Output Maximal Dimension for the Disturbance Decoupling Problem

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Abstract

Given a linear time invariant system with a disturbance we describe a method of finding the maximal dimensional output for a generic Disturbance Decoupling Problem.

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1 Introduction

Given the control linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Qq(t) \\ y(t) &= Cx(t) \end{aligned}$$

the term $q(t)$ represents a disturbance which is assumed not to be directly measurable by the controller. Then, the *Disturbance Decoupling Problem* (DDP) consists in finding (if possible) a state feedback F, such that the disturbance $q(t)$ has no influence on the output. That is to say, F must be such that for any initial state the corresponding output $y(t)$ of the system

$$\begin{aligned} \dot{x}(t) &= (A + BF)x(t) + Bu(t) + Qq(t) \\ y(t) &= Cx(t) \end{aligned}$$

does not depends on $q(t)$.

A solution of this problem can be found for example in [5], Th.4.2, where the following result is proved: the DDP has a solution if and only if

$$\mathfrak{Y}(A, B; \text{Ker } C) \supset \text{Im } Q$$

where $\mathfrak{Y}(A,B;\mathrm{Ker}\,C)$ is the unique maximal (A,B)-invariant subspace contained in $\mathrm{Ker}\,C$. We recall that a subspace S is said to be (A,B)-invariant if $A(S)\subset S+\mathrm{Im}\,B$.

In this note we tackle the problem of showing a method of finding, for a generic Q the maximal dimension p of the output space insensitive to the disturbance $q(t)$. We will call briefly this problem "A maximal solution of the DDP". In [4] is showed through an example a method of solving this problem when the pair (A,B) is controllable. In this note we solve this problem when (A,B) is a general pair. For this we make use of the bundle structure of the set of conditioned invariant subspaces given in [2].

We fix the following notation. $\mathcal{M}_{p,q}$ will denote the set of complex matrices having p rows and q columns. We write $\mathcal{M}_p$ for $\mathcal{M}_{p,p}$. If $A\in\mathcal{M}_p$ we say that A is a p-matrix. $\mathcal{M}^*_{p,q}$ is the set of full rank matrices in $\mathcal{M}_{p,q}$.

If $A\in\mathcal{M}_{p,q}$, A^* means the conjugate transpose of A. I_n will be the identity matrix. If $A\in\mathcal{M}_{p,q}$, $[A]$ means the subspace generated by the columns of A.

If E is a vector space, $\mathrm{Gr}_d(E)$ is the Grassman manifold of d-dimensional subspaces of E.

A Brunovsky (dual) matrix, B-matrix in what follows, is a matrix of the form $\binom{N}{E}$ where $N=\mathrm{diag}\,\{N_0,N_\infty\}$, $E=\mathrm{diag}\,\{E_0,0\}$, being N_∞ a Jordan matrix and (E_0,N_0) a Brunovsky observable pair, that is to say, $N_0=\mathrm{diag}\,\{N_1,\ldots,N_r\}$ each N_i being the standard lower nilpotent k_i-matrix, $E_0=\mathrm{diag}\,\{E_1,\ldots,E_r\}$, each E_i being a k_i-row matrix, $E_i=(0,\ldots,0,1)$, $1\le i\le r$. The integers $k_1,\ldots,k_r$ are called the *observability indices* of $\binom{N}{E}$.

2 A maximal solution of the DDP

(2.1) Consider a linear system

$$\begin{aligned}\dot{x}(t) &= Ax(t)+Bu(t)+Qq(t)\\ y(t) &= Cx(t)\end{aligned}$$

where $A\in\mathcal{M}_n$, $B\in\mathcal{M}_{n,m}$, $Q\in\mathcal{M}_{n,q}$ and $C\in\mathcal{M}_{p,n}$.

With the notation in the introduction, our problem can be stated as follows: *Find, for a generic Q, the maximal dimension of Im C so that $\mathfrak{Y}(A,B;Ker\,C)\supset Im\,Q$.*

Taking into account that

$$\mathrm{Im}\,Q\subset\mathfrak{Y}(A,B;\mathrm{Ker}\,C)\subset\mathrm{Ker}\,C\,,$$

finding the maximal dimension for Im C is equivalent to obtain the minimal dimension for Ker C so that Ker C is an (A,B)-invariant subspace containing Im Q. In turns, this is equivalent to *finding the minimal dimension of Ker C such that $(Ker\,C)^\perp$ is a (B^*,A^*)-invariant subspace of maximal dimension contained in $(Im\,Q)^\perp$.* We recall that a subspace S is said to be (B^*,A^*)-conditioned invariant (or $\binom{A^*}{B^*}$-invariant as in [3]) if $S^\perp$ is (A,B)-invariant.

We are going to describe a method of obtaining this dimension. We will divide our procedure in several steps:

(i) Since the matrix Q is supposed to be *generic*, that is to say, belonging to an open and dense subset of $\mathcal{M}_{n,q}$ we can assume that Q is a full rank matrix. Then, through a convenient change of bases, Q takes the form

$$Q = \begin{pmatrix} -P^* \\ \\ I_q \end{pmatrix}$$

where $P \in \mathcal{M}_{q,n-q}$. Matrices A, B and C modify accordingly to this change of bases, but we will keep on with the same notation for these new matrices.

Then,

$$(\text{Im } Q)^\perp = \begin{bmatrix} I_{n-q} \\ \\ P \end{bmatrix}.$$

Call this subspace F.

(ii) Let $\binom{N}{E}$ be the Brunovsky (dual) form of the pair $\binom{A^*}{B^*}$ (see for example [3] (6.6.3)), $\binom{M}{F}$ a B-matrix and Inv $((B^*, A^*); (M, F))$ the set of $(B^*\ A^*)$-conditioned invariant subspaces W such that the restriction of $\binom{A^*}{B^*}$ to W has a fixed Brunovsky form $\binom{M}{F}$ (see [2] for the definition of this restriction).

We denote by $\underline{k} = (k_1 \geq k_2 \geq \ldots \geq k_r)$ the observability indices of $\binom{N}{E}$ and for every eigenvalue λ of N_∞, $\eta_1(\lambda) \geq \eta_2(\lambda) \geq \ldots$ is the corresponding Segre characteristic. Analogously, $\underline{h} = (h_1 \geq h_2 \geq \ldots \geq h_s)$ and $\varepsilon_1(\lambda) \geq \varepsilon_2(\lambda) \geq \ldots$ are respectively the observability indices of $\binom{M}{F}$ and the Segre characteristic of λ as eigenvalue of M_∞. We assume that $\binom{M}{F}$ is *compatible* with $\binom{N}{E}$, that is to say that Inv $((B^*, A^*); (M, F))$ is not empty. We recall that a B-matrix $\binom{M}{F}$ is compatible with $\binom{N}{E}$ if and only if the following conditions hold (see for example [1]):

(a) $s < r$, and $h_i \leq k_i$ for $i = 0, 1, 2, \ldots$

(b) the eigenvalues of M_∞ are also eigenvalues of N_∞, and for each one the corresponging Segre characteristics $(\eta_1(\lambda), \eta_2(\lambda), \ldots)$ and $(\varepsilon_1(\lambda), \varepsilon_2(\lambda), \ldots)$ verify: $\eta_i(\lambda) \leq \varepsilon_i(\lambda)$, for $i = 1, 2, \ldots$.

In [2] (4.6) (see also [4] (2.9)) the following formula is obtained:

$$\begin{aligned} \dim((B^*, A^*); (M, F)) &= \sum_{i=1}^{h}\sum_{j=1}^{k} m_i(r_{i+j-1} - s_{i+j-1}) + \\ &+ \sum_{\lambda}\sum_{i} e_i(\lambda)(c_i(\lambda) - e_i(\lambda)) + (k_\infty - h_\infty)s \end{aligned}$$

where, $m_i = s_i - s_{i+1}$, $\underline{r} = (r_1 \geq \ldots \geq r_k)$ is the conjugate partition of $\underline{k} = (k_1 \geq \ldots \geq k_r)$, $\underline{s} = (s \geq \ldots \geq s_h)$ is the conjugate partition of $\underline{h} = (h_1 \geq \ldots \geq h_s)$, λ runs over the eigenvalues of M_∞, $c_1(\lambda) \geq c_2(\lambda) \geq \ldots$, $e_1(\lambda) \geq e_2(\lambda) \geq \ldots$ are the conjugate partitions of the Segre characteristics $\eta_1(\lambda) \geq \eta_2(\lambda) \geq \ldots$, $\varepsilon_1(\lambda) \geq \varepsilon_2(\lambda) \geq \ldots$ of λ in N_∞ and M_∞, respectively, and k_∞, h_∞ are the size of N_∞ and M_∞, respectively. Set

$$\dim((B^*, A^*); (M, F)) = \delta .$$

(iii) Now, we have to find, for the Q given in (i), the maximal dimension d so that there exists a d-dimensional $(B^* \ A^*)$-conditioned invariant subspace contained in F. This leads to consider the intersection

$$\mathrm{Gr}_d(F) \cap \mathrm{Inv}\ ((B^*, A^*); (M, F))$$

for all possible $\binom{M}{F}$ compatibles with $\binom{N}{E}$ such that $M \in \mathcal{M}_d$.

We know by Thom Transversality theorem that if

$$\dim \mathrm{Gr}_d(F) + \delta < \dim \mathrm{Gr}_d(\mathbb{C}^n)$$

the intersection $\mathrm{Gr}_d(F) \cap \mathrm{Inv}\ ((B^*, A^*); (M, F))$ will be empty for Q generic. So, we have to look for the greatest d such that

$$d(n-q-d) + \delta \geq d(n-d)$$

for some compatible $\binom{M}{F}$, or equivalently

$$\delta \geq dq\,.$$

Then, we have the following result

Theorem 1 *With the above notation, the maximal dimension of the (B^*, A^*)-conditioned invariant subspaces contained in $(Im\ Q)^\perp$ is the greatest d such that $\delta \geq dq$ and the set of such subspaces is a manifold of dimension $\delta - dq$. Besides of this, the Brunovsky reduced form of the restriction of $(B^* \ A^*)$ to these subspaces is of the same type of $\binom{M}{F}$ (that is to say, they have the same observability indices and for each eigenvalue of the Jordan part the Segre characteristic is the same).*

Example 2 We are going to apply this theorem to the following situation: $\underline{k} = (5, 2)$, so that $\underline{r} = (2\ 2\ 1\ 1\ 1)$, $c = 3$ and $q = 2$. We begin for the possible maximal dimension: $d = 8$. In this case the only possible cases are:

$$(2\ 2\ 1\ 1\ 1),\ 1\,; \quad (2\ 2\ 1\ 1),\ 2\,; \quad \text{and} \quad (2\ 2),\ 3\,,$$

for $\underline{s}$ and e, respectively, and it easily checked that in any case is $\delta < 15$. Continuing this way we see that for $d = 4$ and $s = (2, 2)$, $e = 0$ one has that $\delta = 8$. Hence, in this example the maximal solution of the DDP is obtained for $d = 4$. Notice that if $s = (2, 1)$, $e = 1$, then $\delta = 10$. That is to say, the maximal dimension is unique but not the Brunovsky form of the restriction to the corresponding invariant subspaces.

(2.2) It is also possible to give a method for obtaining a parametrization of the manifold

$$\mathrm{Gr}_d(F) \cap \mathrm{Inv}\ ((B^*, A^*); (M, F))\,.$$

However, only when the pair (A, B) is controllable, or equivalently $\binom{A^*}{B^*}$ is observable, the procedure can be given in an explicit way. So we limit ourselves to consider this case. We will use the explicit description of a coordinate system given in [4] (3.3). There it is shown that, with the notation in (2.1), for any set of positive integers pairwise different $\{n_{ij},\ 1 \le i \le h,\ i \le j \le m_{h-i+1}\}$ such that for $i = 1, 2, \ldots, h$

$$1 \le n_{i1} < n_{i2} < \ldots < n_{im_{h-i+1}} \le r_{h-i+1}$$

if $Y = (Y_{ij}),\ 1 \le i \le k,\ 1 \le j \le h$ is a matrix verifying the conditions below, then the free parameters in Y define a coordinate system in Inv $((B^*, A^*); (M, F))$. The conditions are the following:

1. $Y_{ij} \in \mathcal{M}_{r_i, s_j}$.

2. $Y_{ij} = 0$ if $i < j$.

3. For $i = 1, 2, \ldots, h$, Y_{ii} is partitioned into blocks $Y_{ii} = (L^1_{i\beta}),\ 1 \le \beta \le h - i + 1$ in such a way that for $\beta = 1, 2, \ldots, h - i + 1$, $L^1_{i\beta} \in \mathcal{M}_{r_i, m_{h-\beta+1}}$ is a matrix whose last $r_i - r_{h-\beta+1}$ rows are zero, the rows $n_{ij},\ 1 \le i \le \beta - 1,\ 1 \le j \le m_{h-i+1}$ are also zero, and the rows $n_{\beta 1}, n_{\beta 2}, \ldots, n_{\beta m_{h-\beta+1}}$ are unit vectors $e^\beta_1, e^\beta_2, \ldots, e^\beta_{m_{h-\beta+1}}$:

$$e^\beta_j = (0, \ldots, 0, \overset{(j}{1}, 0, \ldots, 0) \in (\mathbb{F})^{m_{h-\beta+1}}.$$

4. For $i > j$, Y_{ij} is partitioned into blocks $Y_{ij} = (L^{i-j+1}_{i\beta}),\ 1 \le \beta \le h - j + 1$ in such a way that $L^{i-j+1}_{i\beta} \in \mathcal{M}_{r_i, m_{h-\beta+1}}$ is a matrix whose last $r_i - r_{h-\beta+i-j+1}$ rows are zero and for $\beta \ge i - j + 1$, the rows $n_{pq},\ 1 \le p \le \beta - i + j,\ 1 \le q \le m_{h-p+1}$ are also zero. (Notice that Y_{ij} follows from $Y_{i-j+1,1}$).

If Y is such a matrix, the coordinate chart is simply given by the mapping

$$\psi : \mathbb{C}^\delta \longrightarrow \text{Inv } ((B^*, A^*); (M, F))$$

defined by ψ (free parameters in Y) $= [Y]$.

Then, taking into account that $F = \left[\begin{smallmatrix} I_{n-q} \\ P \end{smallmatrix}\right]$, it turns out that a subspace S belongs to $\mathrm{Gr}_d(F) \cap$ Inv $((B^*, A^*); (M, F))$, where we recall that d and (M, F) have been obtained in (iii), if and only if there exists $X \in \mathcal{M}^*_{n-q,d}$ such that

$$Y = \begin{pmatrix} I_{n-q} \\ P \end{pmatrix} X.$$

So, if we write $Y = \left(\begin{smallmatrix} Y_1 \\ Y_2 \end{smallmatrix}\right)$ where $Y_1 \in \mathcal{M}_{n-q,d}$, then $X = Y_1$ and

$$Y_2 - PY = 0.$$

This is a linear system of $q \cdot d$ equations with δ unknowns. Hence, for Q generic the solutions depend on $(\delta - q) \cdot \delta$ free parameters, which is just the dimension of the manifold $\mathrm{Gr}_d(F) \cap$ Inv $((B^*, A^*); (M, F))$, so we have proved the following result:

Proposition 3 *With the above notation, the mapping which assigns to each set of $(\delta - q) \cdot d$ free parameters in the linear system $Y_2 - PY_1 = 0$ the subspace $[Y]$, where $Y = \left(\begin{smallmatrix} Y_1 \\ Y_2 \end{smallmatrix}\right)$ is the corresponding solution of this system, defines a coordinate chart in $Gr_d(F) \cap$ Inv $((B^*, A^*); (M, F))$.*

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