On the stratifications of the set of (A, B)-invariant and controllability subspaces.

F. Puerta, X. Puerta
ETSEIB, EUPB
Departament de Matemàtica Aplicada I
Diagonal 647 -08028 Barcelona- Spain
puerta@ma1.upc.es, coll@ma1.upc.es

Abstract

We introduce a new stratification of the set of (A, B)-invariant subspaces of a fixed dimension of a general pair (A, B) and show an application on the description of the set of controlability subspaces.

Introduction

Given a pair of matrices (A, B) with $A \in K^{n \times n}$ and $B \in K^{n \times m}$ (K denotes the field of real or complex numbers) representing, for example, a linear system, the geometry of the set of (A, B)-invariant subspaces of a fixed dimension d has been recently a subject of interest due to its applications in System Theory (see [2], [3], [4], [5], [6], [8], [9], [10], [11]). We denote this set by $\mathrm{Inv}_d(A, B)$.

This set is not, in general, a smooth manifold. However it can be stratifyed in a different ways, according to the Brunovsky indices of different kind of restrictions. In fact, one way of stratifying $\mathrm{Inv}_d(A, B)$ consists of considering the dual pair (B^t, A^t) and then, the stratification of the set of (B^t, A^t)-invariant subspaces of dimension $n - d$ obtained by fixing the Brunovsky indices of the restriction (see [4] and [5]) induces by duality a stratification in the set $\mathrm{Inv}_d(A, B)$. Another, and more interesting possibility of stratifying this set arises when an appropiate definition of restriction of a pair (A, B) is given (see section 2). Then, by considering the partition of $\mathrm{Inv}_d(A, B)$ obtained by fixing the Brunovsky form of this restriction, we get a new stratification of this set. For this, a first and fundamental step is the knowledge of which Brunovsky forms are possible for this restriction. This problem ity are given. Then, the stratification of $\text{Inv}_d(A, B)$ obtained in this note precises the geometrical structure of the set of (A, B)-invariant subspaces having the same Brunovsky form for the restriction (theorem 3.8). Likewise an explicit description of these subspaces is given (theorem 3.4).

$K^{p\times q}$ denotes the set of K matrices having p-rows and q-columns. For shortness we denote simpliy a basis $(u_1, \ldots, u_p)$ by u if no confussion is possible. Then if f is a linear map, $f(u)$ means the family $(f(u_1), \ldots, f(u_p))$. If u is a set of vectors $[u]$ means the subspace spanned by u. If E is a vector space, E^* is its dual space and if $\mathcal{S}$ is a subspace of E we denote by $\widetilde{\mathcal{S}}$ the anihilator of $\mathcal{S}$ in E^*, that is to say, the set of $\omega \in E^*$ such that $\omega(\mathcal{S}) = \{0\}$. $G_d(K^n)$ denotes the Grassmann manifold of d-dimensional linear subspaces of K^n.

1 Preliminaries

Given a pair of matrices (A, B) with $A \in K^{n\times n}$ and $B \in K^{n\times m}$, we recall that $\mathcal{S} \in \text{Gr}_k(K^n)$ is an (A, B)-invariant subspace if $\mathcal{S} \subset \mathcal{S} + \text{Im}B$. We denote by $\text{Inv}_d(A, B)$ the set of d-dimensional (A, B)-invariant subspaces of K^n. Likewise, given a pair (C, A) with $A \in K^{n\times n}$ and $C \in K^{p\times n}$, $\mathcal{S} \in \text{Gr}_k(K^n)$ is a (C, A)-invariant subspace ($\binom{A}{C}$-invariant in ...) if $A(\mathcal{S}) \cap \ker C \subset \mathcal{S}$. We denote by $\text{Inv}_d(C, A)$ the set of d-dimensional (C, A)-invariant subspaces of K^n. Since theese notions are dual one of the other, the map $\mathcal{S} \mapsto \mathcal{S}^\perp$ (with regard to the usual metric) is a bijection between $\text{Inv}_d(A, B)$ and $\text{Inv}_{n-d}(B^t, A^t)$. Therefore a stratification of $\text{Inv}_{n-d}(B^t, A^t)$ induces a stratification of $\text{Inv}_d(A, B)$ and conversely.

In [4] and [5] a stratification of $\text{Inv}_d(C, A)$ is defined by fixing the Brunovsky indices of the restriction. Now we recall how the restriction of (C, A) on a (C, A)-invariant subspace is defined.

We associate to each pair (C, A) the linear map $f : K^n \to K^{n+m}$ defined by $f(x) = (Ax, Cx)$ and the injective map $i : K^n \to K^{n+m}$ defined by $i(x) = (x, 0)$. Conversely, for each pair of linear maps (f, i) from an n-dimensional vector space $\mathcal{F}$ to a $n + m$-dimensional vector space $\mathcal{K}$ with i injective, taking a basis u of $\mathcal{F}$ and a basis of $\mathcal{K}$ of the form $(i(u), v)$ and taking the matrix of f with regard these bases, we obtain a two block matrix $\binom{A}{C}$ with $A \in K^{n\times n}$ and $C \in K^{p\times n}$. We call (C, A) a *matrix representation* of (f, i). In fact, there is no loss of generality in the following discursion assuming $\mathcal{F} \subset \mathcal{K}$ and i the inclusion map. We identify, then, K^n with $K^n \times \{0\} \subset K^{n+p}$ and we omit i in the pair (f, i) calling, simply, (C, A) a matrix representation of f.

We have the following proposition.

a pair (C', A') is matrix representation of f if and only if (C', A') is output injection equivalent to (C, A). In particular, there exists a matrix representation of f in the Brunovsky canonical form.

The above proposition motivates the following definition.

Definition 1.2 *If f is a linear map $f : \mathcal{F} \to \mathcal{K}$ with $\mathcal{F} \subset \mathcal{K}$ we call the* Brunovsky indices of f *to the Brunovsky indices of any of its matrix representations. Likewise, we say that f is* observable *if any of its matrix representations is observable.*

We have the following traslation of the notion of (C, A)-invariance in terms of f (we recall that we have identifyed K^n with $K^n \times \{0\} \subset K^{n+p}$).

Proposition 1.3 *Let f defined by $f(x) = (Ax, Cx)$. Then, a subspace $\mathcal{S}$ of K^n is (C, A)-invariant if and only if $f(\mathcal{S}) \cap K^n \subset \mathcal{S}$. We also call $\mathcal{S}$ a f-*invariant *subspace.*

Given a linear map $f : \mathcal{F} \to \mathcal{K}$ with $\mathcal{F} \subset \mathcal{K}$, every subspace $\mathcal{S}$ of $\mathcal{F}$ defines two linear maps $\overline{f}$ and $\widetilde{f}$ defined by the conmutance of the following diagram

$$(1) \quad \begin{array}{ccc} \mathcal{S} & \xrightarrow{\widetilde{f}} & \mathcal{S} + f(\mathcal{S}) \\ \cap & & \cap \\ \mathcal{F} & \xrightarrow{f} & \mathcal{K} \\ \downarrow & & \downarrow \\ \mathcal{F}/\mathcal{S} & \xrightarrow{\overline{f}} & \mathcal{K}/(\mathcal{S} + f(\mathcal{S})) \end{array}$$

where the vertical arrows are the natural projections, $\widetilde{f}$ is the restriction of f to $\mathcal{S}$ and $\overline{f}$ the corresponding induced map on the quotient.

If $\mathcal{S}$ is f-invariant it is easy to prove that $\mathcal{S} = (f(\mathcal{S}) + \mathcal{S}) \cap \mathcal{F}$. Then we have

$$\mathcal{F}/\mathcal{S} = \frac{\mathcal{F}}{(f(\mathcal{S}) + \mathcal{S}) \cap \mathcal{F}} \cong \frac{\mathcal{F} + f(\mathcal{S})}{\mathcal{S} + f(\mathcal{S})} \subset \frac{\mathcal{K}}{\mathcal{S} + f(\mathcal{S})}$$

and therefore $\overline{f}$ is of the same type of f and $\widetilde{f}$, that is to say, a map defined on a subspace.

If we take a basis of $\mathcal{F}$ of the form (u, v) with u a basis of $\mathcal{S}$ and a basis of $\mathcal{K}$ of the form (u, v, w, y) being (u, w) a basis of $\mathcal{S} + f(\mathcal{S})$ the matrix of f with regard

$$\begin{pmatrix} 0 & \overline{A} \\ \widetilde{C} & Y \\ 0 & \overline{C} \end{pmatrix}$$

We can easily see that $(\widetilde{C}, \widetilde{A})$ and $(\overline{C}, \overline{A})$ are matrix representations of $\widetilde{f}$ and $\overline{f}$ respectively.

If f is defined as $f : K^n \to K^{n+m}$ with $f(x) = (Ax, Cx)$ and $(\widetilde{C}, \widetilde{A})$, $(\overline{C}, \overline{A})$ are Brunovsky pairs with $\widetilde{A}, \overline{A} \in K^{d\times d}$ and $\widetilde{C}, \overline{C} \in K^{q\times d}$ $(d \leq n, q \leq p)$ we define the following subsets of $\mathrm{Inv}_d(C, A)$:

$$\mathrm{Inv}_{(\widetilde{C},\widetilde{A})}(C, A) = \{\mathcal{S} \in \mathrm{Inv}_d(C, A) | \quad \widetilde{f} \text{ has Brunovsky representation } (\widetilde{C}, \widetilde{A})\}$$

$$\mathrm{Inv}^{(\overline{C},\overline{A})}(C, A) = \{\mathcal{S} \in \mathrm{Inv}_d(C, A) | \quad \overline{f} \text{ has Brunovsky representation } (\overline{C}, \overline{A})\}$$

One can prove that, while the eigenvalues of the Jordan part of $(\widetilde{C}, \widetilde{A})$ are among those of (C, A) (in particular, if (C, A) is observable, $(\widetilde{C}, \widetilde{A})$ also is), the eigenvalues of $(\overline{C}, \overline{A})$ do not have this property. Therefore, while the partition of $\mathrm{Inv}_d(C, A)$ defined by $\mathrm{Inv}_{(\widetilde{C},\widetilde{A})}(C, A)$ is finite, the one defined by $\mathrm{Inv}^{(\overline{C},\overline{A})}(C, A)$ is not, in general. In [**?**] and [5] the sets $\mathrm{Inv}_{(\widetilde{C},\widetilde{A})}(C, A)$ are described as quotient manifolds embedded in $\mathrm{Gr}_d(K^n)$.

We move now to the dual setting, that is to say, that of pairs (A, B). The image of $\mathrm{Inv}_{(\widetilde{B^t},\widetilde{A^t})}(B^t, A^t)$ by the map $\mathcal{S} \mapsto \mathcal{S}^\perp$ defines a subset, say $\mathrm{Inv}^{(\widetilde{A},\widetilde{B})}(A, B)$, of $\mathrm{Inv}_d(A, B)$ and of course we can define a differential structure on $\mathrm{Inv}^{(\widetilde{A},\widetilde{B})}(A, B)$ through the above bijection. Nevertheless, the set $\mathrm{Inv}^{(\widetilde{A},\widetilde{B})}(A, B)$ has no relation with that obtained by considering the restriction of (A, B) on the (A, B)-invariant subspaces (in a natural sense that we will precise later). The goal of these notes is to stratifify $\mathrm{Inv}_d(A, B)$ acording to the Brunovsky indices of such restriction. These strata are the image of $\mathrm{Inv}^{(\overline{A},\overline{B})}(B^t, A^t)$ by $\mathcal{S} \mapsto \mathcal{S}^\perp$. However, because of the applications in control theory (we give a description of the set of controlability subspaces) we don't study the set $\mathrm{Inv}^{(\overline{A},\overline{B})}(B^t, A^t)$ but directely its image by $\mathcal{S} \mapsto \mathcal{S}^\perp$ in the set of (A, B)-invariant subspaces.

subspaces

In order to define a restriction of (A,B) to an (A,B)-invariant subspace we consider the dual diagram of (1) where the isomorphism holds if $\mathcal{S}$ is f-invariant.

$$
(2) \qquad
\begin{array}{cccl}
\widetilde{\mathcal{S}+f(\mathcal{S})} & \xrightarrow{\overline{f^*}} & \mathcal{S}^* & = \widetilde{\mathcal{S}}/\widetilde{\mathcal{F}} \cong \widetilde{\mathcal{S}+f(\mathcal{S})}/\widetilde{\mathcal{F}+f(\mathcal{S})} \\
\cap & & \cap & \\
\mathcal{K}^* & \xrightarrow{f^*} & \mathcal{F}^* & = \mathcal{K}^*/\widetilde{\mathcal{F}} \\
\downarrow & & \downarrow & \\
\mathcal{K}^*/(\widetilde{\mathcal{S}+f(\mathcal{S})}) & \xrightarrow{\widetilde{f^*}} & \mathcal{K}^*/\widetilde{\mathcal{S}} & = (\mathcal{K}^*/(\widetilde{\mathcal{S}+f(\mathcal{S})}))/(\widetilde{\mathcal{S}}/(\widetilde{\mathcal{S}+f(\mathcal{S})}))
\end{array}
$$

In most cases $\mathcal{K}^*$ and $\mathcal{K}^*/\widetilde{\mathcal{S}}$ will be K^{n+m} and K^n respectively. Because of the inconvenience of representing the state space K^n and the (A,B)-invariant subspaces as quotient spaces, we consider, instead of a map f^* having its image on a quotient of $\mathcal{K}^*$, a pair of linear maps (f,π) defined from K^{n+m} to K^n (or more generally between two $n+m$ and n dimensional vector spaces) with π surjective.

We proceed then, as in the previous section, associating to each pair (A,B) a pair formed by the linear map $f : K^{n+m} \to K^n$ defined by $f(x,y) = Ax + By$ and the surjective map $\pi : K^{n+m} \to K^n$ defined by $\pi(x,y) = x$ (π is the dual map of $x \mapsto (x,0)$). Conversely, for each pair of linear maps (f,π) from an $n+m$-dimensional vector space $\mathcal{K}$ to a n-dimensional vector space $\mathcal{F}$ with π surjective, taking a basis of $\mathcal{K}$ of the form (u,v) where v is a basis of the kernel of π and a basis of $\mathcal{F}$ of the form $\pi(u)$ and taking a the matrix of f with regard these bases, we obtain a two block matrix $(A\,B)$ with $A \in K^{n\times n}$ and $B \in K^{n\times m}$. Notice that the matrix of π with regard the above bases is $(I\,0)$. We call (A,B) a *matrix representation* of (f,π). In other words, (A,B) is a *matrix representation* of (f,π) if and only if there exist isomorophisms $\phi : \mathcal{K} \to K^{n+m}$ and $\psi : \mathcal{F} \to K^n$ such that $f = \psi^{-1}(A\,B)\phi$ and $\pi = \psi^{-1}(I\,0)\phi$.

We have then the following proposition.

Proposition 2.1 *Let (f,π) be as above and (A,B) a matrix representation of (f,π). Then, a pair (A',B') is matrix representation of (f,π) if and only if (A',B') is feedback equivalent to (A,B). In particular, there exists a matrix representation of (f,π) in the Brunovsky canonical form.*

Analogously as in the previous section, we define the notion notion of (A,B)-invariance in terms of the pair of linear maps (f,π) associated to (A,B). We have the following proposition.

a subspace $\mathcal{S}$ of K^n is (A,B)-invariant if and only if $\mathcal{S} \subset \pi(f^{-1}(\mathcal{S}))$.

Given a pair of linear maps (f,π) as above, every subspace $\mathcal{S}$ of K^n defines two pairs of linear maps $(\overline{f},\overline{\pi})$ and $(\widetilde{f},\widetilde{\pi})$ defined by the conmutance of the following diagram

$$\begin{array}{llcl}
\overline{f},\overline{\pi}: & \pi^{-1}(\mathcal{S}) \cap f^{-1}(\mathcal{S}) & \longrightarrow & \mathcal{S} \\
 & \cap & & \cap \\
f,\pi: & K^{n+m} & \longrightarrow & K^n \\
 & \downarrow & & \downarrow \\
\widetilde{f},\widetilde{\pi}: & K^{n+m}/(\pi^{-1}(\mathcal{S}) \cap f^{-1}(\mathcal{S})) & \longrightarrow & K^n/\mathcal{S}
\end{array}$$

where the vertical arrows are the natural projections, $\overline{f}$ and $\overline{\pi}$ are the restrictions of f and π to $\pi^{-1}(\mathcal{S}) \cap f^{-1}(\mathcal{S})$ and $\widetilde{f},\widetilde{\pi}$ the corresponding maps induced on the quotients. We remark that, while $\widetilde{\pi}$ is always surjective, $\overline{\pi}$ do not need to be surjective. Nevertheless, if $\mathcal{S}$ is (A,B)-invariant $\overline{\pi}$ is surjective (and conversely, see proposition 2.2). Therefore, each (A,B)-invariant subspace defines two pairs of linear maps $(\overline{f},\overline{\pi})$ and $(\widetilde{f},\widetilde{\pi})$ of the same type of (f,π). We call $(\overline{f},\overline{\pi})$ the *restriction* of (f,π) and $(\widetilde{f},\widetilde{\pi})$ the *quotient induced map.*

Since the matrix representations of $(\overline{f},\overline{\pi})$ and $(\widetilde{f},\widetilde{\pi})$ are, respectively, feedback equivalent, it makes sense to define the Brunovsky indices of $(\overline{f},\overline{\pi})$ and the Brunovsky indices of $(\widetilde{f},\widetilde{\pi})$ as the Brunovsky indices of any matrix representation of these pairs.

Let (u,v,w,y) be a basis of K^{n+m} with w a basis of $\ker\pi \cap f^{-1}(\mathcal{S})$, (w,y), a basis of $\ker\pi$ and (u,w) a basis of $\pi^{-1}(\mathcal{S}) \cap f^{-1}(\mathcal{S})$. Let $(\pi(u),\pi(v))$ a basis of K^n. Then, the matrix representation of (f,π) with regard these bases is

$$\left(\begin{array}{cc|cc} \overline{A} & X & \overline{B} & Y \\ 0 & \widetilde{A} & 0 & \widetilde{B} \end{array}\right)$$

We can easily see that $(\overline{A},\overline{B})$ and $(\widetilde{A},\widetilde{B})$ are matrix representations of $(\overline{f},\overline{\pi})$ and $(\widetilde{f},\widetilde{\pi})$ respectively.

Now, we define a decomposition of $\mathrm{Inv}_d(A,B)$ according to the Brunovsky indices of $(\overline{f},\overline{\pi})$.

Definition 2.3 *Let (f,π) be the pair of linear maps defined by $f(x,y) = Ax + By$ and $\pi(x,y) = x$. Let $(\overline{A},\overline{B})$ be with $\overline{A} \in K^{d\times d}$ and $\overline{B} \in K^{d\times r}$ a Brunovsky pair. We define*

$$\mathrm{Inv}_{(\overline{A},\overline{B})}(A,B) = \{\mathcal{S} \in \mathrm{Inv}_d(A,B) | (\overline{f},\overline{\pi}) \text{ has } (\overline{A},\overline{B}) \text{ as matrix representation}\}$$

$$\mathrm{Inv}_d(A,B)=\bigcup_{(\overline{A},\overline{B})}\mathrm{Inv}_{(\overline{A},\overline{B})}(A,B)$$

Our aim is to study the geometrical structure of $\mathrm{Inv}_{(\overline{A},\overline{B})}(A,B)$. We recall that in [1] conditions in order to ensure that $\mathrm{Inv}_{(\overline{A},\overline{B})}(A,B)\neq\emptyset$ are given.

Remark 2.4 In contrast with the restriction defined in the dual case if (A,B) is a controllable pair, the pair $(\overline{A},\overline{B})$ do not need to be controlable as the following example shows

$$A=\left(\begin{array}{cc|cc}\lambda & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{array}\right)\qquad (\overline{A},\overline{B})=(\lambda,0)\quad (\widetilde{A},\widetilde{B})=(0,1).$$

Therefore the above union is in general infinite. However, this decomposition induce a finite stratification of the set of controlability subspaces as it is shown in [10].

3 Orbit space structure of $\mathrm{Inv}_{(\overline{A},\overline{B})}(A,B)$

In this section we prove that $\mathrm{Inv}_{(\overline{A},\overline{B})}(A,B)$ is a smooth manifold. To this end we describe $\mathrm{Inv}_{(\overline{A},\overline{B})}(A,B)$ as an orbit space following a similar pattern of [4] and generalizing results of [5].

We assume without loos of generality that the pairs (A,B) and $(\overline{A},\overline{B})$ are in the Brunovsky canonical form. More precisely, $A=\mathrm{diag}\{N,J\}$ and $B=\mathrm{diag}\{E,0\}$ with $N=\mathrm{diag}\{N_{k_1},\dots,N_{k_r}\}$ being N_i the standard lower nilpotent $i\times i$ matrix, $E=\mathrm{diag}\{E_{k_1},\dots,E_{k_r}\}$ being $E_i=(1,0,\dots,0)^t\in K^i$ and J a Jordan matrix. Analogously, $\overline{A}=\mathrm{diag}\{\overline{N},\overline{J}\}$ and $\overline{B}=\mathrm{diag}\{\overline{E},0\}$ with $\overline{N}=\mathrm{diag}\{N_{h_1},\dots,N_{h_s}\}$, $\overline{E}=\mathrm{diag}\{E_{h_1},\dots,E_{h_s}\}$ and $\overline{J}$ a Jordan matrix.

The next results depend on the following lemma

Lemma 3.1 *Let (A,B) and $(\overline{A},\overline{B})$ as above and $X\in K^{n\times d}$. Then, the following conditions are equivalent.*

1. *There exist matrices Y and Z such that* $(A\,B)\begin{pmatrix}X & 0\\ Y & Z\end{pmatrix}=X(\overline{A}\,\overline{B})$

2. $\begin{cases}BB^tX\overline{B} & =X\overline{B}\\ BB^tX\overline{A} & =X\overline{A}-AX\end{cases}$

$$\begin{cases} EE^tX_1\overline{E} & = X_1\overline{E} \\ EE^tX_1\overline{N} & = X_1\overline{N} - NX_1 \\ EE^tX_2\overline{J} & = X_2\overline{J} - NX_2 \\ X_3\overline{J} & = JX_3 \end{cases}$$

From (3) of the above lemma we can derive the form of the blocks X_1, X_2, X_3. In fact, the form of the block X_3 is well known. So, we restrict ourselves to describe X_1 and X_2.

Proposition 3.2 *With the above notation we have*

1. $X_1 = (X_1^{i,j})_{1\leq i\leq r,\, 1\leq j\leq s}$ *with*

$$X_1^{i,j} = \begin{pmatrix} x_{i,j}^1 & \cdots & x_{i,j}^{h_j-k_i+1} & 0 & 0 & \cdots & 0 \\ 0 & x_{i,j}^1 & \cdots & x_{i,j}^{h_j-k_i+1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & x_{i,j}^1 & \cdots & x_{i,j}^{h_j-k_i+1} \end{pmatrix} \quad \textit{if } k_i \leq h_j \textit{ and 0 otherwise}$$

2. $X_2 = (X_2^{i,j})_{1\leq i\leq r,\, 1\leq j\leq l}$ *with*

$$X_2^{i,j} = \begin{pmatrix} y_{i,j}^1 & y_{i,j}^2 & \cdots & y_{i,j}^{e_j} \\ \lambda_j y_{i,j}^1 + y_{i,j}^2 & \lambda_j y_{i,j}^2 + y_{i,j}^3 & \cdots & \lambda_j y_{i,j}^{e_j} \\ \lambda_j^2 y_{i,j}^1 + 2\lambda_j y_{i,j}^2 + y_{i,j}^3 & \lambda_j^2 y_{i,j}^1 + 2\lambda_j y_{i,j}^3 + y_{i,j}^4 & \cdots & \lambda_j^2 y_{i,j}^{e_j} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

being λ_j the eigenvalues of $\overline{J}$ and e_j the size of the corresponding Jordan block.

As a consequence of 3.1 (3) and 3.2 (1), we obtain

Lemma 3.3 *With the notation of the previous lemma, X has full rank implies that $B^tX\overline{B}$ has full rank.*

The following theorem characterizes the elements of $\text{Inv}_{(\overline{A},\overline{B})}(A,B)$.

Theorem 3.4 $\mathcal{S} \in \text{Inv}_{(\overline{A},\overline{B})}(A,B)$ *if and only if* $\mathcal{S} = \text{Im}X$ *with X a full rang matrix satisfying*

$$\begin{cases} BB^tX\overline{B} & = X\overline{B} \\ BB^tX\overline{A} & = X\overline{A} - XA \end{cases}$$

Definition 3.5 *Given* $(A,B)\in K^{n\times n}\times K^{n\times m}$ *and* $(\overline{A},\overline{B})\in K^{d\times d}\times K^{d\times l}$ *Brunovsky pairs of matrices, we define*

$$\begin{aligned}\mathcal{M}_{(\overline{A},\overline{B})}(A,B) &= \{X\in K^{n\times d}\,|\,BB^tX\overline{B}=X\overline{B} \text{ and } BB^tX\overline{A}=X\overline{A}-AX \\ &\qquad \text{having } X \text{ full rank.}\}\\ \mathcal{G}_{(\overline{A},\overline{B})} &= \mathcal{M}_{(\overline{A},\overline{B})}(\overline{A},\overline{B})\end{aligned}$$

If no confusion is possible, we denote $\mathcal{M}_{(\overline{A},\overline{B})}(A,B)$ *and* $\mathcal{G}_{(\overline{A},\overline{B})}$ *by* $\mathcal{M}$ *and* $\mathcal{G}$*, respectively.*

We remark that $\mathcal{M}$ is a submanifold of $K^{n\times d}$. In fact, an open subset of a linear subvariety of $K^{n\times d}$. We have the following theorem

Theorem 3.6 *With the above notation we have*

1. $\mathcal{G}$ *is a Lie subgroup of* $Gl(d)$
2. $\mathcal{G}$ *acts freely on* $\mathcal{M}$ *on the right by matrix multiplication.*

Proof We begin by proving that if $T\in\mathcal{G}$ and $X\in\mathcal{M}$ then $XT\in\mathcal{M}$. In fact, $BB^tXT\overline{B}=BB^tX\overline{B}\,\overline{B}^t\overline{B}=X\overline{B}\,\overline{B}^tT\overline{B}=XT\overline{B}$.

For the second equation, $BB^tXT\overline{A}=BB^tX\overline{B}\,\overline{B}^tT\overline{A}+BB^tX\overline{A}T$. But, $BB^tX\overline{A}T=X\overline{A}T-AXT$ and $BB^tX\overline{B}\,\overline{B}^tT\overline{A}=X\overline{B}\,\overline{B}^tT\overline{A}=X\overline{B}\,\overline{B}^t\overline{B}\,\overline{B}^tT\overline{A}+X\overline{B}\,\overline{B}^t\overline{A}T=X\overline{B}\,\overline{B}^tT\overline{A}=XT\overline{A}-X\overline{A}T$ so that $\overline{B}\,\overline{B}^tXT\overline{A}=XT\overline{A}-AXT$.

Then, if we take $X=T'\in\mathcal{G}$, we have that $T,T'\in\mathcal{G}$ implies $TT'\in\mathcal{G}$. It is easily checked that $I\in\mathcal{G}$. Let us see that $T\in\mathcal{G}$ implies $T^{-1}\in\mathcal{G}$. We first check that $\overline{B}^tT\overline{B}$ is invertible and $(\overline{B}^tT\overline{B})^{-1}=\overline{B}^tT^{-1}\overline{B}$. In fact, we have that $(\overline{B}^tT^{-1}\overline{B})(\overline{B}^tT\overline{B})=\overline{B}^tT^{-1}T\overline{B}=\overline{B}^t\overline{B}=I$. Applying this we have that $\overline{B}\,\overline{B}^tT\overline{B}=T\overline{B}$ implies that $\overline{B}=T\overline{B}\,\overline{B}^tT^{-1}\overline{B}$ and therefore $T^{-1}\overline{B}=\overline{B}\,\overline{B}^tT^{-1}\overline{B}$.

For the second equation, from $\overline{B}\,\overline{B}^tT\overline{A}=T\overline{A}-\overline{A}T$ we obtain $T^{-1}\overline{B}\,\overline{B}^tT\overline{A}T^{-1}=\overline{A}T^{-1}-T^{-1}\overline{A}$ and taking into account that T^{-1} satisfies the first equation, $\overline{B}\,\overline{B}^tT^{-1}\overline{B}\,\overline{B}^tT\overline{A}T^{-1}=\overline{A}T^{-1}-T^{-1}\overline{A}$. Now, applying the first equation for T, it turns that $-\overline{B}\,\overline{B}^tT^{-1}\overline{A}=\overline{A}T^{-1}-T^{-1}\overline{A}$.

This prove that $\mathcal{G}$ is a group acting on the right on $\mathcal{M}$.

$XT = X$, since X has maximal rank, it follows that $T = I$. ■

Denote by $\mathcal{M}/\mathcal{G}$ the set of orbits of $\mathcal{M}$ under the action of $\mathcal{G}$. Then, the proof of the following result is analogous to that of (4.5) in [4].

Proposition 3.7 *The orbit space $\mathcal{M}/\mathcal{G}$ has a differentiable structure such that the projection $\mathcal{M} \to \mathcal{M}/\mathcal{G}$ is a submersion.*

We now state the main theorem of this section

Theorem 3.8

1. *The map $\phi : \mathcal{M} \to Gr_d(K^n)$ defined by $X \mapsto \mathrm{Im}X$ induce a injection $\tilde{\phi} : \mathcal{M}/\mathcal{G} \to Gr_d(K^n)$ having image $\mathrm{Inv}_{(\overline{A},\overline{B})}(A, B)$.*
2. *With the differentiable structure given through the above bijection, $\mathrm{Inv}_{(\overline{A},\overline{B})}(A, B)$ is a submanifold of $Gr_d(K^n)$ of dimension $\dim \mathcal{M} - \dim \mathcal{G}$.*

4 Application to the controllability subspaces

We recall that a subspace $\mathcal{S}$ of K^n is a *controllability subspace* if $\mathcal{S}$ has the form (following the notation of [11])

$$\mathcal{S} = < A + BF|\mathrm{Im}BG >$$

with $F \in K^{n\times m}$ and $G \in K^{m\times l}$ ($< A+BF|\mathrm{Im}BG >= \mathrm{Im}BG+\mathrm{Im}(A+BF)BG+ \cdots + \mathrm{Im}(A + BF)^{n-1}BG$. Since $\mathcal{S}$ is $(A + BF)$-invariant, $\mathcal{S}$ is a (A, B)-invariant subspace. Therefore, the set of controllability subspaces is a subset of the set of (A, B)-invariant subspaces.

In [10] is proved the following theorem.

Theorem 4.1 *With the above notation, $\mathcal{S}$ is a controllability subspace with regard to (A, B) if and only the restriction of (A, B) to $\mathcal{S}$ (the pair $(\overline{f}, \overline{\pi})$) is controllable.*

As a corollary one has that the stratifiaction of the set of (A, B)-invariant subspaces introduced in the last section induces a stratification of the set of controllability subspaces. More precisely one has the following theorem.

Theorem 4.2 *The sets $\mathrm{Inv}_{(\overline{A},\overline{B})}(A, B)$ with $(\overline{A}, \overline{B})$ controllable are a finite stratification of the set of controllability subspaces of (A, B).*

[1] I.Baragaña-I.Zaballa, Block similarity of restrictions to $A,B)$-invariant subspaces *Lin. Alg. Appl., vol 220,31-62* (1995)

[2] P.A. Fuhrmann and U. Helmke, A homeomorphism between observable pairs and (A,B)-invariant subspaces, *Sys. and Contr. Letts., 30, 217-223* (1997)

[3] P.A. Fuhrmann and U. Helmke, On the parametrization of (A,B)-invariant subspaces, Preprint.

[4] J. Ferrer, F. Puerta and X. Puerta, Differentiable structure of the set of controllable $(A,B)^t$-invariant subspaces, *Lin. Alg. Appl., vol 275-276, 161-177.* (1998)

[5] J. Ferrer, F. Puerta and X. Puerta, Stratification of the set of general $(A,B)^t$-invariant subspaces, preprint.

[6] D. Hinrichsen, H.F. Münzer and D. Prätzel-Wolters, Parametrization of (C,A)-invariant subspaces, *Sys. and Contr. Letts., 1, 192-199.* (1981)

[7] I. Gohberg, P. Lancaster and L. Rodman. *Invariant Subspaces of Matrices with Applications*, Wiley, New York, 1986.

[8] X. Puerta and U.Helmke, The topology of the set of conditioned invariant subspaces *System and Control Letters, 40,97-105*(2000)

[9] F. Puerta, X. Puerta and I.Zaballa, A coordinate Atlas of the manifold of observable conditioned invariant subspaces. Preprint

[10] F. Puerta, X. Puerta, On the geometry of the set of controllability subspaces of a controllable pair (A,B), Preprint

[11] W.M. Wonham, *Linear Multivariable Control Theory: a Geometric Approach*, Prentice-Hall, (1974)