma(|u_1 - u_2| + |v_1 - v_2|)$$

$$|\partial_v(\phi_n - \phi)(u_1, v) - \partial_v(\phi_n - \phi)(u_2, v)| \leq \gamma(|u_1 - u_2|)$$

$$|\partial_v(\phi_n - \phi)(u, v) - \partial_v(\phi_n - \phi)(u, 0)| \leq \gamma(|u|^\eta |v|^\beta)$$

The above inequalities imply that $\phi_n \rightarrow \phi$ in E and this completes the proof that E is a Banach space. ■

Lemma 11 *Suppose U satisfies (6) and $\|U\|_1, \|U\|_2 < \infty$. Then for $(u_1, v_1), (u_2, v_2), (u, v)$ in B_ε the following hold:*

$$(i) |U(u, v)| \leq \|U\|_1 \left(\frac{|u|^2}{2} + |u||v| \right)$$

$$(ii) |U(u, v) - U(u, 0)| \leq \|U\|_2 |u||v|$$

$$(iii) |U(u_1, v_1) - U(u_2, v_2)| \leq$$

$$\frac{1}{2} [\|U\|_1 |u_1 - u_2| (|u_1| + |v_1| + |u_2| + |v_2|) + \|U\|_2 |v_1 - v_2| (|u_1| + |u_2|)]$$

Proof:

(i) and (ii) follow easily from (iii).

$$\begin{aligned} |U(u_1, v_1) - U(u_2, v_2)| &= \left| \int_0^1 \frac{d}{dt} U(tu_1 + (1-t)u_2, tv_1 + (1-t)v_2) dt \right| \leq \\ &\int_0^1 |\partial_u U(tu_1 + (1-t)u_2, tv_1 + (1-t)v_2)| |u_1 - u_2| dt + \\ &\int_0^1 |\partial_v U(tu_1 + (1-t)u_2, tv_1 + (1-t)v_2)| |v_1 - v_2| dt \leq \end{aligned}$$

$$\begin{aligned}
& \|U\|_1 |u_1 - u_2| \int_0^1 (t|u_1| + (1-t)|u_2| + t|v_1| + (1-t)|v_2|) dt + \\
& |v_1 - v_2| \int_0^1 |\partial_v U(tu_1 + (1-t)u_2, tv_1 + (1-t)v_2) - \partial_v U(0, tv_1 + (1-t)v_2)| dt \leq \\
& \frac{1}{2} \|U\|_1 |u_1 - u_2| (|u_1| + |u_2| + |v_1| + |v_2|) + \|U\|_2 |v_1 - v_2| \int_0^1 (t|u_1| + (1-t)|u_2|) dt = \\
& \frac{1}{2} [\|U\|_1 |u_1 - u_2| (|u_1| + |u_2| + |v_1| + |v_2|) + \|U\|_2 |v_1 - v_2| (|u_1| + |u_2|)] \quad \blacksquare
\end{aligned}$$

Now we replace u and v by (x, y) and z , respectively. Let \mathbb{X} , \mathbb{Y} , \mathbb{Z} and \mathbb{W} be Banach spaces, B_ε be the ball of radius ε and let $\psi : B_\varepsilon \subset \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \rightarrow \mathbb{W}$ be a C^1 function such that

$$\psi(0, 0, z) \equiv 0, \quad \partial_{(x,y)}\psi = 0, \quad \text{at } (0, 0, 0). \quad (10)$$

We define $\|\psi\| := \max\{\|\psi\|_1, \|\psi\|_2, \|\psi\|_3\}$, where:

$$\begin{aligned}
\|\psi\|_1 &:= \sup \frac{|\partial_{(x,y)}\psi(x_1, y_1, z_1) - \partial_{(x,y)}\psi(x_2, y_2, z_2)|}{|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|}, \\
\|\psi\|_2 &:= \sup \frac{|\partial_z\psi(x_1, y_1, z) - \partial_z\psi(x_2, y_2, z)|}{|x_1 - x_2| + |y_1 - y_2|}, \\
\|\psi\|_3 &:= \sup \frac{|\partial_z\psi(x, y, z) - \partial_z\psi(x, y, 0)|}{(|x| + |y|)^\eta |z|^\beta},
\end{aligned}$$

where we require that the denominators of the above fractions do not vanish, that is, for $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$, $(x_1, y_1) \neq (x_2, y_2)$, $(x, y) \neq (0, 0)$, $z \neq 0$, (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_1, y_1, z) , (x_2, y_2, z) , (x, y, z) and $(x, y, 0)$ in B_ε .

Let $E = E_{\eta, \beta, \varepsilon}$ be the space of the functions $\phi : B_\varepsilon \rightarrow \mathbb{Y}$, such that (10) holds and $\|\phi\| < \infty$.

Theorem 12 *Let $A \in L(\mathbb{X})$, $B, B^{-1} \in L(\mathbb{Y})$, $C \in L(\mathbb{Z})$, such that $|A| \leq a$, $|B| \leq b_1$, $|B^{-1}| \leq \frac{1}{b_2}$, $|C| \leq c$, where $a < b_1 < c < 1$ and $\frac{b_1 c}{b_2} < 1$.*

Let $(\mathcal{X}, \mathcal{Y}) : B_\varepsilon \rightarrow \mathbb{X} \times \mathbb{Y}$ such that $\|\mathcal{X}\|, \|\mathcal{Y}\| < \infty$.

Then there exist $\varepsilon_0 > 0$, $0 < \beta_0 < \eta_0 < 1$, $\beta_0 + \eta_0 < 1$ such that the functional equation:

$$\phi(x, y, z) = B^{-1}\phi(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz) - B^{-1}\mathcal{Y}(x, y, z) \quad (11)$$

has a unique solution in $E = E_{\eta, \beta, \varepsilon}$, if $\varepsilon < \varepsilon_0$, $\beta < \beta_0$, $\eta_0 < \eta < 1$ and $\eta + \beta < 1$.

Remark 13 The very important role of this functional equation in the proof of Theorem 1 will be seen later. But the need of the hypothesis $b_1 c / b_2 < 1$ will appear clearly along the next proof. This hypothesis is the origin of the nonresonance condition appearing in Theorem 1.

Proof:

We consider the linear operator defined as:

$$(T\phi)(x, y, z) := B^{-1}\phi(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz)$$

In the next calculation we are going to show that there exist $\varepsilon > 0$, $0 < \beta < \eta < 1$, such that the space E is invariant under T and that T is a contraction.

For the norm estimates that will follow we will use Lemma 11 and also the following easy consequences:

$$|\mathcal{X}(x, y, z)| \leq 2\varepsilon^2 \|\mathcal{X}\|_1,$$

$$|\mathcal{Y}(x, y, z)| \leq 2\varepsilon^2 \|\mathcal{Y}\|_1,$$

$$|\mathcal{X}(x_1, y_1, z_1) - \mathcal{X}(x_2, y_2, z_2)| \leq \varepsilon[\|\mathcal{X}\|_1 + \|\mathcal{X}\|_2](|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|),$$

$$|\mathcal{Y}(x_1, y_1, z_1) - \mathcal{Y}(x_2, y_2, z_2)| \leq \varepsilon[\|\mathcal{Y}\|_1 + \|\mathcal{Y}\|_2](|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|).$$

Let us consider first $\|T\phi\|_1$.

$$\begin{aligned} \partial_{(x,y)}(T\phi)(x, y, z) &= B^{-1}(\partial_{(x,y)}\phi)(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz) \\ &\quad \cdot \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \partial_{(x,y)} \begin{pmatrix} \mathcal{X}(x, y, z) \\ \mathcal{Y}(x, y, z) \end{pmatrix} \right] \\ |\partial_{(x,y)}(T\phi)(x_1, y_1, z_1) - \partial_{(x,y)}(T\phi)(x_2, y_2, z_2)| &= \\ &|B^{-1}(\partial_{(x,y)}\phi)(Ax_1 + \mathcal{X}(x_1, y_1, z_1), By_1 + \mathcal{Y}(x_1, y_1, z_1), Cz_1) \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \partial_{(x,y)} \begin{pmatrix} \mathcal{X}(x_1, y_1, z_1) \\ \mathcal{Y}(x_1, y_1, z_1) \end{pmatrix} \right] \\ &- B^{-1}(\partial_{(x,y)}\phi)(Ax_2 + \mathcal{X}(x_2, y_2, z_2), By_2 + \mathcal{Y}(x_2, y_2, z_2), Cz_2) \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \partial_{(x,y)} \begin{pmatrix} \mathcal{X}(x_2, y_2, z_2) \\ \mathcal{Y}(x_2, y_2, z_2) \end{pmatrix} \right]| \leq \\ &|B^{-1}| |(\partial_{(x,y)}\phi)(Ax_1 + \mathcal{X}(x_1, y_1, z_1), By_1 + \mathcal{Y}(x_1, y_1, z_1), Cz_1)| \left| \partial_{(x,y)} \begin{pmatrix} \mathcal{X}(x_1, y_1, z_1) \\ \mathcal{Y}(x_1, y_1, z_1) \end{pmatrix} - \partial_{(x,y)} \begin{pmatrix} \mathcal{X}(x_2, y_2, z_2) \\ \mathcal{Y}(x_2, y_2, z_2) \end{pmatrix} \right| + \\ &|B^{-1}| |(\partial_{(x,y)}\phi)(Ax_1 + \mathcal{X}(x_1, y_1, z_1), By_1 + \mathcal{Y}(x_1, y_1, z_1), Cz_1) - (\partial_{(x,y)}\phi)(Ax_2 + \mathcal{X}(x_2, y_2, z_2), By_2 + \mathcal{Y}(x_2, y_2, z_2), Cz_2)| \cdot \\ &\quad \cdot \left| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \partial_{(x,y)} \begin{pmatrix} \mathcal{X}(x_2, y_2, z_2) \\ \mathcal{Y}(x_2, y_2, z_2) \end{pmatrix} \right|. \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{|\partial_{(x,y)}(T\phi)(x_1, y_1, z_1) - \partial_{(x,y)}(T\phi)(x_2, y_2, z_2)|}{|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|} \leq \\ &|B^{-1}| \|\phi\|_1 [|A||x_1| + |B||y_1| + |C||z_1| + |\mathcal{X}(x_1, y_1, z_1)| + |\mathcal{Y}(x_1, y_1, z_1)|] [\|\mathcal{X}\|_1 + \|\mathcal{Y}\|_1] + \\ &|B^{-1}| \|\phi\|_1 \left[\frac{|A||x_1 - x_2| + |B||y_1 - y_2| + |C||z_1 - z_2|}{|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|} + \varepsilon(\|\mathcal{X}\|_1 + \|\mathcal{X}\|_2 + \|\mathcal{Y}\|_1 + \|\mathcal{Y}\|_2) \right]. \end{aligned}$$

$$\begin{aligned} & \cdot \left[\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\| + (\|\mathcal{X}\|_1 + \|\mathcal{Y}\|_1)\varepsilon \right] \leq \\ & \varepsilon |B^{-1}| [\|\mathcal{X}\|_1 + \|\mathcal{Y}\|_1] \|\phi\|_1 [|C| + 2[\|\mathcal{X}\|_1 + \|\mathcal{Y}\|_1]\varepsilon] + |B^{-1}| \|\phi\|_1 [|C| + O(\varepsilon)] \cdot \\ & \cdot [|B| + (\|\mathcal{X}\|_1 + \|\mathcal{Y}\|_1)\varepsilon] \leq \|\phi\|_1 [|B^{-1}| |C| |B| + O(\varepsilon)]. \end{aligned}$$

Our conclusion is that:

$$\|T\phi\|_1 \leq \|\phi\|_1 \left[|B^{-1}| |C| |B| + O(\varepsilon) \right] \leq \|\phi\|_1 \left[\frac{c b_1}{b_2} + O(\varepsilon) \right]. \quad (12)$$

Let us consider now $\|T\phi\|_2$.

$$\partial_z(T\phi)(x, y, z) =$$

$$B^{-1}[\partial_{(x,y)}\phi(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz) \begin{pmatrix} \partial_z \mathcal{X}(x, y, z) \\ \partial_z \mathcal{Y}(x, y, z) \end{pmatrix} + (\partial_z \phi)(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz)C].$$

$$|\partial_z(T\phi)(x_1, y_1, z) - \partial_z(T\phi)(x_2, y_2, z)| \leq$$

$$|B^{-1}(\partial_{(x,y)}\phi)(Ax_1 + \mathcal{X}(x_1, y_1, z), By_1 + \mathcal{Y}(x_1, y_1, z), Cz) \begin{pmatrix} \partial_z \mathcal{X}(x_1, y_1, z) \\ \partial_z \mathcal{Y}(x_1, y_1, z) \end{pmatrix} +$$

$$B^{-1}(\partial_z \phi)(Ax_1 + \mathcal{X}(x_1, y_1, z), By_1 + \mathcal{Y}(x_1, y_1, z), Cz)C$$

$$- B^{-1}(\partial_{(x,y)}\phi)(Ax_2 + \mathcal{X}(x_2, y_2, z), By_2 + \mathcal{Y}(x_2, y_2, z), Cz) \begin{pmatrix} \partial_z \mathcal{X}(x_2, y_2, z) \\ \partial_z \mathcal{Y}(x_2, y_2, z) \end{pmatrix}$$

$$- B^{-1}(\partial_z \phi)(Ax_2 + \mathcal{X}(x_2, y_2, z), By_2 + \mathcal{Y}(x_2, y_2, z), Cz)C|$$

$$\leq |B^{-1}| |(\partial_{(x,y)}\phi)(Ax_1 + \mathcal{X}(x_1, y_1, z), By_1 + \mathcal{Y}(x_1, y_1, z), Cz)| \left| \begin{pmatrix} \partial_z \mathcal{X}(x_1, y_1, z) \\ \partial_z \mathcal{Y}(x_1, y_1, z) \end{pmatrix} - \begin{pmatrix} \partial_z \mathcal{X}(x_2, y_2, z) \\ \partial_z \mathcal{Y}(x_2, y_2, z) \end{pmatrix} \right| +$$

$$|B^{-1}| |(\partial_{(x,y)}\phi)(Ax_1 + \mathcal{X}(x_1, y_1, z), By_1 + \mathcal{Y}(x_1, y_1, z), Cz) - (\partial_{(x,y)}\phi)(Ax_2 + \mathcal{X}(x_2, y_2, z), By_2 + \mathcal{Y}(x_2, y_2, z), Cz)| \cdot$$

$$\cdot \left| \begin{pmatrix} \partial_z \mathcal{X}(x_2, y_2, z) \\ \partial_z \mathcal{Y}(x_2, y_2, z) \end{pmatrix} \right| +$$

$$|B^{-1}| |(\partial_z \phi)(Ax_1 + \mathcal{X}(x_1, y_1, z), By_1 + \mathcal{Y}(x_1, y_1, z), Cz) - (\partial_z \phi)(Ax_2 + \mathcal{X}(x_2, y_2, z), By_2 + \mathcal{Y}(x_2, y_2, z), Cz)| |C| \leq$$

$$|B^{-1}| \|\phi\|_1 (|Ax_1 + \mathcal{X}(x_1, y_1, z)| + |By_1 + \mathcal{Y}(x_1, y_1, z)| + |Cz|) (\|\mathcal{X}\|_2 + \|\mathcal{Y}\|_2) (|x_1 - y_1| + |x_2 - y_2|) +$$

$$|B^{-1}| \|\phi\|_1 [|A| |x_1 - x_2| + |B| |y_1 - y_2| + |\mathcal{X}(x_1, y_1, z) - \mathcal{X}(x_2, y_2, z)| + |\mathcal{Y}(x_1, y_1, z) - \mathcal{Y}(x_2, y_2, z)|] (\|\mathcal{X}\|_2 + \|\mathcal{Y}\|_2) \varepsilon +$$

$$|B^{-1}| \|\phi\|_2 [|A| |x_1 - x_2| + |B| |y_1 - y_2| + |\mathcal{X}(x_1, y_1, z) - \mathcal{X}(x_2, y_2, z)| + |\mathcal{Y}(x_1, y_1, z) - \mathcal{Y}(x_2, y_2, z)|] |C| \leq$$

$$|B^{-1}| \|\phi\|_1 (O(\varepsilon)) (\|\mathcal{X}\|_2 + \|\mathcal{Y}\|_2) (|x_1 - y_1| + |x_2 - y_2|) +$$

$$\begin{aligned}
& |B^{-1}| \|\phi\|_1 |B| [|x_1 - x_2| + |y_1 - y_2|] + \varepsilon (\|\mathcal{X}\|_1 + \|\mathcal{Y}\|_1) (|x_1 - x_2| + |y_1 - y_2|) [\|\mathcal{X}\|_2 + \|\mathcal{Y}\|_2] \varepsilon + \\
& |B^{-1}| |C| \|\phi\|_2 [|B| + O(\varepsilon)] [|x_1 - x_2| + |y_1 - y_2|] \leq \\
& [|B^{-1}| |C| |B| + O(\varepsilon)] \|\phi\|_2 + O(\varepsilon) \|\phi\|_1 [|x_1 - x_2| + |y_1 - y_2|]
\end{aligned}$$

Therefore,

$$\|T\phi\|_2 \leq [|B^{-1}| |C| |B| + O(\varepsilon)] \|\phi\|_2 + O(\varepsilon) \|\phi\|_1 \leq \left[\frac{c b_1}{b_2} + O(\varepsilon) \right] \|\phi\|_2 + O(\varepsilon) \|\phi\|_1. \quad (13)$$

Let us consider now $\|T\phi\|_3$.

$$(T\phi)(x, y, z) := B^{-1} \phi(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz)$$

$$\begin{aligned}
\partial_z(T\phi)(x, y, z) &= B^{-1} (\partial_{(x,y)} \phi)(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz) \begin{pmatrix} \partial_z \mathcal{X}(x, y, z) \\ \partial_z \mathcal{Y}(x, y, z) \end{pmatrix} + \\
& B^{-1} (\partial_z \phi)(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz) C.
\end{aligned}$$

$$\begin{aligned}
& |\partial_z(T\phi)(x, y, z) - \partial_z(T\phi)(x, y, 0)| = \\
& |B^{-1} (\partial_{(x,y)} \phi)(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz) \begin{pmatrix} \partial_z \mathcal{X}(x, y, z) \\ \partial_z \mathcal{Y}(x, y, z) \end{pmatrix} + B^{-1} (\partial_z \phi)(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz) C \\
& - B^{-1} (\partial_{(x,y)} \phi)(Ax + \mathcal{X}(x, y, 0), By + \mathcal{Y}(x, y, 0), 0) \begin{pmatrix} \partial_z \mathcal{X}(x, y, 0) \\ \partial_z \mathcal{Y}(x, y, 0) \end{pmatrix} - B^{-1} (\partial_z \phi)(Ax + \mathcal{X}(x, y, 0), By + \mathcal{Y}(x, y, 0), 0) C| \\
& \leq |B^{-1}| |(\partial_{(x,y)} \phi)(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz)| \left| \begin{pmatrix} \partial_z \mathcal{X}(x, y, z) \\ \partial_z \mathcal{Y}(x, y, z) \end{pmatrix} - \begin{pmatrix} \partial_z \mathcal{X}(x, y, 0) \\ \partial_z \mathcal{Y}(x, y, 0) \end{pmatrix} \right| + \\
& |B^{-1}| |(\partial_{(x,y)} \phi)(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz) - (\partial_{(x,y)} \phi)(Ax + \mathcal{X}(x, y, 0), By + \mathcal{Y}(x, y, 0), 0)| \cdot \\
& \quad \cdot \left| \begin{pmatrix} \partial_z \mathcal{X}(x, y, 0) \\ \partial_z \mathcal{Y}(x, y, 0) \end{pmatrix} \right| + \\
& |B^{-1}| |(\partial_z \phi)(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz) - (\partial_z \phi)(Ax + \mathcal{X}(x, y, 0), By + \mathcal{Y}(x, y, 0), Cz)| |C| + \\
& |B^{-1}| |(\partial_z \phi)(Ax + \mathcal{X}(x, y, 0), By + \mathcal{Y}(x, y, 0), Cz) - (\partial_z \phi)(Ax + \mathcal{X}(x, y, 0), By + \mathcal{Y}(x, y, 0), 0)| |C| \leq \\
& |B^{-1}| \|\phi\|_1 [|A||x| + |\mathcal{X}(x, y, z)| + |B||y| + |\mathcal{Y}(x, y, z)| + |C||z|] [\|\mathcal{X}\|_3 + \|\mathcal{Y}\|_3] [(|x| + |y|)^\eta |z|^\beta] + \\
& |B^{-1}| \|\phi\|_1 [|\mathcal{X}(x, y, z) - \mathcal{X}(x, y, 0)| + |\mathcal{Y}(x, y, z) - \mathcal{Y}(x, y, 0)| + |C||z|] [\|\mathcal{X}\|_2 + \|\mathcal{Y}\|_2] [|x| + |y|] + \\
& |B^{-1}| \|\phi\|_2 [|\mathcal{X}(x, y, z) - \mathcal{X}(x, y, 0)| + |\mathcal{Y}(x, y, z) - \mathcal{Y}(x, y, 0)|] |C| + \\
& |B^{-1}| \|\phi\|_3 [|A||x| + |\mathcal{X}(x, y, 0)| + |B||y| + |\mathcal{Y}(x, y, 0)|]^\eta |Cz|^\beta |C| \leq \\
& O(\varepsilon) \|\phi\|_1 [(|x| + |y|)^\eta |z|^\beta] + \\
& |B^{-1}| \|\phi\|_1 [\varepsilon (\|\mathcal{X}\|_2 + \|\mathcal{Y}\|_2) |z|^{1-\beta} + |C||z|^{1-\beta}] [\|\mathcal{X}\|_2 + \|\mathcal{Y}\|_2] [|x| + |y|]^{1-\eta} [|x| + |y|]^\eta |z|^\beta + \\
& |B^{-1}| |C| \|\phi\|_2 [\|\mathcal{X}\|_2 + \|\mathcal{Y}\|_2] [|x| + |y|] |z| +
\end{aligned}$$

$$|B^{-1}| |C|^\beta \|\phi\|_3 [|B| + O(\varepsilon)]^\eta |C| [|x| + |y|]^\eta |z|^\beta \leq$$

$$O(\varepsilon) \|\phi\|_1 [|x| + |y|]^\eta |z|^\beta + O(\varepsilon) \|\phi\|_2 [|x| + |y|]^\eta |z|^\beta + |B^{-1}| |C|^{1+\beta} \|\phi\|_3 [|B| + O(\varepsilon)]^\eta (|x| + |y|)^\eta |z|^\beta \leq$$

$$[|B^{-1}| |C|^{1+\beta} (|B| + O(\varepsilon))^\eta \|\phi\|_3 + O(\varepsilon) \|\phi\|_2 + O(\varepsilon) \|\phi\|_1] (|x| + |y|)^\eta |z|^\beta,$$

or in other words,

$$\begin{aligned} \|T\phi\|_3 &\leq |B^{-1}| |C|^{1+\beta} (|B| + O(\varepsilon))^\eta \|\phi\|_3 + O(\varepsilon) \|\phi\|_2 + O(\varepsilon) \|\phi\|_1 \leq \\ &\frac{c^{1+\beta}}{b_2} (b_1 + O(\varepsilon))^\eta \|\phi\|_3 + O(\varepsilon) \|\phi\|_2 + O(\varepsilon) \|\phi\|_1. \end{aligned} \quad (14)$$

Now, if we take $1 - \eta$ and ε sufficiently small, then β will be small and the assumptions of our theorem and inequalities (12), (13) and (14) imply that T is a contraction. \blacksquare

4 The induction. Proof of The Linearization Theorem.

The hypotheses of Theorem 1 imply that $\mathbb{X} = \mathbb{X}_n \times \cdots \times \mathbb{X}_1$ and $A = \text{diag}(A_n, \cdots, A_1)$, with $A_i \in L(\mathbb{X}_i)$ and $|\sigma(A_i)| \subset (\nu_i^-, \nu_i^+)$, $i = 1, \cdots, n$. Observe that because of Lemma 7 we have that

$$|A_i| < \nu_i^+, \quad |A_i^{-1}| < \frac{1}{\nu_i^-}, \quad i = 1, \cdots, n. \quad (15)$$

with a suitable equivalent norm.

Observe also that the Banach spaces \mathbb{X}_i inherit from \mathbb{X} the $C^{1,1}$ regularity property (1).

The proof of Theorem 1 follows the same procedure as that of Hartman in [4], for the finite dimensional case. It consists of an induction, by linearizing one component at the time. Without loss of generality, we can assume that the last component, ($i = 1$), is linear. This can always be achieved by introducing a new artificial component.

For the general induction step it is sufficient to consider only three components. Therefore, let \mathbb{X} , \mathbb{Y} , \mathbb{Z} be Banach spaces and $A \in L(\mathbb{X})$, $B \in L(\mathbb{Y})$, $C \in L(\mathbb{Z})$ be bounded linear operators, such that $|A| < a$, $|B| < b_1$, $|B^{-1}| < \frac{1}{b_2}$, $|C| < c$, where $a < b_1 < c < 1$ and $\frac{b_1 c}{b_2} < 1$. Let $(\mathcal{X}, \mathcal{Y}) : B_\varepsilon \subset \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \rightarrow \mathbb{X} \times \mathbb{Y}$. We suppose that the Banach space $\mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$ has the $C^{1,1}$ regularity property (1).

Induction Hypothesis.

Suppose A, B, C satisfy the above conditions. Consider the local map $T : (x, y, z) \mapsto (x^1, y^1, z^1)$, where

$$x^1 = Ax + \mathcal{X}(x, y, z), \quad y^1 = By + \mathcal{Y}(x, y, z), \quad z^1 = Cz, \quad (16)$$

- (i) \mathcal{X}, \mathcal{Y} are C^1 -functions in a neighborhood of the origin and \mathcal{X}, \mathcal{Y} are $o(|x| + |y| + |z|)$ as $(x, y, z) \rightarrow (0, 0, 0)$;
- (ii) $\partial_x \mathcal{X}, \partial_y \mathcal{X}, \partial_x \mathcal{Y}, \partial_y \mathcal{Y}$ are Lipschitz continuous with respect to (x, y, z) ;
- (iii) $\partial_z \mathcal{X}, \partial_z \mathcal{Y}$ are uniformly Lipschitz continuous with respect to (x, y) ;
- (iv) $\partial_z \mathcal{X}, \partial_z \mathcal{Y}$ are uniformly Hölder continuous with respect to z at $z = 0$, with exponent $\delta_1 \in (0, 1]$, that is,

$$\sup_{z \neq 0} \frac{|\partial_z \mathcal{X}(x, y, z) - \partial_z \mathcal{X}(x, y, 0)|}{|z|^{\delta_1}} < \infty, \quad \sup_{z \neq 0} \frac{|\partial_z \mathcal{Y}(x, y, z) - \partial_z \mathcal{Y}(x, y, 0)|}{|z|^{\delta_1}} < \infty.$$

- (v) $\partial_z \mathcal{X}, \partial_z \mathcal{Y}$ are $O(|x| + |y| + |z|)$, as $|x| + |y| + |z| \rightarrow 0$.

Induction Assertion. There exists a local map $R : (x, y, z) \mapsto (u, v, w)$, such that,

$$u = x - x(z), \quad v = y - y(x, y, z), \quad w = z \quad (17)$$

where $x(z), y(x, y, z)$ are of class C^1 in a neighborhood of the origin, are $o(|z|), o(|x| + |y| + |z|)$ as $z \rightarrow 0, (x, y, z) \rightarrow (0, 0, 0)$, respectively, and R is such that $RT R^{-1}$ has the form:

$$RT R^{-1} : u^1 = Au + \mathcal{U}(u, v, w), \quad v^1 = Bv, \quad w^1 = Cw, \quad (18)$$

where,

- (i) \mathcal{U} is a C^1 -function in a neighborhood of the origin and \mathcal{U} is $o(|u| + |v| + |w|)$ as $(u, v, w) \rightarrow (0, 0, 0)$;
- (ii) $\partial_u \mathcal{U}$ is Lipschitz continuous with respect to (u, v, w) ;
- (iii) $\partial_v \mathcal{U}, \partial_w \mathcal{U}$ are uniformly Lipschitz continuous with respect to u ;
- (iv) $\partial_v \mathcal{U}, \partial_w \mathcal{U}$ are uniformly Hölder continuous with respect to (v, w) , at $(v, w) = (0, 0)$, with exponent $\delta_2 \in (0, 1]$.
- (v) $\partial_{(v,w)} \mathcal{U}$ is $O(|u| + |v| + |w|)$, as $|u| + |v| + |w| \rightarrow 0$.

For the proof of Theorem 1 we only have to show that the Induction Hypothesis implies the Induction Assertion (perhaps with $0 < \delta_2 \leq \delta_1$).

First of all, we extend our local functions \mathcal{X}, \mathcal{Y} to functions defined on the whole space $\mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$ by writing:

$$\mathcal{X}_r(x, y, z) := \phi(x/r, y/r, z/r) \mathcal{X}(x, y, z).$$

where $r > 0$ is small enough and ϕ is the $C^{1,1}$ function given by (1). Observe that $\mathcal{X}_r(x, y, z) = \mathcal{X}(x, y, z)$ if $|x| + |y| + |z| \leq \frac{r}{2}$ and that $\mathcal{X}_r(x, y, z) = 0$ if $|x| + |y| + |z| \geq r$. Similarly for \mathcal{Y} and \mathcal{Y}_r .

We want to apply Theorem 8 to the map:

$$\begin{cases} x^1 = Ax + \mathcal{X}_r(x, y, z) \\ y^1 = By + \mathcal{Y}_r(x, y, z) \\ z^1 = Cz \end{cases}$$

to obtain an invariant manifold of the form $x = x(z)$, $y = y(z)$. For this we have to check that the quantities :

$$\sup_{(x_1, y_1, z_1) \neq (x_2, y_2, z_2)} \frac{|\partial_{(x,y)} \mathcal{X}_r(x_1, y_1, z_1) - \partial_{(x,y)} \mathcal{X}_r(x_2, y_2, z_2)|}{|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|},$$

$$\sup_{(x_1, y_1) \neq (x_2, y_2)} \frac{|\partial_z \mathcal{X}_r(x_1, y_1, z_1) - \partial_z \mathcal{X}_r(x_2, y_2, z_2)|}{|x_1 - x_2| + |y_1 - y_2|},$$

$$\sup_{z \neq 0} \frac{|\partial_z \mathcal{X}_r(x, y, z) - \partial_z \mathcal{X}_r(x, y, 0)|}{|z|^{\delta_1}}$$

remain bounded as $r \rightarrow 0$. This is needed because Theorem 8 requires the nonlinearities to vanish outside a ball of radius sufficiently small.

The computation of these quantities is straightforward, and will not be presented here. We only want to point out that the hypothesis (v) in the set of induction hypotheses plays a fundamental role here. Also we emphasize that here is the only point where it will be used.

So from Theorem 8 it follows that there exists a invariant manifold M , invariant under T of the form:

$$M : x = x(z), y = y(z) \tag{19}$$

where $x(z)$, $y(z)$ are of class C^1 in a neighborhood of the origin, are $o(|z|)$ as $z \rightarrow 0$, and have Hölder continuous derivative, with exponent $\delta < \delta_1$, at $z = 0$.

Then we consider the following preliminary change of variables $R : (x, y, z) \mapsto (u, v, w)$,

$$u = x - x(z), v = y - y(z), w = z \tag{20}$$

.

A simple calculation shows that $RTR^{-1} : (u, v, w) \mapsto (u^1, v^1, w^1)$ has the form:

$$\begin{cases} u^1 = Au + Ax(w) + \mathcal{X}(u + x(w), v + y(w), w) - x(Cw) \\ v^1 = Bv + By(w) + \mathcal{Y}(u + x(w), v + y(w), w) - y(Cw) \\ w^1 = Cw \end{cases}, \quad (21)$$

where we have written \mathcal{X} and \mathcal{Y} instead of \mathcal{X}_r and \mathcal{Y}_r , respectively.

Since $M : x = x(z), y = y(z)$ is an invariant manifold, then (3) shows that RTR^{-1} has the form:

$$\begin{cases} u^1 = Au + \mathcal{X}^*(u, v, w) \\ v^1 = Bv + \mathcal{Y}^*(u, v, w) \\ w^1 = Cw \end{cases} \quad (22)$$

where

$$\begin{aligned} \mathcal{X}^*(u, v, w) &= \mathcal{X}(u + x(w), v + y(w), w) - \mathcal{X}(x(w), y(w), w), \\ \mathcal{Y}^*(u, v, w) &= \mathcal{Y}(u + x(w), v + y(w), w) - \mathcal{Y}(x(w), y(w), w), \end{aligned} \quad (23)$$

and we see that $\mathcal{X}^*(0, 0, w) = 0, \mathcal{Y}^*(0, 0, w) = 0$

The next calculation shows that with this change of variables the new functions \mathcal{X}^* and \mathcal{Y}^* will be $C^{(1,1)}$ with respect to the two first variables, but they can loose some regularity with respect to the third variable. Precisely, $\|\mathcal{X}^*\|_3, \|\mathcal{Y}^*\|_3 < \infty$ in $E_{\eta, \delta(1-\eta), \varepsilon}$, for all $0 < \eta \leq 1$, with the number δ defined above and ε sufficient small.

First we point out that $x(w), y(w)$ are Lipschitz continuous in w . In fact,

$$|x(w_1) - x(w_2)| = \left| \int_0^1 \frac{d}{d\theta} x(\theta w_1 + (1-\theta)w_2) d\theta \right| \leq \int_0^1 |x'(\theta w_1 + (1-\theta)w_2)| d\theta |w_1 - w_2|$$

Since $\partial_{(x,y)}\mathcal{X}, \partial_{(x,y)}\mathcal{Y}$ are Lipschitz continuous in (x, y, z) it follows that $\partial_{(u,v)}\mathcal{X}^*, \partial_{(u,v)}\mathcal{Y}^*$ are Lipschitz continuous in (u, v, w) :

$$\partial_{(u,v)}\mathcal{X}^*(u, v, w) = \partial_{(x,y)}\mathcal{X}(u + x(w), v + y(w), w).$$

Therefore,

$$\begin{aligned} &|\partial_{(u,v)}\mathcal{X}^*(u_1, v_1, w_1) - \partial_{(u,v)}\mathcal{X}^*(u_2, v_2, w_2)| = \\ &|\partial_{(x,y)}\mathcal{X}(u_1 + x(w_1), v_1 + y(w_1), w_1) - \partial_{(x,y)}\mathcal{X}(u_2 + x(w_2), v_2 + y(w_2), w_2)| \leq \end{aligned}$$

$$M_1(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2| + |x(w_1) - x(w_2)| + |y(w_1) - y(w_2)|).$$

Also

$$\begin{aligned} \partial_w \mathcal{X}^*(u, v, w) &= [\partial_x \mathcal{X}(u + x(w), v + y(w), w) - \partial_x \mathcal{X}(x(w), y(w), w)]x'(w) + \\ &\quad [\partial_y \mathcal{X}(u + x(w), v + y(w), w) - \partial_y \mathcal{X}(x(w), y(w), w)]y'(w) + \\ &\quad \partial_z \mathcal{X}(u + x(w), v + y(w), w) - \partial_z \mathcal{X}(x(w), y(w), w) \end{aligned} \quad (24)$$

and therefore $\partial_w \mathcal{X}^*$ is uniformly Lipschitz continuous in (u, v) and uniformly Hölder continuous in w , at $w = 0$:

$$\begin{aligned} &|\partial_w \mathcal{X}^*(u_1, v_1, w) - \partial_w \mathcal{X}^*(u_2, v_2, w)| \leq \\ &|\partial_x \mathcal{X}(u_1 + x(w), v_1 + y(w), w) - \partial_x \mathcal{X}(u_2 + x(w), v_2 + y(w), w)||x'(w)| + \\ &|\partial_y \mathcal{X}(u_1 + x(w), v_1 + y(w), w) - \partial_y \mathcal{X}(u_2 + x(w), v_2 + y(w), w)||y'(w)| + \\ &|\partial_z \mathcal{X}(u_1 + x(w), v_1 + y(w), w) - \partial_z \mathcal{X}(u_2 + x(w), v_2 + y(w), w)| \leq \\ &M_1(|u_1 - u_2| + |v_1 - v_2|)(|x'(w)| + |y'(w)|) + M_2(|u_1 - u_2| + |v_1 - v_2|). \end{aligned}$$

Therefore $\partial_w \mathcal{X}^*(u, v, w)$ is uniformly Lipschitz continuous on (u, v) .

To prove that $\partial_w \mathcal{X}^*(u, v, w)$ is uniformly Hölder continuous on w , at $w = 0$, it is enough to prove that each adding term from (24) has this property.

It is obvious that the two first terms of (24) have this property. Next we verify that this property holds for the third term:

$$\begin{aligned} &|\partial_z \mathcal{X}(u + x(w), v + y(w), w) - \partial_z \mathcal{X}(x(w), y(w), w) - \partial_z \mathcal{X}(u, v, 0)| \leq \\ &|\partial_z \mathcal{X}(u + x(w), v + y(w), w) - \partial_z \mathcal{X}(u, v, w)| + |\partial_z \mathcal{X}(u, v, w) - \partial_z \mathcal{X}(u, v, 0)| + \\ &|\partial_z \mathcal{X}(x(w), y(w), w) - \partial_z \mathcal{X}(x(w), y(w), 0)| + |\partial_z \mathcal{X}(x(w), y(w), 0)|. \end{aligned}$$

The bound follows easily from the hypotheses.

Therefore $\partial_w \mathcal{X}^*(u, v, w)$ is uniformly Hölder continuous on w , at $w = 0$.

Moreover,

$$|\partial_{(u,v)} \mathcal{X}^*| \leq \text{Const.}(|u| + |v| + |w|), \quad |\partial_w \mathcal{X}^*| \leq \text{Const.}(|u| + |v|) \quad (25)$$

The first inequality follows from:

$$\begin{aligned} &|\partial_{(u,v)} \mathcal{X}^*(u, v, w)| = |\partial_{(x,y)} \mathcal{X}(u + x(w), v + y(w), w)| \leq \\ &M_1(|u| + |v| + |w| + |x(w)| + |y(w)|) \leq \text{Const}(|u| + |v| + |w|). \end{aligned}$$

From (24),

$$\begin{aligned}
|\partial_w \mathcal{X}^*(u, v, w)| &\leq |\partial_x \mathcal{X}(u + x(w), v + y(w), w) - \partial_x \mathcal{X}(x(w), y(w), w)| |x'(w)| + \\
&\quad |\partial_y \mathcal{X}(u + x(w), v + y(w), w) - \partial_y \mathcal{X}(x(w), y(w), w)| |y'(w)| + \\
&\quad |\partial_z \mathcal{X}(u + x(w), v + y(w), w) - \partial_z \mathcal{X}(x(w), y(w), w)| \leq \\
&\quad \text{Const.} [\|\mathcal{X}\|_1 + \|\mathcal{X}\|_2] (|u| + |v|),
\end{aligned}$$

and this shows the second inequality.

Since $\mathcal{X}^*(0, 0, w) = 0$ by (23), the first relation in (25) implies that

$$|\mathcal{X}^*(u, v, w)| \leq \text{Const.} (|u| + |v| + |w|) (|u| + |v|), \quad (26)$$

because

$$\begin{aligned}
|\mathcal{X}^*(u, v, w)| &= \left| \int_0^1 \frac{d}{dt} \mathcal{X}^*(tu, tv, w) dt \right| \leq \int_0^1 |\partial_{(u,v)} \mathcal{X}^*(tu, tv, w)| dt (|u| + |v|) \leq \\
&\quad \text{Const.} (|u| + |v| + |w|) (|u| + |v|).
\end{aligned}$$

The above estimates also imply:

$$|\partial_w \mathcal{X}^*(u, v, w) - \partial_w \mathcal{X}^*(u, v, 0)| \leq \text{Const.} (|u| + |v|)^\eta |w|^{\delta(1-\eta)} \quad (27)$$

for all η , $0 < \eta \leq 1$.

In fact,

$$\begin{aligned}
&|\partial_w \mathcal{X}^*(u, v, w) - \partial_w \mathcal{X}^*(u, v, 0)| = \\
&|\partial_w \mathcal{X}^*(u, v, w) - \partial_w \mathcal{X}^*(u, v, 0)|^\eta |\partial_w \mathcal{X}^*(u, v, w) - \partial_w \mathcal{X}^*(u, v, 0)|^{1-\eta} \leq \\
&\quad \text{Const.} (|u| + |v|)^\eta |w|^{\delta(1-\eta)}.
\end{aligned}$$

The same estimates can be proved for \mathcal{Y}^* .

So, if we come back to the notation \mathcal{X} , \mathcal{Y} , x , y , z instead of \mathcal{X}^* , \mathcal{Y}^* , u , v , w , we see that in addition to the hypotheses (i), (iv) and (v), we also have that

$$|\mathcal{X}(x, y, z)| \leq \text{Const.} (|x| + |y| + |z|) (|x| + |y|) \quad (28)$$

$$|\partial_z \mathcal{X}(x, y, z) - \partial_z \mathcal{X}(x, y, 0)| \leq \text{Const.} (|x| + |y|)^\eta |z|^{\delta(1-\eta)} \quad (29)$$

for all η , $0 < \eta \leq 1$ and for some $\delta > 0$, and

$$|\partial_z \mathcal{X}(x, y, z)| \leq \text{Const.} (|x| + |y|) \quad (30)$$

The same inequalities also hold for \mathcal{Y} .

In what follows we will take $\beta := \delta(1-\eta)$, where $0 < \eta \leq 1$. In the forthcoming calculation we will take η close to 1.

Following Hartman [4], we look for the following change of variables:

$$\begin{aligned} R &: u = x, \quad v = y - \phi(x, y, z), \quad w = z \\ R^{-1} &: x = u, \quad y = v + \psi(u, v, w), \quad z = w \end{aligned} \quad (31)$$

Note that from (31)

$$\phi(x, y, z) = \psi(u, v, w) \quad (32)$$

A simple calculation shows that (22) and (31) give $RTR^{-1} : (u, v, w) \mapsto (u^1, v^1, w^1)$:

$$\begin{cases} u^1 = Au + \mathcal{X}(u, v + \psi, w) \\ v^1 = B(v + \psi) + \mathcal{Y}(u, v + \psi, w) - \phi(Au + \mathcal{X}, B(v + \psi) + \mathcal{Y}, Cw), \\ w^1 = Cw \end{cases} \quad (33)$$

where the argument in \mathcal{X} , \mathcal{Y} is $(u, v + \psi, w)$ and in ψ is (u, v, w) .

Thus the relations

$$u^1 = Au + \mathcal{X}(u, v + \psi, w), \quad v^1 = Bv, \quad w^1 = Cw \quad (34)$$

hold if ϕ , ψ satisfy

$$B\psi = \phi(Au + \mathcal{X}, B(v + \psi) + \mathcal{Y}, Cw) - \mathcal{Y}(u, v + \psi, w).$$

By (31), (32), this means that

$$B\phi(x, y, z) = \phi(Ax + \mathcal{X}(x, y, z), By + \mathcal{Y}(x, y, z), Cz) - \mathcal{Y}(x, y, z) \quad (35)$$

The existence of solution of this functional equation was previously established in Theorem 12, for $|x| + |y| + |z| < \varepsilon$. We have checked that the nonlinearities \mathcal{X} and \mathcal{Y} satisfy the hypotheses required in that theorem, in particular the inequality (27).

Now, to prove that

$$\mathcal{U}(u, v, w) := \mathcal{X}(u, v + \psi(u, v, w), w)$$

satisfies the properties (i) to (v), of the Induction Assertion, we have to prove first that the function $\psi(u, v, w)$ inherits some properties of $\phi(x, y, z)$, namely the properties (a) to (e) in the next paragraphs.

At this point one should recall the definitions of $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_3$ from Theorem 12.

(a) $\psi(0, 0, w) \equiv 0$.

From the fact that

$$\psi(u, v, w) = \phi(u, v + \psi(u, v, w), w),$$

we obtain that:

$$|\psi(0, 0, w)| = |\phi(0, \psi(0, 0, w), w) - \phi(0, 0, w)| \leq \|\phi\|_1 |\psi(0, 0, w)| (|\psi(0, 0, w)| + |w|),$$

by Lemma 11. Since ψ is continuous and $\psi(0, 0, 0) = 0$, we obtain the stated result.

(b) $|\psi(u_1, v_1, w_1) - \psi(u_2, v_2, w_2)| \leq O(\varepsilon)(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|)$

$$|\psi(u_1, v_1, w_1) - \psi(u_2, v_2, w_2)| = |\phi(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) - \phi(u_2, v_2 + \psi(u_2, v_2, w_2), w_2)|$$

$$\leq \varepsilon [\|\phi\|_1 + \|\phi\|_2] (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2| + |\psi(u_1, v_1, w_1) - \psi(u_2, v_2, w_2)|),$$

also by using Lemma 11. If we take ε sufficiently small the above estimate implies the stated result.

(c) $\|\psi\|_1 < \infty$.

$$\partial_u \psi(u, v, w) = \partial_x \phi(u, v + \psi(u, v, w), w) + \partial_y \phi(u, v + \psi(u, v, w), w) \partial_u \psi(u, v, w).$$

From this it follows that $|\partial_u \psi(u, v, w)|$ is $O(\varepsilon)$, as $|x| + |y| + |z| \rightarrow 0$.

$$\begin{aligned} & |\partial_u \psi(u_1, v_1, w_1) - \partial_u \psi(u_2, v_2, w_2)| \leq \\ & |\partial_x \phi(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) + \partial_y \phi(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) \partial_u \psi(u_1, v_1, w_1) - \\ & \partial_x \phi(u_2, v_2 + \psi(u_2, v_2, w_2), w_2) - \partial_y \phi(u_2, v_2 + \psi(u_2, v_2, w_2), w_2) \partial_u \psi(u_2, v_2, w_2)| \leq \\ & |\partial_x \phi(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) - \partial_x \phi(u_2, v_2 + \psi(u_2, v_2, w_2), w_2)| + \\ & |\partial_y \phi(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) - \partial_y \phi(u_2, v_2 + \psi(u_2, v_2, w_2), w_2)| |\partial_u \psi(u_2, v_2, w_2)| + \\ & |\partial_y \phi(u_1, v_1 + \psi(u_1, v_1, w_1), w_1)| [|\partial_u \psi(u_2, v_2, w_2) - \partial_u \psi(u_1, v_1, w_1)|] \leq \\ & \|\phi\|_1 [1 + O(\varepsilon)] (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|) + \\ & \|\phi\|_1 [1 + O(\varepsilon)] (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|) O(\varepsilon) + \\ & O(\varepsilon) \|\phi\|_1 [|\partial_u \psi(u_2, v_2, w_2) - \partial_u \psi(u_1, v_1, w_1)|] \end{aligned}$$

Therefore

$$\begin{aligned} & |\partial_u \psi(u_1, v_1, w_1) - \partial_u \psi(u_2, v_2, w_2)| \leq \\ & \frac{1}{1 - O(\varepsilon) \|\phi\|_1} \|\phi\|_1 [1 + O(\varepsilon)] (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|). \end{aligned}$$

Let us consider now $\partial_v \psi$.

$$\partial_v \psi(u, v, w) = \partial_y \phi(u, v + \psi(u, v, w), w) + \partial_y \phi(u, v + \psi(u, v, w), w) \partial_v \psi(u, v, w)$$

$$\begin{aligned} |\partial_v \psi(u, v, w)| &\leq |\partial_y \phi(u, v + \psi(u, v, w), w)| + |\partial_y \phi(u, v + \psi(u, v, w), w) \partial_v \psi(u, v, w)| \\ &\leq O(\varepsilon) \|\phi\|_1 + O(\varepsilon) \|\phi\|_1 |\partial_v \psi(u, v, w)| \end{aligned}$$

This implies that $|\partial_v \psi(u, v, w)| = O(\varepsilon)$.

$$|\partial_v \psi(u_1, v_1, w_1) - \partial_v \psi(u_2, v_2, w_2)| \leq$$

$$\begin{aligned} &|\partial_y \phi(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) + \partial_y \phi(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) \partial_v \psi(u_1, v_1, w_1) - \\ &\partial_y \phi(u_2, v_2 + \psi(u_2, v_2, w_2), w_2) - \partial_y \phi(u_2, v_2 + \psi(u_2, v_2, w_2), w_2) \partial_v \psi(u_2, v_2, w_2)| \leq \\ &|\partial_y \phi(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) - \partial_y \phi(u_2, v_2 + \psi(u_2, v_2, w_2), w_2)| + \\ &|\partial_y \phi(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) - \partial_y \phi(u_2, v_2 + \psi(u_2, v_2, w_2), w_2)| |\partial_v \psi(u_2, v_2, w_2)| + \\ &|\partial_y \phi(u_1, v_1 + \psi(u_1, v_1, w_1), w_1)| |\partial_v \psi(u_1, v_1, w_1) - \partial_v \psi(u_2, v_2, w_2)| \leq \\ &(1 + O(\varepsilon)) \|\phi\|_1 (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|) + \\ &(1 + O(\varepsilon)) \|\phi\|_1 (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|) O(\varepsilon) + \\ &O(\varepsilon) |\partial_v \psi(u_1, v_1, w_1) - \partial_v \psi(u_2, v_2, w_2)|. \end{aligned}$$

Therefore,

$$\begin{aligned} &|\partial_v \psi(u_1, v_1, w_1) - \partial_v \psi(u_2, v_2, w_2)| \leq \\ &\frac{1}{1 - O(\varepsilon)} (1 + O(\varepsilon)) \|\phi\|_1 (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|). \end{aligned}$$

(d) $\|\psi\|_2 < \infty$.

$$\partial_w \psi(u, v, w) = \partial_z \phi(u, v + \psi(u, v, w), w) + \partial_y \phi(u, v + \psi(u, v, w), w) \partial_w \psi(u, v, w)$$

From this it follows that $|\partial_w \psi(u, v, w)|$ is $O(\varepsilon)$, as $|u| + |v| + |w| \rightarrow 0$.

$$|\partial_w \psi(u_1, v_1, w) - \partial_w \psi(u_2, v_2, w)| \leq$$

$$\begin{aligned} &|\partial_z \phi(u_1, v_1 + \psi(u_1, v_1, w), w) + \partial_y \phi(u_1, v_1 + \psi(u_1, v_1, w), w) \partial_w \psi(u_1, v_1, w) - \\ &\partial_z \phi(u_2, v_2 + \psi(u_2, v_2, w), w) - \partial_y \phi(u_2, v_2 + \psi(u_2, v_2, w), w) \partial_w \psi(u_2, v_2, w)| \leq \\ &|\partial_z \phi(u_1, v_1 + \psi(u_1, v_1, w), w) - \partial_z \phi(u_2, v_2 + \psi(u_2, v_2, w), w)| + \\ &|\partial_y \phi(u_1, v_1 + \psi(u_1, v_1, w), w) - \partial_y \phi(u_2, v_2 + \psi(u_2, v_2, w), w)| |\partial_w \psi(u_2, v_2, w)| + \\ &|\partial_y \phi(u_1, v_1 + \psi(u_1, v_1, w), w)| |\partial_w \psi(u_2, v_2, w) - \partial_w \psi(u_1, v_1, w)| \leq \\ &\|\phi\|_2 [1 + O(\varepsilon)] (|u_1 - u_2| + |v_1 - v_2|) + \\ &\|\phi\|_1 [1 + O(\varepsilon)] (|u_1 - u_2| + |v_1 - v_2|) O(\varepsilon) + \\ &O(\varepsilon) \|\phi\|_1 |\partial_w \psi(u_2, v_2, w) - \partial_w \psi(u_1, v_1, w)| \\ &|\partial_w \psi(u_2, v_2, w) - \partial_w \psi(u_1, v_1, w)| \leq \\ &\frac{1}{1 - O(\varepsilon)} [\|\phi\|_2 (1 + O(\varepsilon)) + O(\varepsilon) \|\phi\|_1] (|u_1 - u_2| + |v_1 - v_2|). \end{aligned}$$

(e) $\partial_w \psi$ is uniformly Hölder continuous at $w = 0$, with the exponent β defined above.

$$\partial_w \psi(u, v, w) = \partial_z \phi(u, v + \psi(u, v, w), w) + \partial_y \phi(u, v + \psi(u, v, w), w) \partial_w \psi(u, v, w)$$

$$\begin{aligned} & |\partial_w \psi(u, v, w) - \partial_w \psi(u, v, 0)| \leq \\ & |\partial_z \phi(u, v + \psi(u, v, w), w) + \partial_y \phi(u, v + \psi(u, v, w), w) \partial_w \psi(u, v, w) - \\ & \partial_z \phi(u, v + \psi(u, v, 0), 0) - \partial_y \phi(u, v + \psi(u, v, 0), 0) \partial_w \psi(u, v, 0)| \leq \\ & |\partial_z \phi(u, v + \psi(u, v, w), w) - \partial_z \phi(u, v + \psi(u, v, 0), w)| + \\ & |\partial_z \phi(u, v + \psi(u, v, 0), w) - \partial_z \phi(u, v + \psi(u, v, 0), 0)| + \\ & |\partial_y \phi(u, v + \psi(u, v, w), w) - \partial_y \phi(u, v + \psi(u, v, 0), 0)| |\partial_w \psi(u, v, 0)| + \\ & |\partial_y \phi(u, v + \psi(u, v, w), w)| |\partial_w \psi(u, v, w) - \partial_w \psi(u, v, 0)| \leq \\ & \|\phi\|_2 |\psi(u, v, w) - \psi(u, v, 0)| + \|\phi\|_3 O(\varepsilon^\eta) |w|^\beta + \\ & \|\phi\|_1 (|\psi(u, v, w) - \psi(u, v, 0)| + |w|) O(\varepsilon) + O(\varepsilon) |\partial_w \psi(u, v, w) - \partial_w \psi(u, v, 0)| \leq \\ & \|\phi\|_2 O(\varepsilon) |w| + \|\phi\|_3 O(\varepsilon^\eta) |w|^\beta + \|\phi\|_1 (O(\varepsilon) |w| + |w|) O(\varepsilon) + O(\varepsilon) |\partial_w \psi(u, v, w) - \partial_w \psi(u, v, 0)|. \end{aligned}$$

Thus,

$$|\partial_w \psi(u, v, w) - \partial_w \psi(u, v, 0)| \leq \frac{1}{1 - O(\varepsilon)} \{ \|\phi\|_2 O(\varepsilon) + O(\varepsilon) \|\phi\|_1 |w| + \|\phi\|_3 O(\varepsilon^\eta) |w|^\beta \}.$$

It remains to check (i)-(v) of the Induction Assertion for $\mathcal{U}(u, v, w) = \mathcal{X}(u, v + \psi(u, v, w), w)$. From the above properties proved for ψ it follows that (i) is obvious. In fact, from Lemma 11 condition (i), it follows that:

$$\frac{|\mathcal{U}(u, v, w)|}{|u| + |v| + |w|} = \frac{|\mathcal{X}(u, v + \psi(u, v, w), w)|}{|u| + |v| + |w|} \leq$$

$$\|\mathcal{X}\|_1 \frac{(|u| + |v| + |\psi(u, v, w)|)^2 + (|u| + |v| + |\psi(u, v, w)|) |w|}{|u| + |v| + |w|} \rightarrow 0, \text{ as } |u| + |v| + |w| \rightarrow 0.$$

In order to verify (ii), note that

$$\partial_u \mathcal{U}(u, v, w) = \partial_x \mathcal{X}(u, v + \psi(u, v, w), w) + \partial_y \mathcal{X}(u, v + \psi(u, v, w), w) \partial_u \psi(u, v, w) \quad (36)$$

Let us prove that $\partial_x \mathcal{X}(u, v + \psi(u, v, w), w)$ is Lipschitz continuous in (u, v, w) .

$$|\partial_x \mathcal{X}(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) - \partial_x \mathcal{X}(u_2, v_2 + \psi(u_2, v_2, w_2), w_2)|$$

$$\leq \|\mathcal{X}\|_1 (|u_1 - u_2| + |v_1 - v_2| + |\psi(u_1, v_1, w_1) - \psi(u_2, v_2, w_2)| + |w_1 - w_2|) \leq \|\mathcal{X}\|_1 (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2| + O(\varepsilon) (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|))$$

Let us prove that $\partial_y \mathcal{X}(u, v + \psi(u, v, w), w) \partial_u \psi(u, v, w)$ is Lipschitz continuous in (u, v, w) .

$$\begin{aligned} & |\partial_y \mathcal{X}(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) \psi_u(u_1, v_1, w_1) - \partial_y \mathcal{X}(u_2, v_2 + \psi(u_2, v_2, w_2), w_2) \psi_u(u_2, v_2, w_2)| \\ & \leq |\partial_y \mathcal{X}(u_1, v_1 + \psi(u_1, v_1, w_1), w_1) - \partial_y \mathcal{X}(u_2, v_2 + \psi(u_2, v_2, w_2), w_2)| |\psi_u(u_1, v_1, w_1)| \\ & + |\partial_y \mathcal{X}(u_2, v_2 + \psi(u_2, v_2, w_2), w_2)| |\psi_u(u_1, v_1, w_1) - \psi_u(u_2, v_2, w_2)| \leq \end{aligned}$$

$$O(\varepsilon) \|\mathcal{X}\|_1 (|u_1 - u_2| + |v_1 - v_2| + |\psi(u_1, v_1, w_1) - \psi(u_2, v_2, w_2)| + |w_1 - w_2|) +$$

$O(\varepsilon)|\psi_u(u_1, v_1, w_1), w_1) - \psi_u(u_2, v_2, w_2)| \leq$

$O(\varepsilon)\|\mathcal{X}\|_1(1 + O(\varepsilon))(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|) + O(\varepsilon)|\psi_u(u_1, v_1, w_1), w_1) - \psi_u(u_2, v_2, w_2)|.$

Thus (ii) holds.

Note that

$$\begin{aligned} \partial_v \mathcal{U}(u, v, w) &= \partial_y \mathcal{X}(u, v + \psi(u, v, w), w)(I + \partial_v \psi(u, v, w)) = \\ &\partial_y \mathcal{X}(u, v + \psi(u, v, w), w) + \partial_y \mathcal{X}(u, v + \psi(u, v, w), w) \partial_v \psi(u, v, w). \end{aligned}$$

As above one can prove that $\partial_y \mathcal{X}(u, v + \psi(u, v, w), w)$ and $\partial_y \mathcal{X}(u, v + \psi(u, v, w), w) \partial_v \psi(u, v, w)$ are Lipschitz continuous on (u, v, w) .

This proves that $\partial_v \mathcal{U}$ is Lipschitz continuous with respect to (u, v, w) and this implies that the properties (iii), (iv) and (v) of the Induction Assertion that concern to $\partial_v \mathcal{U}$, also hold.

Similarly,

$$\partial_w \mathcal{U}(u, v, w) = \partial_y \mathcal{X}(u, v + \psi(u, v, w), w) \partial_w \psi(u, v, w) + \partial_z \mathcal{X}(u, v + \psi(u, v, w), w).$$

Therefore $\partial_w \mathcal{U}$ is uniformly Lipschitz continuous in (u, v) .

Let us prove that $\partial_w \mathcal{U}$ is Hölder continuous on w , at $w = 0$.

We consider first the term $\partial_z \mathcal{X}(u, v + \psi(u, v, w), w)$.

$$\begin{aligned} &|\partial_z \mathcal{X}(u, v + \psi(u, v, w), w) - \partial_z \mathcal{X}(u, v + \psi(u, v, 0), 0)| \leq \\ &|\partial_z \mathcal{X}(u, v + \psi(u, v, w), w) - \partial_z \mathcal{X}(u, v + \psi(u, v, w), 0)| + \\ &|\partial_z \mathcal{X}(u, v + \psi(u, v, w), 0) - \partial_z \mathcal{X}(u, v + \psi(u, v, 0), 0)| \leq \\ &|\partial_z \mathcal{X}(u, v + \psi(u, v, w), w) - \partial_z \mathcal{X}(u, v + \psi(u, v, w), 0)| + \|\mathcal{X}\|_2 |\psi(u, v, w) - \psi(u, v, 0)| \\ &\leq \|\mathcal{X}\|_3 (|u| + |v + \psi(u, v, w)|)^\eta |w|^\beta + \|\mathcal{X}\|_2 O(\varepsilon) |w| \end{aligned}$$

Let us consider now the term $\partial_y \mathcal{X}(u, v + \psi(u, v, w), w) \partial_w \psi(u, v, w)$.

$$\begin{aligned} &|\partial_y \mathcal{X}(u, v + \psi(u, v, w), w) \partial_w \psi(u, v, w) - \partial_y \mathcal{X}(u, v + \psi(u, v, 0), 0) \partial_w \psi(u, v, 0)| \leq \\ &|\partial_y \mathcal{X}(u, v + \psi(u, v, w), w) - \partial_y \mathcal{X}(u, v + \psi(u, v, 0), 0)| |\partial_w \psi(u, v, w)| + \\ &|\partial_y \mathcal{X}(u, v + \psi(u, v, 0), 0)| |\partial_w \psi(u, v, w) - \partial_w \psi(u, v, 0)| \leq \end{aligned}$$

$$\|\mathcal{X}\|_1 (O(\varepsilon)|w| + |w|) |O(\varepsilon) + O(\varepsilon^\eta)| |w|^\beta.$$

This proves (iii) and (iv) concerning $\partial_w \mathcal{U}$.

Concerning (v), from (25), we have

$$|\partial_w \mathcal{U}(u, v, w)| \leq |\partial_y \mathcal{X}(u, v + \psi(u, v, w), w)| |\partial_w \psi(u, v, w)| + |\partial_z \mathcal{X}(u, v + \psi(u, v, w), w)|$$

$$\leq \text{Const.}(|u| + |v| + |\psi(u, v, w)|) = O(|u| + |v| + |w|).$$

This completes the proof of Theorem 1. ■

5 Applications.

Following [10] we consider the next abstract equation, that represents the functional formulation of many damped wave equations:

$$\ddot{u} - 2\gamma\dot{u} + \mathcal{A}u = 0. \quad (37)$$

It is defined in a Hilbert Space E , γ is a negative real number and \mathcal{A} is a self-adjoint positive operator on E ,

$$0 < \mu := \inf_{u \in \mathcal{D}(\mathcal{A})} \frac{\langle \mathcal{A}u, u \rangle}{\langle u, u \rangle}, \quad (38)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of E and $\mathcal{D}(\mathcal{A})$ indicates the domain of \mathcal{A} . As it is well know, μ coincides with the smallest element of the spectrum of \mathcal{A} .

In the following, $E^{1/2}$ will denote the Hilbert space consisting of the domain of the operator $(\mathcal{A} + \xi I)^{1/2}$ endowed with the inner product:

$$\langle u, \hat{u} \rangle_{1/2} := \langle (\mathcal{A} + \xi I)^{1/2}u, (\mathcal{A} + \xi I)^{1/2}\hat{u} \rangle,$$

where ξ is a real number $\xi > -\mu$. Different choices of ξ result in the same vector space with different but equivalent inner products. Define $\mathbb{E} := E^{1/2} \times E$.

Equation (37) is equivalent to the system:

$$\dot{U} = \gamma U + \mathbb{A}u \quad (39)$$

in \mathbb{E} , where

$$\mathbb{E} \ni U = \begin{pmatrix} u \\ w \end{pmatrix} := \begin{pmatrix} u \\ -\gamma u + \dot{u} \end{pmatrix}, \text{ and } \mathbb{A}U = \mathbb{A} \begin{pmatrix} u \\ w \end{pmatrix} := \begin{pmatrix} w \\ \gamma^2 u - \mathcal{A}u \end{pmatrix}$$

and it is known that it defines a C^0 -group $T(t) := e^{\gamma t} e^{\mathbb{A}t}$.

We consider the following inner product in \mathbb{E} , where we choose different values of ξ , according to the values of γ^2 ,

$$\langle U, \hat{U} \rangle_{\mathbb{E}} := \begin{cases} \langle (A - \gamma^2 I)^{1/2} u, (A - \gamma^2 I)^{1/2} \hat{u} \rangle + \langle w, \hat{w} \rangle, & \text{if } \gamma^2 < \mu, \\ \langle (A - (\varepsilon^2 - \mu) I)^{1/2} u, (A - (\varepsilon^2 - \mu) I)^{1/2} \hat{u} \rangle + \langle w, \hat{w} \rangle, & \text{if } \gamma^2 = \mu, \\ \langle (A - (\gamma^2 - 2\mu) I)^{1/2} u, (A - (\gamma^2 - 2\mu) I)^{1/2} \hat{u} \rangle + \langle w, \hat{w} \rangle, & \text{if } \gamma^2 > \mu. \end{cases} \quad (40)$$

where $\varepsilon > 0$ is a small number to be chosen later on.

If $T(t)$ denotes the group generated by $\gamma I + \mathbb{A}$, from [10] it follows that the following estimates hold:

$$\begin{cases} \text{if } \gamma^2 < \mu, & |T(t)| \leq e^{\gamma t}, \text{ for all } t \text{ in } \mathbb{R}, \\ \text{if } \gamma^2 = \mu, & |T(t)| \leq e^{\gamma t + \varepsilon |t|}, \text{ for all } t \text{ in } \mathbb{R}, \\ \text{if } \gamma^2 > \mu, & |T(t)| \leq e^{[\gamma t - \sqrt{\gamma^2 - \mu} |t|]}, \text{ for all } t \text{ in } \mathbb{R}. \end{cases} \quad (41)$$

Consider also the nonlinear equation:

$$\ddot{u} - 2\gamma \dot{u} + \mathcal{A}u = f(u, \dot{u}) \quad (42)$$

where $f \in C^{1,1}(\Omega, E)$, $f(0) = 0$, $f'(0) = 0$ and Ω is a neighborhood of 0 in \mathbb{E} .

Equation (42) is equivalent to the system:

$$\dot{U} = \gamma U + \mathbb{A}u + F(U), \quad (43)$$

in \mathbb{E} , where

$$F(U) := \begin{pmatrix} 0 \\ f(u, w + \gamma u) \end{pmatrix}.$$

Our next purpose is to obtain conditions in such a way that equation (43) be conjugate to (39) in the class C^1 , by using Theorem 1.

Next we obtain conditions on γ and μ in such a way that the nonresonance condition of Theorem 1 holds. If we let $A := T(1)$ then $A^{-1} = T(-1)$.

If $\gamma^2 < \mu$, we choose $\nu^- < e^\gamma < \nu^+$, with ν^- and ν^+ sufficiently close, to obtain:

$$|A| < \nu^+, \quad |A^{-1}| \leq e^{-\gamma} < \frac{1}{\nu^-}, \quad (\nu^+)^2 < \nu^-.$$

where

$$\mathbb{E} := E^{1/2} \times E \ni U_i = \begin{pmatrix} u_i \\ w_i \end{pmatrix} := \begin{pmatrix} u_i \\ -\gamma_i u_i + \hat{u}_i \end{pmatrix}$$

$$\text{and } \mathbb{A}_i U_i = \mathbb{A}_i \begin{pmatrix} u_i \\ w_i \end{pmatrix} := \begin{pmatrix} w_i \\ \gamma_i^2 u_i - \mathcal{A}u_i \end{pmatrix}$$

$$F_i(U_1, \dots, U_n) := \begin{pmatrix} 0 \\ f_i(u_1, w_1 + \gamma_1 u_1, u_2, w_2 + \gamma_2 u_2, \dots, u_n, w_n + \gamma_n u_n) \end{pmatrix},$$

$i = 1, \dots, n$.

As in (40) we define the following inner products:

$$\langle U_i, \hat{U}_i \rangle_{\mathbb{E}} := \begin{cases} \langle (A - \gamma_i^2 I)^{1/2} u_i, (A - \gamma_i^2 I)^{1/2} \hat{u}_i \rangle + \langle w_i, \hat{w}_i \rangle, & \text{if } \gamma_i^2 < \mu, \\ \langle (A - (\varepsilon^2 - \mu) I)^{1/2} u_i, (A - (\varepsilon^2 - \mu) I)^{1/2} \hat{u}_i \rangle + \langle w_i, \hat{w}_i \rangle, & \text{if } \gamma_i^2 = \mu, \\ \langle (A - (\gamma_i^2 - 2\mu) I)^{1/2} u_i, (A - (\gamma_i^2 - 2\mu) I)^{1/2} \hat{u}_i \rangle + \langle w_i, \hat{w}_i \rangle, & \text{if } \gamma_i^2 > \mu. \end{cases} \quad (46)$$

where $\varepsilon > 0$ is a small number as above and \mathbb{E}_i denotes the space \mathbb{E} , endowed with the above inner product, for $i = 1, \dots, n$.

Since $\gamma_n < \gamma_{n-1} < \dots < \gamma_2 < \gamma_1 < 0$, we observe that there is at most one index $i \in \{1, 2, \dots, n\}$, such that $\gamma_i^2 = \mu$.

As above, if $T_i(t)$ denotes the group generated by $\gamma_i I + \mathbb{A}$, $A_i = T_i(1)$, $A_i^{-1} = T_i(-1)$, one can show that the following estimates hold, for $i = 1, \dots, n$:

$$\begin{cases} \text{if } \gamma_i^2 < \mu, |T_i(t)| \leq e^{\gamma_i t}, \text{ for all } t \text{ in } \mathbb{R}, \\ \text{if } \gamma_i^2 = \mu, |T_i(t)| \leq e^{\gamma_i t + \varepsilon |t|}, \text{ for all } t \text{ in } \mathbb{R}, \\ \text{if } \gamma_i^2 > \mu, |T_i(t)| \leq e^{[\gamma_i t - \sqrt{\gamma_i^2 - \mu} |t|]}, \text{ for all } t \text{ in } \mathbb{R}. \end{cases} \quad (47)$$

In order to analyze the nonresonance condition of Theorem 1, let us consider first the case $\gamma_1^2 > \mu$. This implies that $\gamma_i^2 > \mu$, for all $i = 1, \dots, n$ and so,

$$|A_i| \leq e^{[\gamma_i + \sqrt{\gamma_i^2 - \mu}]} < \nu_i^+, |A_i^{-1}| \leq e^{[-\gamma_i + \sqrt{\gamma_i^2 - \mu}]} < \frac{1}{\nu_i^-}, \quad i = 1, \dots, n.$$

In order to find ν_i^- , ν_i^+ such that $\nu_i^+ \nu_1^+ < \nu_i^-$ it is sufficient to have:

$$e^{[\gamma_i + \sqrt{\gamma_i^2 - \mu}]} e^{[\gamma_1 + \sqrt{\gamma_1^2 - \mu}]} < e^{[\gamma_i - \sqrt{\gamma_i^2 - \mu}]}$$

or equivalently,

$$2\sqrt{\gamma_i^2 - \mu} < -\gamma_1 - \sqrt{\gamma_1^2 - \mu}, \quad i = 1, \dots, n,$$

therefore it is sufficient to suppose that $2\sqrt{\gamma_n^2 - \mu} < -\gamma_1 - \sqrt{\gamma_1^2 - \mu}$.

If $\gamma_1^2 \leq \mu$ and there exists $j \in \{1, 2, \dots, n\}$, such that $\gamma_j^2 > \mu$, using the same ideas as above we obtain that the nonresonance condition will be satisfied if $2\sqrt{\gamma_n^2 - \mu} < -\gamma_1$. We observe that in the two previous cases $2\sqrt{\gamma_n^2 - \mu} = \text{diameter } \mathfrak{R}(\sigma(\mathbb{A}_n))$.

If $\gamma_n^2 \leq \mu$ then the nonresonance condition is automatically satisfied.

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